Linear Groups of Degree \( n \) Containing an Involution with two Eigenvalues \(-1\), II

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1. Introduction

In [12], quasiprimitive linear groups \( G \) containing a matrix with exactly two eigenvalues \(-1\), the rest 1 were shown to have a certain property. Either \( G \) was one of a specific list of groups or the product of any two of these matrices with exactly two eigenvalues \(-1\) had order 1, 2, 3, 4, or 5. If the order was 4 the square was either in a normal 2-group of \( G \) or it itself had exactly two eigenvalues \(-1\). This means, using [1, 20], that the group generated by these matrices with two eigenvalues \(-1\) has known composition factors. In this paper these groups are listed explicitly in the following theorem.

**Main Theorem.** Suppose \( G \) is a finite quasiprimitive linear group of degree \( n \geq 6 \) and \( X \) is the corresponding representation. Suppose further that \( G \) contains an involution \( \tau \) for which \( X(\tau) \) has two eigenvalues \(-1\), the rest 1. Then \( G \) can be described as one of the following groups. Here \( Z \) is the center of \( G \) and \( \cdot \) denotes central product [7].

(A) \( G/Z = G_1 \) where \( G_1 \cong A_{n-1} \) or \( S_{n+1} \). The element \( \tau \) is a double transposition such as (12)(34). If \( G_1 \cong A_{n-1} \), \( G = Z \times G_1 \) unless \( n = 6 \).

(B) \( G/Z_1 = G_1 \) where \( G_1 \) is the Weyl Group \( W_n \) of \( E_n \) for \( n = 6, 7, 8 \) or \( G_1 \) is a subgroup of index 2 in \( W_n \) and \( Z_1 \subseteq Z \).

Here \( \tau \) is a product of two reflections in \( W_n - W'_n \). For \( n = 6 \) and 8 there is a class of involutions in \( W_n \) negating a subspace of codimension 2.

Note \( W_6 \) is a proper central extension over a center of order 2 of \( \text{PSO}(8, 2) \), \( W_7 \cong Z_2 \times \text{Sp}(6, 2) \) and \( W_8 \cong \text{PSO}^-(6, 2) \).

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(C) \( G/Z = A \times K \) where \( K \) is a linear group generated projectively by reflections and \( A \cong A_4, S_4, \) or \( A_5. \) Here \( X(G) \) is a subgroup of \( Y \otimes Y_1 \) where \( Y \) is a projective representation of degree 2 and \( Y_1 \) is a quasiprimitive projective representation of \( K \) of degree \( n/2 \) containing reflections. Such groups are listed in [19]. Conjugates of \( \tau \) are in \( K \mod Z \) and are the reflections of \( Y_1(K). \)

(D) \( G/Z = G_1 \) where \( G_1 \cong \text{PSL}(2,7) \) or an extension to \( \text{PGL}(2,7). \) The conjugates of \( \tau \) are in \( \text{PSL}(2,7). \) Here \( n = 6. \) If \( G_1 \cong \text{PSL}(2,7), \) \( G = G_1 \times Z. \)

(E) \( G/Z = G_1 \) with \( G_1 \cong U_3(3) \) or an extension to \( G_3(2). \) The conjugates of \( \tau \) are in \( U_3(3). \) Here there is a unique character of \( U_3(3) \) of degree 6 with Schur Index 2 over the rationals and two irrational characters of degree 7 over \( \mathbb{Q}(i). \) The representations of degree 7 do not extend to \( G_3(2). \) If \( G_1 \cong U_3(3), \) \( G = G_1 \times Z. \)

(F) \( G = G_1 \circ Z \) where \( G_1 \cong \hat{J}_2. \) There are two real projective characters of \( \hat{J}_2 \) with Schur Index 2 over \( \mathbb{Q} \sqrt[3]{5} \) of degree 6. The center has order 2. The conjugates of \( \tau \) are in the center of a Sylow 2-group of \( \hat{J}_2. \) Note \( \hat{J}_2 \) is the Hall–Janko Group of order 604800 and \( \hat{J}_2 \) is a proper double cover.

(G) \( G/Z = G_1 \) with \( G_1 \cong \text{PSL}(3,4) \) or an extension by an automorphism of order 2 which is the product of a field automorphism and a graph automorphism. The conjugates of \( \tau \) are in \( \text{PSL}(3,4). \) There are two projective 6-dimensional representations over a center of order 6 which can be realized in \( \mathbb{Q}(\omega). \) Here \( \omega = e^{2\pi i/3}. \)

(H) \( G/Z = G_1 \) with \( G_1 \cong \text{PSU}(4,3) \) or an extension by a field automorphism. The conjugates of \( \tau \) are in \( \text{PSU}(4,3) \) and are products of two reflections in the extended group. Again there are two representations of dimension 6 realized over \( \mathbb{Q}(\omega) \) with a center of order 6, \( \omega = e^{2\pi i/3}. \)

(I) \( G/Z_1 = G_1 \) where \( G_1 \cong \hat{A}_6 \) or an extension of degree 2 which is the product of a graph and a diagonal automorphism. Here \( n = 6, \) \( G_1 \) has a center of order 3 and \( Z_1 \subseteq Z. \)

(J) \( G \) contains a subgroup \( G_1 \) with \( G = G_1Z, \) \( G_1 \supset H \) where \( H \cong Q_8 \circ D_8 \circ D_8, \) \( D_8 \circ D_8 \circ D_8, \) or \( D_8 \circ D_8 \circ D_8 \circ Z_4. \) Also \( G_1/H \) is isomorphic to a subgroup of \( O^+(6,2) \cong S_6, O^-(6,2), \) or \( S_6(6,2) \) in the respective cases and \( X \mid H \) is irreducible of degree 8.

(K) \( n = 8 \) and \( G \) contains a normal subgroup \( H \cong \text{SL}(2,5) \circ \text{SL}(2,5) \circ \text{SL}(2,5), \) \( G/Z \cong \{A_5 \times A_5 \times A_5\} \) \( \cong S_6. \) Conjugates of \( \tau \) interchange two \( A_5 \)'s and \( X \mid H \cong Y \otimes Y \otimes Y \) where \( Y \) is a 2-dimensional representation of \( \text{SL}(2,5). \)

(L) \( G = G_1 \circ Z \) with \( G_1 \cong \text{PSU}(5,2). \) This is a rational 10-dimensional representation with Schur Index 2 over \( \mathbb{Q} \) of \( G_1. \)

(M) \( G = G_1 \circ Z \) with \( G_1 \) a proper central extension of \( A_7 \) with a center
of order 3. Here \( n = 6 \) and \( X | A_6 \) is the group \( (I) \). The conjugates of \( \tau \) are the involutions. Products of involutions have order 1, 2, 3, 4 or 6. This does not contradict [12] as \( n = 6 \).

Remark. The groups in (A), (B), (H) contain an element of order 3 which fixes a subspace of codimension 2; the remaining groups do not.

The results of [12] enable us to conclude immediately that \( G \) is (A), (B), (H), or (M) or satisfies the conditions of [1, 20]. The work of this paper determines which groups satisfying [1, 20] have a representation with the appropriate property.

It should be noted that all quasiprimitive linear groups with natural representation \( X \) of degree smaller than six represent noncentral involutions \( \tau \) with \( X(\tau) \) having an eigenspace of codimension 1 or 2.

We suppose that \( G \) is a quasiprimitive linear group with \( X \) the natural representation. We assume \( G \) is a counterexample to the theorem. Assume \( \tau \) is an involution with \( X(\tau) \) containing exactly two eigenvalues \(-1\). Such an element will be called a special 2-element. Let \( S(G) \) be the maximal normal solvable subgroup of \( G \). Let \( G = G/S(G) \). We suppose that \( X(\gamma) \) cannot fix a subspace of codimension 2 for any 3-element \( \gamma \) by [8]. As \( G \) is not (A), (B), or (H) we know from [12] that in \( G \), the product of any two special involutions has order 1, 2, 3, 4, or 5. If the order is 4, the square is conjugate in \( G \) to a special involution. We also assume \( \deg X = n \geq 6 \). As quasiprimitive linear groups of degree \( m \) are known for \( m \leq 8 \) we may read the result from [14, 21, 11, 4] if \( n = 6, 7, \) or 8. The quasiprimitive groups of degree 9 are also known but this fact does not help. Note also that \( X(\sigma) \) could not have one eigenvalue \(-1\), the rest ones as all such groups have been determined [19]. Such an element is called a reflection. These groups all contain a 3-element \( \gamma \) such that \( X(\gamma) \) has a fixed space of codimension 2.

2. Reduction to Quasisimple Case

We first suppose \( S(G) \neq Z(G) \). Let \( F(G) \) be the Fitting subgroup of \( G \) [7]. Clearly \( F(G) \) is the Fitting subgroup of \( S(G) \) also. By properties of the Fitting subgroup, \( F(G) \not\subseteq Z(G) \) if \( S(G) \not\subseteq Z(G) \). We show \( F(G)/Z(G) \) is a 2-group and even \( n = 8 \) providing a contradiction as \( G \) is not \((f)\) of the theorem.

Note \( F(G) \) has no noncyclic characteristic abelian subgroups by the quasiprimitivitiy. Suppose \( P \) is a Sylow \( p \)-group of \( F(G) \) which is not abelian. As \( P \) has no characteristic abelian noncyclic subgroups its structure is determined in [7, Theorem 5.4.9]. Let \( P = P_1 \circ P_2 \) where \( P_1 \) is the extra special part as in [7].

Note \( P \) has the characteristic subgroup \( P_1 \) or \( P_1 \circ Z_{p^2} \). Call this group \( Q \). Let \( H = \langle Q, \tau \rangle \) where \( \tau \) is a special 2-element which does not centralize \( Q \).
Note if $K$ is the group generated by all conjugates of a given special 2-element and $X \mid K$ is reducible the argument of [10, Theorem 3] shows $G$ satisfies $C$ of the theorem. We can therefore assume no conjugate of $\tau$ does centralize $Q$ as then $Q$ would be central by Schur's lemma.

Note [7, 5.5.5] that $Q$ has a unique faithful irreducible character of degree $p^m$ where $|P_1| = p^{2m+1}$ once the character on the center has been determined. This extends to two characters $\chi_1, \chi_2$ on $\langle Q, \tau \rangle$. Suppose that in the corresponding representations $Y_i$, $i = 1$ or 2, $Y_i(\tau)$ fixes a subspace of codimension 1 or 2. Then $\chi_i(\tau) = \pm(p^m - 4) \pm(p^m - 2)$. There are $2(p - 1)p^5$ such characters where $\delta = 1$ if $|Z(Q)| = p^2$, $\delta = 0$ if $|Z(Q)| = p$. The character tables for such groups can be easily worked out and the only possibilities are $p = 3$ or 5 with $m = 1$ or $p = 2$ with $m = 1, 2, 3$. It follows equally now that $X \mid Q$ is irreducible or a sum of two irreducible constituents if $\langle Q, \tau \rangle$ has a faithful representation with $\tau$ a reflection. We view $n \leq 8$ and the only groups are those in (J). Note with $n = 8$, $X \mid Q$ is irreducible and so $C(Q) = Z(G)$. In particular $G/Z(G)$ is isomorphic to a subgroup of $\text{Aut}(Q)$.

We can suppose $Z = S(G)$. Let $E(G)$ be the product of all subnormal quasisimple subgroups. Then $E(G)Z$ is the generalized Fitting subgroup in this case. By general properties of this it follows that $C(E(G)Z) \leq Z$ and so $E(G) \neq 1$. Also $E(G) = E_1 \circ E_2 \circ \cdots \circ E_m$ where each $E_i$ is quasisimple. We will show that $m = 1$. Recall a group $H$ is quasisimple if $H'$ is simple. As $E(G)$ is a product of quasisimple groups, we can take $H = E(G)$.

It follows easily from [7-3.4.1, 4.4.5] that $X \mid H = \epsilon \overline{Y}$ where $\overline{Y}$ is a representation of $H$. As $\tau$ is an involution, acting by conjugation it either interchanges or fixes the various $E_i$'s. If it interchanges two $E_i$'s let $K$ be the product of these two. Otherwise let $K$ be any $E_i$ which $\tau$ does not centralize. As $C(E(G)) = Z$, such an $i$ exists. Let $L$ be the product of the $E_j$'s not involved in $K$. Now $E(G) = K \circ L$ and $Y = T_1 \otimes T_2$ where $T_1$ is a representation of $K$ and $T_2$ is a representation of $L$. Note $Y \mid K = (\text{degree } T_2)\hat{T}_1$ is the representation of $\langle K, \tau \rangle$. Recall $X(\tau)$ is special.

If degree $T_2 \geq 2$ it is immediate that degree $T_2$ is 2, $e$ is 1, and $\hat{T}_1(\tau)$ is a reflection. If $K$ is not quasisimple it is immediate from [19] that $K \cong SL(2, 5) \rtimes SL(2, 5)$, $n = 8$ and we get case (K) of the theorem. We can now assume $K$ is quasisimple. As degree $T_2$ is 2, $L$ is quasisimple, and $E(G) = E_1 \circ E_2$. If $\tau$ does not centralize $E_2$, the same argument shows $n = 4$ a contradiction. Also as $n \geq 8$, all conjugates of $\tau$ centralize $E_2$ and we are in case $C$.

We have now shown either $E(G) = E_1$ or $E(G) = E_1 \circ E_2$ and all conjugates of $\tau$ interchange $E_1$ and $E_2$. It is now immediate from [1, 20] in the latter
case that \( \langle E(G), \tau \rangle \) mod its center is \((A_6 \times A_6) \wr Z_2\). As this is generated by four conjugates of \( \tau \) it is clear that \( n \leq 8 \) and we are done.

We may now assume \( E(G) \) is quasisimple. If \( X \mid E(G) \) is reducible, \( X \mid \langle E(G), \tau \rangle \) is reducible and each constituent is generated by reflections. The groups are known by [19] and none have outer automorphisms beyond the reflections except \( S_6 \). This does not extend to a quasiprimitive group in twice the dimension. In particular we may assume \( G \supseteq G_1 = \langle E(G), \tau \rangle \). We find all possible \( G_1 \) and representations of \( G_1 \) with special involutions. We will know that \( G/Z(G) \) is a subgroup of \( \text{Aut} \, G_1/Z(G_1) \) and will be able to describe \( G \) as in the theorem. Note \( X \mid G_1 \) is clearly quasiprimitive as \( X \mid E(G_1) \) is irreducible.

3. \( G' \) Quasisimple

In this section we assume \( G = \langle G', \tau \rangle \) where \( G' \) is quasisimple. In particular \( G' \) is of index 1 or 2 in \( G \). Now the results of [1, 20] show \( G = G/Z(G) \) is one of the groups listed in Table I.

Only a very few of these have representations with an involution represented appropriately. The idea for showing this is to show that a particular Chevalley Group say \( G_n(q) \) is generated by some reasonably small number of the appropriate class of 2-elements and then to show that its smallest nontrivial projective representation over \( \mathbb{C} \) has dimension more than twice the number of generating involutions. To do this we use [18] which gives a list of lower bounds for the smallest nontrivial degrees of projective characters of the appropriate groups. This either eliminates the groups entirely or shows that \( n \) is small. For these we examine the character tables.

Table II lists various groups, the lower bounds from [18] of the smallest projective nontrivial character degree and an upper bound for the number of generators of the group by the appropriate involution. In each case this bound is a rough estimate determined by elementary Chevalley Group techniques.

We now enumerate various cases. The character tables for projective representations of \( J_2 \) appear in [13, 15]. There are exactly two field conjugate representations both of degree 6 which are \((F)\) on the list. Projective representations for \( A_6 \) can be computed. The only possibilities are in \((I)\).

Examination of Table II eliminates \( 2E_6(2), E_6(q), i = 6, 7, 8, F_4(2) \) and \( 3D_4(q) \). Note in these cases the Chevalley Group over \( GF(2) \) has no appropriate representation, nor does the Chevalley Group over \( GF(4) \), as conjugates of the root involutions over \( GF(4) \) appear over \( GF(2) \). Also \( E_6(2) \supseteq E_6(2) \) with conjugates of root involutions in \( E_6(2) \).

Note \( G_2(2) \cong U_4(3) \) [7]. The only representations are \((E)\) as can be noted from the character table in [13]. To show \( G_2(4) \) has no appropriate representation note \( G_2(4) \) contains \( J_2 \) and \( J_2 \) contains a conjugate of the long root involution.
The restriction of $X$ to $J_2$ can contain at most one projective constituent similar to the 6-dimensional representation. The remaining constituents must be trivial. As the 6-dimensional representation has an element of order 6 with eigenvalues \{-\varepsilon, -\varepsilon, -\varepsilon, -\varepsilon, -\varepsilon, -\varepsilon\}, degree $X$ must be 6 or [2, p. 196] would be contradicted. But this contradicts [14] and shows $G \cong G_2(4)$.

The group $A_4(2)$ has order divisible by 31 and so by [3-8.3.4] has no projective representation of degree smaller than 17. It follows from Table II that $G \cong A_4(2)$. The character tables for projective characters of $A_4(2) \cong A_8$ can be computed. Some appear in [17]. The only character is of degree 7 with the appropriate property. However, the root involutions of $A_8(2)$ are the involutions $(1, 2)(3, 4)(5, 6)(7, 8)$ which are not the special involutions in this representation.

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**Table I**

$q = 2, 4$

<table>
<thead>
<tr>
<th>Chevalley notation for $G'$ (if appropriate)</th>
<th>Classical notation (or sporadic) for $G$</th>
<th>Involution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Symmetric</td>
<td>$Sp(2n, q) \cong O(2n + 1, q), n \geq 2$</td>
<td>Transposition $(1, 2)$</td>
</tr>
<tr>
<td>2. $C_n(q) \cong B_n(q), n \geq 2$</td>
<td>$SO(2n + 1, q), n \geq 2$</td>
<td>Transvection or long root</td>
</tr>
<tr>
<td>3. $C_n(3), C_n(5), n \geq 3$</td>
<td>$PO^+(2n, q)$ or $PO^-(2n, q)$ with a reflection adjoined, $n \geq 2$</td>
<td>Reflection</td>
</tr>
<tr>
<td>4. $^2A_n(q), n \geq 2$</td>
<td>$PSU(n + 1, q), n \geq 2$</td>
<td>Transvection</td>
</tr>
<tr>
<td>5. $D_n(q), n \geq 4$</td>
<td>$PO^+(2n, q)$, $n \geq 4$</td>
<td>Transvection</td>
</tr>
<tr>
<td>6. $^2D_n(q), n \geq 4$</td>
<td>$PO^-(2n, q)$, $n \geq 4$</td>
<td>Transvection</td>
</tr>
<tr>
<td>7. $D_n(3), D_n(5), n \geq 4$</td>
<td>$PO^+(2n, 3)$ or $PO^-(2n, 5)$ with a reflection adjoined, $n \geq 2$</td>
<td>Reflection</td>
</tr>
<tr>
<td>8. $^2D_n(3), ^2D_n(5), n \geq 4$</td>
<td>$PO^-(2n, 3)$ or $PO^-(2n, 5)$ with a reflection adjoined, $n \geq 2$</td>
<td>Reflection</td>
</tr>
<tr>
<td>9. $F_i, i = 22, 23, 24$</td>
<td></td>
<td>3-Transposition</td>
</tr>
<tr>
<td>10. $A_n(q), n \geq 1$</td>
<td>$PSL(n + 1, q), n \geq 1$</td>
<td>Root</td>
</tr>
<tr>
<td>11. $G_2(q)$</td>
<td></td>
<td>Long root</td>
</tr>
<tr>
<td>12. $^2D_4(q)$</td>
<td></td>
<td>Long root</td>
</tr>
<tr>
<td>13. $F_4(2)$</td>
<td></td>
<td>Long or short root</td>
</tr>
<tr>
<td>14. $^2E_6(2)$</td>
<td></td>
<td>Long root for $^2E_6(2)$, Root for $E_8(q)$</td>
</tr>
<tr>
<td>15. $A_6$</td>
<td></td>
<td>Double transposition $(12)(34)$</td>
</tr>
<tr>
<td>16. $J_2$</td>
<td></td>
<td>Central involution</td>
</tr>
</tbody>
</table>
There is an element of order 3, namely $(1, 2, 3)$, which fixes a subspace of codimension two and places this representation in $(A)$. Projective characters of $A_d(2) \cong PSL(3, 2) \cong PSL(2, 7)$ are known. The representations are $(D)$. The group $A_d(4)$ is impossible by the table. Some projective characters of $A_d(4) \cong PSL(3, 4)$ appear in [16]. Others can be computed. The only possibility is $(G)$. Note $A_d(4) \cong A_5$ has no representation of degree 6 or more with special 2-elements. This handles the groups $G' \cong A_n(q)$.

The Fischer Groups cannot arise. There are several arguments. Each contains a subgroup $PSU(6, 2)$ which we show later has no representation with an involution fixing a subspace of codimension 2. Conjugates of odd transpositions of $F_2$ are in $PSU(6, 2)$. This completes sections 9 to 16 of Table I.

We now consider the reflection groups over $GF(3)$ or $GF(5)$ which are in Sections 3, 7, 8 of Table I. These are impossible for $n = 4$ by Table II. The group $P_2(5, 3)$ occurs in $(B)$ as $W_6$. The group $P_2^+(6, 3)$ is isomorphic to $PSL(4, 3)$. Its smallest projective nontrivial degree is 63; however, $P_2^+(6, 3)$ can be generated by 7 reflections and so $G' \not\cong PO^+(6, 3)$. Also $P_2-(6, 3)$ is isomorphic to $PSU(4, 3)$. Its projective character table provides the group $(H)$. Now $P_2(7, 3) \supseteq P_2^+(6, 3)$ with some reflections of $P_2(7, 3)$ acting on $P_2^+(6, 3)$ has no such representation. Also note $P_2^+(4, 3)$ is solvable and $P_2^-(4, 3) \cong A_6$ which is covered in $(I)$. For $p = 5$, $P_2^-(4, 5) \cong PSL(2, 25)$ does not have a representation with a special involution nor does an extension. Also $P_2^-(4, 5) \cong PSL(2, 5) \times PSL(2, 5)$ which does not occur here. Finally $P_2(3, 5) \cong PSL(2, 5) \cong A_5$ which does not occur with $n \geq 6$.

We now move to the top of Table I. As $S_n$ is generated by $n - 1$ transpositions we need only consider characters of degree at most $2n - 2$. It is an easy
exercise to show by induction for \( n \geq 6 \) there are no appropriate representations with the transpositions represented as special involutions.

The multiplier of \( A_n \) for \( n \geq 7 \) has order 2. In a double cover of \( S_n \), the transpositions when of order 2 are conjugate to their negatives in a faithful representation and so the degree could only be 4. The groups \( S_5 \) and \( S_6 \) are done by inspection. This covers Section 1 of Table I.

The group \( B_2(4) \) has no such representation. This is seen by consulting a character table. Note \( B_2(4) \) has no multiplier.

The group \( B(2) \cong Sp(6, 2) \) has a representation of degree 7 generated by minus reflections \([5]\) and occurs in \((B)\). In a projective representation over a center of order 2 the long root involutions are conjugate to their negatives and so cannot be special for \( n \geq 6 \). We note from \([18]\) that a projective representation of \( B_q(2) \) has dimension at least 28. Also \( B_q(2) \subseteq B_q(2) \) and contains the appropriate involution. Now \( X \mid Sp(6, 2) \) cannot contain the 7-dimensional representation with multiplicity 1 as there would be an element of order 3 fixing a space of codimension 2 against \([8]\). Also, there cannot be two 7-dimensional representations as then the involution would have 12 eigenvalues \(-1\) and \( n - 12 \) eigenvalues 1. This is impossible as \( n \geq 28 \) and shows \( G \not\cong B_q(2) \).

Note \( B_2(2) \) has \( A_q \) as a composition factor and occurs in \((A)\) or \((I)\) by inspection. This covers Section 2 of Table I.

We now consider \( {}^2A_n(q) \cong PSU(n + 1, q) \). The group \( PSU(3, 4) \) has no projective representation with the appropriate involution which can be checked from character tables. Note the group has no multiplier and by \([20]\), conjugates of \( \tau \) must be in \( PSU(3, 4) \) itself.

Note \( PSU(3, 2) \) is solvable and \( PSU(4, 2) \) or order \( 5 \cdot 3^4 \cdot 2^6 \) gives \( W_6 \) or \( W'_6 \) in \((B)\). Also \( PSU(5, 2) \) has a unique 10-dimensional character with an appropriate involution giving \((L)\). Now \( PSU(6, 2) \) has no projective character of degree smaller than 21 \([18]\). If \( X \) is restricted to \( PSU(5, 2) \) there is one 10-dimensional faithful character and at least an 11-dimensional fixed space. However, \( PSU(6, 2) \) is generated by \( PSU(5, 2) \) with one transvection adjoined.

We now consider the groups in Sections 5 and 6 of Table I. Note the projective character tables for \( D_q(2) \cong P\Omega^+(8, 2)' \), \([6]\), indicate three projective representations of degree 8 and extensions of degree 2 generated by real reflections. Character tables appear in \([6]\). The 8-dimensional representations of \( D_q(2) \) contain an element of order 3 with 6-dimensional fixed space and so it alone cannot occur as a nontrivial constituent when restricting from \( D_q(2) \), \( n \geq 5 \). The extended group could occur with multiplicity two. If the representation had dimension larger than 16 there would be an element of order 6 contradicting Blichfeld \([2, p. 196]\) as the Sylow 3-group contains an element of order 3 with no eigenvalues 1 and there is a center of order 2. Now \( D_q(2) \) has a minimal projective degree \( (2^5 - 1)(2^5 - 1) \) which is larger than 16 and so \( G \not\cong D_q(2) \). Note \( P\Omega^+(6, 2) \cong PSL(4, 2) \) and \( P\Omega^+(4, 2) \cong PSL_2(2) \times PSL_2(2) \) have been handled elsewhere in this paper. As \( Sp(4, 4) \cong O(5, 4) \) does not
have a representation containing a special 2-element we need only consider
$PQ(4, 4) \cong PSL_2(16)$ and $PQ^+(4, 4) \cong PSL_2(4) \times PSL_2(4)$ to handle $q = 4$.
These do not have appropriate representations.

The final case is $PQ(2n, 2)$. If $G \cong \langle PQ(8, 2), \text{transvection} \rangle$, $\pi$ is not in the
simple group. There is a subgroup $\langle PQ(6, 2), \text{transvection} \rangle$ containing a conju-
gate of $\pi$. The centralizer of $\pi$ is isomorphic to $\langle \pi \rangle \times \text{Sp}(6, 2)$. As $\pi$ is conjugate
to some involution other than $\pi$ in $\langle \pi \rangle \times \text{Sp}(6, 2)$, $\pi \times \text{Sp}(6, 2)$ can only be
a sum of two 7-dimensional faithful representations with the remaining con-
stituents trivial. Now $\langle \pi \rangle \times \text{Sp}(6, 2)$ is maximal and so degree $X \leq 16$ a
contradiction to [18] which says degree $X \geq 27$. If $G \cong \langle D_4(2), \text{PSL}(8, 2), G$
is generated by 12 long root involutions. This is again a contradiction to [18]
and it eliminates the final case. Note $PQ(6, 2) \cong \text{PSU}(4, 2)$ and $PQ(4, 2) \cong$
$\text{PSL}_2(4)$ have already been covered.

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