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Bounds on the eigenvalues of graphs with cut vertices or edges ☆

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ABSTRACT

In this paper, we characterize the extremal graph having the maximal Laplacian spectral radius among the connected bipartite graphs with n vertices and k cut vertices, and describe the extremal graph having the minimal least eigenvalue of the adjacency matrices of all the connected graphs with n vertices and k cut edges. We also present lower bounds on the least eigenvalue in terms of the number of cut vertices or cut edges and upper bounds on the Laplacian spectral radius in terms of the number of cut vertices.

1. Introduction

Throughout this paper all graphs are finite and simple. Let $G = (V(G), E(G))$ be a graph with n vertices and $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of G is $A(G) = (a_{ij})_{n\times n}$, where $a_{ij} = 1$ if the edge $v_iv_j \in E(G)$ and $a_{ij} = 0$ otherwise. Let $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees of G. Recall that the matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are called *Laplacian matrix* and *signless Laplacian matrix* of G, respectively. Clearly, $A(G)$, $L(G)$ and $Q(G)$ are real symmetric matrices, which imply that all eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ are real. The largest eigenvalue of $A(G)$ is called the *spectral radius* of G. The least eigenvalue of $A(G)$ is denoted by $\lambda_{min}(G)$. Assume that $x = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^T \in \mathbb{R}^n$ and x is a unit eigenvector of $A(G)$ corresponding to $\lambda_{min}(G)$. Then by the Rayleigh–Ritz Theorem,

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$$\lambda_{min}(G) = \min_{\substack{y \in \mathbb{R}^n \\ ||y||=1}} y^T A(G) y = x^T A(G) x = 2 \sum_{v_i v_j \in E(G)} x_{v_i} x_{v_j} \tag{1.1}$$

and

$$\lambda_{min}(G) x_v = \sum_{u \in N_G(v)} x_u, \quad \text{for each v} \in V(G). \tag{1.2}$$

Similarly, the largest eigenvalue of $L(G)$ is called the *Laplacian spectral radius* of G, denoted by $\mu(G)$, and the *signless Lapalcian spectral radius* (Q-spectral radius), $\rho(G)$, of G is the largest eigenvalue of $Q(G)$. Moreover, by the Perron–Frobenius Theorem, we know that the Q-spectral radius $\rho(G)$ is simple and has a unique (up to a multiplication by a scalar) positive eigenvector if G is connected. We shall refer to such an eigenvector as the *Perron vector* of $Q(G)$. Note that the Q-spectral radius increases if we add an edge to the connected graph G. Let R be the vertex-edge incidence matrix of a graph G. It is well-known that $RR^T = D(G) + A(G)$. Thus, if x is a unit Perron vector of $Q(G)$, then we also have

$$\rho(G) = \max_{\substack{y \in \mathbb{R}^n \\ ||y||=1}} y^T Q(G) y = x^T Q(G) x = x^T RR^T x = \sum_{\substack{v_i v_j \in E(G) \\ 1 \leqslant i < j \leqslant n}} (x_i + x_j)^2. \tag{1.3}$$

A *cut edge* in a connected graph G is an edge whose deletion breaks the graph into two parts. A vertex of a graph is said to be *pendant vertex* if its neighborhood contains exactly one vertex. An edge is called *pendant edge* if one end vertex of it is a pendant vertex. A *cut vertex* in a connected graph G is a vertex whose deletion breaks the graph into two or more parts. Denote by $N_G(v)$ (or $N(v)$ for short) the set of all neighbors of v in G. For other notation in graph theory, we follow [4].

The investigation of the spectral radius of graphs is an important topic in the theory of graph spectra. In [5], Brualdi and Solheid proposed the following problem: Given a set of graphs $\mathcal{G}$, find an upper bound for the spectral radii of graphs in $\mathcal{G}$ and characterize the graphs in which the maximal spectral radius is attained. Recently, the similar problems on the Laplacian spectral radius, the signless Laplacian spectral radius and the least eigenvalue of graphs have also attracted researchers' attention. These problems has been extensively studied, see [1–3,14,17–19] for example.

The aim of this paper is twofold. First, we investigate the extremal graphs having the maximal Laplacian spectral radius of $\mathscr{B}_n^k$, the set of all connected bipartite graphs with n vertices and k cut vertices. Second, we study the structures of the extremal graphs with the minimal least eigenvalue of $\mathscr{G}_{n,k}$, the set of all connected graphs with n vertices and k cut edges. The paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we characterize the extremal graph having the maximal Laplacian spectral radius of $\mathscr{B}_n^k$. In Section 4, we study the extremal graph having the minimal least eigenvalue of $\mathscr{G}_{n,k}$ and obtain a lower bound for the least eigenvalue of a connected graph in terms of the number of cut edges.

2. Preliminaries

In this section, we state some necessary results and notation used in this paper.

Lemma 1 [16]. *For a connected graph, $\mu(G) \leqslant \rho(G)$, with equality if and only if G is a bipartite graph.*

By the lemma above, $\mu(G) = \rho(G)$ for any connected bipartite graph G. Thus we can study $Q(G)$ and its spectral radius $\rho(G)$ instead of $L(G)$ and $\mu(G)$, where $\rho(G)$ is also denoted by ρ for simplicity.

Lemma 2 [12]. *Let G be a connected bipartite graph and H be a subgraph of G. Then $\mu(H) \leqslant \mu(G)$, and equality holds if and only if $G = H$.*

Lemma 3 [8, Theorem 4.7]. *Let G be a graph on n vertices with vertex degrees $d_1, d_2, \ldots, d_n$ and $q_1(G)$ be the largest eigenvalue of $Q(G)$. Then*

$$\min_{v_iv_j \in E(G)} (d_i + d_j) \leqslant q_1(G) \leqslant \max_{v_iv_j \in E(G)} (d_i + d_j), \tag{2.1}$$

For a connected graph G, equality holds in either of these inequalities if and only if G is regular or semi-regular bipartite.

If $f(x)$ is a polynomial in the variable x and let $\rho_1(f)$ denote the largest real root of equation $f(x) = 0$, then the following result is well known: Let $f(x)$ and $g(x)$ be monic polynomials with real roots. If $f(x) < g(x)$ for all $x \geqslant \rho_1(g)$, then $\rho_1(f) > \rho_1(g)$. We will use this fact later.

Lemma 4. *Assume that the complete bipartite graph $K_{m,n}$ has the vertex bipartition (V_1, V_2), where $V_1 = \{v_1, v_2, \ldots, v_n\}$ and $V_2 = \{v_{n+1}, \ldots, v_{n+m}\}$. Suppose that the graph $K_{m,n}^{b,a}$ (see Fig. 1) is obtained from $K_{m,n}$ by adding a pendant edges $\{s_iv_i\}_{i=1}^{a}$ to V_1 and b pendant edges $\{t_iv_{n+i}\}_{i=1}^{b}$ to V_2, where the integers a and b satisfy $0 \leqslant a \leqslant n$ and $0 \leqslant b \leqslant m$ ($a = 0$ or $b = 0$ means that the corresponding pendant edges do not exist). Assume that $m \geqslant 1$, $n \geqslant 1$ and $a + b \geqslant 1$. Then*

(i) *if ρ is the signless Laplacian spectral radius of $K_{m,n}^{b,a}$, then ρ is the largest root of the equation*

$$\begin{aligned} -amn - bmn - m^2n - mn^2 - abx + 2amx + 2m^2x + 2bnx + 5mnx + amnx \\ + bmnx + 2m^2nx + 2n^2x + 2mn^2x - 6mx^2 - amx^2 - 3m^2x^2 - 6nx^2 - bnx^2 \\ - 8mnx^2 - m^2nx^2 - 3n^2x^2 - mn^2x^2 + 4x^3 + 7mx^3 + m^2x^3 + 7nx^3 + 3mnx^3 \\ + n^2x^3 - 4x^4 - 2mx^4 - 2nx^4 + x^5 = 0; \end{aligned}$$

(ii) *if ρ is the signless Laplacian spectral radius of $K_{m,n}^{b,a}$, then $n+m < \rho < n+m+1.1$ for $n+m \geqslant 10$;*

(iii) *for the fixed N and k, if $N \geqslant 10$ and $1 \leqslant k \leqslant \frac{9N}{10}$, then $\rho\left(K_{m,N-m}^{b,k-b}\right) \leqslant \rho\left(K_{1,N-1}^{1,k-1}\right)$ for $1 \leqslant m \leqslant N/2$ and $0 \leqslant b \leqslant k$, where equality holds if and only if $m = b = 1$.*

Proof. (i) Let $x = (x_1, x_2, \ldots, x_{n+m+a+b})^T$ be a unit Perron vector of $Q\left(K_{m,n}^{b,a}\right)$, where x_i corresponds to the vertex v_i for $1 \leqslant i \leqslant n + m$, the vertex s_{i-n-m} for $n + m + 1 \leqslant i \leqslant n + m + a$ and the

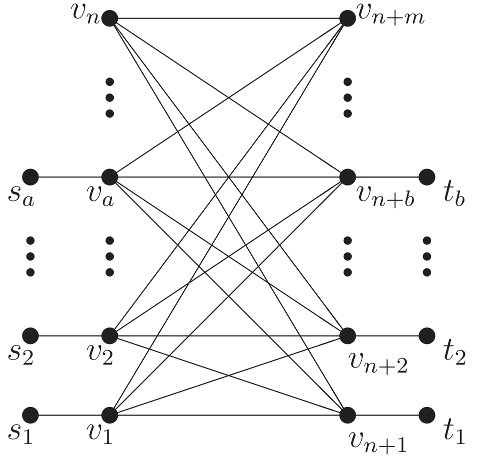

Fig. 1. $K_{m,n}^{b,a}$.

vertex $t_{i-n-m-a}$ for $n+m+a+1 \leqslant i \leqslant n+m+a+b$, respectively. By the symmetry of $K_{m,n}^{b,a}$ and $Q\left(K_{m,n}^{b,a}\right)x = \rho x$, it is not hard to obtain the equation

$$\begin{aligned}&(\rho-n)(\rho-m)[(\rho-1)(\rho-m-1)-1][(\rho-1)(\rho-n-1)-1]\\ &\quad -[m(\rho-1)(\rho-n-1)-m+b\rho][n(\rho-1)(\rho-m-1)-n+a\rho]=0,\end{aligned}$$

that is,

$$\begin{aligned}&\rho(-amn-bmn-m^2n-mn^2-ab\rho+2am\rho+2m^2\rho+2bn\rho+5mn\rho+amn\rho\\ &+bmn\rho+2m^2n\rho+2n^2\rho+2mn^2\rho-6m\rho^2-am\rho^2-3m^2\rho^2-6n\rho^2-bn\rho^2\\ &-8mn\rho^2-m^2n\rho^2-3n^2\rho^2-mn^2\rho^2+4\rho^3+7m\rho^3+m^2\rho^3+7n\rho^3+3mn\rho^3\\ &+n^2\rho^3-4\rho^4-2m\rho^4-2n\rho^4+\rho^5)=0.\end{aligned}$$

Hence we complete the proof of (i). In the following, we suppose that

$$\begin{aligned}f(m,n,a,b,x):=&-amn-bmn-m^2n-mn^2-ab\rho+2amx+2m^2x+2bnx+5mnx\\ &+amnx+bmnx+2m^2nx+2n^2x+2mn^2x-6mx^2-amx^2-3m^2x^2\\ &-6nx^2-bnx^2-8mnx^2-m^2nx^2-3n^2x^2-mn^2x^2+4x^3+7mx^3\\ &+m^2x^3+7nx^3+3mnx^3+n^2x^3-4x^4-2mx^4-2nx^4+x^5.\end{aligned}$$

(ii) By Lemma 2, we have $n+m=\rho(K_{m,n}) < \rho \leqslant \rho\left(K_{m,n}^{m,n}\right)$. By (i), we obtain that $\rho\left(K_{m,n}^{m,n}\right)$ is the largest root of the equation

$$f(m,n,n,m,x)=(x-m)(x-n)(x-2)[x^2-(m+n+2)+m+n]=0.$$

Since $n+m<\rho\left(K_{m,n}^{m,n}\right)$, we have

$$\rho\left(K_{m,n}^{m,n}\right)=\frac{m+n+2+\sqrt{(m+n)^2+4}}{2}<n+m+1.1 \qquad (2.2)$$

for $n+m \geqslant 10$, which is desired.

In what follows, in order to show (iii), we first obtain the following three facts, whose proofs are similar. Thus, here we only give the proof of the first fact and omit the proofs of the other two for brevity.

Fact 1. *For the fixed m, n and k, if $m \leqslant n$ and $n \geqslant 3$, then $\rho\left(K_{m,n}^{b,k-b}\right)$ is a strictly increasing function with respect to b.*

Proof. Since $m \leqslant n$, we have

$$\begin{aligned}f(m,n,k-b-1,b+1,x)-f(m,n,k-b,b,x) &= x[1+2b-k-(2n-m)(x-2)]\\ &\leqslant x[1+m+k-k-(2m-m)(x-2)]\\ &= x[m(3-x)+1]\\ &< 0\end{aligned}$$

for $x > n+m$, which implies that $\rho\left(K_{m,n}^{b+1,k-b-1}\right) > \rho\left(K_{m,n}^{b,k-b}\right)$. Hence $\rho\left(K_{m,n}^{b,k-b}\right)$ is a strictly increasing function with respect to b. $\square$

Fact 2. *For the fixed N and k, if $1 \leqslant k \leqslant \frac{9N}{10}$ and $N \geqslant 10$, then $\rho\left(K_{m,N-m}^{m,k-m}\right)$ is a strictly decreasing function with respect to m for $m \in [1, k] \bigcap [1, \frac{N}{2}]$.*

Fact 3. *For the fixed N and k, $\rho\left(K_{m,N-m}^{k,0}\right)$ is a strictly decreasing function with respect to m for $m \in [k, \frac{N}{2}]$.*

In the following, we begin to show (iii). Since $m \leqslant N - m$, for the fixed m, we can obtain that $\rho\left(K_{m,N-m}^{m,k-m}\right) \geqslant \rho\left(K_{m,N-m}^{b,k-b}\right)$ for $b \leqslant m \leqslant k$ by Fact 1, where equality holds if and only if $b = m$. Further, we divide the discussion into two cases for k. If $\lfloor \frac{N}{2} \rfloor \leqslant k \leqslant \frac{9N}{10}$, then we have

$$\rho\left(K_{1,N-1}^{1,k-1}\right) > \rho\left(K_{2,N-2}^{2,k-2}\right) > \rho\left(K_{3,N-3}^{3,k-3}\right) > \cdots > \rho\left(K_{\lfloor \frac{N}{2} \rfloor, \lceil \frac{N}{2} \rceil}^{\lfloor \frac{N}{2} \rfloor, k-\lfloor \frac{N}{2} \rfloor}\right)$$

in view of Fact 2; If $1 \leqslant k < \lfloor \frac{N}{2} \rfloor$, then we have

$$\rho\left(K_{1,N-1}^{1,k-1}\right) > \rho\left(K_{2,N-2}^{2,k-2}\right) > \rho\left(K_{3,N-3}^{3,k-3}\right) > \cdots > \rho\left(K_{k,N-k}^{k,0}\right)$$

by virtue of Fact 2 and

$$\rho\left(K_{k,N-k}^{k,0}\right) > \rho\left(K_{k+1,N-k-1}^{k,0}\right) > \rho\left(K_{k+2,N-k-2}^{k,0}\right) > \cdots > \rho\left(K_{\lfloor \frac{N}{2} \rfloor, \lceil \frac{N}{2} \rceil}^{k,0}\right)$$

by Fact 3. Thus $\rho\left(K_{1,N-1}^{1,k-1}\right) \geqslant \rho\left(K_{m,N-m}^{b,k-b}\right)$, where equality holds if and only if $b = m = 1$. The proof is complete. $\square$

Let $T_{n,k}$ denote a tree with n vertices and k pendant vertices obtained from a star $K_{1,k}$ by adding k paths of almost equal lengths to each pendant vertex of $K_{1,k}$. The following result was obtained in [13].

Lemma 5 [13]. *Of all trees on n vertices and k pendant vertices, the maximal Laplacian spectral radius is obtained only at $T_{n,k}$.*

Thus the following corollary is immediate.

Corollary 1. *Of all trees on n vertices and k cut vertices, the maximal Laplacian spectral radius is obtained only at $T_{n,n-k}$.*

We also use the next lemma in the proof of Theorem 1 in Section 3.

Lemma 6 [10]. *Let u and v be two adjacent vertices of the connected graph G and for positive integers k and l. Let $G(k, l)$ denote the graph obtained from G by adding pendant paths of length (by the length of a path, we mean the number of its vertices) k at u and length l at v. If $k \geqslant l \geqslant 2$, then $\rho(G(k, l)) > \rho(G(k+1, l-1))$.*

Finally, recall an operation of graphs. A graph is called nontrivial if it contains at least two vertices. Let G_1, G_2 be two disjoint connected graphs, and let $v_1 \in G_1$, $v_2 \in G_2$. The coalescence of G_1 and G_2, denoted by $G_1(v_1) \cdot G_2(v_2)$, is obtained from G_1 and G_2 by identifying v_1 with v_2 and forming a new vertex v; see [7] or [11]. The graph $G_1(v_1) \cdot G_2(v_2)$ is also written as $G_1 v G_2$.

To prove Proposition 1 and Theorem 4 in Section 4, the following two lemmas are needed, respectively.

Lemma 7 [11]. *Let G_1, G_2 be two disjoint nontrivial connected graphs, and let $\{v_1, v_2\} \in V(G_1)$, $u \in V(G_2)$. Let $G = G_1(v_2) \cdot G_2(u)$ and $G^* = G_1(v_1) \cdot G_2(u)$. If there exists an eigenvector x of $A(G)$ corresponding to $\lambda_{min}(G)$ such that $|x_{v_1}| \geqslant |x_{v_2}|$, then $\lambda_{min}(G^*) \leqslant \lambda_{min}(G)$, where the equality holds if and only if x is an eigenvector of $A(G^*)$ corresponding to $\lambda_{min}(G^*)$ and $x_{v_1} = x_{v_2}$ and $\sum_{w \in N_{G_2}(u)} x_w = 0$.*

Lemma 8 [15, Lemma 2.6]. *Let A be an $n \times n$ real symmetric matrix and λ be the least eigenvalue of A. If $\lambda = x^T Ax$, where $x \in \mathbb{R}^n$ is a unit vector, then $Ax = \lambda x$.*

3. Maximizing the Laplacian spectral radius in $\mathscr{B}_n^k$

In this section, we will characterize the extremal graph having the maximal Laplacian spectral radius of all the connected bipartite graphs with n vertices and k cut vertices.

Theorem 1. *Let $\mathscr{B}_n^k$ be the set of the connected bipartite graphs with n vertices and k cut vertices and $G \in \mathscr{B}_n^k$. Then we have the following results.*

(i) *For $1 \leqslant k \leqslant n-2$, $\rho(G) < n-k+1+\frac{1}{n-k-1}$.*
(ii) *For $10 \leqslant n$ and $1 \leqslant k \leqslant \frac{n}{2}$, if G has the maximal Laplacian spectral radius, then G has to be in $\left\{K_{m,n-k-m}^{b,a}\right\}_{m=1}^{n-k-1}$. In particular, for $19 \leqslant n$ and $1 \leqslant k \leqslant \frac{9n}{19}$, if G has the maximal Laplacian spectral radius, then G is unique and $G \cong K_{1,n-k-1}^{1,k-1}$.*

Proof. Note that $G + e \in \mathscr{B}_n^k$ and $\rho(G+e) > \rho(G)$ if we add an edge to G. Thus we can assume that each cut vertex of G connects some blocks and that all of these blocks are complete bipartite graphs. Choose $G \in \mathscr{B}_n^k$ such that $\rho(G)$ is as large as possible. We first show the following three facts.

Fact 4. *There does not exist a cut vertex of G connecting two complete bipartite graphs K_{n_1,m_1} and K_{n_2,m_2} with $\min\{n_1, m_1, n_2, m_2\} \geqslant 2$.*

Proof. Assume, to the contrary, that $V(K_{n_1,m_1}) = \{v_1, v_2, \ldots, v_{n_1+m_1}\}$ and $V(K_{n_2,m_2}) = \{v_{n_1+m_1}, v_{n_1+m_1+1}, \ldots, v_{n_1+m_1+n_2+m_2-1}\}$ with $\min\{n_1, m_1, n_2, m_2\} \geqslant 2$, where $v_{n_1+m_1}$ is a cut vertex. Let $x = (x_1, x_2, \ldots, x_n)^T$ be a unit Perron vector of $Q(G)$, where x_i corresponds to the vertex v_i for $1 \leqslant i \leqslant n$. Without loss of generality, let $x_1 = \min\{x_i : v_i \in N_G(v_{n_1+m_1}), 1 \leqslant i \leqslant n_1+m_1+n_2+m_2-1\}$. Now we can obtain a graph G^* from G by deleting edges $v_1v_i \in E(K_{n_1,m_1})$ and adding edges v_iv_j for $v_j \in N_{K_{n_2,m_2}}(v_{n_1+m_1})$, where $2 \leqslant i \leqslant n_1+m_1-1$ and $n_1+m_1+1 \leqslant j \leqslant n_1+m_1+n_2+m_2-1$. It is obvious that $G^* \in \mathscr{B}_n^k$. Clearly,

$$\begin{aligned} x^T(Q(G^*) - Q(G))x &= \sum_{\substack{v_1v_i \in E(K_{n_1,m_1}),\\ v_j \in N_{K_{n_2,m_2}}(v_{n_1+m_1})}} (x_i + x_j)^2 - \sum_{\substack{v_1v_i \in E(K_{n_1,m_1}),\\ v_i \neq v_{n_1+m_1}}} (x_i + x_1)^2 \\ &> 0. \end{aligned}$$

Thus

$$\begin{aligned} \rho(G^*) &= \max_{\substack{y \in \mathbb{R}^n\\ ||y||=1}} y^T Q(G^*) y \\ &\geqslant x^T Q(G^*) x \\ &> x^T Q(G) x \\ &= \rho(G), \end{aligned}$$

which contradicts the choice of G. So this fact holds. □

Fact 5. *There does not exist a cut vertex of G connecting two complete bipartite graphs K_{n_1,m_1} and K_{1,m_2} with $\min\{n_1, m_1, m_2\} \geqslant 2$, where the cut vertex is obtained by identifying the center of K_{1,m_2} with one vertex of K_{n_1,m_1}.*

Proof. Assume for the contradiction that $V(K_{n_1,m_1}) = \{v_1, v_2, \ldots, v_{n_1+m_1}\}$ and $V(K_{1,m_2}) = \{v_{n_1+m_1}, v_{n_1+m_1+1}, \ldots, v_{n_1+m_1+m_2}\}$ with $\min\{n_1, m_1, m_2\} \geqslant 2$, where the cut vertex $v_{n_1+m_1}$ is the center of K_{1,m_2}. Let $e = v_{n_1+m_1+1}v_i$, where v_i is a vertex in K_{n_1,m_1} nonadjacent to $v_{n_1+m_1}$. Then we have $G + e \in \mathscr{B}_n^k$ and $\rho(G+e) > \rho(G)$, a contradiction. $\square$

Actually, with the help of [9, Theorem 2.9], the next general fact can be obtained by the similar method. We omit its proof for brevity.

Fact 6. *If G contains a block $K_{n1,m1}$ with $n_1 \geqslant m_1 \geqslant 2$, then each cut vertex of G exactly connects two blocks.*

Assume that all the blocks of G are $K_{m_1,n_1}, K_{m_2,n_2}, \ldots, K_{m_{k+1},n_{k+1}}$. Let $a_i := m_i + n_i$. Order the cardinalities of these complete bipartite graphs $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_{k+1} \geqslant 2$. Note that $a_1 = n + k - (a_2 + a_3 + \cdots + a_{k+1}) \leqslant n - k$ and $a_1 + a_2 \leqslant n - k + 2$. Since K_{n_1,m_1} is a block, we have $n_1 = m_1 = 1$ or $n_1 \geqslant 2$ and $m_1 \geqslant 2$. So we divide the following discussion into two cases.

Case 1. If $n_1 = m_1 = 1$, then we obtain that G is a tree because $2 = a_1 \geqslant a_2 \geqslant \cdots \geqslant a_{k+1} \geqslant 2$. Thus we have $\rho(G) \leqslant \rho(T_{n,n-k})$ by Corollary 1, where equality holds if and only if $G \cong T_{n,n-k}$.

Case 2. If $n_1 \geqslant 2$ and $m_1 \geqslant 2$, then by Facts 4 and 5, we can obtain $d_v \leqslant n_1 + 1$ and $d_u \leqslant m_1 + 1$ for any $vu \in E(K_{n_1,m_1})$. On the other hand, if a cut edge uv connects two blocks $K_{mi,ni}$ and $K_{mj,nj}$ with $\min\{n_i, m_i, n_j, m_j\} \geqslant 2$, then

$$d_v + d_u \leqslant a_i - 1 + a_j - 1 \leqslant a_1 + a_2 - 2 \leqslant n - k.$$

Hence, in view of Lemma 3 and Fact 6, we have

$$\rho(G) \leqslant \max_{ij \in E(G)} d_i + d_j \leqslant \max_{vu \in E(K_{n_1,m_1})} d_v + d_u \leqslant n_1 + m_1 + 2 = a_1 + 2.$$

Thus, if $a_1 \leqslant n - k - 2$, then we have $\rho(G) \leqslant n - k < \rho\left(K_{m,n-k-m}^{b,a}\right)$, which contradicts the choice of G. Consequently, we have to consider only the cases $a_1 = n - k - 1$ and $a_1 = n - k$.

If $a_1 = n-k-1$, then $a_2 = 3$ and $a_3 = \cdots = a_{k+1} = 2$. Since we demand that complete bipartite graphs are blocks, it does not exist for $a_2 = 3$. Hence this case does not occur.

If $a_1 = n - k$, then $a_2 = \cdots = a_{k+1} = 2$. Let $V(K_{m,n-k-m}) = \{v_1, v_2, \ldots, v_{n-k}\}$. So $G = K_{m,n-k-m}v_1P_1v_2P_2 \cdots v_{n-k}P_{n-k}$ for some integer $m \in [2, n-k-2]$, where $P_1, \ldots, P_{n-k}$ are disjoint paths, P_i is a path of length of ℓ_i, $V(P_i) \bigcap V(K_{m,n-k-m}) = v_i$ and $\sum_{i=1}^{n-k} l_i = n$. Thus, we have $|\ell_i - \ell_j| \leqslant 1$ by Lemma 6 if v_i and v_j lie in two different bipartite sets of the vertices of $K_{m,n-k-m}$, which implies $|\ell_i - \ell_j| \leqslant 2$ for v_i and v_j lying in the same bipartite set of the vertices of $K_{m,n-k-m}$.

Consequently, in view of Cases 1 and 2, we obtain that G can be written as $K_{m,n-k-m}v_1 P_1v_2P_2 \cdots v_{n-k}P_{n-k}$ for some integer $m \in [1, n-k-1]$, where $|\ell_i - \ell_j| \leqslant 2$ for any $1 \leqslant i, j \leqslant n-k$.

We can easily obtain that (i) follows by computing the limit point of the $\rho(G)$ for the fixed $n - k$, the method of which is similar to that of Theorem 4 [19]. Next we continue to show that (ii) holds.

We claim that $|\ell_i - \ell_j| \leqslant 1$ for $1 \leqslant i, j \leqslant n - k$. Let $K_{m,n-k-m}$ with vertex bipartition (V_1, V_2). Thus, it suffices to show that $\ell_i - \ell_j = 2$ is impossible for $m \geqslant 2$. Otherwise, without loss of generality, we can assume that $\ell_i = 3$ and $\ell_j = 1$ for some two vertices $v_i, v_j \in V_1$ since $k \leqslant \frac{n}{2}$. Let the vertex set of P_i be $\{u_1, u_2, u_3\}$, where u_1 and $u_3 = v_i$ are two end vertices of P_i. Let the vertex set of P_j be $\{s\}$, where $s = v_j$. Let $x = (x_1, \ldots, x_n)^T$ be a Perron vector of $Q(G)$, where x_i corresponds to the vertex u_i for $1 \leqslant i \leqslant 3$, the vertex s for $i = 4$ and the remainder vertices. From $Q(G)x = \rho x$, we have $x_2 = (\rho-1)x_1, x_3 = (\rho-2)x_2-x_1, \rho x_3 = (m+1)x_3+x_2+\sum_{v\in V_2} x_v$ and $\rho x_4 = (m+1)x_4+\sum_{v\in V_2} x_v$. Thus we obtain

$$\left[(\rho - m - 1)\left(\rho - 2 - \frac{1}{\rho - 1}\right) - 1\right] x_2 = (\rho - m - 1)x_4.$$

In the following, we want to show $x_2 < x_4$. Consequently, we only need to prove

$$(\rho - m - 1)\left(\rho - 2 - \frac{1}{\rho - 1}\right) - 1 > \rho - m - 1,$$

that is,

$$(\rho-1)\left(\rho-2-\frac{1}{\rho-1}\right)-\rho-m\left(\rho-3-\frac{1}{\rho-1}\right)>0.$$

Since $m \leqslant n-k-2<\rho-2$ and $\rho-3-\frac{1}{\rho-1}>0$, it suffices to show

$$(\rho-1)\left(\rho-2-\frac{1}{\rho-1}\right)-\rho-(\rho-2)\left(\rho-3-\frac{1}{\rho-1}\right)\geqslant 0,$$

which is equivalent to

$$\rho-\frac{1}{\rho-1}\geqslant 4.$$

It follows from $\rho>n-k>\frac{n}{2}\geqslant 5$. Consequently, we have $x_4>x_2$. Now we can obtain a new graph G^* from G by deleting the edge u_1u_2 and adding a new edge u_1s. So

$$x^T(Q(G^*)-Q(G))x=(2x_1+x_2+x_4)(x_4-x_2)>0.$$

Thus

$$\begin{aligned}\rho(G^*) &= \max_{\substack{y\in\mathbb{R}^n\\ ||y||=1}} y^TQ(G^*)y\\ &\geqslant x^TQ(G^*)x\\ &> x^TQ(G)x\\ &= \rho(G),\end{aligned}$$

a contraction. So we have $|\ell_i-\ell_j|\leqslant 1$ for $1\leqslant i,j\leqslant n-k$. Hence we obtain $G\in\left\{K_{m,n-k-m}^{b,a}\right\}_{m=1}^{n-k-1}$, which is the first result of (ii). Further, the second result follows from Lemma 4 (iii). The proof is complete. □

Remark 1. By a simple calculation or simplifying the Equation of *Lemma 4 (i)*, we know that $\mu\left(K_{1,n-k-1}^{1,k-1}\right)$ is the largest root of the equation

$$x^3-(n+4-k)x^2+(3n-3k+4)x-n=0.$$

By the proofs of Theorem 1 and Lemma 4 (iii), we also have the following result.

Theorem 2. *Let $\mathcal{T}_n^k$ be the set of trees with n vertices and k cut vertices and $\mathscr{B}_n^k$ be the set of connected bipartite graphs with n vertices and k cut vertices. Assume $G\in\mathscr{B}_n^k\backslash\mathcal{T}_n^k$ for $n\geqslant 19$. If G has the maximal Laplacian spectral radius, then G is unique for $1\leqslant k\leqslant\frac{9n}{19}$, and $G\cong K_{2,n-k-2}^{2,k-2}$ for $2\leqslant k\leqslant\frac{9n}{19}$ and $G\cong K_{2,n-3}^{1,0}$ for $k=1$.*

The next result immediately follows from (ii) of Theorem 1 and Equality (2.2).

Theorem 3. *Let $\mathscr{B}_n^k$ be the set of the connected bipartite graphs with n $(n\geqslant 10)$ vertices and k $\left(1\leqslant k<\frac{n}{2}\right)$ cut vertices and $G\in\mathscr{B}_n^k$. Then $\rho(G)<\frac{n-k+2+\sqrt{(n-k)^2+4}}{2}$.*

Let $\mathscr{G}_n^k$ denote the set of the connected graphs with n vertices and k cut vertices. In [17], Theorem 3.3 tells that if a graph $G\in\mathscr{G}_n^k$ and G has the minimal least eigenvalue, then G is a bipartite graph.

Note that the spectral radius of a graph is at most a half of its signless Lapalacian spectral radius [6]. Thus, we have the following corollary in view of Theorems 1 and 3.

Corollary 2. *Let $\mathscr{G}_n^k$ denote the set of the connected graphs with n vertices and k cut vertices and $G \in \mathscr{G}_n^k$. Then*

$$\lambda_{min}(G) > -\frac{1}{2}\left(n-k+1+\frac{1}{n-k-1}\right)$$

for $1 \leqslant k \leqslant n-2$. *Further,*

$$\lambda_{min}(G) > -\frac{n-k+2+\sqrt{(n-k)^2+4}}{4}$$

for $1 \leqslant k < \frac{n}{2}$.

4. The least eigenvalues of graphs with *n* vertices and *k* cut edges

The purpose of this section is to investigate the extremal graph having the minimal least eigenvalue of all the connected graphs with n vertices and k cut edges. We also present a lower bound for the least eigenvalue of a connected graph in terms of the number of cut edges.

Let $\mathscr{G}_{n,k}$ denote the set of the connected graphs with n vertices and k cut edges. For convenience, a graph is called minimizing in $\mathscr{G}_{n,k}$ if its least eigenvalue attains the minimum among all graphs in $\mathscr{G}_{n,k}$.

Proposition 1. *Let G be a minimizing graph in $\mathscr{G}_{n,k}$ and let x be an eigenvector of $A(G)$ corresponding to $\lambda_{min}(G)$. Assume that $e = uv$ is any cut edge of G and $G - e = G_1 \bigcup G_2$, where $u \in V(G_1)$ and $v \in V(G_2)$. Then*

(i) $|x_u| = \max_{w\in V(G_1)} |x_w|$ *and* $|x_v| = \max_{w\in V(G_2)} |x_w|$;
(ii) $|V(G_1)| = 1$ *or* $|V(G_2)| = 1$. *In addition, if* $|V(G_1)| > 1$, *then* $|x_u| > |x_w|$ *for all* $w \in V(G_1) \setminus \{u\}$.

Proof. Let $K_2 = uv$. Assume that $H = G_1(u) \cdot K_2(u)$ and $H^* = G_2(v) \cdot K_2(v)$.

(i) If there exists a vertex $w \in V(G_1)$ with $|x_u| < |x_w|$, then we view G_1 and H^* as G_1 and G_2 of Lemma 7, respectively. Thus we get a graph $G^* \in \mathscr{G}_{n,k}$ and $\lambda_{min}(G^*) < \lambda_{min}(G)$, a contradiction. Thus $|x_u| = \max_{w\in V(G_1)} |x_w|$. Similarly, we have $|x_v| = \max_{w\in V(G_2)} |x_w|$.

(ii) Assume, to the contrary, that $|V(G_1)| > 1$ and $|V(G_2)| > 1$. If we view H and G_2 as G_1 and G_2 of Lemma 7, respectively, then we can obtain $x_u = x_v$ and $\sum_{w\in N_{G_2}(v)} x_w = 0$; If we view H^* and G_1 as G_1 and G_2 of Lemma 7, respectively, then we get $x_u = x_v$ and $\sum_{w\in N_{G_1}(u)} x_w = 0$. Otherwise, we have a graph $G^* \in \mathscr{G}_{n,k}$ and $\lambda_{min}(G^*) < \lambda_{min}(G)$, a contradiction. Thus $x_u = x_v$, $\sum_{w\in N_{G_1}(u)} x_w = 0$ and $\sum_{w\in N_{G_2}(v)} x_w = 0$. Hence we get $\lambda_{min}(G)x_u = x_v$ by the eigenvalue Equation (1.2) for the vertex u. If $x_u = x_v = 0$, then we obtain that x is a zero vector by (i), which is impossible. Thus $x_u = x_v \neq 0$, which implies $\lambda_{min}(G) = 1$, a contradiction. Therefore, we have $|V(G_1)| = 1$ or $|V(G_2)| = 1$. Assume that $|V(G_1)| > 1$ and there is a vertex $w \in V(G_1)$ with $|x_u| \leqslant |x_w|$. Then we view G_1 and H^* as G_1 and G_2 of Lemma 7, respectively. Thus we get a graph $G^* \in \mathscr{G}_{n,k}$ and $\lambda_{min}(G^*) \leqslant \lambda_{min}(G)$. Since G is a minimizing graph, we obtain that the equality holds, which implies $x_u = x_w = x_v = 0$. Hence x is a zero vector, a contradiction. Consequently, if $|V(G_1)| > 1$, then $|x_u| > |x_w|$ for all $w \in V(G_1) \setminus \{u\}$. □

Remark 2. Let G be a minimizing graph in $\mathscr{G}_{n,k}$ and let x be an eigenvector of $A(G)$ corresponding to $\lambda_{min}(G)$. By Proposition 1, we obtain that all k cut edges of G are pendant edges and all cut edges of G are appended at some vertex u with $|x_u|$ being maximal. Thus we can assume that G is obtained by

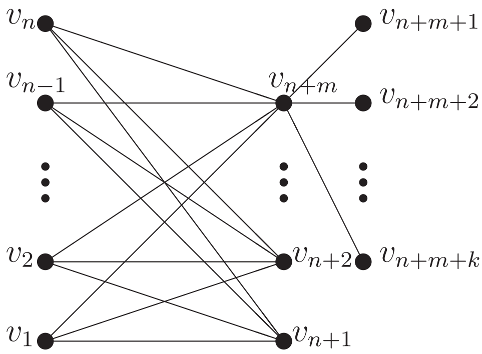

Fig. 2. $K_{m,n}^k$.

identifying some vertex u of a graph G_1 with the center of the star $K_{1,k}$. Denote by $V^+ = \{v \in V(G_1) : x_v > 0\}$, $V^- = \{v \in V(G_1) : x_v < 0\}$ and $V^0 = \{v \in V(G_1) : x_v = 0\}$. Let $\mathscr{H}_{n,k}$ denote the set of the graphs in $\mathscr{G}_{n,k}$ with only one of $|V^+|$ and $|V^-|$ being equal to 1 and $|V^0| = 0$.

Let $K_{m,n}^k$ denote the graph obtained by identifying the center of a star $K_{1,k}$ ($k \geqslant 1$) with one vertex of degree n of $K_{m,n}$, where k, m and n are integers (see Fig. 2).

Theorem 4. *Let G be a minimizing graph in $\mathscr{G}_{n,k}$. Then*

(i) $G = K_{1,n}$ *for* $k = n - 1$*;*
(ii) $G = K_{1,n-1} + e$ *for* $k = n - 3$*;*
(iii) $G \in \left\{K_{p,n-k-p}^k\right\}_{p=2}^{n-k-2} \bigcup \mathscr{H}_{n,k}$ *for* $1 \leqslant k \leqslant n - 4$*, where* $\mathscr{H}_{n,k}$ *is defined in Remark 2.*

Proof. Assume that G is a minimizing graph in $\mathscr{G}_{n,k}$. Let $\lambda_{min}(G)$ be the least eigenvalue and x be a unit eigenvector of $A(G)$ corresponding to $\lambda_{min}(G)$. By Remark 2, we can assume that G is obtained by identifying some vertex u of a graph G_1 with the center of the star $K_{1,k}$. Thus it is obvious for (i) and (ii). In what follows, we will prove that (iii) holds. Denote by $V^+ = \{v \in V(G_1) : x_v > 0\}$, $V^- = \{v \in V(G_1) : x_v < 0\}$ and $V^0 = \{v \in V(G_1) : x_v = 0\}$. It is clear that $V^+ \neq \emptyset$ and $V^- \neq \emptyset$. Then each vertex of V^+ has to be adjacent to each vertex of V^-, otherwise we would obtain a graph $G^* \in \mathscr{G}_{n,k}$ by adding such edges with $\lambda_{min}(G^*) < \lambda_{min}(G)$, a contradiction. In the following, we first prove $V^0 = \emptyset$. Assume to the contrary that $V^0 \neq \emptyset$.

Case 1. If $|V^+| > 1$ or $|V^-| > 1$, without loss of generality, letting $|V^+| > 1$, then we can obtain a bipartite graph $G^* \in \mathscr{G}_{n,k}$ by deleting all edges in V^0, V^+, V^- and between V^0 and V^- (if they exist) and adding all possible edges between V^0 and V^+. However, $\lambda_{min}(G^*) \leqslant x^TA(G^*)x \leqslant x^TA(G)x = \lambda_{min}(G)$, which implies $\lambda_{min}(G^*) = \lambda_{min}(G)$ since G is a minimizing graph. Hence by Lemma 8, we obtain that x is also an eigenvector of $A(G^*)$ corresponding to $\lambda_{min}(G^*)$, which is impossible as x contains no zero entries for any connected bipartite graph.

Case 2. If $|V^+| = |V^-| = 1$, then we can assume that $V^+ = \{v_1\}$, $V^- = \{v_2\}$ and $u_1 \in V^0$. Now we would get a bipartite graph $G^* \in \mathscr{G}_{n,k}$ by deleting all the edges whose end vertex (vertices) is in V^0, and adding the edge u_1v_2 and all edges between $V^0 \setminus \{u_1\}$ and $\{u_1, v_1\}$. However, $\lambda_{min}(G^*) \leqslant x^TA(G^*)x = x^TA(G)x = \lambda_{min}(G)$, which implies $\lambda_{min}(G^*) = \lambda_{min}(G)$ since G is a minimizing graph. Hence x is also an eigenvector of $A(G^*)$ corresponding to $\lambda_{min}(G^*)$, a contradiction.

Thus we have $V^0 = \emptyset$. We will divide the next proof into two cases.

Case 1. If $|V^+| \geqslant 2$ and $|V^-| \geqslant 2$, then the graph G has no edges joining vertices within V^+ or V^-. Otherwise by deleting such edges we would obtain a graph $G^* \in \mathscr{G}_{n,k}$ with $x^TA(G^*)x < x^TA(G)x$ and hence $\lambda_{min}(G^*) < \lambda_{min}(G)$, a contradiction. Therefore G_1 contains a complete bipartite subgraph with the vertex bipartition (V^+, V^-). Consequently, $G \in \left\{K_{p,n-k-p}^k\right\}_{p=2}^{n-k-2}$.

Case 2. If only one of $|V^+|$ and $|V^-|$ equals to 1, without loss of generality, assuming that $|V^+| = 1$ and $v \in V^+$, then we know that each vertex of V^- must be adjacent to vertex v and G_1 has no cut edge. Thus $G \in \mathscr{H}_{n,k}$. This completes the proof. □

Remark 3. Assume that $G \in \mathscr{H}_{n,k}$ and $|V^+| = 1$. If we delete all edges within V^-, then we get a tree T. Hence we have $\lambda_{min}(G) > \lambda_{min}(T) \geqslant -\sqrt{n-1}$.

Remark 4. Given the fixed $k \in \left[1, \frac{2n-6}{3}\right]$, one can obtain

$$\lambda_{min}(K^k_{p,n-k-p}) = -\sqrt{\frac{k + p(n-k-p) + \sqrt{[k - p(n-k-p)]^2 + 4k(n-k-p)}}{2}},$$

which is strictly increasing and decreasing with respect to p for $p \in \left[\frac{n-k-1}{2}, n-k-2\right]$ and $p \in \left[2, \frac{n-k-1}{2}\right)$, respectively. Thus, if $1 \leqslant k \leqslant \frac{2n-6}{3}$, then

$$\begin{aligned} \lambda_{min}(K^k_{p,n-k-p}) &\geqslant \min\left\{\lambda_{min}\left(K^k_{\lfloor\frac{n-k}{2}\rfloor,\lceil\frac{n-k}{2}\rceil}\right), \lambda_{min}\left(K^k_{\lfloor\frac{n-k}{2}\rfloor-1,\lceil\frac{n-k}{2}\rceil+1}\right)\right\} \\ &= \lambda_{min}\left(K^k_{\lfloor\frac{n-k}{2}\rfloor,\lceil\frac{n-k}{2}\rceil}\right) \end{aligned}$$

for $p \in [2, n-k-2]$, where equality holds if and only if $p = \lfloor\frac{n-k}{2}\rfloor$.

Finally, by a direct computation, one can easily show that $\lambda_{min}\left(K^k_{\lfloor\frac{n-k}{2}\rfloor,\lceil\frac{n-k}{2}\rceil}\right) < -\sqrt{n-1}$ for $1 \leqslant k \leqslant \frac{2n-6}{3}$. Consequently, we immediately have the next result in view of Theorem 4, Remarks 3 and 4.

Theorem 5. *Let G be a connected graph of order n with k cut edges, where $1 \leqslant k \leqslant \frac{2n-6}{3}$. Then $\lambda_{min}(G) \geqslant \lambda_{min}\left(K^k_{\lfloor\frac{n-k}{2}\rfloor,\lceil\frac{n-k}{2}\rceil}\right)$, where equality holds if and only if $G \cong K^k_{\lfloor\frac{n-k}{2}\rfloor,\lceil\frac{n-k}{2}\rceil}$.*

Note that the spectral radius of a bipartite graph G equals $-\lambda_{min}(G)$ (see [7]). Since $K^k_{\lfloor\frac{n-k}{2}\rfloor,\lceil\frac{n-k}{2}\rceil}$ is a connected bipartite graph, as an application of Theorem 5, the following result is immediate.

Theorem 6. *Let $1 \leqslant k \leqslant \frac{2n-6}{3}$. Of all the connected bipartite graphs of order n with k cut edges, the maximal spectral radius is attained only at $K^k_{\lfloor\frac{n-k}{2}\rfloor,\lceil\frac{n-k}{2}\rceil}$.*

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