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Existence theorem and blow-up criterion of the strong solutions to the two-fluid MHD equation in $\mathbb{R}^3$

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Abstract

We first give the local well-posedness of strong solutions to the Cauchy problem of the 3D two-fluid MHD equations, and then study the blow-up criterion of the strong solutions. By means of the Fourier frequency localization and Bony's paraproduct decomposition, it is proved that the strong solution (u,b) can be extended after $t=T$ if either $u\in L^q_T(\dot B^0_{p,\infty})$ with $\frac{2}{q}+\frac{3}{p}\leqslant 1$ and $b\in L^1_T(\dot B^0_{\infty,\infty})$ or $(\omega,J)\in L^q_T(\dot B^0_{p,\infty})$ with $\frac{2}{q}+\frac{3}{p}\leqslant 2$, where $\omega(t)=\nabla\times u$ denotes the vorticity of the velocity and $J=\nabla\times b$ stands for the current density.

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1. Introduction

We are concerned with the following two-fluid magnetohydrodynamics equations in $\mathbb{R}^3$:

$$\begin{cases} u_t-\nu\Delta u+u\cdot\nabla u-b\cdot\nabla b+\nabla\left(p+\frac{1}{2}b^2\right)=0,\\ b_t-\alpha\Delta b_t-\eta\Delta b+u\cdot\nabla b-b\cdot\nabla u+h\nabla\times(J\times b)=0,\\ \nabla\cdot u=\nabla\cdot b=0,\\ u(0,x)=u_0(x),\quad b(0,x)=b_0(x),\end{cases}\tag{1.1}$$

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where $x \in \mathbb{R}^3, t \geqslant 0$, ν, η, α, h stand for kinematic viscosity, the resistivity, the electron inertia term and the Hall coefficient respectively, u, b describe the flow velocity vector and the magnetic field vector respectively, and $J = \nabla \times b$ is the current density, p is a scalar pressure, and u_0 and b_0 are the given initial velocity and initial magnetic field with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. This model describes some important physical phenomena. In particular, for a plasma composed of two types of fluids and formed by ions and electrons, this model can explain the phenomena of fast magnetic reconnection such as in solar flares which cannot be characterized appropriately by the one-fluid magnetohydrodynamics. It is generally accepted now that the two-fluid magnetohydrodynamics is more complete than the classical one-fluid magnetohydrodynamics (MHD) model (see [5,6,17] and references therein). This is the reason why the two-fluid MHD equations are studied.

In general, the coefficient α is very small. Meanwhile, the Hall current term $h\nabla \times (J \times b)$ is also small in dense plasmas, so at large scales its effect is less important than that of the velocity. Neglecting both of them, that is, formally letting $\alpha = h = 0$, Eqs. (1.1) reduce to the classical MHD equations. Further, if we also omit the kinematic viscosity ν and the resistivity η, that is, formally let $\alpha = h = \nu = \eta = 0$, we then obtain the classical ideal MHD equations. Both the MHD and the ideal MHD equations, which are called one-fluid magnetohydrodynamics, have been studied extensively and are similar in many aspects to the Navier–Stokes equations and Euler equations, respectively.

It is well known [18] that the classical MHD equations are locally well-posed for any given initial data $u_0, b_0 \in H^s(\mathbb{R}^3)$, $s \geqslant 3$. In the case of the two-fluid MHD equations, Núñez [16] has proved the existence and uniqueness of local solutions to the system for either Dirichlet or periodic boundary conditions. His result is

Theorem A. *If $u_0 \in V$, $b_0 \in D(A)$, then there exists an interval $[0,T]$ such that the two-fluid MHD equations (1.1) have a unique solution (u,b) in $[0,T]$. Moreover*

$$u \in C\big((0,T),V\big) \cap L^2\big((0,T),D(A)\big),$$
$$b \in C\big((0,T),D(A)\big).$$

Here in the Dirichlet case, $\Omega \subset \mathbb{R}^3$ is bounded and smooth, and

$$H = \big\{f \in L^2(\Omega)^3;\ \nabla \cdot f = 0,\ f \cdot n|_{\partial\Omega} = 0\big\}, \qquad V = H_0^1(\Omega)^3 \cap H,$$
$$(A) = H^2(\Omega)^3 \cap V.$$

While, in the periodic case, $\Omega \subset \mathbb{R}^3$ is a box, and

$$H = \left\{f \in L^2(\Omega)^3;\ \int_\Omega f(x)\,dx = 0,\ \nabla \cdot f = 0,\ f \cdot n \text{ antiperiodic at opposite sides of } \Omega\right\},$$
$$V = H^1(\Omega)^3 \cap H, \qquad D(A) = H^2(\Omega)^3 \cap V.$$

The method of Núñez's proof seems not to apply to the Cauchy Problem (1.1) since the Poincaré inequality plays a basic role in the proof. The first purpose of this paper is to show the local well-posedness of strong solutions to Eqs. (1.1) in $\mathbb{R}^3$ by Fourier localization together with Picard's method.

Theorem 1.1. *If* $(u_0, b_0) \in H^s(\mathbb{R}^3) \times H^{s+1}(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$, $s \geqslant 3$, *then there exists an interval* $[0, T]$ *such that the two-fluid MHD equations* (1.1) *have a unique solution* (u, b) *in* $C([0, T], (H^s \times H^{s+1})(\mathbb{R}^3))$. *Moreover, u satisfies*

$$u \in L^2\big([0, T], H^{s+1}\big(\mathbb{R}^3\big)\big).$$

Strong solutions we obtain here exist only locally. In general, even for the classical MHD equations, it is not known whether the smooth solution of the Cauchy problem exists for all time though Duvaut and Lions [10] constructed a class of global weak solutions. An interesting question is whether smooth solutions will blow up at $t = T$ or, in other words, solutions can be extended to $[0, T')$ for $T' > T$ with the same regularity. In particular, we want to obtain conditions under which the smooth solution loses its regularity at $t = T$ or the solution can be extended beyond $t = T$.

As we known, for the 3D incompressible Navier–Stokes equations, Giga [11] and Kozono and Taniuchi [13] obtained criterions on extension of strong solutions, that is, strong solutions can be continued beyond $t = T$ provided one of following conditions holds:

(1) $u \in L^q(0, T; L^p(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leqslant 1, 3 < p \leqslant \infty$,
(2) $u \in L^2(\varepsilon_0, T; BMO(\mathbb{R}^3))$,
(3) $\nabla \times u \in L^1(\varepsilon_0, T; BMO(\mathbb{R}^3))$, for $0 \leqslant \varepsilon_0 < T$, where BMO is the space of bounded mean oscillation functions. On the other hand, many authors (see [2,14] and references therein) have studied the regularity criterion for the weak solution such as:
(4) $\nabla u \in L^q(0, T; L^p(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leqslant 2$ and $\frac{3}{2} < p \leqslant \infty$,
(5) $\nabla \times u \in L^q(0, T; \dot{B}^0_{p,\infty}(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leqslant 2$ and $3 \leqslant p \leqslant \infty$, where $\dot{B}^0_{p,\infty}$ is homogeneous Besov space (see Section 2).

Caflisch, Klapper and Steele [7] extended the well-known result of Beale, Kato and Majda [1] for incompressible Euler equations to the cases of the 3D ideal MHD equations. Precisely, they showed that if the smooth solution (u, b) satisfies the condition

$$\int_0^T \big(\big\|\nabla \times u(t)\big\|_\infty + \big\|\nabla \times b(t)\big\|_\infty\big)\, dt < \infty, \tag{1.2}$$

then the solution (u, b) can be extended beyond $t = T$. In other words, let $[0, T)$ be the maximal existence time interval for the smooth solution (u, b) to the 3D ideal MHD equations. Then (u, b) blows up at T iff

$$\lim_{\varepsilon \to 0} \int_{\varepsilon_0}^{T-\varepsilon} \big(\big\|\nabla \times u(t)\big\|_\infty + \big\|\nabla \times b(t)\big\|_\infty\big)\, dt = \infty, \quad \forall 0 \leqslant \varepsilon_0 < T. \tag{1.3}$$

Recently, the blow-up criterion (1.3) has been extended to mixed time–space Besov spaces by the Fourier localization method (see [8,22]). For the classical MHD equations, Wu [20] showed that if the velocity and the magnetic field (u, b) satisfy

$$\int_0^T \left(\|\nabla u(t)\|_2^4 + \|\nabla b(t)\|_2^4\right) dt < \infty \tag{1.4}$$

or

$$\int_0^T \left(\|u(t)\|_\infty^2 + \|b(t)\|_\infty^2 dt\right) < \infty, \tag{1.5}$$

then the solution remains smooth. Later, He and Xin [12] or Zhou [23] obtained some integrability condition of the velocity u alone, or of the gradient of the velocity ∇u alone to characterize the regularity criterion for solutions to the classical MHD equations:

$$\int_0^T \|u(t)\|_p^q \, dt < \infty, \quad \frac{2}{q} + \frac{3}{p} \leqslant 1, \ 3 < p \leqslant \infty, \tag{1.6}$$

or

$$\int_0^T \|\nabla u(t)\|_p^q \, dt < \infty, \quad \frac{2}{q} + \frac{3}{p} \leqslant 2, \ \frac{3}{2} < p \leqslant \infty. \tag{1.7}$$

Other relevant results can be found in [12,21,23].

As mentioned above, there are similarities between the one-fluid MHD equations and the Navier–Stokes equations. It is natural to ask whether similar results hold for the two-fluid MHD equations. The second purpose of this paper is to derive a similar blow-up criterion for the strong solution to the 3D two-fluid MHD equations. However, it seems to be difficult to obtain blow-up criterions using only the velocity u like (1.6) and (1.7). Roughly speaking, for the classical MHD equations (i.e. $\alpha = h = 0$), for given u, the magnetic induction equation is linear, so b can be dominated by ∇u in some ways. However, for the 3D two-fluid MHD equations (i.e. $\alpha, h \neq 0$), the magnetic induction equation is nonlinear with the nonlinear current term $\nabla \times (J \times b)$, so the "good" term $-\alpha \Delta b_t - \eta \Delta b$ cannot compensate for the "bad" effect caused by this nonlinear term. This is why our blow-up criterion is given in terms of both the velocity and the magnetic field. We expect to establish a blow-up condition either on vorticity of (u, b) or on (u, b) in terms of Besov spaces as in [14], whose proof is based on the logarithmic Sobolev inequalities. However, in order to obtain the blow-up criterion on (u, b) itself, it seems that the logarithmic Sobolev inequalities do not work. More precisely, from the logarithmic Sobolev inequalities, one can deduce the following estimate of the solutions

$$f(t) \leqslant C \exp\left(\int_0^t g(t')\left(\log f(t')\right)^k dt'\right)$$

for some $k > 1$, which does not imply that $f(t)$ will blow up in the finite time. To overcome this difficulty, we make use of the method of Fourier frequency localization and Bony's paraproduct decomposition which enable us to obtain more precise nonlinear estimates. On the other hand, for the blow-up condition on the vorticity of (u,b), our method gives a priori estimate with one exponential growth, but the logarithmic Sobolev inequalities only give a priori estimate with a double exponential growth. We now state our blow-up result.

Theorem 1.2. *Assume that the initial solenoidal velocity and magnetic field $u_0 \in H^s(\mathbb{R}^3)$, $b_0 \in H^{s+1}(\mathbb{R}^3)$, $s \geqslant 3$. Suppose that $(u,b) \in C([0,T],(H^s \times H^{s+1})(\mathbb{R}^3))$ is the strong solution to* (1.1)*. If either*

$$\int_0^T \left(\|u(t)\|^q_{\dot{B}^0_{p,\infty}} + \|b(t)\|_{\dot{B}^0_{\infty,\infty}}\right) dt < \infty \quad \text{with } \frac{2}{q} + \frac{3}{p} \leqslant 1,\ 3 < p \leqslant \infty, \tag{1.8}$$

or

$$\int_0^T \left(\|\omega(t)\|^q_{\dot{B}^0_{p,\infty}} + \|J(t)\|^q_{\dot{B}^0_{p,\infty}}\right) dt < \infty \quad \text{with } \frac{2}{q} + \frac{3}{p} \leqslant 2,\ 3 \leqslant p \leqslant \infty, \tag{1.9}$$

then the solution (u,b) can be extended beyond $t=T$. In other words, the solution blows up at $t=T$ iff either

$$\lim_{\varepsilon\to 0} \int_{\varepsilon_0}^{T-\varepsilon} \left(\|u(t)\|^q_{\dot{B}^0_{p,\infty}} + \|b(t)\|_{\dot{B}^0_{\infty,\infty}}\right) dt = \infty \quad \text{with } \frac{2}{q} + \frac{3}{p} \leqslant 1,\ 3 < p \leqslant \infty, \tag{1.10}$$

or

$$\lim_{\varepsilon\to 0} \int_{\varepsilon_0}^{T-\varepsilon} \left(\|\omega(t)\|^q_{\dot{B}^0_{p,\infty}} + \|J(t)\|^q_{\dot{B}^0_{p,\infty}}\right) dt = \infty \quad \text{with } \frac{2}{q} + \frac{3}{p} \leqslant 2,\ 3 \leqslant p \leqslant \infty, \tag{1.11}$$

where $\omega(t) = \nabla \times u$ denotes the vorticity of the velocity and $J = \nabla \times b$ denotes the current density.

Remark 1.1. When $\alpha = h = 0$, it is known that if (u,b) solves (1.1) then so does the pair of family (u_λ, b_λ) for all $\lambda > 0$, where $u_\lambda = \lambda u(\lambda x, \lambda^2 t)$, $b_\lambda = \lambda b(\lambda x, \lambda^2 t)$. Moreover, $\|u_\lambda\|_{L^q(\mathbb{R}^+;L^p(\mathbb{R}^3))} = \|u\|_{L^q(\mathbb{R}^+;L^p(\mathbb{R}^3))}$ holds if and only if $\frac{2}{q} + \frac{3}{p} = 1$. However, in the case of $\alpha, h \neq 0$, the second equation of (1.1) does not have such scaling invariance under the transformation $(u,b) \mapsto (u_\lambda, b_\lambda)$. This is why we cannot set up a similar blow-up condition for the magnetic field b as in the 3D MHD equations.

Remark 1.2. In the conditions (1.9) and (1.11), the integrability range of ω can be $\frac{3}{2} < p \leqslant \infty$ by the Sobolev embedding theorem. On the other hand, by means of the Hölder inequality

$$\|b\|_{L^1_T(\dot{B}^0_{\infty,\infty})} \leqslant T^{1-\frac{1}{\tilde{q}}} \|b\|_{L^{\tilde{q}}_T(\dot{B}^0_{\infty,\infty})}, \quad 1 \leqslant \tilde{q} \leqslant \infty,$$

the condition on b in (1.8) can be extended to $b \in L_T^{\tilde q}(\dot B^0_{\infty,\infty})$, $1 \leqslant \tilde q \leqslant \infty$. For the case $p=\infty$, the two-fluid system seems to get a benefit from the term $-\alpha \Delta b_t$. For the classical MHD equations, we have the restriction condition $\tilde q \geqslant 2$ in the case $p=\infty$ (see [20]).

Remark 1.3. By means of the Sobolev embedding theorem $L^p \hookrightarrow \dot B^0_{p,\infty}$, $1 \leqslant p \leqslant \infty$, the corresponding result to Theorem 1.2 can be obtained in the framework of Lebesgue spaces, that is, if either

$$\begin{cases} u \in L_T^q\big(L^p\big) \quad \text{with } \dfrac{2}{q} + \dfrac{3}{p} \leqslant 1,\ 3 < p \leqslant \infty, \\ b \in L_T^1\big(L^\infty\big), \end{cases}$$

or

$$(\omega, J) \in L_T^q\big(L^p\big) \quad \text{with } \frac{2}{q} + \frac{3}{p} \leqslant 2,\ 3 \leqslant p \leqslant \infty,$$

then the solution (u,b) of (1.1) can be extended beyond $t = T$, where $L_T^q(X)$ denotes $L^q((0,T);X)$.

Notation. Throughout the paper, C stands for a generic constant. We will use the notation $A \lesssim B$ to denote the relation $A \leqslant CB$ and the notation $A \approx B$ to denote the relations $A \lesssim B$ and $B \lesssim A$. Further, $\|\cdot\|_p$ denotes the norm of the Lebesgue space L^p and $\|(f_1, f_2, \dots, f_i)\|_X^a$ denotes $\|f_1\|_X^a + \cdots + \|f_i\|_X^a$.

2. Preliminaries

Let us recall the Littlewood–Paley decomposition. Let $\mathcal S(\mathbb R^3)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal S(\mathbb R^3)$, its Fourier transform $\mathcal F f = \hat f$ is defined by

$$\hat f(\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb R^3} e^{-ix\cdot\xi} f(x)\,dx.$$

Choose two nonnegative radial functions $\chi, \varphi \in \mathcal S(\mathbb R^3)$ supported respectively in $\mathcal B = \{\xi \in \mathbb R^3, |\xi| \leqslant \frac{4}{3}\}$ and $\mathcal C = \{\xi \in \mathbb R^3, \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$ such that

$$\chi(\xi) + \sum_{j \geqslant 0} \varphi\big(2^{-j}\xi\big) = 1, \quad \xi \in \mathbb R^3,$$

$$\sum_{j \in \mathbb Z} \varphi\big(2^{-j}\xi\big) = 1, \quad \xi \in \mathbb R^3 \backslash \{0\}.$$

Set $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ and let $h = \mathcal F^{-1}\varphi$ and $\tilde h = \mathcal F^{-1}\chi$. Define the frequency localization operators:

$$\Delta_j f = \varphi\big(2^{-j}D\big)f = 2^{3j}\int_{\mathbb{R}^3} h\big(2^j y\big)f(x-y)\,dy,$$

$$S_j f = \sum_{k\leqslant j-1}\Delta_k f = \chi\big(2^{-j}D\big)f = 2^{3j}\int_{\mathbb{R}^3}\tilde{h}\big(2^j y\big)f(x-y)\,dy.$$

Formally, $\Delta_j = S_j - S_{j-1}$ is a frequency projection into the annulus $\{|\xi|\approx 2^j\}$, and S_j is a frequency projection into the ball $\{|\xi|\lesssim 2^j\}$. One easily verifies that with the above choice of φ

$$\Delta_j\Delta_k f\equiv 0\quad \text{if } |j-k|\geqslant 2\quad\text{and}\quad \Delta_j(S_{k-1}f\,\Delta_k f)\equiv 0\quad \text{if } |j-k|\geqslant 5. \tag{2.1}$$

We now introduce the following definition of Besov spaces.

Definition 2.1. Let $s\in\mathbb{R}$, $1\leqslant p,q\leqslant\infty$. The homogenous Besov space $\dot{B}^s_{p,q}$ is defined by

$$\dot{B}^s_{p,q} = \big\{f\in\mathcal{Z}'\big(\mathbb{R}^3\big):\ \|f\|_{\dot{B}^s_{p,q}}<\infty\big\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \big(\sum_{j\in\mathbb{Z}} 2^{jsq}\|\Delta_j f\|_p^q\big)^{1/q}, & \text{for } q<\infty,\\ \sup_{j\in\mathbb{Z}} 2^{js}\|\Delta_j f\|_p, & \text{for } q=\infty,\end{cases}$$

and $\mathcal{Z}'(\mathbb{R}^3)$ can be identified by the quotient space $\mathcal{S}'/\mathcal{P}$ with the space $\mathcal{P}$ of polynomials.

Definition 2.2. Let $s\in\mathbb{R}$, $1\leqslant p,q\leqslant\infty$. The inhomogeneous Besov space $B^s_{p,q}$ is defined by

$$B^s_{p,q} = \big\{f\in\mathcal{S}'\big(\mathbb{R}^3\big):\ \|f\|_{B^s_{p,q}}<\infty\big\},$$

where

$$\|f\|_{B^s_{p,q}} = \begin{cases} \big(\sum_{j\geqslant 0} 2^{jsq}\|\Delta_j f\|_p^q\big)^{1/q} + \|S_0(f)\|_p, & \text{for } q<\infty,\\ \sup_{j\geqslant 0} 2^{js}\|\Delta_j f\|_p + \|S_0(f)\|_p, & \text{for } q=\infty.\end{cases}$$

If $s>0$, then $B^s_{p,q} = L^p\cap\dot{B}^s_{p,q}$ and $\|f\|_{B^s_{p,q}}\approx\|f\|_p+\|f\|_{\dot{B}^s_{p,q}}$. We refer to [3,19] for details.

Let us state some basic properties about the Besov spaces.

Proposition 2.1. (i) *When $p=q=2$, the homogeneous Sobolev spaces $\dot{H}^s$ and $\dot{B}^2_{2,2}$ are equal and the two norms are equivalent*:

$$\|f\|_{\dot{H}^s}\approx\|f\|_{\dot{B}^2_{2,2}}.$$

Similar properties hold for the inhomogeneous Sobolev spaces H^s and $B^2_{2,2}$.

(ii) *We have the equivalence of norms*

$$\|D^k f\|_{\dot B^s_{p,q}} \approx \|f\|_{\dot B^{s+k}_{p,q}}, \quad \text{for } k\in\mathbb{Z}^+.$$

(iii) *Interpolation*: *for* $s_1, s_2 \in \mathbb{R}$ *and* $\theta\in[0,1]$, *one has*

$$\|f\|_{\dot B^{\theta s_1+(1-\theta)s_2}_{p,q}} \leqslant \|f\|^{\theta}_{\dot B^{s_1}_{p,q}} \|f\|^{(1-\theta)}_{\dot B^{s_2}_{p,q}}.$$

Similar interpolation inequality holds for inhomogeneous Besov spaces.

The proofs of (i)–(iii) are standard and can be found in [3,19].

3. Local existence and uniqueness

We now prove Theorem 1.1.

The proof of local existence. It involves the method of successive approximation. Define the sequence $\{u^{(n)}, b^{(n)}\}_{n\in\mathbb{N}_0}$ by the following linear system:

$$\begin{cases} u_t^{(n+1)} - \nu\Delta u^{(n+1)} = -u^{(n)}\cdot\nabla u^{(n)} + b^{(n)}\cdot\nabla b^{(n)} - \nabla\left(p^{(n)} + \frac{1}{2}b^{2(n)}\right), \\ b_t^{(n+1)} - \alpha\Delta b_t^{(n+1)} - \eta\Delta b^{(n+1)} = -u^{(n)}\cdot\nabla b^{(n)} + b^{(n)}\cdot\nabla u^{(n)} - h\nabla\times\left(J^{(n)}\times b^{(n)}\right), \\ \nabla\cdot u^{(n+1)} = \nabla\cdot b^{(n+1)} = 0, \\ \left(u^{(n+1)}, b^{(n+1)}\right)\big|_{t=0} = S_{n+2}(u_0, b_0), \end{cases} \tag{3.1}$$

where we set $(u^{(0)}, b^{(0)}) = (0,0)$, so $p^{(0)}=0$. We first derive the L^2 estimate of solutions. By the divergence free condition, the embedding relation $H^s \hookrightarrow L^\infty$ and the ϵ-Young inequality it is easy to see that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|\left(u^{(n+1)}, b^{(n+1)}, \alpha^{\frac12}\nabla b^{(n+1)}\right)(t)\right\|_2^2 + \nu\left\|\nabla u^{(n+1)}(t)\right\|_2^2 + \eta\left\|\nabla b^{(n+1)}(t)\right\|_2^2 \\ &\quad\leqslant \left(\left\|u^{(n)}\right\|_2^2 + \left\|b^{(n)}\right\|_2^2 + h\left\|J^{(n)}\right\|_2\left\|b^{(n)}\right\|_2\right)\left(\left\|\nabla u^{(n+1)}\right\|_\infty + \left\|\nabla b^{(n+1)}\right\|_\infty\right) \\ &\quad\leqslant \frac{\nu}{2}\left\|\nabla u^{(n+1)}\right\|_{H^s} + \frac{\eta}{2}\left\|\nabla b^{(n+1)}\right\|_{H^s} + C\left\|\left(u^{(n)}, b^{(n)}, \alpha^{\frac12}\nabla b^{(n)}\right)\right\|_2^4. \end{aligned} \tag{3.2}$$

Now we derive the $\dot H^s$ estimate. Apply the operator Δ_k to Eqs. (3.1), multiply the first one by $\Delta_k u^{(n+1)}$ and the second one by $\Delta_k b^{(n+1)}$, integrate by parts to get, on noting that $\operatorname{div} u^{(n+1)} = \operatorname{div} b^{(n+1)} = 0$, that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|\left(\Delta_k u^{(n+1)}, \Delta_k b^{(n+1)}, \alpha^{\frac12}\nabla\Delta_k b^{(n+1)}\right)(t)\right\|_2^2 + \nu\left\|\nabla\Delta_k u^{(n+1)}(t)\right\|_2^2 + \eta\left\|\nabla\Delta_k b^{(n+1)}(t)\right\|_2^2 \\ &= \left\langle\Delta_k\left(u^{(n)}\otimes u^{(n)}\right), \nabla\Delta_k u^{(n+1)}\right\rangle - \left\langle\Delta_k\left(b^{(n)}\otimes b^{(n)}\right), \nabla\Delta_k u^{(n+1)}\right\rangle \\ &\quad + \left\langle\Delta_k\left(u^{(n)}\otimes b^{(n)} - b^{(n)}\otimes u^{(n)}\right), \nabla\Delta_k b^{(n+1)}\right\rangle - h\left\langle\Delta_k\left(J^{(n)}\otimes b^{(n)}\right), \nabla\times\Delta_k b^{(n+1)}\right\rangle, \end{aligned} \tag{3.3}$$

where $\langle\cdot,\cdot\rangle$ stands for the inner product. Multiplying 2^{2ks} on both sides of (3.3) and summing up over $k \in \mathbb{Z}$ yield that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big\|\big(u^{(n+1)}, b^{(n+1)}, \alpha^{\frac{1}{2}}\nabla b^{(n+1)}\big)(t)\big\|_{\dot{H}^s} + \nu\big\|\nabla u^{(n+1)}(t)\big\|_{\dot{H}^s} + \eta\big\|\nabla b^{(n+1)}(t)\big\|_{\dot{H}^s}\\ &\quad\leqslant \sum_{i=1}^{4} \Pi_i,\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}\sum_{i=1}^{4}\Pi_i \triangleq{}& \sum_{k\in\mathbb{Z}} 2^{2ks}\big\|\Delta_k\big(u^{(n)}\otimes u^{(n)}\big)\big\|_2\big\|\Delta_k\nabla u^{(n+1)}\big\|_2\\ &+\sum_{k\in\mathbb{Z}} 2^{2ks}\big\|\Delta_k\big(b^{(n)}\otimes b^{n}\big)\big\|_2\big\|\Delta_k\nabla u^{(n+1)}\big\|_2\\ &+\sum_{k\in\mathbb{Z}} 2^{2ks}\big\|\Delta_k\big(u^{(n)}\otimes b^{(n)}+b^{(n)}\otimes u^{n}\big)\big\|_2\big\|\Delta_k\nabla b^{(n+1)}\big\|_2\\ &+h\sum_{k\in\mathbb{Z}} 2^{2ks}\big\|\Delta_k\big(J^{(n)}\otimes b^{(n)}\big)\big\|_2\big\|\Delta_k\nabla b^{(n+1)}\big\|_2.\end{aligned}$$

Using the Schwartz inequality, Lemma A.2 and the embedding results that $H^s \hookrightarrow \dot{H}^s$ and $H^s \hookrightarrow L^\infty$, we obtain that

$$\begin{aligned}\Pi_1(t) &\leqslant \big\|u^{(n)}u^{(n)}\big\|_{\dot{H}^s}\big\|\nabla u^{(n+1)}\big\|_{\dot{H}^s} \leqslant C\big\|u^{(n)}\big\|_{L^\infty}\big\|u^{(n)}\big\|_{\dot{H}^s}\big\|\nabla u^{(n+1)}\big\|_{\dot{H}^s}\\ &\leqslant \big\|u^{(n)}\big\|_{H^s}^2\big\|\nabla u^{(n+1)}\big\|_{H^s} \leqslant \frac{\nu}{4}\big\|\nabla u^{(n+1)}\big\|_{H^s}^2 + C_\nu\big\|u^{(n)}\big\|_{H^s}^4.\end{aligned} \tag{3.5}$$

Similarly, we have

$$\sum_{i=2}^{4}\Pi_i(t) \leqslant \frac{\nu}{4}\big\|\nabla u^{(n+1)}\big\|_{H^s}^2 + \frac{\eta}{4}\big\|\nabla b^{(n+1)}\big\|_{H^s}^2 + C_{\alpha,\nu,\eta,h}\big\|\big(u^{(n)}, b^{(n)}, \alpha^{\frac{1}{2}}\nabla b^{(n)}\big)\big\|_{H^s}^4. \tag{3.6}$$

Set $E_s^{(n)}(t) \triangleq \|(u^{(n)}, b^{(n)}, \alpha^{\frac{1}{2}}\nabla b^{(n)})\|_{H^s}^2$, $n \in \mathbb{N}_0$. Adding (3.2) and (3.4), and using (3.5) and (3.6) yield that

$$\frac{d}{dt}E_s^{(n+1)}(t) + \nu\big\|\nabla u^{(n+1)}(t)\big\|_{H^s}^2 + \eta\big\|\nabla b^{(n+1)}(t)\big\|_{H^s}^2 \leqslant \widetilde{C}\big\|\big(u^{(n)}, b^{(n)}, \alpha^{\frac{1}{2}}\nabla b^{(n)}\big)\big\|_{H^s}^4,$$

where $\widetilde{C} = C_{\alpha,\nu,\eta,h}$. Integrating the above inequality with respect to t gives

$$\sup_{t\in[0,T]} E_s^{(n+1)}(t) + \int_0^T \nu\big\|\nabla u^{(n+1)}(t)\big\|_{H^s}^2 + \eta\big\|\nabla b^{(n+1)}(t)\big\|_{H^s}^2\,dt$$

$$\leqslant \left\| S_{n+2}\left(u_0, b_0, \alpha^{\frac{1}{2}}\nabla b_0\right)\right\|_{H^s}^2 + \widetilde{C}\int_0^T \left\|u^{(n)}(t)\right\|_{H^s}^4 + \left\|b^{(n)}(t)\right\|_{H^s}^4 + \alpha^2\left\|\nabla b^{(n)}(t)\right\|_{H^s}^4\,dt$$

$$\leqslant C_0\left\|\left(u_0, b_0, \alpha^{\frac{1}{2}}\nabla b_0\right)\right\|_{H^s}^2 + \widetilde{C}T\Big(\sup_{t\in[0,T]} E_s^{(n)}(t)\Big)^2.$$

Thus, by the standard induction argument, it follows that

$$\left\|\left(u^{(n+1)}, b^{(n+1)}, \alpha^{\frac{1}{2}}\nabla b^{(n+1)}\right)(t)\right\|_{L_T^\infty(H^s)} + \nu^{\frac{1}{2}}\left\|\nabla u^{(n+1)}(t)\right\|_{L_T^2(H^s)} + \eta^{\frac{1}{2}}\left\|\nabla b^{(n+1)}(t)\right\|_{L_T^2(H^s)}$$
$$\leqslant 2C_0\left\|\left(u_0, b_0, \alpha^{\frac{1}{2}}\nabla b_0\right)\right\|_{H^s} \tag{3.7}$$

for all $n \in \mathbb{N}_0$, and for $T \in [0, T_0]$, where we set

$$T_0 = \frac{1}{4C_0\widetilde{C}\|(u_0, b_0, \alpha^{\frac{1}{2}}\nabla b_0)\|_{H^s}^2}.$$

Next we show that there exists a positive time T_1 ($\leqslant T$) independent of n such that $\{u^{(n)}, b^{(n)}\}$ is a Cauchy sequence in the space

$$\mathcal{X}_{T_1}^{s-1} \triangleq \left\{\left(f, g, \alpha^{\frac{1}{2}}\nabla g\right) \in L_{T_1}^\infty\left(H^{s-1}\right), \left(\nu^{\frac{1}{2}}\nabla f, \eta^{\frac{1}{2}}\nabla g\right) \in L_{T_1}^2\left(H^{s-1}\right)\right\}.$$

Let $\delta u^{(n+1)} = u^{(n+1)} - u^{(n)}$, $\delta b^{(n+1)} = b^{(n+1)} - b^{(n)}$, $\delta p^{(n+1)} = p^{(n+1)} - p^{(n)}$ and $\delta(b^2)^{(n)} = b^{2(n)} - b^{2(n-1)}$ satisfy that

$$\begin{cases} \delta u_t^{(n+1)} - \nu\Delta\delta u^{(n+1)} = F_1 + F_2 + \cdots + F_5, \\ \delta b_t^{(n+1)} - \alpha\Delta\delta b_t^{(n+1)} - \eta\Delta\delta b^{(n+1)} = G_1 + G_2 + \cdots + G_6, \\ \left(\delta u^{(n+1)}, \delta b^{(n+1)}\right)\big|_{t=0} = \Delta_{n+1}(u_0, b_0), \end{cases} \tag{3.8}$$

where

$$\sum_{j=1}^{5} F_i \triangleq -\delta u^{(n)}\cdot\nabla u^{(n)} - u^{(n-1)}\cdot\nabla\delta u^{(n)} + \delta b^{(n)}\cdot\nabla b^{(n)} + b^{(n-1)}\cdot\nabla\delta b^{(n)}$$
$$- \nabla\left(\delta p^{(n)} + \frac{1}{2}\delta b^{2(n)}\right),$$

$$\sum_{j=1}^{6} G_i \triangleq -\delta u^{(n)}\cdot\nabla b^{(n)} - u^{(n-1)}\cdot\nabla\delta b^{(n)} + \delta b^{(n)}\cdot\nabla u^{(n)} + b^{(n-1)}\cdot\nabla\delta u^{(n)}$$
$$- h\nabla\times\left(\delta J^{(n)}\times b^{(n)}\right) - h\nabla\times\left(J^{(n-1)}\times\delta b^{(n)}\right).$$

Applying the divergence free condition to F_2, F_3 and F_5 yields that

$$\left|\left\langle \sum_{j=1}^{5} F_i, \delta u^{(n+1)} \right\rangle\right| \leqslant \|\delta u^{(n)}\|_2 \|\nabla u^{(n)}\|_\infty \|\delta u^{(n+1)}\|_2 + \|u^{(n-1)}\|_\infty \|\delta u^{(n)}\|_2 \|\nabla \delta u^{(n+1)}\|_2$$
$$+ \|\delta b^{(n)}\|_2 \|\nabla b^{(n)}\|_\infty \|\delta u^{(n+1)}\|_2 + \|b^{(n-1)}\|_\infty \|\delta b^{(n)}\|_2 \|\nabla \delta u^{(n+1)}\|_2^2$$
$$\leqslant \nu \|\nabla \delta u^{(n+1)}\|_2^2 + C\big(\|(u^{(n)}, b^{(n)})\|_{H^s} + \|(u^{(n-1)}, b^{(n-1)})\|_{H^s}^2\big)$$
$$\times \|(\delta u^{(n)}, \delta b^{(n)}, \delta u^{(n+1)})\|_2^2. \tag{3.9}$$

Similarly, we have

$$\left|\left\langle \sum_{j=1}^{6} G_i, \delta u^{(n+1)} \right\rangle\right| \leqslant C \|(u^{(n)}, b^{(n)}, u^{(n-1)}, b^{(n-1)})\|_{H^s}$$
$$\times \big(\|(\delta u^{(n)}, \delta b^{(n)}, \delta b^{(n+1)}, \alpha^{\frac{1}{2}} \nabla \delta b^{(n)}, \alpha^{\frac{1}{2}} \nabla \delta b^{(n+1)})\|_2^2\big). \tag{3.10}$$

Set $\delta E^{(n)}(t) \triangleq \|(\delta u^{(n)}, \delta b^{(n)}, \alpha^{\frac{1}{2}} \nabla \delta b^{(n)})\|_2^2$. By the L^2 energy estimate combined with (3.9) and (3.10) it is derived that

$$\frac{d}{dt} \delta E^{(n+1)}(t) + \nu \|\nabla \delta u^{(n+1)}(t)\|_2^2 + \eta \|\nabla \delta b^{(n+1)}(t)\|_2^2 \leqslant C_1\big(\delta E^{(n)}(t) + \delta E^{(n+1)}(t)\big),$$

where $C_1 = C_{\alpha,\nu,\eta,h,\|(u_0,h_0,\sqrt{\alpha}\nabla h_0)\|_{H^s}^2}$. Integrating the above inequality with respect to t gives

$$\sup_{t\in[0,T]} \delta E^{(n+1)}(t) + \int_0^T \nu \|\nabla \delta u^{(n+1)}(t)\|_2^2 + \eta \|\nabla \delta b^{(n+1)}(t)\|_2^2 \, dt$$
$$\leqslant C_2 2^{-2(n+1)s} \|(u_0, h_0, \sqrt{\alpha} \nabla h_0)\|_{H^s}^2 + C_1 T \sup_{t\in[0,T]} \big(\delta E^{(n)}(t) + \delta E^{(n+1)}(t)\big), \tag{3.11}$$

using the fact that

$$\|\Delta_{n+1}(u_0, b_0, \alpha^{\frac{1}{2}} \nabla b_0)\|_2 \leqslant C_2 2^{-(n+1)s} \|(u_0, h_0, \sqrt{\alpha} \nabla h_0)\|_{H^s}.$$

Thus, if $C_1 T \leqslant \frac{1}{4}$, then we have

$$\|(\delta u^{(n+1)}, \delta b^{(n+1)}, \alpha^{\frac{1}{2}} \nabla \delta b^{(n+1)})\|_{L_T^\infty(L^2)} + \nu^{\frac{1}{2}} \|\nabla \delta u^{(n+1)}(t)\|_{L_T^2(L^2)} + \eta^{\frac{1}{2}} \|\nabla \delta b^{(n+1)}(t)\|_{L_T^2(L^2)}$$
$$\leqslant 2C_2 2^{-(n+1)s}, \quad n \in \mathbb{N}_0.$$

This, together with the interpolation $\|f\|_{H^{s-1}} \leqslant \|f\|_2^{\frac{1}{s}} \|f\|_{H^s}^{1-\frac{1}{s}}$ and (3.7), implies that

$$\|(\delta u^{(n+1)}, \delta b^{(n+1)}, \alpha^{\frac{1}{2}} \nabla \delta b^{(n+1)})\|_{L_T^\infty(H^{s-1})} + \nu^{\frac{1}{2}} \|\nabla \delta u^{(n+1)}(t)\|_{L_T^2(H^{s-1})}$$
$$+ \eta^{\frac{1}{2}} \|\nabla \delta b^{(n+1)}(t)\|_{L_T^2(H^{s-1})}$$
$$\leqslant 2C_2 C_0 2^{-(n+1)} \|(u_0, b_0, \alpha^{\frac{1}{2}} \nabla b_0)\|_{H^s}. \tag{3.12}$$

By a standard argument, it can been shown that for $T_1 \leqslant \min\{T_0, \frac{1}{4C_1}\}$, the sequence $\{u^{(n)}, b^{(n)}\}$ converges to (u, b) in $\mathcal{X}_{T_1}^{s-1}$ which is an equation to Eq. (1.1). Moreover, (u, b) satisfies that

$$\begin{aligned}&\big\|\big(u, b, \alpha^{\frac{1}{2}}\nabla b\big)(t)\big\|_{L^\infty_{T_1}(H^s)} + \nu^{\frac{1}{2}}\big\|\nabla u(t)\big\|_{L^2_{T_1}(H^s)} + \eta^{\frac{1}{2}}\big\|\nabla b(t)\big\|_{L^2_{T_1}(H^s)}\\ &\quad\leqslant 2C_0\big\|\big(u_0, b_0, \alpha^{\frac{1}{2}}\nabla b_0\big)\big\|_{H^s}. \qquad \square \end{aligned} \tag{3.13}$$

The proof of the uniqueness. Suppose $(u', b') \in L^\infty_T(H^s)$ is another solution to (1.1). Let $\delta u = u - u'$ and $\delta b = b - b'$. Then $(\delta\theta, \delta u)$ satisfies the following equations

$$\begin{cases} \delta u_t - \nu\Delta\delta u = -\delta u\cdot\nabla u - u'\cdot\nabla\delta u + \delta b\cdot\nabla b + b'\cdot\nabla\delta b - \nabla\left(\delta p + \dfrac{1}{2}\delta b^2\right),\\ \delta b_t - \alpha\Delta\delta b_t - \eta\Delta\delta b = -\delta u\cdot\nabla b - u'\cdot\nabla\delta b + \delta b\cdot\nabla u + b'\cdot\nabla\delta u - h\nabla\times(\delta J\times b)\\ \qquad\qquad - h\big(\nabla\times(J'\times\delta b)\big). \end{cases}$$

By the divergence free condition and integrating by part, we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big\|\big(\delta u, \delta b, \alpha^{\frac{1}{2}}\nabla\delta b\big)\big\|_2^2 + \nu\|\nabla\delta u\|_2^2 + \eta\|\nabla\delta b\|_2^2\\ &\quad= -\langle\delta u\cdot\nabla u, \delta u\rangle + \langle\delta b\cdot\nabla b, \delta u\rangle - \langle\delta u\cdot\nabla b, \delta b\rangle + \langle\delta b\cdot\nabla u, \delta b\rangle - h\langle J'\times\delta b, \nabla\times\delta b\rangle\\ &\quad\leqslant \|\delta u\|_2^2\|\nabla u\|_\infty + 2\|\delta b\|_2\|\delta u\|_2\|\nabla b\|_\infty + \|\delta b\|_2^2\|\nabla u\|_\infty + h\|J'\|_\infty\|\delta b\|_2\|\nabla\times\delta b\|_2\\ &\quad\leqslant C\big\|(u, b, b')\big\|_{H^s}\big(\|\delta u\|_2^2 + \|\delta b\|_2^2 + \alpha\|\nabla\delta b\|_2^2\big).\end{aligned}$$

Thus we have

$$\big\|\big(\delta u, \delta b, \alpha^{\frac{1}{2}}\nabla\delta b\big)\big\|_2 \leqslant C_3T\big\|\big(\delta u, \delta b, \alpha^{\frac{1}{2}}\nabla\delta b\big)\big\|_2,$$

where $C_3 = C_{\|(u_0, b_0, \sqrt{\alpha}\nabla b_0)\|_{H^s}}$. This implies that for sufficiently small $T \leqslant T_2$, $\|(\delta u, \delta b, \alpha^{\frac{1}{2}}\nabla\delta b)\|_2 \equiv 0$. Then a standard argument shows the uniqueness of local solutions in $L^\infty_T(H^s)$. This completes the proof of Theorem 1.1. $\square$

4. Blow-up criterion

In this section, we prove Theorem 1.2 which establishes the blow-up criterion for the smooth solution to (1.1). The proof is broken down into two cases.

Case I. The proof of blow-up criterion under condition (1.8). We first derive a priori estimate of the smooth solution to (1.1). Arguing similarly as in deriving (3.3), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\Delta_k u\|_2^2 + \|\Delta_k b\|_2^2 + \alpha\|\nabla\Delta_k b\|_2^2\big) + \nu\|\nabla\Delta_k u\|_2^2 + \eta\|\nabla\Delta_k b\|_2^2\\ &\quad= -\big\langle\Delta_k(u\cdot\nabla u), \Delta_k u\big\rangle + \big\langle\Delta_k(b\cdot\nabla b), \Delta_k u\big\rangle - \big\langle\Delta_k(u\cdot\nabla b), \Delta_k b\big\rangle\\ &\qquad + \big\langle\Delta_k(b\cdot\nabla u), \Delta_k b\big\rangle - h\big\langle\Delta_k\big(\nabla\times(J\times b)\big), \Delta_k b\big\rangle.\end{aligned} \tag{4.1}$$

Noting that $\int_{\mathbb{R}^3}(b\times\Delta_k J)\Delta_k J\,dx=0$, it follows that

$$-\langle\Delta_k(\nabla\times(J\times b)),\Delta_k b\rangle=\langle\Delta_k(b\times J),\Delta_k(\nabla\times b)\rangle=\langle(\Delta_k(b\times J)-b\times\Delta_k J),\Delta_k J\rangle.$$

Substituting this into (4.1) and making use of the fact that $\operatorname{div} u=\operatorname{div} b=0$, we obtain by integrating by parts that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Delta_k u\|_2^2+\|\Delta_k b\|_2^2+\alpha\|\nabla\Delta_k b\|_2^2\right)+\nu\|\nabla\Delta_k u\|_2^2+\eta\|\nabla\Delta_k b\|_2^2\\
&\quad=-\langle(\Delta_k(u\cdot\nabla u)-u\cdot\nabla\Delta_k u),\Delta_k u\rangle+\langle(\Delta_k(b\cdot\nabla b)-b\cdot\nabla\Delta_k b),\Delta_k u\rangle\\
&\qquad+\langle(\Delta_k(b\cdot\nabla u)-b\cdot\nabla\Delta_k u),\Delta_k b\rangle-\langle(\Delta_k(u\cdot\nabla b)-u\cdot\nabla\Delta_k b),\Delta_k b\rangle\\
&\qquad+h\langle(\Delta_k(b\times J)-b\times\Delta_k J),\Delta_k J\rangle.
\end{aligned}\tag{4.2}$$

Write the commutator $[f,\Delta_k]\cdot\nabla g$ for $f\cdot\nabla\Delta_k g-\Delta_k(f\cdot\nabla g)$, multiply both sides of (4.2) by 2^{2ks}, and sum the resulting equation over $k\in\mathbb{Z}$ to deduce that

$$\begin{aligned}
&\frac{d}{dt}\left\|\left(u,b,\alpha^{\frac12}\nabla b\right)(t)\right\|_{\dot H^s}+2\nu\|\nabla u(t)\|_{\dot H^s}+\eta\|\nabla b(t)\|_{\dot H^s}\\
&\quad\leqslant\sum_{k\in\mathbb{Z}}2^{2ks}\left\|[u,\Delta_k]\cdot\nabla u\right\|_2\|\Delta_k u\|_2\\
&\qquad+\sum_{k\in\mathbb{Z}}2^{2ks}\left(\left\|[b,\Delta_k]\cdot\nabla b\right\|_2\|\Delta_k u\|_2+\left\|[b,\Delta_k]\cdot\nabla u\right\|_2\|\Delta_k b\|_2\right)\\
&\qquad+\sum_{k\in\mathbb{Z}}2^{2ks}\left\|[u,\Delta_k]\cdot\nabla b\right\|_2\|\Delta_k b\|_2+h\sum_{k\in\mathbb{Z}}2^{2ks}\left\|[b\times,\Delta_k]J\right\|_2\|\Delta_k J\|_2\\
&\quad\triangleq \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}.
\end{aligned}\tag{4.3}$$

Making use of the Schwartz inequality, and applying Lemma A.3 with $\sigma=s-1$, $\sigma_1=\sigma_2=-1$ and $p_1=p_2=p$ to the commutator, it follows that for $3<p\leqslant\infty$,

$$\begin{aligned}
|\mathrm{I}|&\leqslant C\left\|2^{k(s-1)}\left\|[u,\Delta_k]\cdot\nabla u\right\|_2\right\|_{\ell^2(\mathbb{Z})}\|u\|_{\dot B^{s+1}_{2,2}}\\
&\leqslant C\|u\|_{\dot B^0_{p,\infty}}\|u\|_{\dot B^{s+\frac3p}_{2,2}}\|u\|_{\dot B^{s+1}_{2,2}}\leqslant C\|u\|_{\dot B^0_{p,\infty}}\|u\|_{\dot H^s}^{1-\frac3p}\|\nabla u\|_{\dot H^s}^{1+\frac3p}\\
&\leqslant\nu\|\nabla u\|_{\dot H^s}^2+C\|u\|_{\dot B^0_{p,\infty}}^{\frac{2p}{p-3}}\|u\|_{\dot H^s}^2,
\end{aligned}\tag{4.4}$$

where the use has been made of the equivalent norms of $\dot B^\sigma_{2,2}$ and $\dot H^\sigma$ for $\sigma\in\mathbb{R}$ and the interpolation theorem in deriving the third inequality, and of the Young inequality to obtain the last inequality. Similarly, for $3<p\leqslant\infty$ we have the estimates

$$\begin{aligned}
|\mathrm{II}|&\leqslant C\left\|2^{ks}\left\|[b,\Delta_k]\cdot\nabla b\right\|_2\right\|_{\ell^2(\mathbb{Z})}\|u\|_{\dot B^s_{2,2}}+\left\|2^{k(s-1)}\left\|[b,\Delta_k]\cdot\nabla u\right\|_2\right\|_{\ell^2(\mathbb{Z})}\|b\|_{\dot B^{s+1}_{2,2}}\\
&\leqslant C\|b\|_{\dot B^0_{\infty,\infty}}\|b\|_{\dot B^{s+1}_{2,2}}\|u\|_{\dot B^s_{2,2}}+C\left(\|b\|_{\dot B^0_{\infty,\infty}}\|u\|_{\dot B^s_{2,2}}+\|u\|_{\dot B^0_{p,\infty}}\|b\|_{\dot B^{s+\frac3p}_{2,2}}\right)\|b\|_{\dot B^{s+1}_{2,2}}
\end{aligned}$$

$$\leqslant C\big(\|u\|_{\dot B^0_{p,\infty}}\|b\|_{\dot H^s}^{1-\frac{3}{p}}\|\nabla b\|_{\dot H^s}^{1+\frac{3}{p}}+\|b\|_{\dot B^0_{\infty,\infty}}\|u\|_{\dot H^s}\|\nabla b\|_{\dot H^s}\big)$$

$$\leqslant \frac{\eta}{2}\|\nabla b\|^2_{\dot H^s}+C\big(\|u\|^{\frac{2p}{p-3}}_{\dot B^0_{p,\infty}}+\|b\|_{\dot B^0_{\infty,\infty}}\big)\big(\|u\|^2_{\dot H^s}+\|b\|^2_{\dot H^s}+\alpha\|\nabla b\|^2_{\dot H^s}\big), \tag{4.5}$$

and

$$|\mathrm{III}|\leqslant C\big\|2^{k(s-1)}\big\|[u,\Delta_k]\cdot\nabla b\big\|_2\big\|_{\ell^2(\mathbb Z)}\|b\|_{\dot B^{s+1}_{2,2}}$$

$$\leqslant C\big(\|u\|_{\dot B^0_{p,\infty}}\|b\|_{\dot H^{s+\frac{3}{p}}}+\|b\|_{\dot B^0_{\infty,\infty}}\|u\|_{\dot H^s}\big)\|b\|_{\dot H^{s+1}}$$

$$\leqslant \frac{\eta}{2}\|\nabla b\|^2_{\dot H^s}+C\big(\|u\|^{\frac{2p}{p-3}}_{\dot B^0_{p,\infty}}+\|b\|_{\dot B^0_{\infty,\infty}}\big)\big(\|u\|^2_{\dot H^s}+\|b\|^2_{\dot H^s}+\alpha\|\nabla b\|^2_{\dot H^s}\big) \tag{4.6}$$

and

$$|\mathrm{IV}|\leqslant Ch\big\|2^{ks}\big\|[b\times,\Delta_k]J\big\|_2\big\|_{\ell^2(\mathbb Z)}\|J\|_{\dot B^s_{2,2}}$$

$$\leqslant Ch\|b\|_{\dot B^0_{\infty,\infty}}\|b\|_{\dot B^{s+1}_{2,2}}\|J\|_{\dot B^s_{2,2}}\leqslant Ch\|b\|_{\dot B^0_{\infty,\infty}}\|\nabla b\|^2_{\dot H^s}. \tag{4.7}$$

Integrating (4.3) with respect to t and using (4.4)–(4.7), we deduce that for $3<p\leqslant\infty$,

$$\sup_{t\in[0,T]}\big\|\big(u,b,\alpha^{\frac12}\nabla b\big)(t)\big\|^2_{\dot H^s}+\int_0^T\big(\nu\big\|\nabla u(t)\big\|^2_{\dot H^s}+\eta\big\|\nabla b(t)\big\|^2_{\dot H^s}\big)\,dt$$

$$\leqslant\big\|\big(u_0,b_0,\alpha^{\frac12}\nabla b_0\big)\big\|^2_{\dot H^s}+C\int_0^T\big(\big\|u(t)\big\|^{\frac{2p}{p-3}}_{\dot B^0_{p,\infty}}+\big\|b(t)\big\|_{\dot B^0_{\infty,\infty}}\big)\big\|\big(u,b,\alpha^{\frac12}\nabla b\big)(t)\big\|^2_{\dot H^s}\,dt.$$

Note that

$$0<\frac{2p}{p-3}\leqslant q\quad\text{if }\frac{2}{q}+\frac{3}{p}\leqslant 1.$$

Then, the Gronwall inequality yields

$$\sup_{t\in[0,T]}\big(\big\|u(t)\big\|^2_{\dot H^s}+\big\|b(t)\big\|^2_{\dot H^s}+\alpha\big\|\nabla b(t)\big\|^2_{\dot H^s}\big)+\int_0^T\big(\nu\big\|\nabla u(t)\big\|^2_{\dot H^s}+\eta\big\|\nabla b(t)\big\|^2_{\dot H^s}\big)\,dt$$

$$\leqslant C\big(\|u_0\|^2_{\dot H^s}+\|b_0\|^2_{\dot H^s}+\alpha\|\nabla b_0\|^2_{\dot H^s}\big)\exp\big(\big\|u(t)\big\|^{\frac{2p}{p-3}}_{L^q_T(\dot B^0_{p,\infty})}T^{\frac{p}{p-3}(1-\frac{2}{q}-\frac{3}{p})}+\big\|b(t)\big\|_{L^1_T(\dot B^0_{\infty,\infty})}\big). \tag{4.8}$$

On the other hand, by the energy estimate we have

$$\sup_{t\in[0,T]}\left(\|u(t)\|_{L^2}^2+\|b(t)\|_2^2+\alpha\|\nabla b(t)\|_2^2\right)+\int_0^T\left(\nu\|\nabla u(t)\|_2^2+\eta\|\nabla b(t)\|_2^2\right)dt$$
$$\leqslant\|u_0\|_2^2+\|b_0\|_2^2+\alpha\|\nabla b_0\|_2^2. \tag{4.9}$$

Combining (4.8) and (4.9), and by the standard argument of continuation of local solutions, it is easy to show that if (1.8) holds, then the solution remains smooth.

Case II. The proof of blow-up criterion under condition (1.9). Let us return to (4.3). Applying the Schwartz inequality and Lemma A.3 with $\sigma=s-\frac{3}{2p}$, $\sigma_1=\sigma_2=0$ and $p_1=p_2=p$ to the commutator, it follows on using the equivalent norms of $\dot B^\sigma_{2,2}$ and $\dot H^\sigma$ for $\sigma\in\mathbb{R}$, the interpolation theorem and Young inequality, that for $\frac32<p<\infty$,

$$\begin{aligned}|\mathrm{I}|&\leqslant C\left\|2^{k(s-\frac{3}{2p})}\left\|[u,\Delta_k]\cdot\nabla u\right\|_2\right\|_{\ell^2(\mathbb{Z})}\|u\|_{\dot B^{s+\frac{3}{2p}}_{2,2}}\\&\leqslant C\|\nabla u\|_{\dot B^0_{p,\infty}}\|u\|^2_{\dot B^{s+\frac{3}{2p}}_{2,2}}\leqslant C\|\nabla u\|_{\dot B^0_{p,\infty}}\|u\|_{\dot H^s}^{\frac{2p-3}{p}}\|\nabla u\|_{\dot H^s}^{\frac{3}{p}}\\&\leqslant\frac{\nu}{2}\|\nabla u\|^2_{\dot H^s}+C\|\nabla u\|^{\frac{2p}{2p-3}}_{\dot B^0_{p,\infty}}\|u\|^2_{\dot H^s}.\end{aligned}\tag{4.10}$$

Similar arguments as in deriving (4.10) can be used to get that

$$\begin{aligned}|\mathrm{II}|+|\mathrm{III}|&\leqslant C\left\|2^{k(s-\frac{3}{2p})}\left\|[b,\Delta_k]\cdot\nabla b\right\|_2\right\|_{\ell^2(\mathbb{Z})}\|u\|_{\dot B^{s+\frac{3}{2p}}_{2,2}}+\left\|2^{k(s-\frac{3}{2p})}\left(\left\|[b,\Delta_k]\cdot\nabla u\right\|_2\right.\right.\\&\quad\left.\left.+\left\|[u,\Delta_k]\cdot\nabla b\right\|_2\right)\right\|_{\ell^2(\mathbb{Z})}\|b\|_{\dot B^{s+\frac{3}{2p}}_{2,2}}\\&\leqslant C\left(\|\nabla b\|_{\dot B^0_{p,\infty}}\|u\|_{\dot B^{s+\frac{3}{2p}}_{2,2}}+\|\nabla u\|_{\dot B^0_{p,\infty}}\|b\|_{\dot B^{s+\frac{3}{2p}}_{2,2}}\right)\|b\|_{\dot B^{s+\frac{3}{2p}}_{2,2}}\\&\leqslant C\left(\|\nabla u\|_{\dot B^0_{p,\infty}}+\|\nabla b\|_{\dot B^0_{p,\infty}}\right)\left(\|b\|_{\dot H^s}^{2-\frac{3}{p}}\|\nabla b\|_{\dot H^s}^{\frac{3}{p}}+\|b\|_{\dot H^s}^{1-\frac{3}{2p}}\|\nabla b\|_{\dot H^s}^{\frac{3}{2p}}\|u\|_{\dot H^s}^{1-\frac{3}{2p}}\|\nabla u\|_{\dot H^s}^{\frac{3}{2p}}\right)\\&\leqslant\left(\frac{\nu}{2}\|\nabla u\|^2_{\dot H^s}+\frac{\eta}{2}\|\nabla b\|^2_{\dot H^s}\right)+C\left(\|\nabla u\|^{\frac{2p}{2p-3}}_{\dot B^0_{p,\infty}}+\|\nabla b\|^{\frac{2p}{2p-3}}_{\dot B^0_{p,\infty}}\right)\left(\|u\|^2_{\dot H^s}+\|b\|^2_{\dot H^s}\right)\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}|\mathrm{IV}|&\leqslant Ch\left\|2^{ks}\left\|[b\times,\Delta_k]J\right\|_2\right\|_{\ell^2(\mathbb{Z})}\|J\|_{\dot B^s_{2,2}}\\&\leqslant Ch\|\nabla b\|_{\dot B^0_{p,\infty}}\|b\|_{\dot B^{s+\frac{3}{p}}_{2,2}}\|J\|_{\dot B^s_{2,2}}\leqslant Ch\|\nabla b\|_{\dot B^0_{p,\infty}}\|b\|_{\dot H^s}^{1-\frac{3}{p}}\|\nabla b\|_{\dot H^s}^{\frac{3}{p}}\|J\|_{\dot H^s}\\&\leqslant\frac{\eta}{2}\|\nabla b\|^2_{\dot H^s}+C\|\nabla b\|^{\frac{2p}{2p-3}}_{\dot B^0_{p,\infty}}\left(\|b\|^2_{\dot H^s}+\alpha\|\nabla b\|^2_{\dot H^s}\right).\end{aligned}\tag{4.12}$$

Integrating (4.3) with respect to t and utilizing (4.10)–(4.12) lead to the result that for $3 \leqslant p < \infty$,

$$\sup_{t\in[0,T]} \big\|\big(u, b, \alpha^{\frac{1}{2}}\nabla b\big)(t)\big\|_{\dot H^s}^2 + \int_0^T \big(\nu\big\|\nabla u(t)\big\|_{\dot H^s}^2 + \eta\big\|\nabla b(t)\big\|_{\dot H^s}^2\big)\,dt$$
$$\leqslant \big\|\big(u_0, b_0, \alpha^{\frac{1}{2}}\nabla b_0\big)\big\|_{\dot H^s}^2 + C\int_0^T \big(\big\|\nabla u(t)\big\|_{\dot B^0_{p,\infty}}^{\frac{2p}{2p-3}} + \big\|\nabla b(t)\big\|_{\dot B^0_{p,\infty}}^{\frac{2p}{2p-3}}\big)\big\|\big(u, b, \alpha^{\frac{1}{2}}\nabla b\big)(t)\big\|_{\dot H^s}^2\,dt. \tag{4.13}$$

On the other hand, by the Biot Savart law [15] we have

$$\nabla u = (-\Delta)^{-1}\nabla\nabla\times\omega, \qquad \nabla b = (-\Delta)^{-1}\nabla\nabla\times J,$$

where $\omega = \nabla\times u$, $J = \nabla\times b$. It follows from the boundedness of singular integral operators on homogeneous Besov spaces that

$$\big\|(\nabla u, \nabla b)\big\|_{\dot B^\sigma_{p,r}} \leqslant C\big\|(\omega, J)\big\|_{\dot B^\sigma_{p,r}} \quad \text{for } \sigma\in\mathbb{R},\ (p,q)\in[1,\infty]\times[1,\infty]. \tag{4.14}$$

Inserting (4.14) into (4.13), we get

$$\sup_{t\in[0,T]} \big\|\big(u, b, \alpha^{\frac{1}{2}}\nabla b\big)(t)\big\|_{\dot H^s}^2 + \int_0^T \big(\nu\big\|\nabla u(t)\big\|_{\dot H^s}^2 + \eta\big\|\nabla b(t)\big\|_{\dot H^s}^2\big)\,dt$$
$$\leqslant \big\|\big(u_0, b_0, \alpha^{\frac{1}{2}}\nabla b_0\big)\big\|_{\dot H^s}^2 + C\int_0^T \big(\big\|\omega(t)\big\|_{\dot B^0_{p,\infty}}^{\frac{2p}{2p-3}} + \big\|J(t)\big\|_{\dot B^0_{p,\infty}}^{\frac{2p}{2p-3}}\big)\big\|\big(u, b, \alpha^{\frac{1}{2}}\nabla b\big)(t)\big\|_{\dot H^s}^2\,dt.$$

Note that

$$0 < \frac{2p}{2p-3} \leqslant q \quad \text{if } \frac{2}{q} + \frac{3}{p} \leqslant 2.$$

Then, the Gronwall inequality implies that for $3 \leqslant p < \infty$,

$$\sup_{t\in[0,T]} \big\|\big(u, b, \alpha^{\frac{1}{2}}\nabla b\big)(t)\big\|_{\dot H^s}^2 \leqslant \big\|\big(u_0, b_0, \alpha^{\frac{1}{2}}\nabla b_0\big)\big\|_{\dot H^s}^2 \exp\left(C\int_0^T \big(\big\|(\omega, J)(t)\big\|_{\dot B^0_{p,\infty}}^{\frac{2p}{2p-3}}\big)\,dt\right)$$
$$\leqslant \big\|\big(u_0, b_0, \alpha^{\frac{1}{2}}\nabla b_0\big)\big\|_{\dot H^s}^2 \exp\big(\big\|(\omega, J)(t)\big\|_{L^q_T(\dot B^0_{p,\infty})}^{\frac{2p}{2p-3}} T^{\frac{p}{2p-3}(2-\frac{2}{q}-\frac{3}{p})}\big). \tag{4.15}$$

For the case $p=\infty$, we apply Lemma A.3 with $p_1=p_2=\infty$, $\sigma_1=\sigma_2=0$ to the commutator to obtain that

$$|\mathrm{I}|+|\mathrm{II}|+|\mathrm{III}|+|\mathrm{IV}| \leqslant C\big(\|\nabla u\|_{L^\infty}+\|\nabla b\|_{L^\infty}\big)\big(\|u\|_{\dot H^s}^2+\|b\|_{\dot H^s}^2+\|\nabla b\|_{\dot H^s}^2\big).$$

Using the above estimate along with (4.3) it follows from the Gronwall inequality that

$$\sup_{t\in[0,T]}\big\|\big(u,b,\alpha^{\frac12}\nabla b\big)(t)\big\|_{\dot H^s}^2 \leqslant \big\|\big(u_0,b_0,\alpha^{\frac12}\nabla b_0\big)\big\|_{\dot H^s}^2 \exp\left(C\int_0^T \big\|(\nabla u,\nabla b)(t)\big\|_{L^\infty}\,dt\right). \tag{4.16}$$

The logarithmic Sobolev inequality (see (2.2) in [14]) and (4.14) allow us to get that

$$\begin{aligned}\big\|(\nabla u,\nabla b)\big\|_{L^\infty} &\leqslant C\big(1+\big\|(\nabla u,\nabla b)\big\|_{\dot B^0_{\infty,\infty}}\log\big(\big\|(u,b)\big\|_{H^s}+e\big)\big)\\ &\leqslant C\big(1+\big\|(\omega,J)\big\|_{\dot B^0_{\infty,\infty}}\log\big(\big\|(u,b)\big\|_{H^s}+e\big)\big).\end{aligned} \tag{4.17}$$

Plugging (4.17) into (4.16), combining with (4.9) and setting

$$Z(t)\triangleq \log\big(\big\|\big(u(t),b(t),\alpha^{\frac12}\nabla b(t)\big)\big\|_{H^s}+e\big),$$

we deduce that

$$\sup_{t\in[0,T]} Z(t)\leqslant Z(0)+CT+C\int_0^T\big\|(\omega,J)(t)\big\|_{\dot B^0_{\infty,\infty}}Z(t)\,dt.$$

Then the Gronwall inequality yields that

$$\sup_{t\in[0,T]} Z(t)\leqslant \big(Z(0)+CT\big)\exp\big(C\big\|(\omega,J)(t)\big\|_{L^q_T(\dot B^0_{\infty,\infty})}T^{(1-\frac1q)}\big).$$

This implies that

$$\begin{aligned}&\sup_{t\in[0,T]}\big(\big\|u(t)\big\|_{H^s}+\big\|b(t)\big\|_{H^s}+\alpha^{\frac12}\big\|\nabla b(t)\big\|_{H^s}\big)\\ &\quad\leqslant \big(\|u_0\|_{H^s}+\|b_0\|_{H^s}+\alpha^{\frac12}\|\nabla b_0\|_{H^s}+e\big)^{A(T)}\exp\big(CTA(T)\big),\end{aligned} \tag{4.18}$$

where

$$A(T)\triangleq \exp\big(C\big\|(\omega,J)(t)\big\|_{L^q_T(\dot B^0_{\infty,\infty})}T^{(1-\frac1q)}\big).$$

Using the standard argument of continuation of local solutions, we easily prove that if (1.9) holds, the solution remains smooth. The proof of Theorem 1.2 is thus complete.

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Appendix A

Let us recall the paradifferential calculus which enables us to define a generalized product between distributions, which is continuous in many functional spaces where the usual product does not make sense (see [4]). The paraproduct between u and v is defined by

$$T_u v \triangleq \sum_{j\in\mathbb{Z}} S_{j-1}u\Delta_j v.$$

We then have the following formal decomposition:

$$uv = T_u v + T_v u + R(u,v) \tag{A.1}$$

with

$$R(u,v) = \sum_{j\in\mathbb{Z}} \Delta_j u \widetilde{\Delta}_j v \quad \text{and} \quad \widetilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}.$$

The decomposition (A.1) is called Bony's paraproduct decomposition.

We first introduce the well-known Bernstein inequality which will be used repeatedly in the proof of the commutator estimate.

Lemma A.1. *Let k be in $\mathbb{N}$. Let (R_1, R_2) satisfy $0 < R_1 < R_2$. There exists a constant C depending only on R_1, R_2, d such that for all $1 \leqslant p \leqslant q \leqslant \infty$ and $f \in L^p(\mathbb{R}^d)$ we have*

$$\sup_{|\gamma|=k} \left\|\partial^\gamma f\right\|_q \leqslant C^{k+1}\lambda^{k+d(\frac{1}{p}-\frac{1}{q})}\|f\|_p \quad \textit{if } \operatorname{supp}\hat{f} \subset \mathcal{B}(0, R_1\lambda), \tag{A.2}$$

$$C^{-k-1}\lambda^k\|f\|_p \leqslant C \sup_{|\gamma|=k}\left\|\partial^\gamma f\right\|_p \quad \textit{if } \operatorname{supp}\hat{f} \subset \mathcal{C}(0, R_1\lambda, R_2\lambda). \tag{A.3}$$

The proof can be found in [9].

Lemma A.2. *Let $s > 0$, $f, g \in L^\infty \cap \dot{H}^s$. Then $fg \in L^\infty \cap \dot{H}^s$ and*

$$\|fg\|_{\dot{H}^s} \leqslant C\big(\|f\|_\infty\|g\|_{\dot{H}^s} + \|g\|_\infty\|f\|_{\dot{H}^s}\big).$$

For a proof see [9].

Lemma A.3. *Let* $1 \leqslant p_1, p_2 \leqslant \infty$, $\sigma > 0$, $\frac{d}{p_i} - \sigma_i > 0\ (i = 1, 2)$ *and assume that* $\sigma - \sigma_2 + \frac{d}{p_2} > 0$. *Then the following inequality holds*:

$$\Big(\sum_{j\in\mathbb{Z}} 2^{2j\sigma} \big\|[f, \Delta_j]\nabla g\big\|_2^2\Big)^{\frac{1}{2}} \lesssim \|\nabla f\|_{\dot B^{\sigma_1}_{p_1,\infty}} \|g\|_{\dot B_{2,2}^{\sigma-\sigma_1+\frac{d}{p_1}}} + \|\nabla g\|_{\dot B^{\sigma_2}_{p_2,\infty}} \|f\|_{\dot B_{2,2}^{\sigma-\sigma_2+\frac{d}{p_2}}}. \tag{A.4}$$

If $\sigma_1 = 0$, $p_1 = \infty$, $\|\nabla f\|_{\dot B^{\sigma_1}_{p_1,\infty}}$ *has to be replaced by* $\|\nabla f\|_{L^\infty}$, *and if* $\sigma_2 = 0$, $p_2 = \infty$, *then* $\|\nabla g\|_{\dot B^{\sigma_2}_{p_2,\infty}}$ *has to be replaced by* $\|\nabla g\|_{L^\infty}$. *In* (A.4)

$$[f, \Delta_j]\nabla g = f\Delta_j(\nabla g) - \Delta_j(f\nabla g).$$

Proof. The proof is standard. By Bony's decomposition, we have

$$[f, \Delta_j]\nabla g = [T_f, \Delta_j]\nabla g + T'_{\Delta_j\nabla g} f - \Delta_j T_{\nabla g} f - \Delta_j R(f, \nabla g), \tag{A.5}$$

where $T'_u v$ stands for $T_u v + R(u, v)$.

By (2.1), we have

$$[T_f, \Delta_j]\nabla g = \sum_{j'\sim j} 2^{jd} \int_{\mathbb{R}^3} h\big(2^j(x-y)\big)\big(S_{j'-1}f(x) - S_{j'-1}f(y)\big)\Delta_{j'}\nabla g(y)\,dy,$$

where $j' \sim j$ means that $|j'-j| \leqslant 4$. Since $\frac{d}{p_1} - \sigma_1 > 0$, by Lemma A.1 and the Hölder inequality we infer that

$$\|S_{j'-1}\nabla f\|_\infty \leqslant C \sum_{j''\leqslant j'-2} 2^{j''\sigma_1}\|\Delta_{j''}\nabla f\|_{p_1} 2^{j''(\frac{d}{p_1}-\sigma_1)} \leqslant C\|\nabla f\|_{\dot B^{\sigma_1}_{p_1,\infty}} 2^{j'(\frac{d}{p_1}-\sigma_1)}, \tag{A.6}$$

so

$$\begin{aligned}\big\|[T_f, \Delta_j]\nabla g\big\|_2 &\leqslant C2^{-j}\sum_{j'\sim j} \big\|2^j|\cdot|2^{jd}h\big(2^j\cdot\big)\big\|_1 \|S_{j'-1}\nabla f\|_\infty \|\Delta_{j'}\nabla g\|_2 \\ &\leqslant C\|\nabla f\|_{\dot B^{\sigma_1}_{p_1,\infty}} 2^{-j}\sum_{j'\sim j} 2^{j'(\frac{d}{p_1}-\sigma_1)}\|\Delta_{j'}\nabla g\|_2.\end{aligned}$$

This, together with the convolution inequality for series, gives

$$\begin{aligned}\big\|2^{\sigma j}\big\|[T_f, \Delta_j]\nabla g\big\|_2\big\|_{\ell^2(\mathbb{Z})} &\leqslant C\|\nabla f\|_{\dot B^{\sigma_1}_{p_1,\infty}} \Big\|\sum_{j'\sim j} 2^{(j'-j)(1-\sigma)} 2^{j'(\sigma-\sigma_1+\frac{d}{p_1})}\|\Delta_{j'}g\|_2\Big\|_{\ell^2(\mathbb{Z})} \\ &\leqslant C\|\nabla f\|_{\dot B^{\sigma_1}_{p_1,\infty}} \|g\|_{\dot B_{2,2}^{\sigma-\sigma_1+\frac{d}{p_1}}}.\end{aligned} \tag{A.7}$$

Using the definition of $T'_{\Delta_j\nabla g} f$ and (2.1), we can rewrite

$$T'_{\Delta_j\nabla g} f = \sum_{j'\gtrsim j} \Delta_{j'} f S_{j'+2}\Delta_j\nabla g,$$

where $j' \gtrsim j$ means that $j' \geqslant j-2$. By Lemma A.1 and the Hölder inequality, it follows that

$$\|S_{j'+2}\Delta_j \nabla g\|_\infty \leqslant C2^{j\sigma_2}\|\Delta_j \nabla g\|_{p_2} 2^{j(\frac{d}{p_2}-\sigma_2)} \leqslant C\|\nabla g\|_{\dot{B}^{\sigma_2}_{p_2,\infty}} 2^{j(\frac{d}{p_2}-\sigma_2)}.$$

Thus, for $\sigma-\sigma_2+\frac{d}{p_2}>0$, the convolution inequality for series yields that

$$\begin{aligned}\big\|2^{\sigma j}\big\|T'_{\Delta_j\nabla g}f\big\|_2\big\|_{\ell^2(\mathbb{Z})} &\leqslant C\|\nabla g\|_{\dot{B}^{\sigma_1}_{p_2,\infty}}\bigg\|\sum_{j'\gtrsim j}2^{-(j'-j)(\sigma-\sigma_2+\frac{d}{p_2})}2^{j'(\sigma-\sigma_2+\frac{d}{p_2})}\|\Delta_{j'}f\|_2\bigg\|_{\ell^2(\mathbb{Z})}\\ &\leqslant C\|\nabla g\|_{\dot{B}^{\sigma_2}_{p_2,\infty}}\|f\|_{\dot{B}^{\sigma-\sigma_2+\frac{d}{p_2}}_{2,2}}. \end{aligned} \tag{A.8}$$

Similarly as in deriving (A.6), we can show that for $\frac{d}{p_2}-\sigma_2>0$, we get $\|S_{j'-1}\nabla g\|_\infty \leqslant C\|\nabla g\|_{\dot{B}^{\sigma_2}_{p_2,\infty}} 2^{j'(\frac{d}{p_2}-\sigma_2)}$. This, together with the convolution inequality for series, implies that

$$\begin{aligned}\big\|2^{j\sigma}\big\|\Delta_j(T_{\nabla g}f)\big\|_2\big\|_{\ell^2(\mathbb{Z})} &\leqslant \bigg\|2^{j\sigma}\sum_{j'\sim j}\big\|\Delta_j(\Delta_{j'}fS_{j'-1}\nabla g)\big\|_2\bigg\|_{\ell^2(\mathbb{Z})}\\ &\leqslant C\|\nabla g\|_{\dot{B}^{\sigma_2}_{p_2,\infty}}\bigg\|\sum_{j'\sim j}2^{j'(\frac{d}{p_2}-\sigma_2+\sigma)}\|\Delta_{j'}f\|_2 2^{(j-j')\sigma}\bigg\|_{\ell^2(\mathbb{Z})}\\ &\leqslant C\|\nabla g\|_{\dot{B}^{\sigma_2}_{p_2,\infty}}\|f\|_{\dot{B}^{\sigma-\sigma_2+\frac{d}{p_2}}_{2,2}}. \end{aligned} \tag{A.9}$$

Finally, for $\sigma>0$ we have

$$\begin{aligned}\big\|2^{j\sigma}\big\|\Delta_jR(f,\nabla g)\big\|_2\big\|_{\ell^2(\mathbb{Z})} &\leqslant \bigg\|\sum_{j'\gtrsim j}2^{j\sigma}\big\|\Delta_j(\Delta_{j'}f\widetilde{\Delta}_{j'}\nabla g)\big\|_2\bigg\|_{\ell^2(\mathbb{Z})}\\ &\leqslant C\|\nabla g\|_{\dot{B}^{\sigma_2}_{p_2,\infty}}\bigg\|\sum_{j'\gtrsim j}\|\Delta_{j'}f\|_2 2^{j'(\frac{d}{p_2}-\sigma_2+\sigma)}2^{(j-j')\sigma}\bigg\|_{\ell^2(\mathbb{Z})}\\ &\leqslant C\|\nabla g\|_{\dot{B}^{\sigma_2}_{p_2,\infty}}\|f\|_{\dot{B}^{\sigma-\sigma_2+\frac{d}{p_2}}_{2,2}}. \end{aligned} \tag{A.10}$$

Combining (A.7)–(A.10) gives the desired inequality (A.4). □

References

[1] J.T. Beale, T. Kato, A.J. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, Comm. Math. Phys. 94 (1984) 61–66.

[2] H. Beirao da Veiga, A new regularity class for the Navier–Stokes equations in $\mathbb{R}^n$, Chinese Ann. Math. Ser. B 16 (1995) 407–412.

[3] J. Bergh, J. Löfstrom, Interpolation Spaces, An Introduction, Springer-Verlag, New York, 1976.

[4] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Ann. Sci. École Norm. Sup. 14 (1981) 209–246.

[5] D. Biskamp, Nonlinear Magnetohydrodynamics, Cambridge University Press, Cambridge, UK, 1993.

[6] D. Biskamp, Magnetic Reconnection in Plasmas, Cambridge University Press, Cambridge, UK, 2000.
[7] R.E. Caflisch, I. Klapper, G. Steele, Remarks on singularities, dimension and energy dissipation for ideal hydrodynamics and MHD, Comm. Math. Phys. 184 (1997) 443–455.
[8] M. Cannone, Q.-L. Chen, C.-X. Miao, A losing estimate for the Ideal MHD equations with application to blow-up criterion, SIAM J. Math. Anal. 38 (2007) 1847–1859.
[9] J.-Y. Chemin, Perfect Incompressible Fluids, Oxford University Press, New York, 1998.
[10] G. Duvaut, J.L. Lions, Inéquation en thermoélasticite et magnétohydrodynamique, Arch. Ration. Mech. Anal. 46 (1972) 241–279.
[11] Y. Giga, Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier–Stokes system, J. Differential Equations 62 (1986) 182–212.
[12] C. He, Z. Xin, On the regularity of weak solutions to the magnetohydrodynamic equations, J. Differential Equations 213 (2005) 235–254.
[13] H. Kozono, Y. Taniuchi, Bilinear estimates in BMO and the Navier–Stokes equations, Math. Z. 235 (2000) 173–194.
[14] H. Kozono, T. Ogawa, Y. Taniuchi, The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations, Math. Z. 242 (2002) 251–278.
[15] A.J. Majda, Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables, Appl. Math. Sci., vol. 53, Springer-Verlag, New York, 1984.
[16] M. Núñez, Existence theorems for two-fluid magnetohydrodynamics, J. Math. Phys. 46 (2005) 083101.
[17] E.R. Priest, T.G. Forbes, Magnetic Reconnection: MHD Theory and Applications, Cambridge University Press, Cambridge, UK, 2000.
[18] M. Sermange, R. Temam, Some mathematical questions related to the MHD equations, Comm. Pure Appl. Math. 36 (1983) 635–664.
[19] H. Triebel, Theory of Function Spaces, Monogr. Math., vol. 78, Birkhäuser, Basel, 1983.
[20] J. Wu, Bounds and new approaches for the 3D MHD equations, J. Nonlinear Sci. 12 (2002) 395–413.
[21] J. Wu, Regularity results for weak solutions of the 3D MHD equations, Discrete Contin. Dyn. Syst. 10 (2004) 543–556.
[22] Z. Zhang, X. Liu, On the blow-up criterion of smooth solutions to the 3D Ideal MHD equations, Acta Math. Appl. Sin. E 20 (2004) 695–700.
[23] Y. Zhou, Remarks on the regularity for the 3D MHD equations, Discrete Contin. Dyn. Syst. 12 (2005) 881–886.