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New Relations and Identities for Generalized Hypergeometric Coefficients*

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Generalized hypergeometric coefficients $\langle {}_p\mathscr{F}_q(\mathbf{a};\mathbf{b})|\lambda\rangle$ enter into the problem of constructing matrix elements of tensor operators in the unitary groups and are the expansion coefficients of a multivariable symmetric function generalization ${}_p\mathscr{F}_q(\mathbf{a};\mathbf{b};z)$, $z=(z_1,z_2,\ldots,z_t)$, of the Gauss hypergeometric function in terms of the Schur functions $e_\lambda(z)$, where $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_t)$ is an arbitrary partition. As befits their group-theoretic origin, identities for these generalized hypergeometric coefficients characteristically involve series summed over the Littlewood–Richardson numbers $g(\mu\nu\lambda)$. Identities that may be interpreted as generalizations of the Bailey, Saalschütz, ... identities are developed in this paper. Of particular interest is an identity which develops in a natural way a group-theoretically defined expansion over new inhomogeneous symmetric functions. While the relations obtained here are essential for the development of the properties of tensor operators, they are also of interest from the viewpoint of special functions. © 1992 Academic Press, Inc.

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I. Introduction: Background, Motivation and Summary

The principal results of the present paper are the proofs of two fundamental identities for generalized hypergeometric coefficients, completing a series of papers [1–7], partly based on the presumed validity of these results. We shall attempt a self-contained treatment so that the present paper can be read independently of earlier results.

This work, and the subject of generalized hypergeometric coefficients itself [8–24], had its origins in the problem of constructing *tensor operators* for compact symmetry groups, a problem that is itself subsumed in the general problem of *vertex operator algebras* in quantal field theory, a subject of great current interest in both mathematics and physics [25].

Our particular interest has been the construction of the irreducible tensor operators of the unitary symmetry group SU(3) which have been shown [7] to be uniquely determined from their characteristic null space, which in turn is defined [2, 3, 6, 7] by polynomial invariants denoted by G_q^t, $q \in \mathbb{N} = \{0, 1, 2, \ldots\}$, $t = 1, 2, \ldots, q$. The explicit determination of this set of polynomials, $\{G_q^t\}$, is a major step toward the goal of obtaining algebraic expressions for all SU(3) tensor operators and their associated invariant structures (called $3nj$-coefficients). The analysis of these invariant G_q^t polynomials provided the motivation for a far-reaching symmetric (Schur) function generalization [10] of the Gauss hypergeometric function, ${}_2F_1$, and its ${}_pF_q$ extensions. Equivalently, it is the properties of the *generalized hypergeometric coefficients*, denoted $\langle {}_2\mathscr{F}_1(a, b; c) | \mu \rangle$ in [10–12] and defined below in Eq. (1.4), that are the subject of this paper.

This paper can be read without knowledge of the tensor operator problem for SU(3). Its results stand independently in the arena of special function theory, as the relations given below (Eqs. (1.13)–(1.15)) generalizing Saalschütz's formula, Bailey's identity, and the binomial addition theorem clearly show. The main results of this paper (referred to above) are the proofs of Eq. (1.11) and Theorem 4.8 which are new generalizations of Bailey's identity (in addition to relation (1.13)), as the special case, Eq. (1.22), for $t = 1$ shows. Theorem 4.8, together with Eq. (4.18) expresses, in one master statement, a number of important relations for the theory of symmetric functions, as developed and proved in [12].

While the results for special functions given here are presented for their own intrinsic interest, as noted above, it is useful, for continuity with our previous work on various SU(3) coefficients, to place these topics in the appropriate context. The canonical resolution of the multiplicity problem for irreducible unit (normalized) tensor operators in SU(3) is equivalent to a map from the set of all such SU(3) tensor operators to a set of SU(3) invariant functions known as the *denominator functions* and denoted $D^2(\Gamma_t; x)$. Here Γ_t denotes an *operator pattern* labelling a canonical SU(3) unit tensor operator and $x = (x_1, x_2, x_3)$ denotes a lattice point in the Möbius plane associated with the irreducible representation (irrep) spaces in which the unit tensor operators act. The details of this construction for SU(3) have been described in [1–7]. The original form one obtains for the denominator function is extremely complicated and thoroughly unwieldy. Considerable progress has been made in bringing this function to a more comprehensible form involving the ratio of two successive polynomials in the set

$$\left\{G_q^t(\Delta; x) | t = 0, 1, \dots, q\right\}. \tag{1.1}$$

Here $q \in \mathbb{N} = \{0, 1, 2, \dots\}$, $\Delta = (\Delta_1, \Delta_2, \Delta_3)$ denotes the common shift coordinates associated with the set of operator patterns $\{\Gamma_t | t = 1, 2, \dots, q\}$, and $x = (x_1, x_2, x_3)$ the barycentric coordinates of an arbitrary point in the Möbius plane (we arbitrarily define $G_q^0 = 1$ for all q).

The relationships between the polynomials $\{G_q^t(\Delta; x)\}$ and the denominator functions $\{D^2(\Gamma_t; x)\}$ are discussed in detail in [3–7]. (In particular, the properties of the $\{G_q^t(\Delta; x)\}$ polynomials have been developed in [6–7].) All of these results have culminated in [7], where it is proved that

$$G_q^t(\Delta; x) = \mathscr{G}_q^t(\Delta; x), \tag{1.2}$$

where the new polynomials $\mathscr{G}_q^t(\Delta; x)$ on the right-hand side are explicit polynomials, both in the shift parameters $\Delta = (\Delta_1, \Delta_2, \Delta_3)$ and in the barycentric coordinates $x = (x_1, x_2, x_3)$. (The nontrivial content of relation (1.2) is that it had not been possible previously to give G_q^t explicitly as a polynomial.) The proof of Eq. (1.2) has been achieved in [7], *but subject to the condition that the polynomials* $\mathscr{G}_q^t$ *obey a certain symmetry, called determinantal symmetry, an already proved property of the* G_q^t. Thus, the validity of Eq. (1.2) depends still on a proof of determinantal symmetry of the known $\mathscr{G}_q^t$.

It was shown in [7] that determinantal symmetry of the polynomial $\mathscr{G}_q^t$ is implied by known (proved) symmetries plus one other additional symmetry, which will be proved below. We have been able to show in [7] that this additional symmetry is equivalent to proving a new identity, which is most conveniently formulated in terms of yet another set of functions, denoted by A_λ, and enumerated by partitions λ having t parts, including zero as a part,

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t), \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t \geq 0, \tag{1.3}$$

where each λ_s $(s = 1, 2, \dots, t)$ is a nonnegative integer. We denote the (infinite) set of all partitions having t parts by $\mathbb{P}_t$. The polynomials A_λ, $\lambda \in \mathbb{P}_t$, are defined over five parameters (a, b, c, d, e) that are simply related to the shift parameters $(\Delta_1, \Delta_2, \Delta_3)$, to the barycentric coordinates

(x_1, x_2, x_3), and to q and t. (The precise relation is given by Eq. (5.9a) of [7], but this detailed result is not needed here.) Of interest here are the properties of the polynomials A_λ, especially their symmetries, as expressed in terms of the parameters (a, b, c, d, e), which will now be regarded as indeterminates. Accordingly, let us begin by defining fully the polynomials A_λ, stating their proved symmetries and the one additional symmetry—proved in the present paper—which is needed for establishing the determinantal symmetry of the polynomials $\mathscr{G}_q^t$, and, hence, the fundamental relation (1.2).

The polynomials A_λ are defined in terms of a set of coefficients, hereafter called *generalized hypergeometric coefficients*, first introduced in [10] and defined by

$$\begin{aligned}&\langle {}_2\mathscr{F}_1(a, b; c)|\lambda\rangle\\&\quad = M^{-1}(\lambda)\prod_{s=1}^{t}(a-s+1)_{\lambda_s}(b-s+1)_{\lambda_s}/(c-s+1)_{\lambda_s}. \qquad (1.4)\end{aligned}$$

In this definition, (a, b, c) are arbitrary parameters (indeterminates) and λ is a partition of the form (1.3). The Kummer symbol $(x)_k$ for $k = 0, 1, \ldots$ and indeterminate x denotes the rising factorial

$$(x)_k = x(x+1)\cdots(x+k-1) \qquad (1.5a)$$

with $(x)_0 = 1$ for all x. For arbitrary $y \in \mathbb{R}$, we extend definition (1.5a) by using the gamma function,

$$(x)_y = \Gamma(x+y)/\Gamma(x). \qquad (1.5b)$$

The quantity $M^{-1}(\lambda)$ is defined for each partition λ by

$$M^{-1}(\lambda) = \operatorname{Dim}\lambda \Big/ \prod_{s=1}^{t}(t-s+1)_{\lambda_s}, \qquad (1.6)$$

where Dim λ denotes the dimension of irrep λ of the unitary group $U(t)$, which, using Weyl's formula, is given by

$$\operatorname{Dim}\lambda = \prod_{1\le r<s\le t}(\lambda_r - \lambda_s + s - r)/1!2!\cdots(t-1)!, \qquad (1.7)$$

with Dim $\equiv 1$ for $t = 1$. ($M(\lambda)$—called the measure of the Young frame λ— has an interesting tableau interpretation (see [26] and [27]) in terms of the hook graph of λ, and in this context is denoted by H^λ.)

The hypergeometric coefficients (1.4) satisfy several remarkable relations (proved in [10–12]), which we will summarize below, since these identities will be required in the present paper. But let us first define the polynomials A_λ and then state the basic problem addressed in this paper.

Let (a, b, c, d, e) denote a five-tuple of indeterminates and $\lambda, \mu, \nu, \ldots$ partitions containing t parts, as described in Eq. (1.3). Then the polynomial A_λ is defined by

$$A_\lambda\begin{pmatrix} a, b, d, e \\ c \end{pmatrix} \equiv \prod_{s=1}^{t} (a + b + c - s + 1)_{\lambda_s}(d + e + c - s + 1)_{\lambda_s}$$
$$\times \sum_{\mu\nu} g(\mu\nu\lambda)\langle {}_2\mathcal{F}_1(a, b; a + b + c)|\mu\rangle$$
$$\times \langle {}_2\mathcal{F}_1(d + c, e + c; d + e + c)|\nu\rangle. \tag{1.8}$$

In this definition, the symbol $g(\mu\nu\lambda)$ denotes the number of times irrep λ of $U(t)$ occurs in the direct product $\mu \times \nu$ of irreps μ and ν (the so-called Littlewood–Richardson numbers). Each partition μ and ν in the summation in (1.8) satisfies, for each given partition λ, the relations

$$\mu_s \le \lambda_s, \qquad \nu_s \le \lambda_s, \qquad s = 1, 2, \ldots, t. \tag{1.9}$$

It follows from this property and

$$(x)_k/(x)_l = (x + l)_{k-l} \tag{1.10}$$

for $k, l \in \mathbb{N}$ and $k \ge l$ that factors from the multiplicative term in front of the sum in definition (1.8) always cancel all the denominator factors in the ${}_2\mathcal{F}_1$-coefficients under the summation; that is, *the function A_λ is indeed a polynomial in the indeterminates* (a, b, c, d, e).

The occurrence of summations over partitions in which the Littlewood–Richardson numbers occur is a striking feature of the generalizations of hypergeometric functions given in [10, 11]. This characteristic carries over to a variety of relations between hypergeometric coefficients obtained in [12] and to the new relations obtained here for A_λ. We believe this is indicative of the nontrivial nature of these generalizations, which themselves are rooted in the underlying group-theoretical origin of the SU(3) Wigner–Clebsch–Gordan coefficients.

The polynomial A_λ is clearly invariant under the interchange of a and b as well as under the interchange of d and e. The symmetry of A_λ needed, however, to prove the determinantal symmetry of the polynomials $\mathcal{G}_q^t$ of interest is

THEOREM 1.1 (proven in Section VI). *The polynomial A_λ is invariant under the interchange of b and e; that is,*

$$A_\lambda\begin{pmatrix} a, b, d, e \\ c \end{pmatrix} = A_\lambda\begin{pmatrix} a, e, d, b \\ c \end{pmatrix}. \tag{1.11}$$

The transpositions (ab), (de)—which are obvious symmetries—plus the transposition (be) suffice to generate the group S_4 of permutations of the parameters a, b, d, e. It is this invariance of the polynomials A_λ (for all partitions λ) under the action of the group S_4 that implies the determinantal symmetry of the explicit SU(3) invariant denominator polynomials, $\mathscr{G}_q^t$, needed in the proof of Eq. (1.2).

The invariance of $A_\lambda\binom{a,b,d,e}{c}$ under interchange of b and e may appear, at first glance, to be rather straightforward, but this apparent simplicity is very deceptive! We found the construction of a valid proof to be unusually elusive and difficult. None of the known identities among the generalized ${}_2\mathscr{F}_1$ hypergeometric coefficients given in [10–12] seem to bear directly on the problem, although these relations are indeed needed for the proof of special cases (for example, $d = 0$ below; see Eq. (1.21b)).

Let us now state three general relations satisfied by the generalized hypergeometric coefficients (see [10–12]). For this purpose, we first note the definition:

$$\begin{aligned}\langle {}_1\mathscr{F}_0(a)|\lambda\rangle &= \langle {}_2\mathscr{F}_1(a,b;b)|\lambda\rangle\\ &= \operatorname{Dim}\lambda \prod_{s=1}^{t} \frac{(a-s+1)_{\lambda_s}}{(t-s+1)_{\lambda_s}}.\end{aligned} \tag{1.12}$$

The relations are

(i) *Generalization of Bailey's identity.*

$$\begin{aligned}\sum_{\mu\nu} g(\mu\nu\lambda)&\langle {}_2\mathscr{F}_1(c-a, c-b; c)|\mu\rangle\langle {}_2\mathscr{F}_1(c'-a', c'-b'; c')|\nu\rangle\\ &= \sum_{\mu\nu} g(\mu\nu\lambda)\langle {}_2\mathscr{F}_1(a,b;c)|\mu\rangle\langle {}_2\mathscr{F}_1(a',b';c')|\nu\rangle,\end{aligned} \tag{1.13a}$$

where

$$c - a - b + c' - a' - b' = 0; \tag{1.13b}$$

(ii) *Generalization of Saalschütz's identity.*

$$\begin{aligned}\sum_{\mu\nu} g(\mu\nu\lambda)&\langle {}_2\mathscr{F}_1(a,b;c)|\mu\rangle\langle {}_1\mathscr{F}_0(c-a-b)|\nu\rangle\\ &= \langle {}_2\mathscr{F}_1(c-a, c-b; c)|\lambda\rangle;\end{aligned} \tag{1.14}$$

(iii) *Generalized addition rule of binomial type.*

$$\sum_{\mu\nu} g(\mu\nu\lambda)\langle {}_1\mathscr{F}_0(x)|\mu\rangle\langle {}_1\mathscr{F}_0(y)|\nu\rangle = \langle {}_1\mathscr{F}_0(x+y)|\lambda\rangle. \tag{1.15}$$

Relation (1.13) is called a generalization of Bailey's identity because for $t = 1$ we have

$$\sum_{\mu+\nu=\lambda} \langle {}_2\mathscr{F}_1(a,b;c)|\mu\rangle\langle {}_2\mathscr{F}_1(a',b';c'|\nu\rangle$$
$$= \frac{(a)_\lambda(b)_\lambda}{\lambda!(c)_\lambda}\,{}_4F_3\left(\begin{matrix} a',b',1-c'-\lambda,-\lambda; \\ c',1-a-\lambda,1-b-\lambda \end{matrix}\right)$$
$$= \frac{(a')_\lambda(b')_\lambda}{\lambda!(c')_\lambda}\,{}_4F_3\left(\begin{matrix} a,b,1-c'-\lambda,-\lambda; \\ c,1-a'-\lambda,1-b'-\lambda \end{matrix}\right), \qquad (1.16)$$

in which $c - a - b + c' - a' - b' = 0$. The second identity, used above in (1.16), between the two ${}_3F_3$ hypergeometric series (of unit argument) is the reversal identity (reverse the order of terms in the finite series expression). Using (1.16) in the identity (1.13) for $t = 1$ now gives Bailey's identity (see p. 56 of Bailey [28]).

Relation (1.14) is called a generalization of Saalschütz's identity because for $t = 1$ it reduces to the well-known Saalschütz identity (see Bailey [28]) given by

$$\sum_{\substack{\mu,\nu \\ \mu+\nu=\lambda}} \frac{(a)_\mu(b)_\mu(c-a-b)_\nu}{(c)_\mu\mu!\nu!} = \frac{(c-a)_\lambda(c-b)_\lambda}{(c)_\lambda\lambda!}. \qquad (1.17)$$

Relation (1.15) is called a generalized binomial addition rule because for $t = 1$ it reduces to the well-known rule

$$\sum_{\mu+\nu=\lambda} \frac{(x)_\mu(y)_\nu}{\mu!\nu!} = \frac{(x+y)_\lambda}{\lambda!},$$

which itself implies the classical binomial identity:

$$\sum_{\mu+\nu=\lambda} \frac{x^\mu y^\nu}{\mu!\nu!} = \frac{(x+y)^\lambda}{\lambda!}.$$

The polynomials A_λ possess yet another symmetry that is proved directly from the defining relation, Eq. (1.8), and the generalization of Bailey's identity, Eq. (1.13). Using this property, we easily prove

LEMMA 1.1. *The polynomials A_λ are invariant under the linear transformation L defined by*

$$\begin{aligned} L:\quad & a \mapsto a+c,\ b \mapsto b+c, \\ & d \mapsto d+c,\ e \mapsto e+c, \\ & c \mapsto -c. \end{aligned} \qquad (1.18)$$

Remark. A basis set of independent invariants under the transformation L is:

Linear invariants,

$$\zeta_1 = a + \tfrac{1}{2}c, \qquad \zeta_2 = b + \tfrac{1}{2}c,$$
$$\xi_3 = d + \tfrac{1}{2}c, \qquad \zeta_4 = e + \tfrac{1}{2}c; \tag{1.19a}$$

Quadratic invariant,

$$\xi = c^2. \tag{1.19b}$$

In particular, the transformation L leaves invariant the combinations $a + b + c = \zeta_1 + \zeta_2$ and $c + d + e = \zeta_3 + \zeta_4$, which occur in Eq. (1.8).

It is useful to give some special cases of the polynomials A_λ in which the invariance under the transposition (be) is evident, and other instances that support this property (see [12]):

LEMMA 1.2. *The polynomials A_λ are invariant under the interchange of b and e if at least one of the conditions $a = 0$, $d = 0$, $c = -a$, $c = -d$ holds.*

Proof. The proof of this lemma is a straightforward application of the generalized Saalschütz identity, Eq. (1.14), to each of the four cases.

Before proceeding with the proof, it is useful and straightforward to extend the definition of the generalized hypergeometric function ${}_2\mathscr{F}_1$ in [10] to ${}_p\mathscr{F}_q$ (see also Shukla [11]). In particular, we will need the generalized hypergeometric coefficient,

$$\langle {}_3\mathscr{F}_0(a,b,c)|\lambda\rangle \equiv M^{-1}(\lambda)\prod_{s=1}^{t}(a-s+1)_{\lambda_s}(b-s+1)_{\lambda_s}(c-s+1)_{\lambda_s}, \tag{1.20}$$

which is clearly completely symmetric in a, b, c.

The polynomials A_λ for the special cases in this lemma all have a common form, as we now give. For $a = 0$ or $d = 0$ (similarly, for $b = 0$ or $e = 0$), the polynomial A_λ reduces to

$$A_\lambda\begin{pmatrix}0, b, d, e\\ c\end{pmatrix} = \langle {}_3\mathscr{F}_0(b+c, d+c, e+c)|\lambda\rangle, \tag{1.21a}$$

$$A_\lambda\begin{pmatrix}a, b, 0, e\\ c\end{pmatrix} = \langle {}_3\mathscr{F}_0(a+c, b+c, e+c)|\lambda\rangle, \tag{1.21b}$$

while for $a + c = 0$ or $c + d = 0$, it becomes

$$A_\lambda\begin{pmatrix} a, b, d, e \\ -a \end{pmatrix} = \langle {}_3\mathscr{F}_0(b, d, e)|\lambda\rangle, \tag{1.21c}$$

$$A_\lambda\begin{pmatrix} a, b, d, e \\ -d \end{pmatrix} = \langle {}_3\mathscr{F}_0(a, b, e)|\lambda\rangle. \tag{1.21d}$$

In all of these special cases, symmetry under $b \leftrightarrow e$ exchange is evident. □

There are two further important (but still special) cases, where invariance of A_λ under b and e interchange may be proved directly. These cases are more difficult to prove than those above and are valid for general values of the indeterminates (a, b, c, d, e), but only for special partitions λ. These special cases were significant in suggesting methods of general proof. Accordingly, we give these results here.

For $t = 1$ and $\lambda_1 = n \in \mathbb{N}$, the polynomial A_n can be expressed in terms of the standard ${}_4F_3$ hypergeometric series. Directly from the defining relation, Eq. (1.8), and the ${}_4F_3$ relation, Eq. (1.16), we find

$$A_n\begin{pmatrix} a, b, d, e \\ c \end{pmatrix} = \left[\frac{(a+b+c)_n(c+d)_n(c+e)_n}{n!}\right] \times {}_4F_3\begin{pmatrix} a, b, 1-c-d-e-n, -n; \\ a+b+c, 1-c-d-n, 1-c-e-n \end{pmatrix}. \tag{1.22a}$$

Using Bailey's identity, we can give a different form for this result:

$$A_n\begin{pmatrix} a, b, d, e \\ c \end{pmatrix} = \left[\frac{(a+b+c)_n(a+c+d)_n(a+c+e)_n}{n!}\right] \times {}_4F_3\begin{pmatrix} a, a+c, a+b+d+e+2c+n-1, -n \\ a+b+c, a+c+d, a+c+e \end{pmatrix}. \tag{1.22b}$$

The form given in Eq. (1.22b) explicitly shows the invariance of A_n under interchange of b and e, since hypergeometric series are invariant under permutations of their denominator parameters. Thus, it is the Bailey transformation that implies the desired symmetry. A subtle hint toward generalization is also contained in relation (1.22b): Ignoring the multiplicative factors in front of the ${}_4F_3$ series, one finds that the dependence of A_n on the integer $n \in \mathbb{N}$ occurs in the kth term in the explicit summation expression for the ${}_4F_3$ series as

$$(-n)_k(n + \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 - 1)_k, \qquad k = 0, 1, \ldots, n. \tag{1.23}$$

(See Eqs. (1.19)). While the significance of this result is hardly obvious at this point, it turns out to be a crucial structural property (developed below as R-symmetry) in the generalization of Eqs. (1.22) to A_λ. (We actually found this symmetry in a very different way.)

The second special case of A_λ showing explicitly the invariance under b and e interchange is for $\lambda_1 = \lambda_2 = \cdots = \lambda_t = \lambda$, which we denote (using a convenient computational notation) by

$$(\lambda, \lambda, \dots, \lambda) \equiv (\dot{\lambda}). \tag{1.24}$$

For this special partition we are able to transform the original definition (1.8) to the form

LEMMA 1.3. *For the special partition $\dot{\lambda}$, the polynomial $A_{\dot{\lambda}}\left(\begin{smallmatrix} a,b,d,e \\ c \end{smallmatrix}\right)$ has the form*:

$$\begin{aligned}
&A_{\dot{\lambda}}\begin{pmatrix} a,b,d,e \\ c \end{pmatrix} \\
&\quad = \prod_{s=1}^{t} (a+b+c-s+1)_\lambda (a+d+c-s+1)_\lambda (a+e+c-s+1)_\lambda \\
&\qquad \times \sum_{\mu} [M(\mu)M(\bar{\mu})]^{-1} \prod_{s=1}^{t} \frac{(-1)^{\mu_s}(a-s+1)_{\mu_s}(a+c-s+1)_{\mu_s}}{(a+b+c-s+1)_{\mu_s}(a+d+c-s+1)_{\mu_s}} \\
&\qquad \times \frac{(\lambda+2c+a+b+d+e-t-s+1)_{\mu_s}}{(a+e+c-s+1)_{\mu_s}}.
\end{aligned} \tag{1.25}$$

In this result, $\bar{\mu}$ denotes the partition defined in terms of μ by

$$\bar{\mu}_s = \lambda - \mu_{t-s+1}, \tag{1.26a}$$

so that

$$M^{-1}(\bar{\mu}) = \mathrm{Dim}\,\mu \Big/ \prod_{s=1}^{t} (s)_{\lambda-\mu_s}. \tag{1.26b}$$

The summation in Eq. (1.25) is over all partitions μ satisfying

$$0 \le \mu_s \le \lambda, \qquad s = 1, 2, \dots, t. \tag{1.27}$$

Relation (1.25) for $A_{\dot{\lambda}}$ shows explicitly the invariance under the interchange of b and e.

Proof. The derivation of relation (1.25) proceeds straightforwardly from the general definition (1.8) using

$$g(\mu, \nu, \dot{\lambda}) = \delta_{\nu,\bar{\mu}}. \tag{1.28}$$

The basic identity needed is the following one for arbitrary parameters (a,b,c,a',b',c'):

$$\sum_{\mu}\langle {}_2\mathscr{F}_1(a,b;c)|\mu\rangle\langle {}_2\mathscr{F}_1(a',b';c')|\bar{\mu}\rangle$$
$$= (-1)^{t\lambda}\left[\prod_{s=1}^{t}\frac{(a-s+1)_{\lambda}}{(c'-s+1)_{\lambda}}\right]$$
$$\times\sum_{\mu}\langle {}_2\mathscr{F}_1(-c'-\lambda+t,b;c)|\mu\rangle\langle {}_2\mathscr{F}_1(a',b';-a-\lambda+t)|\bar{\mu}\rangle. \tag{1.29}$$

The transformation of the left-hand side of this relation to the right-hand side is effected by using the identity

$$\prod_{s=1}^{t}\frac{(a-s+1)_{\mu_s}}{(c'-s+1)_{\bar{\mu}_s}} = (-1)^{\lambda t}\prod_{s=1}^{t}\frac{(a-s+1)_{\lambda}}{(c'-s+1)_{\lambda}}$$
$$\times\prod_{s=1}^{t}\frac{(-c'-\lambda+t-s+1)_{\mu_s}}{(-a-\lambda+t-s+1)_{\bar{\mu}_s}}. \tag{1.30}$$

The verification of this identity uses the relation (1.10).

We next replace c by $a+b+c$ and then set $a'=c+d$, $b'=c+e$, $c'=c+d+e$ in relation (1.29). This transforms relation (1.8), for $\lambda=\dot{\lambda}$, to

$$A_{\dot{\lambda}}\begin{pmatrix} a,b,d,e \\ c \end{pmatrix}$$
$$= (-1)^{\lambda t}\left[\prod_{s=1}^{t}(a-s+1)_{\lambda}(a+b+c-s+1)_{\lambda}\right]$$
$$\times\sum_{\mu}\langle {}_2\mathscr{F}_1(-c-d-e-\lambda+t,b;a+b+c)|\mu\rangle$$
$$\times\langle {}_2\mathscr{F}_1(c+d,c+e;-a-\lambda+t)|\bar{\mu}\rangle$$
$$= (-1)^{\lambda t}\prod_{s=1}^{t}(a-s+1)_{\lambda}(a+b+c-s+1)_{\lambda}$$
$$\times\sum_{\mu}\langle {}_2\mathscr{F}_1(2c+a+b+d+e+\lambda-t,a+c;a+b+c)|\mu\rangle$$
$$\times\langle {}_2\mathscr{F}_1(-a-c-d-\lambda+t,-a-c-e-\lambda+t;-a-\lambda+t)|\bar{\mu}\rangle. \tag{1.31}$$

The second identity in this relation has been obtained by applying the generalization of Bailey's identity [Eqs. (1.13)] for $\lambda=\dot{\lambda}$ to the first summation, after noting that the parameters do satisfy the required relation (1.13b) (with parameters renamed). Several elementary operations with rising factorials may now be used to bring the final expression in relation (1.31) to the stated form, Eq. (1.25), which shows explicitly the invariance under b, e interchange. □

The results given above in Lemma 1.2, Eqs. (1.20)–(1.22), and Eq. (1.25) constitute our initial evidence for believing A_λ to be symmetric in b and e, in the general case. We give in Section IV a conjectured new form of A_λ (denoted there by $\mathscr{A}_\lambda$) that shows explicitly the invariance under interchange of b and e, *and which reduces correctly to the special cases just cited*.

We have been able to prove the validity of the new (conjectured) form only by using yet another symmetry, which generalizes the factor given in Eq. (1.23). Thus, the proof of the invariance of A_λ under b and e interchange has been replaced by the proof of a quite different symmetry. We believe this reformulation of the problem is a significant step forward, not only for the problem at hand, but also for the new results in symmetric function theory stemming from it (discussed in Section IV).

The plan of this paper is as follows: in Section II, we reformulate the polynomial A_λ by evaluating the Littlewood–Richardson numbers. (This simplifies actual computations, although the formulae are lengthier.) Thus, the results here in the Introduction and in Section II constitute a complete statement of the problem addressed in this paper; namely, to prove the set of identities $A_\lambda = \mathscr{A}_\lambda$ for all parameter values (a, b, c, d, e), for all partitions $\lambda \in \mathbb{P}_t$, for all $t \in \mathbb{N}$. The strategy of the proof, which uses Carlson's theorem, is discussed in Section III. Here the relevant objects are a set of polynomials, denoted P_k^t, which are defined in terms of the A_λ. The detailed properties of these polynomials are developed in Section III. In Sections IV and V, a parallel course of development is followed for a new set of polynomials, denoted $\mathscr{P}_k^t$, the definition of which is suggested by the established properties of the original P_k^t. The conjectured new form $\mathscr{A}_\lambda$ of A_λ is now given. In Section VI, we complete the proof that $A_\lambda = \mathscr{A}_\lambda$, thereby establishing not only $b \leftrightarrow e$ symmetry, but a number of new hypergeometric coefficient identities as well.

II. Reformulation of A_λ

The purpose of this section is to reformulate the original definition of the polynomial $A_\lambda\binom{a,b,d,e}{c}$ as given in Eq. (1.8) into a new expression that not only brings out more clearly the symmetries inherent in the polynomial, but is actually more amenable to calculation.

The first step is to reformulate the Littlewood–Richardson numbers $g(\mu\nu\lambda)$. Here we use the fact that, for given partitions μ and ν, we have

$$g(\mu\nu\lambda) = 0, \qquad \text{unless } \nu = \lambda - \beta, \tag{2.1}$$

where β denotes the content of a standard Young–Weyl tableaux of shape μ. Equivalently, β can be defined as the weight of a Gel'fand–Weyl pattern as we now discuss. Let us denote the partition μ by

$$\mu \equiv (m_{1t}, m_{2t}, \ldots, m_{tt}). \tag{2.2}$$

A Gel'fand–Weyl pattern (m) is defined to be a triangular array of integers in the form:

$$(m) \equiv \begin{pmatrix} m_{1,t} & & m_{2,t} & & \cdots & & m_{t-1,t} & & m_{t,t} \\ & m_{1,t-1} & & & \cdots & & & m_{t-1,t-1} & \\ & & \ddots & & & & \cdot^{\cdot^{\cdot}} & & \\ & & & m_{12} & & m_{22} & & & \\ & & & & m_{11} & & & & \end{pmatrix}. \tag{2.3a}$$

The key structural element of this pattern is that the allowed integers must fulfill the *betweenness conditions*, that is,

$$m_{1j} \geq m_{1,j-1} \geq m_{2j} \geq m_{2,j-1} \geq \cdots \geq m_{j-1,j} \geq m_{j-1,j-1} \geq m_{j,j}, \tag{2.3b}$$

for $j = n, n-1, \ldots, 2$. These inequalities express the condition that row $j-1$ of the pattern, $(m_{1,j-1}, m_{2,j-1}, \ldots, m_{j-1,j-1}) \in \mathbb{P}_{j-1}$ is a partition whose parts fall between those of row j of the pattern, which is the partition $(m_{1j}, m_{2j}, \ldots, m_{jj}) \in \mathbb{P}_j$. The number of triangular patterns (2.3a) for fixed partition μ is given by the Weyl dimension formula Dim μ (see Eq. (1.7)). The use of Gel'fand–Weyl patterns for labelling the irreducible representations of the group $U(t)$ is well known. The one-to-one correspondence between the set of Gel'fand–Weyl patterns belonging to a partition and the set of Young–Weyl standard tableaux of shape μ is also well known. It is described in detail in several places (see, for example, [29–30]. Here we will use Gel'fand–Weyl patterns exclusively.

The *weight* of a Gel'fand–Weyl pattern (m) is defined to be the t-tuple

$$w(m) \equiv (w_1(m), w_2(m), \ldots, w_t(m)),$$

with

$$w_j(m) \equiv \left(\sum_{i=1}^{j} m_{ij} \right) - \left(\sum_{i=1}^{j-1} m_{i,j-1} \right), \qquad j = 1, \ldots, t. \tag{2.4}$$

(We define $m_{ij} = 0$ if $i > j$.)

Note that, whereas the Gel'fand patterns are all distinct, the weights $w(m)$ generally have multiplicities; that is, different patterns (m) may have the same weight. We denote a given weight by α and the set of all distinct weights by $W(\mu)$, the weights of the irrep μ without repetitions.

The multiplicity of a weight $\alpha \in W(\mu)$ is called the Kostka number, denoted $K(\mu, \alpha)$. Clearly $K(\mu, \alpha)$ is the number of distinct Gel'fand–Weyl patterns (m) with fixed partition μ and with $w(m) = \alpha$. (This provides a convenient algorithm for actually computing these numbers.) It is useful to define

$$K(\mu, \alpha) = 0 \qquad \text{for } \alpha \notin W(\mu). \tag{2.5}$$

For $\nu = \lambda - \beta$, the Littlewood–Richardson number $g(\mu, \lambda - \beta, \lambda)$, $\beta \in W(\mu)$, can be expressed as a sum over the set of Kostka numbers,

$$\{ K(\mu, \alpha) | \alpha \in W(\mu) \}. \tag{2.6}$$

Using these notations, we have the following identity (see [31]):

$$g(\mu, \lambda - \beta, \lambda) \equiv \sum_{\pi \in S_t} \varepsilon_\pi K(\mu, \beta + \pi(\lambda + \delta) - (\lambda + \delta)). \tag{2.7}$$

Here S_t denotes the symmetric group on the integers $(1, 2, \ldots, t)$, π an element of S_t with signature ε_π, and $\delta = (t - 1, \ldots, 1, 0)$. The action of $\pi \in S_t$ on an arbitrary t-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_t)$ is defined by

$$\pi: \gamma \mapsto \pi\gamma = (\gamma_{\pi_1}, \gamma_{\pi_2}, \ldots, \gamma_{\pi_t}) \tag{2.8a}$$

for

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & t \\ \pi_1 & \pi_2 & \cdots & \pi_t \end{pmatrix}. \tag{2.8b}$$

The summation in Eq. (2.7) is carried out over all $t!$ elements $\pi \in S_t$. Finally, $\lambda + \delta$ denotes the t-tuple of partial hooks:

$$\lambda + \delta = (\lambda_1 + t - 1, \ldots, \lambda_{t-1} + 1, \lambda_t). \tag{2.8c}$$

The second step in our reformulation of A_λ, as defined by Eq. (1.8), is to extend the definition of the hypergeometric coefficients $\langle {}_2\mathcal{F}_1(a, b; c) | \lambda \rangle$ to arbitrary t-tuples $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_t)$ of nonnegative integers; that is, to arbitrary t-tuples in place of partitions λ. Thus, we define

$$\begin{aligned} &{}_2\mathscr{F}_1(a,b;c)|\gamma\rangle \\ &\quad = \frac{D(\gamma+\delta)}{1!2!\cdots(t-1)!}\prod_{s=1}^{t}\frac{(a-s+1)_{\gamma_s}(b-s+1)_{\gamma_s}}{(t-s+1)_{\gamma_s}(c-s+1)_{\gamma_s}}, \end{aligned} \tag{2.9}$$

where $D(\gamma+\delta)$ denotes the Vandermonde determinant in the t variables $(\gamma_1+t-1,\ldots,\gamma_{t-1}+1,\gamma_t)$. If γ is a partition, the extended hypergeometric coefficient (2.9) reduces to the earlier definition (1.4) (see also Eqs. (1.5) and (1.7)).

The extended coefficient (2.9) obeys the rule

$$\langle{}_2\mathscr{F}_1(a,b;c)|\gamma\circ\pi\rangle = \varepsilon_\pi\langle{}_2\mathscr{F}_1(a,b;c)|\gamma\rangle, \qquad \text{each } \pi\in S_t, \tag{2.10}$$

where here the action of the permutation π on γ, denoted $\gamma\circ\pi$, is defined by

$$\gamma\circ\pi = \pi(\gamma+\delta)-\delta, \qquad \text{each } \pi\in S_t. \tag{2.11}$$

The signature ε_π of $\pi\in S_t$ in Eq. (2.10) comes from

$$\begin{aligned} D(\gamma\circ\pi+\delta) &= D(\pi(\gamma+\delta)) \\ &= \varepsilon_\pi D(\gamma+\delta). \end{aligned} \tag{2.12}$$

Factors of the form $\prod_{s=1}^{t}(u-s+1)_{\gamma_s}$ are invariant under $\gamma\to\gamma\circ\pi=\gamma'$; that is,

$$\prod_{s=1}^{t}(u-s+1)_{\gamma_s} = \prod_{s=1}^{t}(u-s+1)_{\gamma'_s}, \tag{2.13}$$

as direct calculation shows (since the individual terms are simply rearranged in the product).

Consider next the definition of A_λ given by Eq. (1.8). In this relation, we now replace ν by $\lambda-\beta$, and the summation over ν by a summation over all β such that $\beta\in W(\mu)$ and $\lambda-\beta$ is a partition. In the resulting relation, we also replace $g(\mu,\lambda-\beta,\lambda)$ by its expression in terms of Kostka numbers, Eq. (2.7), and the hypergeometric coefficient $\langle{}_2\mathscr{F}_1(d+c,e+c;d+e+c)|\lambda-\beta\rangle$ by its extended form, Eq. (2.9).

The polynomial A_λ is defined in terms of the partition λ, but the parts λ_s of λ do not occur symmetrically. The symmetric variables (as we shall show) are the *partial hooks* p_s. We replace the partition λ by the t-tuple $p=(p_1,p_2,\ldots,p_t)$ of partial hooks defined in terms of the partition

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ by

$$p \equiv \lambda + \delta, \qquad \delta = (t-1, \ldots, 1, 0); \tag{2.14a}$$

that is,

$$p_s \equiv \lambda_s + t - s, \qquad s = 1, 2, \ldots, t. \tag{2.14b}$$

In terms of the partial hooks, $p = \lambda + \delta$, the Kostka numbers and the Vandermonde determinant occurring in the new expression for Eq. (1.8) are $K(\mu, \beta - p + \pi p)$ and $D(p - \beta)$. Carrying out these steps, we obtain the intermediate form given by

$$\begin{aligned} A_\lambda\begin{pmatrix} a, b, d, e \\ c \end{pmatrix} &= \prod_{s=1}^{t} (a+b+c-s+1)_{\lambda_s}(d+e+c-s+1)_{\lambda_s} \\ &\quad \times \sum_{\mu} \langle {}_2\mathscr{F}_1(a, b; a+b+c)|\mu\rangle \sum_{\beta}{}' \sum_{\pi \in S_t} K(\mu, \beta - p + \pi p) \\ &\quad \times \langle {}_2\mathscr{F}_1(d+c, e+c; d+e+c)|(p - \beta - \delta)\circ\pi^{-1}\rangle. \end{aligned} \tag{2.15}$$

In this result, the summation of β is over the subset of $\beta \in W(\mu)$ such that $\lambda - \beta = p - \beta - \delta$ is a partition (this restriction is denoted by the prime on the summation symbol).

For each $\pi \in S_t$ and each $\beta \in W(\mu)$, we define (uniquely) for each p, the t-tuple α by

$$\pi(p - \alpha) = p - \beta. \tag{2.16a}$$

Using the properties

$$\begin{aligned} K(\mu, \beta - p + \pi p) &= K(\mu, \pi\alpha) = K(\mu, \alpha), \\ (p - \beta - \delta)\circ\pi^{-1} &= \pi^{-1}\pi(p - \alpha) - \delta = \lambda - \alpha, \end{aligned} \tag{2.16b}$$

we now bring the intermediate form (2.15) to the following expression:

$$\begin{aligned} A_\lambda\begin{pmatrix} a, b, d, e \\ c \end{pmatrix} = &\prod_{s=1}^{t} (a+b+c-s+1)_{\lambda_s}(d+e+c-s+1)_{\lambda_s} \\ &\times \sum_{\mu} \langle {}_2\mathscr{F}_1(a, b; a+b+c)|\mu\rangle \\ &\times \sum_{\alpha} K(\mu, \alpha)\langle {}_2\mathscr{F}_1(d+c, e+c; d+e+c)|\lambda - \alpha\rangle. \end{aligned} \tag{2.17}$$

The summation over μ is over all partitions μ such that $0 \leq \mu_s \leq \lambda_s$, $s = 1, 2, \ldots, t$. For each such μ, the summation is over all $\alpha \in W(\mu)$. This last step requires proof, which we next give.

Proof. In Eq. (2.15), β runs over the subset of $\beta \in W(\mu)$ such that $\lambda - \beta$ is a partition, and π runs over all $t!$ elements in S_t. To justify that the final summation in Eq. (2.17) is over all $\alpha \in W(\mu)$ in Eq. (2.15), we first eliminate β in favor of α by using Eq. (2.16a), so that (symbolically)

$$\sum_{\beta} \sum_{\pi \in S_t} \to \sum_{\pi \in S_t} \sum_{\alpha \in A_p(\pi)} .$$

The set $A_p(\pi)$ is the set of weights $\{\alpha\}$ defined for each given π and p by

$$A_p(\pi) = \left\{ \alpha \in W(\mu) \;\middle|\; \begin{array}{c} \pi(p - \alpha) - \delta \in W(\mu); \\ p - \alpha \text{ has distinct components} \end{array} \right\}.$$

The sets $A_p(\pi)$ and $A_p(\pi')$ are disjoint for $\pi \neq \pi'$, since any common element α must satisfy $\pi(p - \alpha) = \pi'(p - \alpha)$, which implies $\pi = \pi'$, since the components of $p - \alpha$ are distinct. Finally, for each $\alpha \in W(\mu)$, we have either $p - \alpha$ has at least two components equal, or all components are distinct, in which case there exists a unique permutation π' such that $\pi'(p - \alpha) - \delta \in W(\mu)$; the permutation π' is the one that brings the components of $p - \alpha$ to decreasing order (as read from left to right). Thus, each $\alpha \in W(\mu)$ is either such that $D(p - \alpha) = 0$ or α belongs to one of the sets in the family $\{A_p(\pi) | \pi \in S_t\}$. We have thus proved

$$\prod_{\pi \in S_t} A_p(\pi) = \{\alpha \in W(\mu) | D(p - \alpha) \neq 0\}.$$

This result proves that the summation in the right-hand side of (2.15) may be replaced by a summation $\alpha \in W(\mu)$. □

An equivalent, but computationally very convenient, form of Eq. (2.17) is obtained by eliminating the Kostka numbers and replacing the sum over distinct weights $\alpha \in W(\mu)$ by a sum over *all* Gel'fand–Weyl patterns $(m) \in \mu$ (that is, all patterns (m) whose top row is μ). In this way we obtain

$$\begin{aligned} A_\lambda\begin{pmatrix} a, b, d, e \\ c \end{pmatrix} = {} & \prod_{s=1}^{t} (a + b + c - s + 1)_{\lambda_s} (d + e + c - s + 1)_{\lambda_s} \\ & \times \sum_{\mu} \langle {}_2\mathscr{F}_1(a, b; a + b + c) | \mu \rangle \\ & \times \sum_{(m) \in \mu} \langle {}_2\mathscr{F}_1(d + c, e + c; d + e + c) | \lambda - W(m) \rangle . \end{aligned} \tag{2.18}$$

It is useful to rewrite Eq. (2.17) in still another way that parallels relation (1.22a) for $t = 1$. We use the identity for indeterminate y,

$$\frac{(y-s+1)_{\lambda_s}}{(y-s+1)_{\lambda_s-\alpha_s}} = (p_s + y - t - \alpha_s + 1)_{\alpha_s}, \tag{2.19}$$

where $p_s = \lambda_s + t - s$. We also use relations (1.6) and (1.7) to express $M^{-1}(\lambda)$ and $M^{-1}(\lambda - \alpha)$ in terms of Vandermonde determinants. In this way, we bring A_λ to the form

$$\begin{aligned} A_\lambda\begin{pmatrix} a, b, d, e \\ c \end{pmatrix} = M^{-1}(\lambda) \prod_{s=1}^{t} &(d + c - s + 1)_{\lambda_s}(e + c - s + 1)_{\lambda_s} \\ &\times (a + b + c - s + 1)_{\lambda_s} \\ &\times \sum_{\mu} \langle {}_2\mathscr{F}_1(a, b; a + b + c)|\mu\rangle G^t_\mu(c, d, e; p), \end{aligned} \tag{2.20a}$$

where G^t_μ is defined *for arbitrary variables* $z = (z_1, z_2, \ldots, z_t)$ by

$$\begin{aligned} G^t_\mu(c, d, e; z) &= D^{-1}(z) \sum_{\alpha} K(\mu, \alpha) D(z - \alpha) \\ &\times \prod_{s=1}^{t} \frac{(z_s - \alpha_s + 1)_{\alpha_s}(z_s + c + d + e - t - \alpha_s + 1)_{\alpha_s}}{(z_s + c + d - t - \alpha_s + 1)_{\alpha_s}(z_s + c + e - t - \alpha_s + 1)_{\alpha_s}}. \end{aligned} \tag{2.20b}$$

It is useful to introduce this new function G^t_μ, since it has an important structural property given by the following lemma.

LEMMA 2.1. *The function* $G^t_\mu(c, d, e; z)$ *has the property*

$$G^t_\mu(c, d, e; p) = 0, \qquad \textit{unless } \mu_s \leq \lambda_s;\ s = 1, \ldots, t, \tag{2.21}$$

for $z = p = \lambda + \delta$.

Proof. This property is a consequence of the fact that the summation over the weights α in Eq. (2.20b) has the form

$$\sum_{\alpha} K(\mu, \alpha) D(p - \alpha) S(\alpha, p),$$

where $K(\mu, \pi\alpha) = K(\mu, \alpha)$, $D(\pi(p - \alpha)) = \varepsilon_\pi D(p - \alpha)$, and $S(\pi\alpha, \pi p) = S(\alpha, p)$. This sum is nonzero if and only if the components of $p - \alpha$ are distinct and in this case there exists a unique permutation $\pi' \in S_t$ that brings the components of $p - \alpha$ to strictly decreasing order. Accounting for $D(\pi' p) = \varepsilon_{\pi'} D(p)$, we see that $G^t_\mu(c, d, e; p)$ can always be expressed, for given p, in a modified form where the summation over α extends only over those $\alpha \in W(\mu)$ such that

$$p_1 - \alpha_1 > p_2 - \alpha_2 > \cdots > p_t - \alpha_t.$$

The rising factorial $(z_s - \alpha_s + 1)_{\alpha_s}$ appearing in Eq. (2.20b) has the property (which follows from the definition given in Eq. (1.5a)):

$$(p_s - \alpha_s + 1)_{\alpha_s} = 0, \qquad \text{unless } \alpha_s \leq p_s,\ s = 1, \ldots t.$$

These two properties, in turn, imply that the only weights $\alpha \in W(\mu)$ that contribute to the summation are those satisfying the conditions

$$\alpha_1 \leq \lambda_1, \alpha_2 \leq \lambda_2, \ldots, \alpha_t \leq \lambda_t,$$
$$\lambda_1 - \alpha_1 \geq \lambda_2 - \alpha_1 \geq \cdots \geq \lambda_t - \alpha_t \geq 0.$$

The only partitions $\mu \in \mathbb{P}_t$ that have weights α satisfying these conditions are those for which

$$\mu_1 \leq \lambda_1, \mu_2 \leq \lambda_2, \ldots, \mu_t \leq \lambda_t.$$

Thus, the summation over α in Eq. (2.20b) yields zero for $z = p = \lambda + \delta$ unless $\mu_s \leq \lambda_s$, $s = 1, \ldots, t$, as stated in Eq. (2.21). □

Remarks. (1) For arbitrary variables z, the (formal) function $G_t(a, b, c, d, e; z)$ defined by

$$G_t(a, b, c, d, e; z) = \sum_{\mu} \langle {}_2\mathscr{F}_1(a, b; a + b + c)|\mu\rangle G_\mu^t(c, d, e; z) \tag{2.22}$$

is an infinite series.

(2) Equation (2.20a) is in a form analogous to Eq. (1.22a). Comparing these two relations for $t = 1$ and setting $\lambda_1 = n \in \mathbb{N}$ shows that

$$\begin{aligned} G_1(a, b, c, d, e; n) &= \sum_k \langle {}_2\mathscr{F}_1(a, b; a + b + c)|k\rangle G_k^1(c, d, e; n) \\ &= {}_4F_3\left(\begin{matrix} a, b, 1 - c - d - e - n, -n; \\ a + b + c, 1 - c - d - n, 1 - c - e - n \end{matrix}\right). \end{aligned} \tag{2.23}$$

The extension of this result given by Eq. (2.22) for $t = 1$ is the replacement of n by an arbitrary variable z_1, in which case the hypergeometric series is nonterminating.

(3) The function G_μ^t defined by Eq. (2.20b) is always a finite series for arbitrary variables z. Indeed, it is a symmetric function in the variables z

as easily follows from $K(\mu, \pi\alpha) = K(\mu, \alpha)$, $D(\pi(z - \alpha) = \varepsilon_\pi D(z - \alpha)$, $D(\pi z) = \varepsilon_\pi D(z)$, each $\pi \in S_t$, where we also observe that permutations of the weights α leave the sum over α invariant. Thus,

$$G_\mu^t(c, d, e; \pi z) = G_\mu^t(c, d, e; z), \qquad \text{each } \pi \in S_t. \tag{2.24}$$

III. The Polynomials P_k^t and Their Properties

All our attempts at a *direct* attack on proving the main theorem (S_4 symmetry of the polynomial A_λ) were unsuccessful. In such a situation, the standard strategy (cf. Polya's heuristics [32]) is to retreat to the study of simpler subproblems, in the hope that this indirect approach may offer insights that will carry the day, as indeed proved to be the case here.

We show below that the polynomial $A_\lambda\binom{a,b,d,e}{c}$ simplifies remarkably when a is a nonpositive integer. *But there is an even greater advantage in this specialization: aside from a multiplicative factor* (which can be removed) *the result is a symmetric polynomial in the partial hooks* $\{p_s \equiv \lambda_s + t - s\}$, *as well as a polynomial in the remaining four parameters* (b, c, d, e). Study of these special cases of $A_\lambda\binom{a,b,d,e}{c}$ for $a = -k$, $k \in \mathbb{N}$ will prove to be quite rewarding and (by appeal to Carlson's theorem or even the Lagrange interpolation formula) will allow at the end full recovery of the polynomial A_λ itself. This is the motivation behind our introduction of yet another family of polynomials, which we denote by $P_k^t(b, d, e; c; p)$ and now define.

Definition. The functions $P_k^t(b, d, e; c; p)$ are defined by

$$\begin{aligned} P_k^t(b, d, e; c; p) \equiv M(\lambda) \cdot A_\lambda\binom{-k, b, d, e}{c} \\ \times \Bigg[\prod_{s=1}^{t} (c + b - s + 1)_{\lambda_s - k} (c + d - s + 1)_{\lambda_s - k} \\ \times (c + e - s + 1)_{\lambda_s - k} \Bigg]^{-1}, \end{aligned} \tag{3.1a}$$

where $k, t \in \mathbb{N}$, b, d, e, c are indeterminates and λ is a partition in t-parts. The parts λ_s of the partition λ and the partial hooks p_s are related by

$$p_s = \lambda_s + t - s, \qquad 1 \le s \le t. \tag{3.1b}$$

Let us now establish some properties of the P_k^t which help in showing why these objects are of interest.

LEMMA 3.1. *For $a = -k$ the function $A_\lambda\binom{a,b,d,e}{c}$ has the factors*:

$$\prod_{s=1}^{t}(c+b-s+1)_{\lambda_s-k}(c+d-s+1)_{\lambda_s-k}(c+e-s+1)_{\lambda_s-k}. \quad (3.2)$$

Proof. The condition $a = -k$ forces the partition μ in the summation in Eq. (2.20a) to those partitions for which $\mu_s \leq k$, in consequence of the zeros of the factors $(-k-s+1)_{\mu_s}$ of $\langle {}_2\mathscr{F}_1(-k, b; -k+b+c)$. By examining the terms that enter the sum, the form claimed in the lemma is seen to be valid. □

Remark. Using the symmetries discussed in Section I, a similar result obtains for b, d, or $e = -k$. For $c + a = -k$, a result of the same form obtains but the factors are in b, d, e (not in $c+b$, $c+d$, $c+e$ as above), and similarly for $c+b=-k$, $c+d=-k$, $c+e=-k$.

The usefulness of the partial hook variables is established by the next lemma.

LEMMA 3.2. *The function $A_\lambda\binom{a,b,d,e}{c}$ regarded as a function of the parameters $\{p_s\}$ obeys the symmetry: $A_\lambda \overset{\pi}{\to} \varepsilon_\pi A_\lambda$ for every permutation π of the variables $\{p_s\}$.*

Proof. This result has essentially been established in Section II, where it was remarked (see Eq. (2.24)) that the terms appearing in the sum defining $A_\lambda\binom{a,b,d,e}{c}$, Eq. (2.20a), are invariant under permutations of the partial hooks. This leaves only the multiplicative factors outside the sum in Eq. (2.20a) to be considered. These factors are

$$M^{-1}(\lambda)\prod_{s=1}^{t}(d+c-s+1)_{\lambda_s}(e+c-s+1)_{\lambda_s}(a+b+c-s+1)_{\lambda_s}.$$

From the definition of the Kummer notation we see that

$$(d+c-s+1)_{\lambda_s} \equiv \frac{\Gamma(d+c-s+1+\lambda_s)}{\Gamma(d+c-s+1)} = \frac{\Gamma(d+c+1-t+p_s)}{\Gamma(d+c-s+1)},$$

and similarly for the remaining terms under the product. Clearly any permutation simply rearranges the terms in the product leaving the product itself invariant.

The remaining factor $M^{-1}(\lambda)$ has the definition:

$$\begin{aligned} M(\lambda) &= \prod_{s=1}^{t}(\lambda_s+t-s)! \Big/ \prod_{r<s}(\lambda_r-\lambda_s+s-r), \\ &= \prod_{s=1}^{t} p_s! \Big/ \prod_{r<s}(p_r-p_s). \end{aligned} \quad (3.3)$$

Hence,

$$M(\lambda) \xrightarrow{\pi} \varepsilon_\pi M(\lambda), \qquad \text{each } \pi \in S_t,$$

and the lemma follows. □

Using Lemma 3.2, definition (3.1) of P_k^t, and the form of A_λ given by Eq. (2.20a) (with $a = -k$), we can combine various factors,

$$\frac{(c+d-s+1)_{\lambda_s}}{(c+d-s+1)_{\lambda_s-k}(\lambda_s+c+d-\alpha_s-s+1)_{\alpha_s}} = (\lambda_s+c+d-k-s+1)_{k-\alpha_s},$$

with a similar expression with e replacing d. Similarly, we have

$$\frac{(-k+b+c-s+1)_{\lambda_s}}{(b+c-s+1)_{\lambda_s-k}(-k+b+c-s+1)_{\mu_s}} = (-k+b+c+\mu_s-s+1)_{k-\mu_s}.$$

Carrying out these substitutions explicitly in definition (3.1) of P_k^t and recognizing that the partial hook variables p_s can be extended to arbitrary variables z_s, we establish

THEOREM 3.1. *The functions $P_k^t(b,d,e;c;z)$ for $k, t \in \mathbb{N}$ are polynomials in the indeterminates b, d, e, c and symmetric polynomials in the variables $z = (z_1, z_2, \ldots, z_t)$. The explicit polynomials P_k^t are given by*

$$\begin{aligned} &P_k^t(b,d,e;c;z) \\ &\quad = \sum_\mu \langle {}_1\mathscr{F}_0(-k)|\mu\rangle \prod_{s=1}^t (b-s+1)_{\mu_s}(b+c-k+\mu_s-s+1)_{k-\mu_s} \\ &\qquad \times F_{k,\mu}^t(c,d,e;z), \end{aligned} \tag{3.4a}$$

where the polynomials $F_{k,\mu}^t$ in c, d, e, and $z = (z_1, z_2, \ldots, z_t)$ are defined by

$$\begin{aligned} &F_{k,\mu}^t(c,d,e;z) \\ &\quad = D^{-1}(z)\sum_\alpha K(\mu,\alpha)D(z-\alpha) \\ &\qquad \times \prod_{s=1}^t (z_s-\alpha_s+1)_{\alpha_s}(z_s+c+d+e-t-\alpha_s+1)_{\alpha_s} \\ &\qquad \times (z_s+c+d-k-t+1)_{k-\alpha_s}(z_s+c+e-k-t+1)_{k-\alpha_s}. \end{aligned} \tag{3.4b}$$

It is the factor $\langle {}_1\mathscr{F}_0(-k)|\mu\rangle$ that restricts the summation over μ to $\mu_s \le k$ $(s = 1, 2, \ldots, t)$ in the definition (3.4) of the polynomials P_k^t. Accordingly, these polynomials are well defined for arbitrary indeterminates $z = (z_1, z_2, \ldots, z_t)$; that is, we may replace the integer-valued partial hooks p in the definition (3.1) by z.

The introduction of the auxiliary polynomials $F_{k,\mu}^t$ (given by Eq. (3.4b)) into the definition (3.4a) of the polynomials P_k^t is for the purpose of making the structure of the latter easier to discuss. The polynomials $F_{k,\mu}^t$ carry the full (symmetric) dependence of the polynomials P_k^t on the variables $(z_1, z_2, \ldots, z_t)$. The properties of the $F_{k,\mu}^t$ given in Eqs. (3.7)–(3.8) and Lemma 3.3 below underlie similar results (Eq. (3.9) and Lemma 3.3) for the P_k^t themselves.

The polynomials $P_k^t(b, d, e; c; z)$ defined by Eqs. (3.4) are explicit, but they do not show, in general, the desired $b \leftrightarrow e$ symmetry. Nonetheless, they possess a number of properties that are important for establishing $b \leftrightarrow e$ symmetry. These properties are inferred either from those of the A_λ already proved in Sections I and II, or directly from the definition (3.4).

We begin by giving several properties of the polynomials $F_{k,\mu}^t$:

Permutational symmetry in z.

$$F_{k,\mu}^t(c, d, e; \pi z) = F_{k,\mu}^t(c, d, e; z), \qquad \text{each } \pi \in S_t. \tag{3.5}$$

Zeros.

$$F_{k,\mu}^t(c, d, e; p) = 0, \qquad \text{unless } \mu_s \le \lambda_s,\ s = 1, \ldots, t, \tag{3.6}$$

where $p = \lambda + \delta$.

Reduction formulas.

$$\begin{aligned} &F_{k,(\mu_1,\ldots,\mu_t)}^t(c, d, e; z_1, \ldots, z_{t-1}, 0) \\ &\quad = \delta_{\mu_t,0}(c + d - k - t + 1)_k(c + e - k - t + 1)_k \\ &\qquad \times F_{k,(\mu_1,\ldots,\mu_{t-1})}^{t-1}(c, d, e; z_1 - 1, \ldots, z_{t-1} - 1), \end{aligned} \tag{3.7}$$

$$\begin{aligned} F_{k,(\mu_1,\ldots,\mu_t)}^t(c, d, e; z_1, \ldots, z_t) &= \left[\prod_{s=1}^{t} z_s(z_s + c + d + e - t)\right] \\ &\times F_{k-1,(\mu_1-1,\ldots,\mu_t-1)}^t(c, d, e; z_1 - 1, \ldots, z_t - 1), \qquad \mu_t \ge 1. \end{aligned} \tag{3.8}$$

Proof of Eqs. (3.5)–(3.8). The permutational symmetry is obvious from the definition (3.4b), using the permutational symmetries of the variables in the Vandermonde determinants and of the weights α in the Kostka numbers. The zeros given by Eq. (3.6) are just a restatement of property

(2.21) of the functions G^t_μ. Relation (3.7) is proved directly from the definition (3.4), using the following results:

$$(\alpha_t + 1)_{\alpha_t} = 0, \qquad \text{unless } \alpha_t = 0,$$
$$\alpha_t = 0 \qquad \text{implies } \mu_t = 0,$$
$$\prod_{s=1}^{t-1} \frac{(z_s - \alpha_s)}{z_s}(z_s - \alpha_s + 1)_{\alpha_s} = \prod_{s=1}^{t-1}(z_s - \alpha_s)_{\alpha_s},$$
$$D(z_1, \dots, z_{t-1}, 0) = D(z_1, \dots, z_{t-1})\left(\prod_{s=1}^{t-1} z_s\right).$$

Relation (3.8) is similarly proved directly from the definition (3.4) by noting that the functions $D(z)$, $D(z - \alpha)$, $K(\mu, \alpha)$ are all invariant to the shifts of z, α, and μ by $(1, 1, \dots, 1)$. □

Relation (3.8), applied to the polynomials P^t_k defined by Eqs. (3.4) implies

$$\begin{aligned} P^t_k(b, c, d; e; z_1, \dots, z_{t-1}, 0) &= [(-k + c + b - t + 1)_k(-k + c + d - t + 1)_k \\ &\qquad \times(-k + c + e - t + 1)_k] \\ &\quad \times P^{t-1}_k(b, c, d; e; z_1 - 1, \dots, z_{t-1} - 1). \end{aligned} \tag{3.9}$$

This relation, in turn, when used in Eq. (3.1) for A_λ, yields

$$A_{(\lambda_1, \dots, \lambda_{t-1}, 0)}\begin{pmatrix} -k, b, d, e \\ c \end{pmatrix} = A_{(\lambda_1, \dots, \lambda_{t-1})}\begin{pmatrix} -k, b, d, e \\ c \end{pmatrix}. \tag{3.10}$$

The following lemma summarizes easily verified symmetry properties of the functions $F^t_{k,\mu}$ and the polynomials P^t_k:

LEMMA 3.3. *The functions $F^t_{k,\mu}(b, c, d, e; z)$ are invariant under the interchange of d and e and under the linear transformation J defined by*

$$\begin{aligned} J\colon\ & b \mapsto b, \quad c \mapsto c, \\ & d \mapsto -c - d + t, \quad e \mapsto -c - e + t, \\ & z_s \mapsto z_s + c + d + e - t, \quad s = 1, 2, \dots, t. \end{aligned} \tag{3.11}$$

These symmetries are also true for polynomials $P^t_k(b, d, e; c; z)$.

Remark. The reduction formula (3.9) and J-symmetry for the polynomials P^t_k turn out to have key roles in our proof of $b \leftrightarrow e$ symmetry of these polynomials.

We need another important general property of the polynomials P_k^t in the sequel. For the statement of this property, we define the quantity $Q_{k,\lambda}^t$ by

$$
\begin{aligned}
&Q_{k,\lambda}^t(b,c,d,e) \\
&\quad = \prod_{s=1}^{t} (\lambda_s + c + b - k - s + 1)_{k-\lambda_s}(\lambda_s + c + d - k - s + 1)_{k-\lambda_s} \\
&\qquad \times (\lambda_s + c + e - k - s + 1)_{k-\lambda_s}. \qquad (3.12)
\end{aligned}
$$

LEMMA 3.4. *The polynomials P_k^t, evaluated on the integer-valued variables $p_s = \lambda_s + t - s$, $s = 1,2,\ldots,t$, may be written in the factored form*

$$P_k^t(b,d,e;c;p) = Q_{k,\lambda}^t(b,c,d,e)R_{k,\lambda}^t(b,c,d,e) \qquad (3.13a)$$

for all partitions $\lambda \in \mathbb{P}_t$ such that

$$0 \le \lambda_s \le k, \qquad s = 1,2,\ldots,t, \qquad (3.13b)$$

where $R_{k,\lambda}^t(b,c,d,e)$ is a polynomial in the parameters b,c,d,e.

Proof. Set $z = p = \lambda + \delta$ in Eqs. (3.4) and use property (3.7) of the functions $F_{k,\mu}^t$. The summation over μ in Eq. (3.4a) for P_k^t is then over all μ such that $0 \le \mu_s \le \lambda_s$, $s = 1,2,\ldots,t$. Accordingly, in this expression for P_k^t, we may write

$$
\begin{aligned}
&\prod_{s=1}^{t} (c + b - k + \mu_s - s + 1)_{k-\mu_s}(p_s + c + d - k - t + 1)_{k-\alpha_s} \\
&\quad = Q_{k,\lambda}^t(b,c,d,e)\Bigg[\prod_{s=1}^{t} (c + b - s + 1)_{\lambda_s-\mu_s} \\
&\qquad \times (c + d - s + 1)_{\lambda_s-\alpha_s}(c + e - s + 1)_{\lambda_s-\alpha_s}\Bigg].
\end{aligned}
$$

From this, we find that the explicit form of $R_{k,\lambda}^t$ in Eq. (3.13a) is

$$
\begin{aligned}
&R_{k,\lambda}^t(b,c,d,e) \\
&\quad = \sum_{\mu} \langle {}_1\mathscr{F}_0(-k)|\mu\rangle \prod_{s=1}^{t} (b - s + 1)_{\mu_s}(c + b - s + 1)_{\lambda_s-\mu_s} \\
&\quad \times F_{\lambda,\mu}^t(c,d,e), \qquad (3.14a)
\end{aligned}
$$

where

$$F^t_{\lambda,\mu}(c,d,e) = D^{-1}(p)\sum_{\alpha} K(\mu,\alpha)\prod_{s=1}^{t}(p_s - \alpha_s + 1)_{\alpha_s}$$
$$\times(p_s + c + d + e - t - \alpha_s + 1)_{\alpha_s}(c + d - s + 1)_{\lambda_s - \alpha_s}$$
$$\times(c + e - s + 1)_{\lambda - \alpha_s}. \tag{3.14b}$$

Since the summation is over all μ such that $0 \le \mu_s \le \lambda_s$, $s = 1,\ldots,t$, the function $R^t_{k,\lambda}$ is clearly a polynomial of finite degree in each of the parameters b, c, d, e. □

There are still five important properties of the polynomials P^t_k that need to be noted. Four of these, Eqs. (3.15)–(3.18), are just the transfer of the proved properties (1.21b), (1.21d), (1.18), and (1.25) of the A_λ to properties of the polynomials P^t_k; the fifth is a relation that is derived by using an elegant formula involving the Kostka numbers and the Vandermonde determinant. This formula was proved in [31] and leads here to an explicit expression for the coefficient of the term $(z_1 z_2 \ldots z_t)^{2k}$ in the polynomial P^t_k. We state, in sequence, the five properties in question indicating below their proofs.

The $d = 0$ polynomials:

$$P^t_k(b,0,e;c;z) = \prod_{s=1}^{t}(c - k - s + 1)_k(z_s + c + b - k - t + 1)_k$$
$$\times(z_s + c + e - k - t + 1)_k. \tag{3.15}$$

Polynomials with $d = -c = m$, $m \in \mathbb{N}$:

$$P^t_k(b,m,e;-m;p)$$
$$= \prod_{s=1}^{t}\frac{(-k - s + 1)_k(-p_s - b + t)_{k+m}(-p_s - e + t)_{k+m}}{(b - m - s + 1)_m(e - m - s + 1)_m} \tag{3.16a}$$

for all λ_s such that

$$0 \le \lambda_s \le k, \qquad s = 1,2,\ldots,t. \tag{3.16b}$$

The shifted-parameter polynomials:

$$\begin{aligned}
&P^t_{k+m}(b,d,e;m;z)\\
&\quad=\prod_{s=1}^{t}(b-s+1)_m(d-s+1)_m(e-s+1)_m\\
&\qquad\times P^t_k(b+m,d+m,e+m;-m;z),
\end{aligned}\tag{3.17}$$

where $m = -k, -k+1, \ldots, 0, 1, 2, \ldots$.

The specialized variable polynomials:

$$\begin{aligned}
&P^t_k(b,d,e;c;y+t-1,\ldots,y+1,y)\\
&\quad=\sum_{\mu}\langle{}_1\mathscr{F}_0(-k)|\mu\rangle\Bigg[\prod_{s=1}^{t}(c-k-s+1)_{\mu_s}(c+b-k+\mu_s-s+1)_{k-\mu_s}\\
&\qquad\qquad\times(c+d-k+\mu_s-s+1)_{k-\mu_s}(c+e-k+\mu_s-s+1)_{k-\mu_s}\Bigg]\\
&\qquad\times Q_\mu(\gamma;y),
\end{aligned}\tag{3.18a}$$

where $Q_\mu(\gamma;y)$ is the polynomial in the single indeterminate y and the parameter γ defined for each partition μ by

$$Q_\mu(\gamma;y)\equiv \operatorname{Dim}\mu\prod_{s=1}^{t}(y-\mu_s+s)_{\mu_s}(-y-\gamma-t-\mu_s+s+1)_{\mu_s}\tag{3.18b}$$

with

$$\gamma=2c+b+d+e-k-2t+1.\tag{3.18c}$$

Highest-term monomial:

$$\text{Coefficient in } P^t_k \text{ of } (z_1z_2\ldots z_t)^{2k}=\prod_{s=1}^{t}(c-k-s+1)_k.\tag{3.19}$$

Proof of Relation (3.15). This is a straightforward application of Eq. (1.21b) for $A_\lambda(a=-k, d=0)$ and the definition of $P^t_k(d=0)$ given by Eqs. (3.4), using only the property

$$(x)_{m+n}=(x)_m(x+m)_n$$

of rising factorials. □

Proof of Relation (3.16). The proof follows the same steps used in the proof above of Eq. (3.15). Now, however, the factor

$$\prod_{s=1}^{t}(-s+1)_{\lambda_s-k} \tag{3.20a}$$

which divides $A_\lambda(a=-k, d=-c=m)$ can be zero. To avoid division by 0 coming from this factor, we impose

$$0 \le \lambda_s \le k, \qquad s=1,2,\ldots,t. \tag{3.20b}$$

Then the factors

$$1/(-s+1)_{\lambda_s-k} = (-\lambda_s+k-s+1)_{k-\lambda_s} \tag{3.20c}$$

are well defined. This leads easily to

$$P_k^t(b,m,e;-m;p) = \prod_{s=1}^{t}\frac{(-k-s+1)_k(b-s+1)_{\lambda_s}(e-s+1)_{\lambda_s}}{(b-m-s+1)_{\lambda_s-k}(e-m-s+1)_{\lambda_s-k}}. \tag{3.20d}$$

In this expression, we use the following identity (and a similar one with b replaced by e) to obtain relation (3.16):

$$\begin{aligned}\frac{(b-s+1)_{\lambda_s}}{(b-m-s+1)_{\lambda_s-k}} &= \frac{(\lambda_s+b-m-k-s+1)_{k+m}}{(b-m-s+1)_m}\\ &= \frac{(-1)^{k-m}(-\lambda_s+s-b)_{k+m}}{(b-m-s+1)_m}.\end{aligned} \tag{3.20e}$$

(Note that $\lambda_s - s = p_s - t$.) □

Remark. Relation (3.16a) seems to contradict the fact that $P_k^t(b,d,e;c;z)$ is polynomial in the parameters (b,d,e,c) as well as in the variables $(z_1,\ldots,z_t)$. We discuss this in Section VI and show that, despite appearances, relation (3.16a) is indeed a polynomial in the parameters b and e.

Proof of Relation (3.17). This relation is a direct consequence of the definition (3.1) of the set of polynomials $\{P_k^t|k=0,1,\ldots\}$ in terms of the A_λ and its L-symmetry expressed in Lemma 1.1. □

Proof of Relation (3.18). This relation is a direct application of definition (3.1) and the identity (1.25) given in Lemma 1.3, the result being extended to an arbitrary indeterminate y (since each side of the relation is valid for infinitely many integers $\lambda \in \mathbb{N}$). □

Remark. The parameter γ defined by Eq. (3.18c) occurs naturally in the form (3.18a) of P_k^t. This relation for P_k^t has many of the features of the general result (see Theorem 6.1 and definition (4.15)) to be proved. We have been unable to prove the general result given in Theorem 6.1 directly from relations (3.18), but it is highly suggestive of that form. There appears to be no direct way of "lifting" the polynomials $Q_\mu(\gamma; y)$ in the single variable y to the general symmetric polynomial $T_\mu(\gamma; z)$ occurring in Eq. (4.15). We should point out that the result given in Theorem 6.1 was conjectured to be correct prior to the proof above of relations (3.18), this conjecture being based on the special cases of P_k^t given below (Eqs. (3.22)–(3.27)).

Proof of Relation (3.19). The coefficient of $(z_1 z_2 \ldots z_t)^{2k}$ in P_k^t, as given by Eqs. (3.4), is

$$\sum_\mu \langle {}_1\mathscr{F}_0(-k)|\mu\rangle \prod_{s=1}^{t} (b-s+1)_{\mu_s}(b+c-k+\mu_s-s+1)_k \times D^{-1}(z)\sum_\alpha K(\mu,\alpha)D(z-\alpha).$$

The following relation was proved in [31]:

$$\sum_\alpha K(\mu,\alpha)D(z-\alpha) = (\operatorname{Dim}\mu)D(z). \tag{3.21}$$

Using this relation in the above expression gives

$$\begin{aligned}
&\text{coefficient of } (z_1 z_2 \ldots z_t)^k \\
&= \prod_{s=1}^{t} (-1)^k (t-s+1)_k \sum_\mu \langle {}_1\mathscr{F}_0(b)|\mu\rangle\langle {}_1\mathscr{F}_0(-b-c+t)|\bar{\mu}\rangle \\
&= (-1)^{kt} \prod_{s=1}^{t} (t-s+1)_k \langle {}_1\mathscr{F}_0(-c+t)|k\ldots k\rangle \\
&= \prod_{s=1}^{t} (c-k-s+1)_k .
\end{aligned}$$

In obtaining this result, we have rearranged various factors and used the generalized addition rule, Eq. (1.15). □

The properties given above in Eqs. (3.15)–(3.19) are all for general $t, k \in \mathbb{N}$, but with specialized parameters and variables. It is also useful (and difficult) to calculate special cases for t and k, but with general parameters and variables, using the definition (3.1); that is, Eqs. (3.4). The difficulty with this direct procedure (aside from the division by $D(z)$) is that the individual terms, corresponding to partitions μ in the summation, do not show $b \leftrightarrow e$ symmetry. Accordingly, the terms must be assembled in a quite different way than that presented term-wise by Eqs. (3.4). It is this feature of the P_k^t given by Eqs. (3.4) that makes it difficult to work with them. The re-assembled result, showing explicitly $b \leftrightarrow e$ symmetry, is quite remarkable in its structure, as we now illustrate by examples.

Examples of the Polynomials P_k^t

The choice of the normalization given in the definition (3.1) has been made explicitly to yield the result:

$$P_0^t(b, d, e; c; p) = 1.$$

(One may view this result alternatively as defining the function A_λ for the special case $a = 0$. Note that this result is indeed symmetric in b, d, e, as has already been remarked in Section I.)

We have established (by direct computation) the following explicit results as examples of the polynomials P_k^t:

Example 1.

$$P_{k=1}^{t=1} = (c-1)(p_1)(p_1 + \gamma_{1,1}) + (c+b-1)(c+d-1)(c+e-1), \tag{3.22a}$$

where we have introduced a more explicit notation $\gamma_{t,k}$ for the parameter γ of Eq. (3.18c),

$$\gamma_{t,k} \equiv 2c + b + d + e + 1 - 2t - k. \tag{3.22b}$$

This definition of $\gamma_{t,k}$ is designed to make apparent a very important, empirically discovered, symmetry, which we prove below, namely, the involutary (reflection) symmetry called R-symmetry.

Empirically found symmetry. The polynomial P_k^t is invariant under each of the involutions R_s, $s = 1, 2, \ldots, t$, defined by

$$R_s\colon p_s \mapsto -(p_s + \gamma_{t,k}), \qquad p_r \mapsto p_r,\ r = 1, 2, \ldots, t\ (r \neq s). \tag{3.23}$$

Note that this substitution affects a single p_s, not the entire set (although this weaker result is true a posteriori). This symmetry is valid in all the examples to follow.

It is significant that $\gamma_{t,k}$ can be written much more suggestively (using $a = -k$) in the form:

$$\gamma_t = 2c + a + b + d + e + 1 - 2t. \tag{3.24}$$

We remark that in this form, the expression γ_t is totally symmetric in the parameters a, b, d, e and is invariant under the substitution L, Eq. (1.18).

Example 2.

$$\begin{aligned} P_{k=1}^{t=2} = {} & (c-2)_2 p_1(p_1 + \gamma_{2,1}) p_2(p_2 + \gamma_{2,1}) \\ & + (c-1)(c+b-2)(c+d-2)(c+e-2) \\ & \quad \times [(p_1 - 1)(p_1 + 1 + \gamma_{2,1}) + p_2(p_2 + \gamma_{2,1})] \\ & + (c+b-2)_2(c+d-2)_2(c+e-2)_2. \end{aligned} \tag{3.25}$$

Here $\gamma_{2,1}$ has the value: $\gamma_{2,1} = 2c + b + d + e - 4$. This empirically found form demonstrates several interesting features:

- symmetry in b, d, e.
- symmetry in p_1 and p_2 (but *not* explicitly!). That is to say, the "optimal" way to write the polynomial seemingly *breaks* $p_1 \leftrightarrow p_2$ symmetry.
- R-symmetry is explicitly valid.

We have derived a recursion relation for the polynomials $P_{k=1}^{t}$ which proves these three points to be valid for $P_{k=1}^{t}$ for every $t = 1, 2, \ldots$. Both the recursion relation and the explicit result for $P_{k=1}^{t}$ will be given in a wider context below. It is surprisingly difficult (even with algebraic computer software) to obtain further explicit cases!

Example 3.

$$\begin{aligned} P_{k=2}^{t=1} = {} & (c-2)_2(p_1 - 1)_2(p_1 + \gamma_{1,2})_2 \\ & + 2(c-2)(c+b-1)(c+d-1)(c+e-1)p_1(p_1 + \gamma_{1,2}) \\ & + (c+b-2)_2(c+d-2)_2(c+e-2)_2. \end{aligned} \tag{3.26}$$

Here clearly the empirical symmetries noted above are again valid.

Example 4. The case $P_{k=2}^{t=2}$ was calculated by hand—a very laborious task indeed! It was checked on MACSYMA at Los Alamos National Laboratory. This case proved to be the "Rosetta stone" of the problem, since the explicit result is so contrary to anyone's reasonable guess that it demanded, and eventually received, explanation:

$$
\begin{aligned}
P_{k=2}^{t=2} = {} & (c-2)_2(c-3)_2 p_2(p_2-1)(p_2+\gamma_{2,2})(p_2+\gamma_{2,2}+1) \\
& \times p_1(p_1-1)(p_1+\gamma_{2,2})(p_1+\gamma_{2,2}+1) && \begin{pmatrix} 22 \\ 2 \end{pmatrix} \\
& +2(c-2)_2(c-3)_1(c+b-2)(c+d-2)(c+e-2) \\
& \times[p_2(p_2-1)(p_2+\gamma_{2,2})(p_2+\gamma_{2,2}+1)p_1(p_1+\gamma_{2,2}) && \begin{pmatrix} 21 \\ 2 \end{pmatrix} \\
& +p_2(p_2+\gamma_{2,2})p_1(p_1-2)(p_1+\gamma_{2,2})(p_1+\gamma_{2,2}+2)] && \begin{pmatrix} 21 \\ 1 \end{pmatrix} \\
& +1(c-1)_2(c+b-3)_2(c+d-3)_2(c+e-3)_2 \\
& \times[p_2(p_2-1)(p_2+\gamma_{2,2})(p_2+\gamma_{2,2}+1) && \begin{pmatrix} 20 \\ 2 \end{pmatrix} \\
& +p_2(p_2+\gamma_{2,2})(p_1-2)(p_1+\gamma_{2,2}+2) && \begin{pmatrix} 20 \\ 1 \end{pmatrix} \\
& +(p_2-1)(p_1-2)(p_1+\gamma_{2,2}+1)(p_1+\gamma_{2,2}+2)] && \begin{pmatrix} 20 \\ 0 \end{pmatrix} \\
& +3(c-3)_2(c+b-2)_2(c+d-2)_2(c+e-2)_2 \\
& \times[p_2(p_2+\gamma_{2,2})p_1(p_1+\gamma_{2,2})] && \begin{pmatrix} 11 \\ 1 \end{pmatrix} \\
& +2(c-2)(c+b-3)_3(c+d-3)_3(c+e-3)_3 \times [p_2(p_2+\gamma_{2,2}) && \begin{pmatrix} 10 \\ 1 \end{pmatrix} \\
& +(p_1-1)(p_1+\gamma_{2,2}+1)] && \begin{pmatrix} 10 \\ 0 \end{pmatrix} \\
& +(c+b-2)_2(c+d-2)_2(c+e-2)_2 \\
& \times(c+b-3)_2(c+d-3)_2(c+e-3)_2. && \begin{pmatrix} 00 \\ 0 \end{pmatrix}
\end{aligned}
\tag{3.27}
$$

Here $\gamma_{2,2} \equiv 2c + b + d + e - 5$. (The (Gel'fand–Weyl) patterns in the right-hand margin identify terms in the sum, as will be explained below.)

It will be seen at once that this example validates, once again, the three empirical features discussed above. There are two especially curious features of Examples 2 and 4, which are the only ones above with two variables ($t = 2$):

- the peculiar shifts in the variable p_1 versus p_2, and
- the unreasonable appearance of the numerical coefficients 2, 1, 3, 2 in the four middle terms of Example 4. Yet these numerical coefficients are indeed correct.

The hints contained in these few examples as to the properties, and correct form, of the polynomials P_k^t all proved to be significant. (MACSYMA calculation for the next example $P_{k=3}^{t=3}$ proved to be prohibitively long.) In particular the involutary symmetry R is the key to our proof of the main theorem. We will spare the reader at this point the travail (but no little fun as well) of inducing (from these few examples essentially) the conjectured form (given in the next section), which we will prove to be correct in the remaining sections.

R-symmetry was discovered empirically, as described above, but we can now go back and check its validity for the special cases given by Eqs. (3.15), (3.16a), and (3.18). Using $(x)_k = (-1)^k(-x-k+1)_k$, we find that the monomial (3.15) is indeed invariant under each transformation,

$$R_s: \quad z_s \mapsto -z_s - 2c - b - e - 1 + 2t + k, \qquad (3.28)$$
$$z_r \mapsto z_r, \qquad r = 1, 2, \ldots, t \qquad (r \neq s).$$

Similarly, relation (3.16a) possesses R-symmetry (using the γ-value appropriate to the special variables in that relation).

Remarkably, relation (3.18a) also possesses a form of R-symmetry. Since $\lambda = (\dot{\lambda})$ in Eq. (3.18a), there is only a single variable y that occurs and we must interpret R-symmetry as the product $R = R_1 R_2 \ldots R_t$. To prove this R-symmetry, we first permute the variables (using the proved permutational symmetry) and then apply R:

$$\begin{aligned}
&(y + t - 1, \ldots, y + 1, y) \\
&\quad \xrightarrow{\pi} (y, y + 1, \ldots, y + t - 1) \\
&\quad \xrightarrow{R} (-y - \gamma, -y - 1 - \gamma, \ldots, -y - t + 1 - \gamma), \qquad (3.29)
\end{aligned}$$

which is the transformation of the original variables given by

$$y \to -y - \gamma - t + 1.$$

This is just the symmetry shown explicitly by $Q_\mu(\gamma; y)$.

One can derive a recurrence relation for the polynomial $P_1^t(b, d, e; c; z_1, \ldots, z_t)$ $(k = 1)$, which can be iterated to give the polynomials P_1^t for all $t = 1, 2, \ldots$. The recurrence relation is

$$\begin{aligned}
&P_1^t(b, d, e; c; z_1, \ldots, z_t) \\
&\quad = (c - 1) z_t (z_t + \gamma_{t,1}) P_1^{t-1}(b, d, e; c - 1; z_1, \ldots, z_{t-1}) \\
&\qquad + (c + b - t)(c + d - t)(c + e - t) \\
&\qquad\quad \times P_1^{t-1}(b, d, e; c; z_1 - 1, \ldots, z_{t-1} - 1). \qquad (3.30)
\end{aligned}$$

This recurrence relation was originally proved directly from the defining relations, Eqs. (1.8) and (3.1), but this derivation is tedious and nonilluminating and will be omitted.

It suffices here to point out that the correctness of relation (3.30) can be verified by using one of the main theorems of this paper, Theorem 6.2. Applied to the special case at hand ($k = 1$), Theorem 6.1 asserts that

$$P_1^t(b,d,e;c;z) = \mathscr{P}_1^t(b,d,e;c;z) \tag{3.31}$$

for all parameters (b,d,e,c), variables $(z_1, z_2, \ldots, z_t)$, and all $t \in \mathbb{N}$. Starting from P_1^1 given by Eq. (3.22a), one can, in fact, iterate relation (3.30) to obtain the result for $\mathscr{P}_1^t$ given by Eq. (4.15) for $k = 1$.

We have given in this section the principal properties of the polynomials P_k^t that suggest—and are needed to establish—a completely new expression for the P_k^t. This new form is formally introduced in the next section and proved in the remaining two sections.

IV. The Conjectured Polynomials $\mathscr{P}_k^t(b,d,e;c;z)$

We will give in this section a conjectured general form for the polynomials P_k^t discussed in Section III above. We will prove (in Section VI below) that these conjectured polynomials are in fact precisely the original polynomials P_k^t. For logical precision, however, we will distinguish these two sets of polynomials, using $\mathscr{P}_k^t$ for the conjectured set and (as before) P_k^t for the original set.

We will show in this section that the conjectured set $\mathscr{P}_k^t$ allows a determination of a new form for A_λ, which is correspondingly also a conjectured form distinguished typographically as $\mathscr{A}_\lambda(\begin{smallmatrix} a, b, d, e \\ c \end{smallmatrix})$. The identity of A_λ and $\mathscr{A}_\lambda$ is established in Section VI.

Of particular interest is the fact that for both these new forms ($\mathscr{P}_k^t$ and $\mathscr{A}_\lambda$) symmetry under $b \leftrightarrow e$ interchange is fully explicit. Moreover, these new forms involve, in a natural way, a new structure: *inhomogeneous symmetric polynomials* in the partial hooks, $\{p_s\}$. We first develop these new symmetric functions, and then give the conjectured forms.

The new symmetric polynomials are defined in terms of the set of Gel'fand–Weyl patterns $\{(m)\}$ having as their top row of $U(n)$ irrep labels $\mu = (\mu_1, \mu_2, \ldots, \mu_t) \equiv (m_{1t}, m_{2t}, \ldots, m_{tt})$ (see Eq. (2.3a)). The variables in these new symmetric functions are the n-tuples $z = (z_1, z_2, \ldots, z_t)$ of indeterminates.

DEFINITION. With each Gel'fand–Weyl pattern (m), as described by Eqs. (2.3), we associate a polynomial $t_{(m)}(z)$,

$$t_{(m)}(z) = \prod_{j=1}^{t} \prod_{i=1}^{j} (z_j - m_{ij} - j + i + 1)_{m_{ij} - m_{i,j-1}}, \tag{4.1}$$

where, by definition, $m_{i,j} = 0$ for $j < i$.

The symmetric polynomials $t_\mu(z)$ are then obtained by summing the polynomials $t_{(m)}(z)$ over all patterns (m):

$$t_\mu(z) = \sum_{(m) \in G_\mu} t_{(m)}(z). \tag{4.2}$$

Here G_μ denotes the set of all n-rowed Gel'fand–Weyl patterns having partition μ for their top row.

Let us now state some of the most important properties of the polynomials $t_\mu(z)$. (Proofs are given in [33].)

THEOREM 4.1. *The polynomials $t_\mu(z)$ are symmetric polynomials in the variables $(z_1, z_2, \dots, z_t)$ for each partition $\mu = (\mu_1, \mu_2, \dots, \mu_t)$.*

This property is not at all obvious from the definitions (4.1) and (4.2) because of the curious shifts, $-(m_{ij} + j - i - 1)$, in the rising factorials in expression (4.1).

THEOREM 4.2. *The set of symmetric polynomials $\{t_\mu(z) | \mu \in \mathbb{P}_t\}$ forms a $\mathbb{Z}$-basis of the ring Λ_t of symmetric polynomials.*

THEOREM 4.3. *Let $\lambda \in \mathbb{P}_t$ and define p by*

$$p_s = \lambda_s + t - s, \qquad s = 1, 2, \dots, t.$$

Then, for each $r = 1, 2, \dots, t$, we have

$$t_\mu(z_1, \dots, z_r, p_{r+1}, \dots, p_t) = 0,$$

unless $\mu_s \le \lambda_s$, $s = r + 1, \dots, t$.

Let us give two useful properties of the symmetric polynomials t_μ needed in the sequel:

First property.

$$t_{(\mu_1, \mu_2, \dots, \mu_t)}(z_1, z_2, \dots, z_t) = (z_1 z_2 \dots z_t)\, t_{(\mu_1 - 1, \mu_2 - 1, \dots, \mu_t - 1)}(z_1 - 1, z_2 - 1, \dots, z_t - 1), \tag{4.3}$$

for $\mu_t \ge 1$.

Second property.

$$t_{(\mu_1,\mu_2,\ldots,\mu_t)}(z_1, z_2, \ldots, z_{t-1}, 0)$$
$$= \delta_{\mu_t,0} t_{(\mu_1,\mu_2,\ldots,\mu_{t-1})}(z_1 - 1, z_2 - 1, \ldots, z_{t-1} - 1). \quad (4.4)$$

Let us proceed now to define the symmetric polynomials $T_\mu(\gamma; z)$ of direct interest in this paper. These new polynomials are designed not only to be symmetric functions but also to exhibit R-symmetry, an involution on the partial hooks: $p_s \to -(\gamma + p_s)$, with $\gamma \equiv 2c + a + b + d + e + 1 - 2t$. Since the parameter γ has no significance in this general definition, we replace it now by an arbitrary parameter α and let the partial hooks p_s be arbitrary indeterminates z_s.

Definition. The definition of $T_\mu(\alpha; z)$ is given directly in terms of the polynomials $t_{(m)}$ defined in Eq. (4.1),

$$T_\mu(\alpha; z) = \sum_{(m)\in G_\mu} t_{(m)}(z) t_{(m)}(-z - \alpha), \quad (4.5a)$$

in which

$$z + \alpha = (z_1 + \alpha, z_2 + \alpha, \ldots, z_t + \alpha). \quad (4.5b)$$

It is again not obvious that the polynomials $T_\mu(\alpha; z)$ are symmetric in the variables $(z_1, \ldots, z_t)$. (The proof of this fact is given in [33].)

One property of the $T_\mu(\alpha; z)$ that is obvious from the definition (4.5) is the following: Define the involutory transformation R_s by

$$\begin{aligned} R_s\colon \quad & z_s \mapsto -z_s - \alpha \qquad (r = s), \\ & z_r \mapsto z_r, \qquad r = 1, 2, \ldots, t \ (r \neq s). \end{aligned} \quad (4.6)$$

Here s may be $1, 2, \ldots, t$; that is, there are t such transformations (4.6) in all. Then, $T_\mu(\alpha; z)$ is invariant under the action of each transformation R_s:

$$R_s\colon \ T_\mu(\alpha; z) \mapsto T_\mu(\alpha; z), \qquad s = 1, 2, \ldots, t. \quad (4.7)$$

Let us next summarize some important properties of the polynomials $T_\mu(\alpha; z)$ given in [33, 34].

Theorem 4.4. *The polynomials $T_\mu(\alpha; z)$ are symmetric polynomials in the variables $(z_1, z_2, \ldots, z_t)$ for each partition $\mu = (\mu_1, \mu_2, \ldots, \mu_t) \in \mathbb{P}_t$.*

The symmetric polynomials $T_\mu(\alpha; z)$ are, of course, related to Schur functions. The simplest expression of this relation is given, however, not in terms of the standard $e_\mu(z)$ (see, for example, [27, 35]) but in terms of modified Schur functions satisfying the symmetries R_s and defined by

$$e_\mu(u), \qquad u_s = z_s(-z_s - \alpha), \qquad s = 1, 2, \ldots, t. \tag{4.8}$$

Each variable u_s is then an invariant under each of the transformations R_s, $s = 1, 2, \ldots, t$, hence, under the group H_t of transformations generated by the R_s. These Schur functions are then not only invariants under the action of the symmetric group S_t but also under the action of the group H_t. Clearly, the $\{e_\lambda(u)\}$ are a basis of all polynomials in α and z that are invariant under S_t and H_t. In particular, the symmetric polynomials $T_\mu(\alpha; z)$ can be expanded in terms of the basis $\{e_\lambda(u)\}$. The leading term in this expansion is $e_\mu(u)$, since it is easily verified that

$$T_\mu(\zeta\alpha; \zeta z) = \zeta^{2(\mu_1 + \cdots + \mu_t)} e_\mu(u) + (\text{lower order terms in } \zeta). \tag{4.9}$$

THEOREM 4.5. *The set of symmetric polynomials $\{T_\mu(\alpha; z) | \mu \in \mathbb{P}_t\}$ forms a $\mathbb{Z}$-basis of the ring of polynomials invariant under the groups S_t and H_t.*

The symmetric polynomials $\{T_\mu(\alpha; z)\}$ satisfy two important recurrence-like relations given by

$$\begin{aligned} &T_{(\mu_1, \mu_2, \ldots, \mu_t)}(\alpha; z_1, z_2, \ldots, z_t) \\ &\quad = (u_1 u_2 \ldots u_t) \\ &\qquad \times T_{(\mu_1 - 1, \mu_2 - 1, \ldots, \mu_t - 1)}(\alpha + 2; z_1 - 1, z_2 - 1, \ldots, z_t - 1), \\ &\hspace{22em} \mu_t \geq 1, \end{aligned} \tag{4.10}$$

$$\begin{aligned} &T_{(\mu_1, \mu_2, \ldots, \mu_t)}(\alpha; z_1, z_2, \ldots, z_{t-1}, 0) \\ &\quad = \delta_{\mu_t, 0} T_{(\mu_1, \mu_2, \ldots, \mu_{t-1})}(\alpha + 2; z_1 - 1, z_2 - 1, \ldots, z_{t-1} - 1). \end{aligned} \tag{4.11}$$

These relations are used to establish (see [33, 34]) the next result on zeros of the symmetric polynomials $T_\mu(\alpha; z)$:

THEOREM 4.6. *Let $z = (z_1, z_2, \ldots, z_t) = (p_1, p_2, \ldots, p_t)$, where $p_s = \lambda_s + t - s$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t) \in \mathbb{P}_t$. Then*

$$T_\mu(\alpha; p) = 0, \qquad \text{unless } \mu_s \leq \lambda_s,\ s = 1, 2, \ldots, t. \tag{4.12}$$

The last result we require from [33] is the following expansion:

THEOREM 4.7. *Let x and y be arbitrary parameters. Then the following expansion is valid,*

$$\prod_{s=1}^{t}(x+1-z_s)_k(y+1-z_s)_k$$
$$= \sum_{\mu}\langle {}_1\mathscr{F}_0(-k)|\mu\rangle\left[\prod_{s=1}^{t}(x-t+s+1)_{k-\mu_s}(y-t+s+1)_{k-\mu_s}\right]$$
$$\times T_\mu(a;z), \tag{4.13a}$$

where

$$a = -x-y-k-1. \tag{4.13b}$$

Remark. The coefficient $\langle {}_1\mathscr{F}_0(-k)|\mu\rangle$ in this result is obtained by setting $a=-k$ and $\lambda=\mu$ in Eq. (1.12). It may also be written in the following form, which shows that it is *integral* (see [33]),

$$\langle {}_1\mathscr{F}_0(-k)|\mu\rangle = \begin{cases}(-1)^{\mu_1+\cdots+\mu_t}\operatorname{Dim}[\tilde{\mu}0^{k-\mu_1}] & \text{for } \mu_1 \le k,\\ 0 & \text{for } \mu_1 > k,\end{cases} \tag{4.14a}$$

where $\tilde{\mu}$ is the partition conjugate to μ; specifically,

$$\tilde{\mu} = \left[t^{\mu_t},(t-1)^{\mu_{t-1}-\mu_t},\ldots,1^{\mu_1-\mu_2}\right], \tag{4.14b}$$

in which $n^k = (\dot{n})$ denotes that integer n is repeated k times.

We have now given the complete definition of the $T_\mu(\gamma;z)$ which we require in defining the new polynomials $\mathscr{P}_k^t$; we now replace the parameter α in definition (4.5) by the special parameter $\gamma=\gamma_{t,k}$ defined by Eq. (3.22b):

The new polynomials $\mathscr{P}_k^t$.

$$\mathscr{P}_k^t(b,d,e;c;z)$$
$$= \sum_{\mu}\langle {}_1\mathscr{F}_0(-k)|\mu\rangle \prod_{s=1}^{t}(c-k-s+1)_{\mu_s}$$
$$\times\left[\prod_{s=1}^{t}(c+b-k+\mu_s-s+1)_{k-\mu_s}(c+d-k+\mu_s-s+1)_{k-\mu_s}\right.$$
$$\left.\times(c+e-k+\mu_s-s+1)_{k-\mu_s}\right]T_\mu(\gamma;z). \tag{4.15}$$

Here $T_\mu(\gamma;z)$ are the symmetric polynomials defined in Eq. (4.5) above, with $\gamma = 2c - k + b + d + e + 1 - 2t$ and with general variables $z = (z_1, z_2, \ldots, z_t)$. Since the hypergeometric coefficient $\langle {}_1\mathscr{F}_0(-k)|\mu\rangle$ is zero for $\mu_s > k$, $s = 1, 2, \ldots, t$, the summation over partitions $\mu \in \mathbb{P}_t$ is automatically limited to those for which $0 \leq \mu_s \leq k$, $s = 1, 2, \ldots, t$.

Remarks. 1. Definition (4.15) reduces correctly to yield all the explicit cases discussed in Section III, although in some cases the proof is nontrivial, as we discuss in more detail below. In particular, one sees that the Gel'fand–Weyl patterns given for Example 4 ($P_{k=2}^{t=2}$) label exactly the terms in the sum for T_μ, in Eq. (4.5a). Moreover, the dimension factors, $\mathrm{Dim}[\tilde{\mu}\dot{0}]$ (see Eq. (4.14a)), easily yield exactly the curious integers 2, 1, 3, 2 found in this example.

2. The form of Eq. (4.15) shows that $\mathscr{P}_k^t$ is actually a *group-theoretically defined polynomial* whose terms correspond one-to-one with every Gel'fand–Weyl pattern (m) in $U(t)$ with $(\dot{0}) \leq \mu = (m_{1t}, m_{2t}, \ldots, m_{t,t}) \leq (\dot{k})$.

3. Relation (4.13a) extends formally to negative values of k; that is, the following expansion is valid as a formal series:

$$\frac{1}{\prod_{s=1}^t (z_s + x)_k (z_s + y)_k} = \sum_\mu \frac{\langle {}_1\mathscr{F}_0(k)|\mu\rangle T_\mu(\alpha; z)}{\prod_{s=1}^t (x + t - s)_{k+\mu_s} (y + t - s)_{k+\mu_s}},$$
$$\text{each } k \in \mathbb{N}, \quad (4.16a)$$

where now

$$\alpha = -x - y + k - 1. \quad (4.16b)$$

This formal identity is a consequence of the invariance of the function in the left-hand side under the actions of the groups S_t and H_t, and of the basis property of the $\{T_\mu(\alpha; z)\}$ for all such functions. Observe that the summation over μ in relation (4.16a) no longer terminates.

Let us turn next to the definition of the new functions $\mathscr{A}_\lambda\binom{a,b,d,e}{c}$. For the values $a = -k$, all $k \in \mathbb{N}$, we use Eq. (3.1) to define $\mathscr{A}_\lambda$ in terms of $\mathscr{P}_k^t$; that is,

$$\mathscr{A}_\lambda\binom{-k,b,d,e}{c} = M^{-1}(\lambda)\left[\prod_{s=1}^t (c + b - s + 1)_{\lambda_s - k} (c + d - s + 1)_{\lambda_s - k} \right.$$
$$\left. \times (c + e - s + 1)_{\lambda_s - k}\right] \mathscr{P}_k^t(b, c, d; e; p),$$
$$\text{all } k = 0, 1, 2, \ldots. \quad (4.17)$$

A problem occurs when one seeks to extend definition (4.17) to general values of the parameter a. To resolve this problem, we use

Carlson's theorem (see p. 36 of [28]), which asserts that (under certain mild growth conditions) knowledge of a function at all integer points $\{0, 1, 2, \dots\}$ is equivalent to knowledge everywhere. For the case at hand, Eq. (4.17), the application of Carlson's theorem is particularly simple: one simply replaces $-k$ in Eq. (4.17) by a. That this is the correct procedure is seen by combining the multiplicative factors in Eq. (4.17) with factors under the summation in the definition (4.15) of $\mathscr{P}_k^t$ as follows:

$$\begin{aligned}(c+b-k+\mu_s-s+1)_{k-\mu_s}&(c+b-s+1)_{\lambda_s-k}\\ &= (c+b-k+\mu_s-s+1)_{\lambda_s-\mu_s}\end{aligned}$$

with similar factors in d and e. One now sees that the dependence of $\mathscr{A}_\lambda\binom{-k,b,d,e}{c}$ on k is, in fact, polynomial in k. Thus, the unique extension of relation (4.17) to a polynomial in the parameter a is given by

$$\begin{aligned}\mathscr{A}_\lambda\binom{a,b,d,e}{c} &= M^{-1}(\lambda)\sum_\mu M^{-1}(\mu)\\ &\times\Bigg[\prod_{s=1}^t (a-s+1)_{\mu_s}(a+c-s+1)_{\mu_s}\\ &\times(a+b+c+\mu_s-s+1)_{\lambda_s-\mu_s}(a+c+d+\mu_s-s+1)_{\lambda_s-\mu_s}\\ &\times(a+c+e+\mu_s-s+1)_{\lambda_s-\mu_s}\Bigg]T_\mu(\gamma;p),\end{aligned}\tag{4.18}$$

where

(a) the sum is over all partitions $\mu = (\mu_1, \mu_2, \dots, \mu_t) \in \mathbb{P}_t$,
(b) $\gamma \equiv 2c + a + b + d + e + 1 - 2t$,
(c) $p = (p_1, p_2, \dots, p_t)$ with $p_s = \lambda_s + t - s$.

The relationship of this general result for a to the polynomials $\mathscr{P}_k^t$ is recovered by setting $a = -k$.

Remarks. 1. The form given by Eq. (4.18) is obviously symmetric in b, d, e. By construction, Eq. (4.18) also shows the symmetry R defined in Eq. (3.23). The symmetry of Eq. (4.18) under all permutations of the partial hooks $\{p_s\}$ is also valid, but not obviously so (see Theorem 4.1).

2. One might wonder as to what makes the sum in (4.18) finite. The answer is that the symmetric functions $T_\mu(\gamma; p)$, according to Theorem 4.6, are such that the sum terminates at limits set by the values of the λ_s. Thus, $\mathscr{A}_\lambda$ is polynomial in a, b, c, d, e.

The strategy of the present paper is now to prove our main proposition (the symmetry of A_λ for $b \leftrightarrow e$) in the form:

THEOREM 4.8. *The polynomials A_λ defined by Eq.* (1.8) *and $\mathscr{A}_\lambda$ defined by Eq.* (4.18) *are identical.*

Since A_λ and $\mathscr{A}_\lambda$ are the unique extensions of the polynomials P_k^t and $\mathscr{P}_k^t$, respectively (as shown above for $\mathscr{A}_\lambda$ and with a similar result for A_λ), the proof of this theorem is equivalent to proving the lemma:

LEMMA 4.1. *The polynomials $P_k^t(b, d, e; c; z)$ defined by Eqs.* (3.4) *and $\mathscr{P}_k^t(b, d, e; c; z)$ defined by Eq.* (4.15) *are identical.*

The proof of this lemma is a sizeable task because the expressions for P_k^t and $\mathscr{P}_k^t$ are *structurally quite dissimilar*. This is carried out in Section VI below.

We continue this section by proving the relation $A_\lambda = \mathscr{A}_\lambda$ for the special cases of A_λ considered in Section I.

LEMMA 4.2. *The polynomials $\mathscr{A}_\lambda$ are invariant under the linear transformation L defined by Eq.* (1.18).

Proof. This is an obvious property of definition (4.18) and the fact that γ is invariant under L. □

LEMMA 4.3. *The polynomials A_λ and $\mathscr{A}_\lambda$ are equal if at least one of the conditions $a = 0$, $d = 0$, $c = -a$, $c = -d$ holds.*

Proof. These are the special cases noted in Lemma 1.2 and Eqs. (1.20)–(1.21). For $a = 0$ or $c = -a$, only the $\mu = (0, \ldots, 0)$ term in the summation in Eq. (4.18) is nonzero. Since

$$T_{(0,\ldots,0)}(\gamma; p) = 1,$$

one easily verifies the lemma.

For $d = 0$ or $c = -d$, the proof reduces easily to that of proving the following equality for arbitrary parameters u, v, a:

$$\begin{aligned}\prod_{s=1}^{t} (u - s + 1)_{\lambda_s}(v - s + 1)_{\lambda_s} &\\ = \sum_{\mu} \langle {}_1\mathscr{F}_0(a)|\mu\rangle \Bigg[\prod_{s=1}^{t} (u + a + \mu_s - s + 1)_{\lambda_s - \mu_s} &\\ \times (v + a + \mu_s - s + 1)_{\lambda_s - \mu_s} \Bigg] &\\ \times T_\mu(\alpha; p), &\qquad (4.19a)\end{aligned}$$

where

$$\alpha = a + u + v - 2t + 1. \tag{4.19b}$$

The relation given by Eqs. (4.13) in Theorem 4.7 is the key to the proof of relations (4.19): It is used to prove that $P_k^t(b, 0, e; c; p) = \mathscr{P}_k^t(b, 0, e; c; p)$, as explicitly shown in Eq. (5.3) of Section V. We will use this result here, since its proof is easy to give. It leads in Eq. (4.18) to

$$\mathscr{A}_\lambda\begin{pmatrix} -k, b, 0, e \\ c \end{pmatrix} = \langle {}_3\mathscr{F}_0(c - k, c + b, c + e)|\lambda\rangle, \tag{4.20a}$$

which extends to

$$\mathscr{A}_\lambda\begin{pmatrix} a, b, 0, e \\ c \end{pmatrix} = \langle {}_3\mathscr{F}_0(c + a, c + b, c + e)|\lambda\rangle. \tag{4.20b}$$

This result not only agrees directly with Eq. (1.21b) for $A_\lambda(d = 0)$, but also yields Eq. (4.19a) when used in Eq. (4.17). Relation (4.19) is quite surprising in that the left-hand side is independent of a. □

LEMMA 4.4. *The polynomials A_λ and $\mathscr{A}_\lambda$ are equal for $t = 1$.*

Proof. For $t = 1$ in Eq. (4.17), we have

$$\lambda_1 = n, \qquad \mu_1 = k, \qquad p_1 = n, \qquad M^{-1}(n) = 1/n!,$$
$$\langle {}_1\mathscr{F}_0(a)|k\rangle = (a)_k/k!,$$
$$T_k(\gamma; n) = (n - k + 1)_k(-n - \gamma - k + 1)_k$$
$$= (-n)_k(n + r)_k,$$

where

$$\gamma = 2c + a + b + d + e - 1.$$

Making these substitutions in Eq. (4.17), we obtain exactly relation (1.22b) for A_n. □

LEMMA 4.5. *The polynomials A_λ and $\mathscr{A}_\lambda$ are equal for $\lambda = (\dot{\lambda})$.*

Proof. The relations needed here are

$$M^{-1}(\dot{\lambda}) = \frac{M^{-1}(\bar{\mu})}{(\operatorname{Dim}\mu)\prod_{s=1}^{t}(\lambda - \mu_s + s)_{\mu_s}}, \tag{4.21}$$

$$T_\mu(\alpha; \lambda + t - 1, \ldots, \lambda + 1, \lambda)$$
$$= (\operatorname{Dim}\mu)\prod_{s=1}^{t}(\lambda - \mu_s + s)_{\mu_s}(-\lambda - \alpha - t + 1 - \mu_s + s)_{\mu_s}, \tag{4.22a}$$

so that

$$M^{-1}(\dot{\lambda})T_{\mu}(\alpha;\lambda+t-1,\ldots,\lambda+1,\lambda)$$
$$=M^{-1}(\bar{\mu})\prod_{s=1}^{t}(-1)^{\mu_s}(\lambda+\alpha+t-s)_{\mu_s}. \qquad (4.22\text{b})$$

Using this last relation in Eq. (4.18) with $\lambda = (\dot{\lambda})$ and $\alpha = \gamma$ gives exactly relation (1.25) for $A_{\dot{\lambda}}$. Equation (4.21) is proved straightforwardly from the definition (1.6) of $M^{-1}(\lambda)$ and Eq. (1.26b) for $M^{-1}(\bar{\mu})$. The proof of Eq. (4.22a) requires more work.

Equations (4.10) and (4.11) may be used to prove relation (4.22a). We first iterate Eq. (4.10) k times to obtain

$$T_{\mu}(\alpha;z)=\prod_{s=1}^{t}(z_s-k+1)_k(-z_s-\alpha-k+1)_k$$
$$\times T_{(\mu_1-k,\ldots,\mu_t-k)}(\alpha+2k;z_1-k,\ldots,z_t-k). \quad (4.23)$$

We now choose $k=\mu_t$ and set $z_s=\lambda+t-s$, where $\lambda\in\mathbb{N}$ and $s=1,2,\ldots,t$. We obtain

$$T_{\mu}(\alpha;\lambda+t-1,\ldots,\lambda+1,\lambda)$$
$$=\prod_{s=1}^{t}(\lambda+t-\mu_t-s+1)_{\mu_t}(-\lambda-\alpha-t-\mu_t+s+1)_{\mu_t}$$
$$\times T_{(\mu_1-\mu_t,\ldots,\mu_{t-1}-\mu_t,0)}(\alpha+2\mu_t;\lambda-\mu_t+t-1,\ldots,\lambda-\mu_t). \quad (4.24)$$

We can prove Eq. (4.22a) by showing that the following relation is true:

$$T_{(\mu_1,\ldots,\mu_{t-1},0)}(\alpha;\lambda+t-1,\ldots,\lambda+1,\lambda)$$
$$=\frac{\mathrm{Dim}(\mu_1,\ldots,\mu_{t-1},0)}{\mathrm{Dim}(\mu_1,\ldots,\mu_{t-1})}$$
$$\times T_{(\mu_1,\ldots,\mu_{t-1})}(\alpha+1;\lambda+t-2,\ldots,\lambda+1,\lambda). \quad (4.25)$$

Note that this relation is implied by Eq. (4.22a). Conversely, relation (4.25) implies relation (4.22a). To show this we use induction on t and the proved relation (4.24). We begin by noting that Eq. (4.22a) is correct for $t=1$ (see [33] for tables of special cases of $T_{\mu}(\alpha;z)$). We now assume that Eq. (4.22a) is correct for t replaced by $t-1$ and use this result in the right-hand side of Eq. (4.25). In the expression thus obtained for $T_{(\mu_1,\ldots,\mu_{t-1},0)}(\alpha;\lambda+t-1,\ldots,\lambda+1,\lambda)$, we make the parameter shifts

$\lambda \to \lambda - \mu_t, \mu_s \to \mu_s - \mu_t$ $(s = 1, 2, \ldots, t-1)$, $\alpha \to \alpha + 2\mu_t$, and substitute the result into the right-hand side of Eq. (4.24). This result obtained for $T_\mu(\alpha; \lambda + t - 1, \ldots, \lambda + 1, \lambda)$ from Eq. (4.24) agrees exactly with that given by Eq. (4.22a); that is, the induction step from $t - 1$ to t is completed. Thus, the validity of Eq. (4.25) implies that of Eq. (4.22a).

The above result shows that Eqs. (4.22a) and (4.25) are equivalent; each implies the other. We next prove Eq. (4.25) (hence, (4.22a)) by induction on t, again using relation (4.24). The validity of Eq. (4.25) for $t = 1$ is established directly from the definition of the $T_\mu(\alpha; z)$. We assume (4.25) to be valid as given and show it to be valid for t replaced by $t + 1$. We need to prove

$$\begin{aligned} &T_{(\mu_1, \ldots, \mu_t, 0)}(\alpha; \lambda + t, \ldots, \lambda + 1, \lambda) \\ &\quad = \frac{\mathrm{Dim}(\mu_1, \ldots, \mu_t, 0)}{\mathrm{Dim}(\mu_1, \ldots, \mu_t)} T_{(\mu_1, \ldots, \mu_t)}(\alpha + 1; \lambda + t - 1, \ldots, \lambda + 1, \lambda). \end{aligned} \tag{4.26a}$$

To prove this relation, we apply relation (4.24) to the right-hand side. We then apply Eq. (4.25) (induction hypothesis) to the resulting relation, making the appropriate shifts in parameters. The result is

$$\begin{aligned} &T_{(\mu_1, \ldots, \mu_t, 0)}(\alpha; \lambda + t, \ldots, \lambda + 1, \lambda) \\ &\quad = \frac{\mathrm{Dim}(\mu_1, \ldots, \mu_t, 0)}{\mathrm{Dim}(\mu_1, \ldots, \mu_{t-1})} \prod_{s=1}^{t} (\lambda - \mu_t + s)_{\mu_t} (-\lambda - \alpha - t - \mu_t + s)_{\mu_t} \\ &\quad\quad \times T_{(\mu_1 - \mu_t, \ldots, \mu_{t-1} - \mu_t)}(\alpha + 2 + 2\mu_t; \lambda - \mu_t + t - 2, \ldots, \lambda - \mu_t). \end{aligned} \tag{4.26b}$$

But we have already proved above that Eq. (4.25) implies relation (4.22) for all $\mu \in \mathbb{P}_t$ for all t. Accordingly, we can use this explicit result in the right-hand side of Eq. (4.26b), after shifting parameters. This gives

$$\begin{aligned} &T_{(\mu_1, \ldots, \mu_t, 0)}(\alpha; \lambda + t, \ldots, \lambda + 1, \lambda) \\ &\quad = \mathrm{Dim}(\mu_1, \ldots, \mu_t, 0) \prod_{s=1}^{t} (\lambda - \mu_s + s)_{\mu_s} (\lambda - \alpha - t - \mu_s + s)_{\mu_s}, \end{aligned} \tag{4.26c}$$

which, in turn, implies Eq. (4.26a). Thus, the induction loop closes and Eq. (4.25) is proved. □

The polynomial $\mathscr{A}_\lambda$ possesses some further properties that are useful to summarize:

LEMMA 4.6. *The polynomial* $\mathscr{A}_\lambda$ *is invariant under all permutations of* b, d, e.

Proof. This is an obvious property of definition (4.18). □

LEMMA 4.7. *The polynomial* $\mathscr{A}_\lambda$ *satisfies*

$$\mathscr{A}_{(\lambda_1,\ldots,\lambda_{t-1},0)}\begin{pmatrix} a,b,d,e \\ c \end{pmatrix} = \mathscr{A}_{(\lambda_1,\ldots,\lambda_{t-1})}\begin{pmatrix} a,b,d,e \\ c \end{pmatrix}. \tag{4.27}$$

Proof. This important property follows from the definition (4.18) by setting $p_t = \lambda_t = 0$, using $M^{-1}(\lambda_1,\ldots,\lambda_{t-1},0) = M^{-1}(\lambda_1,\ldots,\lambda_{t-1})$, and identity (4.11). □

We conclude this section by writing the definition (4.18) of $\mathscr{A}_\lambda$ in another form that shows a relationship to generalized hypergeometric coefficients. The first form involves a straightforward generalization of the hypergeometric coefficients to p numerator parameters and q denominator parameters, which we have anticipated in our use of ${}_3\mathscr{F}_0$ earlier. This generalization of the hypergeometric coefficients is

$$\langle {}_p\mathscr{F}_q(a_1,\ldots,a_p;b_1,\ldots,b_q)|\mu\rangle = M^{-1}(\mu)\prod_{s=1}^{t}\left[\frac{\prod_{i=1}^{p}(a_i-s+1)_{\mu_s}}{\prod_{j=1}^{q}(b_j-s+1)_{\mu_s}}\right]. \tag{4.28}$$

The corresponding (formal) *generalized hypergeometric function* associated with the coefficients is then defined with the aid of Schur functions by

$${}_p\mathscr{F}_q(\mathbf{a};\mathbf{b};z) = \sum_\mu \langle {}_p\mathscr{F}_q(\mathbf{a};\mathbf{b})|\mu\rangle e_\mu(z), \tag{4.29}$$

where $\mathbf{a} = (a_1,\ldots,a_p)$, $\mathbf{b} = (b_1,\ldots,b_q)$, and $z = (z_1,\ldots,z_t)$.

Using property (1.10) of rising factorials and the definition of the generalized hypergeometric coefficients, we can write $\mathscr{A}_\lambda$ as defined by Eq. (4.18) in the form

$$\mathscr{A}_\lambda\begin{pmatrix} a,b,d,e \\ c \end{pmatrix} = M^{-1}(\lambda)\left[\prod_{s=1}^{t}\prod_{i=1}^{3}(b_i-s+1)_{\lambda_s}\right]{}_2\mathscr{F}_3^T(\mathbf{a};\mathbf{b};p). \tag{4.30}$$

Here ${}_2\mathscr{F}_3^T$ is defined by

$${}_2\mathscr{F}_3^T(\mathbf{a};\mathbf{b};p) = \sum_\mu \langle {}_2\mathscr{F}_3(\mathbf{a};\mathbf{b})|\mu\rangle T_\mu(\gamma;p), \tag{4.31a}$$

where

$$\mathbf{a} = (a_1, a_2), \qquad \mathbf{b} = (b_1, b_2, b_3) \tag{4.31b}$$

with the parameters defined by

$$\begin{aligned}
&a_1 = a, \qquad a_2 = a + c,\\
&b_1 = a + b + c, \qquad b_2 = a + c + d, \qquad b_3 = a + c + e,\\
&p_s = \lambda_s + t - s, \qquad s = 1, 2, \ldots, t.\\
&\gamma = 2c + a + b + d + e - 2t + 1 = b_1 + b_2 + b_3 - a_1 - a_2 - 2t + 1.
\end{aligned} \tag{4.31c}$$

For $t = 1$ [see Eq. (1.22b)], we have, for $\lambda_1 = n$, the identity

$${}_2\mathscr{F}_3^{\,T}(a_1, a_2; b_1, b_2, b_3; n) = {}_4F_3\begin{pmatrix} a_1, a_2, \gamma, -n \\ b_1, b_2, b_3 \end{pmatrix}, \tag{4.32}$$

where $\gamma = b_1 + b_2 + b_3 - a_1 - a_2 - 1 + n$. Here the hypergeometric series ${}_4F_3$ of unit argument terminates because of the occurrence of the numerator parameter $-n$. As remarked earlier, the summation over μ in the general definition of ${}_2\mathscr{F}_3^{\,T}$ given by Eq. (4.31a) terminates because $T_\mu(\gamma; p) = 0$, unless $\mu_s \leq \lambda_s$, $s = 1, \ldots, t$ (see Theorem 4.6).

Since $T_\mu(\gamma; p)$ is polynomial in the integer-valued variables $p = (p_1, p_2, \ldots, p_t)$, the natural extension of definition (4.31a) is to replace these integral variables by arbitrary *variables* $z = (z_1, \ldots, z_t)$, and define

$${}_2\mathscr{F}_3^{\,T}(\mathbf{a}; \mathbf{b}; z) = \sum_\mu \langle {}_2\mathscr{F}_3(\mathbf{a}; \mathbf{b}) | \mu \rangle T_\mu(\gamma; z), \tag{4.33}$$

where γ is defined in terms of the a_i, b_j, and t by Eq. (4.31c).

The (formal) series in the right-hand side of definition (4.33) no longer terminates just as the series (4.32) would no longer terminate should we replace the integer n by an arbitrary parameter z_1.

From the viewpoint of relation (4.32), one might regard the functions ${}_2\mathscr{F}_3^{\,T}$ defined by Eq. (4.33) as generalizing the ${}_4\mathscr{F}_3$ hypergeometric series. It is much more natural, however, to regard ${}_2\mathscr{F}_3^{\,T}$ as generalizing ${}_2\mathscr{F}_3$ defined by Eq. (4.29) *in that the Schur functions* $\{e_\mu(z)\}$ *have been replaced by the new class of symmetric polynomials* $\{T_\mu(\alpha; z)\}$. (The notation is designed to suggested this.)

V. Properties of the Polynomials $\mathscr{P}_k^t$

We have given several properties and special cases of the polynomials P_k^t in Section III: These include relations (3.9) and (3.15)–(3.19), together with the four special cases $k, t = 1, 2$ given in Eqs. (3.22)–(3.27). The verification that $\mathscr{P}_k^t$ for $k, t = 1, 2$ agrees with P_k^t is tedious but direct (indeed, these special cases provided the initial motivation for the new form).

The purpose of the present section is to establish the above-mentioned properties of the polynomials P_k^t directly for the new polynomials $\mathscr{P}_k^t$. Relation (3.9) for $\mathscr{P}_k^t$ follows easily and directly from the reduction property (4.11) of the basis polynomials $T_\mu(\alpha; z)$. Thus, we obtain

$$\begin{aligned}\mathscr{P}_k^t(b, c, d; e, z_1, \ldots, z_{t-1}, 0) \\ &= [(a + c + b - t + 1)_k(a + c + d - t + 1)_k(a + c + e - t + 1)_k] \\ &\quad \times \mathscr{P}_k^{t-1}(b, c, d; e; z_1 - 1, \ldots, z_{t-1} - 1). \qquad (5.1)\end{aligned}$$

This relation, in turn, when used in definition (4.17) for $\mathscr{A}_\lambda$ yields

$$\mathscr{A}_{(\lambda_1, \ldots, \lambda_{t-1}, 0)}\begin{pmatrix} -k, b, d, e \\ c \end{pmatrix} = \mathscr{A}_{(\lambda_1, \ldots, \lambda_{t-1})}\begin{pmatrix} -k, b, d, e \\ c \end{pmatrix}. \qquad (5.2)$$

LEMMA 5.1. *The new polynomial* $\mathscr{P}_k^t$ *satisfies the same identities, Eqs.* (3.15)–(3.19), *as the polynomial* P_k^t.

Proof. We prove each of these results, in turn.

The $d = 0$ *polynomials.* For the integer-valued variables $z_s = p_s = \lambda_s + t - s$, the equality $\mathscr{P}_k^t(b, 0, e; c; p) = P_k^t(b, 0, e; c; p)$ is a consequence of Lemma 4.3 and the definitions of these polynomials in terms of $\mathscr{A}_\lambda$ and A_λ, respectively (cf. Eqs. (4.15) and (3.1)). Its validity for arbitrary variables z may be proved directly from the definition (4.15) and the general result given in Eq. (4.13a). Thus, setting $d = 0$ in Eq. (4.15), we see that the following terms combine and may be brought in front of the summation over μ:

$$(c - k - s + 1)_{\mu_s}(c - k + \mu_s - s + 1)_{k-\mu_s} = (c - k - s + 1)_k.$$

The summation over μ is then exactly that given by the right-hand side of Eq. (4.13a) for $x = -c - b + t - 1$, $y = -c - e + t - 1$, after using $(u)_l = (-1)^l(-u - l + 1)_l$. Application of identity (4.13a) now gives exactly the right-hand side of Eq. (3.15), after using $(u)_l = (-1)^l(-u - l + 1)_l$ again; that is, we have demonstrated

$$\mathscr{P}_k^t(b, 0, e; c; z) = P_k^t(b, 0, e; c; z). \qquad (5.3)$$

The polynomials with $d = -c = m$, $m \in \mathbb{N}$. We set $d = -c = m$ and $z = p$ in Eq. (4.15) and combine factors $(-k - s + 1)_{\mu_s}(-k + \mu_s - s + 1)_{k-\mu_s} = (-k - s + 1)_k$, where the first factor comes from the hypergeometric coefficient $\langle {}_1\mathscr{F}_0(-k)|\mu\rangle$. The summation limits over μ must now be given explicitly, namely, $(\dot{0}) \leq \mu \leq (\dot{k})$. For each $m \in \mathbb{N}$, we can write

$$(b-m-k+\mu_s-s+1)_{k-\mu_s}=\frac{(-1)^{k+m-\mu_s}(-b+s)_{k+m-\mu_s}}{(b-m-s+1)_m},$$

with a similar term in e. We thus obtain

$$\begin{aligned}
&\mathscr{P}_k^t(b,m,e;-m;p)\\
&\quad=\prod_{s=1}^{t}[(-k-s+1)_k/(b-m-s+1)_m(e-m-s+1)_m]\\
&\qquad\times\sum_{(\dot{0})\leq\mu\leq(\dot{k})}\langle {}_1\mathscr{F}_0(-k-m)|\mu\rangle\\
&\qquad\times\left[\prod_{s=1}^{t}(-b+s)_{k+m-\mu_s}(-e+s)_{k+m-\mu_s}\right]T_\mu(\gamma';p),
\end{aligned}\tag{5.4a}$$

where

$$\gamma'=b+e-k-m-2t+1.\tag{5.4b}$$

Comparing the summation term in Eq. (5.4a) with the right-hand side of Eq. (4.13a), we see that they agree for $x=-b+t-1$, $y=-e+t-1$, $z_s=p_s=\lambda_s+t-s$, and with k replaced by $k+m$, *except that the upper limit of the summation is different*; it is $\mu=(\dot{k})$ in Eq. (5.4a) and $\mu=(\dot{k})+(\dot{m})$ in Eq. (4.13a) for the special case being considered. This is where the restriction $0\leq\lambda_s\leq k$ comes in. Because $T_\mu(\gamma';p)=0$, unless $\mu_s\leq\lambda_s$, the summation over μ in Eq. (4.13a), for the case being considered (having k replaced by $k+m$) also terminates at the upper limit $\mu=(\dot{k})$ (for $0\leq\lambda_s\leq k$). Thus, *under the restriction* $0\leq\lambda_s\leq k$, Eq. (5.4a) becomes

$$\begin{aligned}
&\mathscr{P}_k^t(b,m,e;-m;p)\\
&\quad=\prod_{s=1}^{t}\frac{(-k-s+1)_k(-b+t-p_s)_{k+m}(-e+t-p_s)_{k+m}}{(b-m-s+1)_m(e-m-s+1)_m}\\
&\quad=P_k^t(b,m,e;-m;p)
\end{aligned}\tag{5.5}$$

by Eqs. (3.16). □

The shifted-parameter polynomials. The proof of the relation

$$\begin{aligned}\mathscr{P}^t_{k+m}(b, d, e; m; z) &= \prod_{s=1}^{t} (b - s + 1)_m (d - s + 1)_m (e - s + 1)_m \\ &\quad \times \mathscr{P}^t_k(b + m, d + m, e + m; -m; z) \qquad (5.6)\end{aligned}$$

for $m = -k, -k + 1, \ldots, 0, 1, 2, \ldots$ is a straightforward application of the definition (4.15). Note that the γ-parameter is the same for the parameters of $\mathscr{P}^t_{k+m}$ and those of $\mathscr{P}^t_k$ occurring in relation (5.6). □

The specialized variable polynomials. The proof of

$$\begin{aligned}\mathscr{P}^t_k(b, d, e; c; y + t - s, \ldots, y + 1, y) \\ = P^t_k(b, d, e; c; y + t - 1, \ldots, y + 1, y) \qquad (5.7)\end{aligned}$$

is a direct consequence of $\mathscr{A}_{\dot{\lambda}} = A_{\dot{\lambda}}$ (Lemma 4.5) for all nonnegative integer values of y. Since both $\mathscr{P}^t_k$ and P^t_k are polynomials in y, identity (5.7) is true for arbitrary y. □

Highest-term polynomial. The term of highest degree in $T_\mu(\gamma; z)$ is $(-1)^{\mu_1 + \cdots + \mu_t} z_1^{2\mu_1} z_2^{2\mu_2} \cdots z_t^{2\mu_t}$; hence the term of highest degree in $\mathscr{P}^t_k$ in definition (4.15) occurs (uniquely) for $\mu = (\dot{k})$. Since $\langle {}_1\mathscr{F}_0(-k) | \dot{k} \rangle = (-1)^{kt}$, we obtain

$$\text{coefficient of } (z_1 z_2 \cdots z_t)^{2k} \text{ in } \mathscr{P}^t_k = \prod_{s=1}^{t} (c - k - s + 1)_k. \ \square \qquad (5.8)$$

Remarks. (1) Two properties that as yet we have not proved to be shared by the polynomials P^t_k and $\mathscr{P}^t_k$ is the J-symmetry, Eq. (3.11), of the former and the R-symmetry, Eq. (4.6), of the latter. J-symmetry of the $\mathscr{P}^t_k$ and R-symmetry of the P^t_k are both nontrivial results, which are established only by the proof (in the next section) that the two polynomials are identical: $\mathscr{P}^t_k = P^t_k$.

(2) The proof given above in Eq. (5.5) that $\mathscr{P}^t_k(b, m, e; -m; p) = P^t_k(b, m, e; -m; p)$ $(m \in \mathbb{N})$ shows that the P^t_k are polynomial in all the variables b, m, e, and $p_s = \lambda_s + t - s$, this being so despite the form of Eq. (3.16a). This is so *because the denominator divides the numerator* in Eq. (3.16a), as may be shown by use of R-symmetry of the numerator and the integer-valued property of the variables p_s. The explicit polynomial form of $P^t_k(b, m, e; -m, p)$ is, of course, $\mathscr{P}^t_k(b, m, e; -m; p)$, as obtained directly from Eq. (4.15).

(3) We have given more shared properties of the P_k^t and $\mathscr{P}_k^t$ than strictly needed for the proof that P_k^t and $\mathscr{P}_k^t$ are identical (see Section VI). The difficulty in finding this proof led us to enumerate as many mutual properties as could be established directly.

VI. Proof of the Main Proposition: $A_\lambda = \mathscr{A}_\lambda$

Before proceeding to the proof of the identity $P_k^t = \mathscr{P}_k^t$, hence of $A_\lambda = \mathscr{A}_\lambda$, we require one additional basic result:

Theorem 6.1. *If $P_k^t(b, d, e; c; z)$ is invariant under each transformation R_s defined for each $s = 1, 2, \ldots, t$ by*

$$R_s\colon \quad z_s \mapsto -z_s - \gamma, \qquad z_r \mapsto z_r, \quad r = 1, 2, \ldots, t \ (s \neq r), \tag{6.1a}$$

then

$$P_k^t(b, d, e; c; z) = \mathscr{P}_k^t(b, d, e; c; z) \tag{6.1b}$$

for all parameters b, d, e, c and all variables z_s.

Proof. We have earlier referred to invariance under the group of transformations generated by R_s, $s = 1, 2, \ldots, t$, as R-symmetry. We have also shown that the symmetric polynomials $\{T_\mu(\gamma; z)\}$ are a basis for all symmetric polynomials in $(z_1, z_2, \ldots, z_t)$ that also have R-symmetry. Thus, the following expansion is valid under the assumption of the theorem:

$$P_k^t(b, d, e; c; z) = \sum_\mu C_{k,\mu}^t(b, d, e, c) T_\mu(\gamma; z). \tag{6.2}$$

The proof of the theorem is accordingly reduced to that of showing that the coefficients $C_{k,\mu}^t$ are exactly those given in Eq. (4.15).

The main property we need to show is that the coefficients $C_{k,\mu}^t$ factor in the following way:

$$C_{k,\mu}^t(b, d, e, c) = B_{k,\mu}^t(b, c) Q_{k,\mu}^t(b, c, d, e), \tag{6.3a}$$

where $Q_{k,\mu}^t$ is defined by

$$\begin{aligned} Q_{k,\mu}^t(b, c, d, e) = \prod_{s=1}^t &(c + b - k + \mu_s - s + 1)_{k-\mu_s} \\ &\times (c + d - k + \mu_s - s + 1)_{k-\mu_s} \\ &\times (c + e - k + \mu_s - s + 1)_{k-\mu_s}. \end{aligned} \tag{6.3b}$$

The factors $B^t_{k,\mu}$ are to be independent of d, hence, also of e (by the known $d \leftrightarrow e$ symmetry of the P^t_k). Once the form (6.3a) is established, we evaluate $B^t_{k,\mu}$ uniquely by setting $d = 0$, and using identity (5.3). This gives

$$B^t_{k,\mu}(b,c) = \langle {}_1\mathscr{F}_0(-k)|\mu\rangle \prod_{s=1}^{t} (c-k-s+1)_{\mu_s}, \tag{6.4}$$

which shows that $B^t_{k,\mu}$ is also independent of b. Thus, Theorem 6.1 is proved fully by establishing Eq. (6.3).

We prove Eq. (6.3) by using the zeros of the symmetric polynomials as given by Theorem 4.6 (making the appropriate notational changes) and by induction on the (total) ordering of the set $\mathbb{P}_t$ of partitions. We assume (induction hypothesis) that $C^t_{k,\mu}(b,c,d,e)$ has the form (6.3) for all partitions $\mu = (\mu_1,\ldots,\mu_t)$ such that

$$(0,\ldots,0) \le \mu \le \lambda, \tag{6.5}$$

where $\lambda = (\lambda_1,\ldots,\lambda_t)$ is an arbitrary, but fixed partition. We denote by λ^* the smallest partition greater than λ, and set $p = \lambda^* + \delta$ in Eq. (6.2), as well as in the defining relation for P^t_k, Eqs. (3.4). The only terms occurring in the summation in Eq. (6.2) for $z = p = \lambda^* + \delta$ have $\mu_s \le \lambda^*_s$, $s = 1,2,\ldots,t$, and from the definition of λ^*, we uniquely split the summation into two terms:

$$\begin{aligned} P^t_k(b,d,e;c;\lambda^*+\delta) = &\sum_{\mu\le\lambda} C^t_{k,\mu}(b,c,d,e)T_\mu(\gamma;\lambda^*+\delta) \\ &+ C^t_{k,\lambda^*}(b,c,d,e)T_{\lambda^*}(\gamma;\lambda^*+\delta). \end{aligned} \tag{6.6}$$

The induction hypothesis applies to each term in the summation with $\mu \le \lambda$; hence, this summation contains the factor

$$Q^t_{k,\lambda^*}(b,c,d,e), \tag{6.7}$$

since

$$\begin{aligned} &Q^t_{k,\mu}(b,c,d,e) \\ &\quad = Q^t_{k,\lambda^*}(b,c,d,e) \prod_{s=1}^{t} (c+b-s+1)_{\lambda^*_s-\mu_s} \\ &\qquad \times (c+d-s+1)_{\lambda^*_s-\mu_s}(c+e-s+1)_{\lambda^*_s-\mu_s} \end{aligned}$$

for $\mu_s \le \lambda_s \le \lambda^*_s$. By Lemma 3.4, the function $P^t_k(b,d,e;c;\lambda^*+\delta)$ also contains the factor (6.7). Thus $C^t_{k,\lambda^*}(b,c,d,e)$ also contains this factor. Equation (6.6) reduces to the polynomial relation in the parameters

(b, c, d, e) given by

$$P_k^t(b, d, e; c; \lambda^* + \delta) = Q_{k,\lambda^*}^t(b, c, d, e) R_{k,\lambda^*}^t(b, c, d, e), \tag{6.8a}$$

$$\begin{aligned} R_{k,\lambda^*}^t(b, c, d, e) = \sum_{\mu \leq \lambda} & B_{k,\mu}^t(b, c) T_\mu(\gamma; \lambda^* + \delta) \\ & \times \prod_{s=1}^t (c + b - s + 1)_{\lambda_s^* - \mu_s} (c + d - s + 1)_{\lambda_s^* - \mu_s} \\ & \times (c + e - s + 1)_{\lambda_s^* - \mu_s} \\ & + B_{k,\lambda^*}^t(b, c, d, e) T_{\lambda^*}(\gamma; \lambda^* + \delta). \end{aligned} \tag{6.8b}$$

Here we have also put

$$C_{k,\lambda^*}^t(b, c, d, e) = Q_{k,\lambda^*}^t(b, c, d, e) B_{k,\lambda^*}^t(b, c, d, e). \tag{6.8c}$$

We see from Eq. (6.8a) that $R_{k,\lambda^*}^t(b, c, d, e)$ is a polynomial of degree $n^* = \lambda_1^* + \cdots + \lambda_t^*$ in d. Each term in the summation part $\mu \leq \lambda$ of Eq. (6.8b) is also of degree n^* in d, since $T_\mu(\gamma; \lambda^* + \delta)$ is of degree $\mu_1 + \cdots + \mu_t$ in d. Since the degree of $T_{\lambda^*}(\gamma; \lambda^* + \delta)$ in d is n^*, the degree in d of $B_{k,\lambda^*}^t(b, c, d, e)$ must be zero, for otherwise the degree of the last term in relation (6.8c) would exceed that of all other terms. Thus, $B_{k,\lambda^*}^t(b, c, d, e)$ is independent of d, and by symmetry, also of e:

$$B_{k,\lambda^*}^t(b, c, d, e) = B_{k,\lambda^*}^t(b, c). \tag{6.9}$$

This result, when used in Eq. (6.8c), closes the induction loop. The theorem is thus proved if it is true for $\lambda^* = (0, \ldots, 0)$; that is, for the variables $z = (t - 1, \ldots, 1, 0)$.

To verify the validity of the starting point in the above induction proof, we must show that

$$\begin{aligned} P_k^t(b, d, e; c; \dot{0}) = \prod_{s=1}^t & (c + b - k - s + 1)_k \\ & \times (c + d - k - s + 1)_k (c + e - k - s + 1)_k. \end{aligned} \tag{6.10}$$

This result is proved by a straightforward application of Eqs. (3.7) and (3.8) to $\lambda = \dot{0}$, which gives

$$\begin{aligned} F_{k,\mu}^t(c, d, e; t - 1, \ldots, 1, 0) = \delta_{\mu,\dot{0}} \prod_{s=1}^t & (c + d - k - s + 1)_k \\ & \times (c + e - k - s + 1)_k. \end{aligned} \tag{6.11}$$

This result, when used in relation (3.4a) for P_k^t, yields the desired identity, Eq. (6.10). □

We have now assembled in Theorem 6.1 and in the previous sections all the relations needed to prove that $P_k^t = \mathscr{P}_k^t$, hence, that $A_\lambda = \mathscr{A}_\lambda$. There are (at least) two methods of proof. One uses as the starting point, the special relation (5.5),

$$P_k^t(b, m, e; -m; p) = \mathscr{P}_k^t(b, m, e; -m, p) \tag{6.12}$$

for $m \in \mathbb{N}$ and $0 \leq \lambda_s \leq k$ $(p_s = \lambda_s + t - s)$; the other uses the $d = 0$ identity (5.3),

$$P_k^t(b, 0, e; c; z) = \mathscr{P}_k^t(b, 0, e; c; z). \tag{6.13}$$

We give the details of the proof based on identity (6.12), and only outline the proof based on (6.13). We first need

LEMMA 6.1. *The identity*

$$P_k^t(b, -c, e; c; z) = \mathscr{P}_k^t(b, -c, e; c; z) \tag{6.14}$$

is true for all parameters b, c, e and variables z_s.

Proof. Both $P_k^t(b, d, e; c; p)$ and $\mathscr{P}_k^t(b, d, e, c; p)$ are polynomials in the parameters b, d, e, c. In particular, $P_k^t(b, -c, e; c; p)$ and $\mathscr{P}_k^t(b, -c, e; c; p)$ are both polynomials in c (of finite degree). Therefore, the equality (6.12) for all integers $m \in \mathbb{N}$ implies

$$P_k^t(b, -c, e; c; p) = \mathscr{P}_k^t(b, -c, e; c, p) \tag{6.15}$$

for all parameters b, c, e and all integer-valued variables $p_s = \lambda_s + t - s$ for which $0 \leq \lambda_s \leq k$.

We next extend the result (6.15) to arbitrary variables z_s. For this, we observe that each of the polynomials $P_k^t(b, -c, e; c; z)$ and $\mathscr{P}_k^t(b, -c, e; c; z)$ is of degree $2k$ in each variable z_s $(s = 1, 2, \ldots, t)$. Relation (6.15), the proved symmetry of each polynomial under permutations of the $(p_1, p_2, \ldots, p_t)$, and the R-symmetry implicit in the identity (6.15) imply that relation (6.14) is true for all values of the variable z_s given by

$$z_s = 0, 1, \ldots, k, -\gamma, -\gamma - 1, \ldots, -\gamma - k, \tag{6.16}$$

this result being true for each $s = 1, 2, \ldots, t$. Accordingly, the difference of these polynomials,

$$\Delta_k^t(b, c, e; z) = P_k^t(b, -c, e; c; z) - \mathscr{P}_k^t(b, -c, e; c; z), \tag{6.17}$$

is zero at all $2k + 1$ values of z_s given in Eq. (6.16), this result being true for each $s = 1, 2, \ldots, t$. It follows that the polynomial Δ_k^t must contain the factor

$$\prod_{s=1}^{t} (z_s - k)_{k+1}(z_s + \gamma + k)_{k+1}. \tag{6.18}$$

But the degree of Δ_k^t is at most $2k$ in each variable z_s; hence, Δ_k^t cannot have the factor (6.18), which is of degree $2k + 2$ in each z_s. The only way to avoid this contradiction is to have $\Delta_k^t = 0$. □

We next prove

LEMMA 6.2. *The identity*

$$P_k^t(b, t, e; c; z) = \mathscr{P}_k^t(b, -c, -e - c + t; c; z''), \tag{6.19a}$$

in which

$$z_s'' = z_s + c + e, \qquad s = 1, 2, \ldots, t, \tag{6.19b}$$

is true for all parameters (b, e, c), *all variables* $(z_1, z_2, \ldots, z_t)$, *and all* $t, k \in \mathbb{N}$.

Proof. From J-symmetry (see Eqs. (3.11)), we have the identity:

$$P_k^t(b, d, e; c; z) = P_k^t(b, -c - d + t, -c - e + t; c; z'), \tag{6.20a}$$

in which

$$z_s' = z_s + c + d + e - t, \qquad s = 1, 2, \ldots, t. \tag{6.20b}$$

We set $d = t$ in this relation and use Lemma 6.1 to obtain Eqs. (6.19). □

Relation (6.19) is valid for all $t \in \mathbb{N}$. Let us replace t by $n \in \mathbb{N}$ so that relation (6.19) becomes

$$P_k^n(b, n, e; c; z_1, z_2, \ldots, z_n)$$
$$= \mathscr{P}_k^n(b, -c, -e - c + n; c; z_1'', z_2'', \ldots, z_n''), \tag{6.21a}$$
$$z_s'' = z_s + c + e, \qquad s = 1, 2, \ldots, n. \tag{6.21b}$$

The idea now is to take $n > t$ in this relation and use relations (3.9) and (5.1) to "lower" n down to t. Thus, we set $z_n = 0$ in Eq. (6.21a) and apply properties (3.9) and (5.1). The multiplicative factors are nonzero and cancel. The variables $(z_1, z_2, \ldots, z_{n-1})$ are shifted down by one unit, but since they are general, we may redefine these variables, shifting them back up by one. The new identity thus obtained is

$$P_k^{n-1}(b, n, e; c; z_1, \ldots, z_{n-1})$$
$$= \mathscr{P}_k^{n-1}(b, -c, -e-c+n; c; z_1'', z_2'', \ldots, z_{n-1}'').$$

We may repeat this process $n-t$ times to obtain

$$P_k^t(b, n, e; c; z_1, \ldots, z_t) = \mathscr{P}_k^t(b, -c, -e-c+n; c; z_1'', \ldots, z_t'').$$

Since n is an arbitrary integer, with t defined such that

$$n = t, t+1, \ldots,$$

and since P_k^t and $\mathscr{P}_k^t$ are polynomials in d, we have proved

LEMMA 6.3. *The following identity is valid for all parameter values* (b, d, e, c), *all variables* z_s, *and all* $k, t \in \mathbb{N}$,

$$P_k^t(b, d, e; c; z) = \mathscr{P}_k^t(b, -c, -e-c+d; c; z''), \tag{6.22a}$$

where

$$z_s'' = z_s + c + e, \qquad s = 1, 2, \ldots, t. \tag{6.22b}$$

The variables z_s'' in $\mathscr{P}_k^t$ in the right-hand side of Eq. (6.22a) occur in the combination

$$(z_s + c + e)(z_s + b + c + d - k - 2t + 1),$$

which is invariant under the substitution

$$z_s \to -z_s - \gamma = -z_s - 2c + k - b - d - e + 2t - 1.$$

Thus, *the polynomial* $P_k^t(b, d, e; c; z)$ *has R-symmetry*. This result, together with Theorem 6.1, proves

THEOREM 6.2. *The following identity is true for all parameters* b, d, e, c, *all variables* z_s, *and all* $k, t \in \mathbb{N}$:

$$P_k^t(b, d, e; c; z) = \mathscr{P}_k^t(b, d, e; c; z). \tag{6.23}$$

This result is one of the major results of the present paper. An alternative method of proof of Theorem 6.2 may also be given: It begins with Eq. (3.17) for $P_{k+m}^t(b, d, e; m; z)$ and uses J-symmetry (Eq. (3.11)) to transform the right-hand side of Eq. (3.17), after replacing k by $k-m$, to the form

$$P_k^t(b, d, e; m; z) = \prod_{s=1}^{t} (b-s+1)_m (d-s+1)_m (e-s+1)_m$$
$$\times P_{k-m}^t(b+m, t-d, t-e; -m; z'), \tag{6.24a}$$

where

$$z'_s = z_s + m + d + e - t, \qquad s = 1, \ldots, t, \tag{6.24b}$$

$$m = k, k-1, \ldots, 0, -1, -2, \ldots . \tag{6.24c}$$

We now set $d = t$ in this relation to obtain

$$\begin{aligned} &P_k^t(b, t, e; m; z) \\ &\quad = \prod_{s=1}^{t} (b - s + 1)_m (t - s + 1)_m (e - s + 1)_m \\ &\quad\quad \times P_{k-m}^t(b + m, 0, t - e; -m; z''), \end{aligned} \tag{6.25a}$$

where

$$z''_s = z_s + e + m, \qquad s = 1, \ldots, t, \tag{6.25b}$$

$$m = k, k-1, \ldots, 0, -1, -2, \ldots . \tag{6.25c}$$

There seems, at first, to be a difficulty with Eq. (6.25a) because for negative values of m the factor

$$\prod_{s=1}^{t} (t - s + 1)_m \tag{6.26}$$

is undefined. The correct resolution of this difficulty is obtained from the explicit expression for the $d = 0$ polynomial expanded in terms of the basis $T_\mu(\gamma; z)$. Thus, from Eq. (5.3), we have for the case at hand that

$$\begin{aligned} &P_{k-m}^t(b + m, 0, t - e; -m; z'') \\ &\quad = \mathscr{P}_{k-m}^t(b + m, 0, t - e; -m; z'') \\ &\quad = \sum_{(\dot{0}) \le \mu \le (\dot{k})} \langle {}_1\mathscr{F}_0(-k + m) | \mu \rangle \prod_{s=1}^{t} (-k - s + 1)_{\mu_s} \\ &\quad\quad \times \Bigg[\prod_{s=1}^{t} (b + m - k + \mu_s - s + 1)_{k-m-\mu_s} (-k + \mu_s - s + 1)_{k-m-\mu_s} \\ &\quad\quad\quad \times (-e - k + \mu_s + t - s + 1)_{k-m-\mu_s} \Bigg] T_\mu(\gamma''; z''), \end{aligned} \tag{6.27a}$$

where

$$\gamma'' = b - e - k - t + 1. \tag{6.27b}$$

It is crucial to recognize that in this expression the summation over μ

cannot exceed the term $(\dot{k})$ because of the occurrence of the factor $\prod_{s=1}^{t}(-k-s+1)_{\mu_s}$ although for $m = 0, 1, \ldots, k$ it is limited by the factor $\langle {}_1\mathscr{F}_0(-k+m)|\mu\rangle$ to $\mu \leq (k-m, \ldots, k-m)$. Having established the correct limits on the summation μ, we can now combine terms

$$\prod_{s=1}^{t}(-k-s+1)_{\mu_s}(-k+\mu_s-s+1)_{k-m-\mu_s}$$
$$= \prod_{s=1}^{t}(-k-s+1)_{k-m} = \prod_{s=1}^{t}(-1)^{k-m}(t+m-s+1)_{k-m}, \tag{6.28}$$

giving a factor which comes out in front of the summation.

Returning now to Eq. (6.25a), we see that the factors (6.26) and (6.28) also combine:

$$\prod_{s=1}^{t}(t-s+1)_m(-1)^{k-m}(t+m-s+1)_{k-m}$$
$$= \prod_{s=1}^{t}(-1)^{k-m}(t-s+1)_k. \tag{6.29}$$

Relation (6.29) is unambiguously correct for $m = 0, 1, \ldots, k$. It is also correct for negative m, since, for an arbitrary complex number α, the following identity is valid for all integers m (positive, zero, and negative):

$$\prod_{s=1}^{t}(\alpha-s+1)_m(-1)^{k-m}(\alpha+m-s+1)_{k-m}$$
$$= \prod_{s=1}^{t}(-1)^{k-m}(\alpha-s+1)_k. \tag{6.30}$$

Accordingly, the right-hand side of Eq. (6.29) is also correct for negative m, since this is the correct limit of relation (6.30) for $\alpha \to t$.

The b-factor from Eq. (6.25a) may also be combined with the b-factor under the summation in Eq. (6.27a) to give

$$\prod_{s=1}^{t}(b+m-k+\mu_s-s+1)_{k-m-\mu_s}(b-s+1)_m$$
$$= \prod_{s=1}^{t}(b+m-k+\mu_s-s+1)_{k-\mu_s}.$$

A similar result is true for the e-factors. Putting all these results together, we obtain the following explicit form for the polynomial P_k^t, evaluated at $d = t$ and $c = m$:

$$
\begin{aligned}
&P_k^t(b, t, e; m; z) \\
&\quad = \prod_{s=1}^{t} (-1)^{k-m}(t-s+1)_k \sum_{(\dot{0}) \le \mu \le (\dot{k})} \langle {}_1\mathscr{F}_0(-k+m)|\mu\rangle \\
&\quad \times \Bigg[\prod_{s=1}^{t} (b+m-k+\mu_s-s+1)_{k-\mu_s} \\
&\qquad \times (e+m-k+\mu_s-s+1)_{k-\mu_s} \Bigg] T_\mu(\gamma''; z''). \qquad (6.31)
\end{aligned}
$$

The right-hand side of Eq. (6.31) is now recognized to be exactly $\mathscr{P}_k^t(b, -m, -e-m+t; m; z'')$; that is, we have proved the identity

$$P_k^t(b, t, e; m; z) = \mathscr{P}_k^t(b, -m, -e-m+t; m; z''), \qquad (6.32a)$$

where

$$z_s'' = z_s + e + m, \qquad s = 1, 2, \dots, t, \qquad (6.32b)$$

$$m = 0, -1, -2, \dots . \qquad (6.32c)$$

Since Eq. (6.32a) is a polynomial relation that is valid for infinitely many values of $c = m$, it is true for arbitrary parameter c,

$$P_k^t(b, t, e; c; z) = \mathscr{P}_k^t(b, -c, -e-c+t; c; z''), \qquad (6.33a)$$

where now

$$z_s'' = z_s + e + c, \qquad s = 1, 2, \dots, t. \qquad (6.33b)$$

We have thereby regained identity (6.19a) by this alternative method, and the remainder of the proof of Theorem 6.2 is completed as before.

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