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Annotated nonmonotonic rule systems

A. Nerode[a,1], J.B. Remmel[b,*,2], V.S. Subrahmanian[c,3]

[a] *Mathematical Sciences Institute, Cornell University, Ithaca, NY 14853, USA*

[b] *Department of Mathematics, University of California at San Diego, La Jolla, CA 92903, USA*

[c] *Department of Computer Science and the Institute for Advanced Computer Studies, University of Maryland, College Park, MD 20742, USA*

Abstract

Annotated logics were proposed by Subrahmanian as a unified paradigm for representing a wide variety of reasoning tasks including reasoning with uncertainty within a single theoretical framework. Subsequently, Marek, Nerode and Remmel have shown how to provide nonmonotonic extensions of arbitrary languages through their notion of a nonmonotonic rule systems. The primary aim of this paper is to define annotated nonmonotonic rule systems which merge these two frameworks into a general purpose nonmonotonic reasoning framework over arbitrary multiple-valued logics. We then show how Reiter's normal default theories may be generalized to the framework of annotated nonmonotonic rule systems.

1. Introduction

Many sophisticated applications require the ability to draw conclusions in the presence of uncertain information about the world. Furthermore, uncertainty in the real world arises for a number of reasons: first, even though all facts are either true or false in the current state of the world, *our knowledge* of those facts is uncertain. As reasoning is based on our beliefs about the world, we are forced to reason with our uncertain beliefs. Second, uncertainty may arise because we wish to make decisions *now* about events whose outcomes will only be known in the future. This involves uncertainty due to temporal reasons. The purpose of this paper is to present a unified framework for reasoning in the presence of uncertainty as well as incomplete beliefs. Some sample scenarios where such problems arise are the following:

* Corresponding author.

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1. Consider a battlefield commander who is relying on sensor information in order to draw conclusions about what offensive actions he must take. Sensors are notoriously unreliable, so in all likelihood, the battlefield commander will use a set of sensors to make his decisions. He may reason thus: I will take action α_1 if I can conclude that A_1 is true with certainty at least c_1 and A_2 is true with at least certainty c_2. However, if it cannot be established that A_1 is true with certainty at least c_1, but A_2 is true with at least certainty c_2, then I will take action α_2. If neither of these cases hold, I will undertake course of action α_3.

In this example, uncertainty is derived from the inherent unreliability of the sensors. Furthermore, nonmonotonicity is involved as the commander must rely on the inability to establish that certain atoms are true with a given certainty. The commander therefore must use a *mix* of nonmonotonicity and uncertainty in order to draw various conclusions.

2. Another simple example is in the case of *visual reasoning*. Suppose we have a face-recognition system coupled to an airport surveillance system. The face-recognition system attempts to match the faces of travellers using the airport to known terrorists. As most face recognition algorithms specify a *certainty* of match between a surveillance image and an image on file, there could be considerable uncertainty in whether a match has been indeed found. As the security officials do not wish to unnecessarily harrass innocent travelers, they must use some policy of deciding what actions must be taken. Such actions could include:

(a) If the probability of the match is over 90%, then send an armed official to keep close tabs on the suspect.

(b) If the probability of the match is at least 80% and there is no evidence to confirm that the suspect is in the country, then maintain careful human-assisted surveillance.

(c) If the probability of the match is at least 80% and there is evidence to confirm that the suspect is in the country, then send an armed official to keep close tabs on the suspect.

As in the previous case, this example too requires a mix of reasoning with uncertainty and reasoning in the absence of evidence (e.g. case 2(b) above).

Annotated logics were proposed by Subrahmanian [30] as a uniform theoretical framework for reasoning with multiple valued logics. They form the basis for reasoning with uncertainty in applications such as those listed above. In addition, Marek et al. [15] proposed the concept of nonmonotonic rule systems which provide a host of techniques for reasoning in the presence of incomplete beliefs. In this paper, we show how these two powerful frameworks may be unified to provide a single theoretical framework within which problems such as the ones posed above can be articulated and solved. In particular, we develop a general theory of *annotated nonmonotonic rule systems* which gives a theory and algorithms for annotations in an abstract form applying equally to default logic, truth maintenance systems, stable models of logic programs, and even a full system of nonmonotonic predicate logic. Annotated nonmonotonic rule systems are intended to provide a general framework for annotations arising from such concerns as credibility measures, probabilities, timestamps, place of origin stamps, assessments of quality of data, synchronization and timing data, etc. This work may be viewed as a smooth theoretical integration of two important reasoning paradigms that have been introduced during the last decade.

Annotated logics: Assessment of the significance of conclusions deduced from rule bases in AI often depends on adroit use of annotations. Such annotations record items such as degree of credibility, probability, quality of data, timestamps, ownership, or any other items that may be used to draw conclusions. An army general deciding where to commit troops or an investor deciding where to commit funds, relies on annotations. Similarly, religious congregations rely on statements as annotated by authority and age.

As an alternative to our approach, an obvious way to enlarge logic to include annotations would be to add new places in predicates for variables ranging over annotations. This would put the domain of annotations on the same footing as the domain over which the rule variables range. Unfortunately this approach gives rise to an unnecessarily intractable many sorted theory. In 1987 Subrahmanian introduced a more tractable theory of *annotated logic programming* [30]. This was further developed by Kifer and Subrahmanian in Generalized Annotated Programs (GAP) [11, 12]. These programs provide a unifying framework for reasoning about uncertainty and time in logic databases.

Nonmonotonic reasoning:AI is replete with reasoning about beliefs as well as reasoning about rules and facts. The beliefs have to be revised occasionally based on new facts. The mathematical logic of beliefs, rules and facts has been modelled by extensions in the default logic of Reiter [28], by stable models of logic programs by Gelfond and Lifshitz [6], and by extensions in truth maintenance systems by MacDermott and Doyle [23]. To get a birdseye view of the state of these subjects, see the workshop volumes on logic programming and nonmonotonic reasoning [24, 27, 21].

Marek–Nerode–Remmel in 1990 introduced a logic-free algebraic generalization of these systems, nonmonotonic rule systems [15]. Nonmonotonic rule systems (NMRS) capture all the essential mathematical and computational common features of many nonmonotonic reasoning formalisms including general logic programming, default logics, and truth maintenance systems. Indeed, there are simple translations between nonmonotonic rule systems and the systems of these formalisms which allow one to immediately transfer theorems about nonmonotonic rule systems into theorems about general logic programs, default logics, and truth maintenance systems.

A very rich theory of nonmonotonic rule systems has been developed by Marek, Nerode, and Remmel in a whole series of papers [15–19]. For example, they gave a complete recursion theoretic analysis of the complexity of the set of extensions of a NMRS which immediately applies to stable models of general logic programs, extensions of default theories, and extensions of truth maintenance systems. In addition, Marek, Nerode, and Remmel gave a far reaching generalization of Reiter's normal default theories called *FC*-normal NRS which are especially appropriate for belief revision and they gave a forward chaining rule processing algorithm which is an efficient way for computing extensions in a variety of settings and which allows one to extract a maximally consistent set $N' \subseteq N$ and an extension for the nonmonotonic rule system $\langle U, N' \rangle$ even when the original nonmonotonic rule system $\langle U, N \rangle$ is inconsistent in the sense that it has no extensions.

Annotated nonmonotonic rule systems: The beliefs, facts, and rules of nonmonotonic reasoning need the same kind of essentially non-logical annotations as monotonic systems. Subrahmanian made the first attempt to unify annotations and nonmonotonic reasoning in [31], which integrates annotated logics, stable model semantics, and the well-founded semantics of logic programs. Also, Nerode and Subrahmanian [25, 13] based on ideas from the newly emerging area of hybrid systems proposed a concept of *hybrid knowledge base*. Lu et al. [14] added constraints to this formalism.

Our purpose here in introducing annotated nonmonotonic rule systems is to provide a single unifying framework which extends such results to all the nonmonotonic reasoning systems listed above equally, and has a unified set of theorems, algorithms, and complexity results.

Finally, one can easily incorporate constraints as annotations, of the form found in the constraint logic programming (CLP) of Jaffar and Lassez [8, 9]. We will not focus on this aspect in this paper; the addition of constraints has been carried out in the logic programming setting in [13, 14]. The more general constraint logic paradigm of [20] can also easily be integrated into annotated nonmonotonic rule systems.

The outline of this paper is as follows. In Sections 2 and 3, we shall briefly review the theory of nonmonotonic rule systems and describe how one can easily translate some standard nonmonotonic reasoning formalisms such logic programming with negation as failure, default logic, logic programming with classical negation, truth maintenance systems, and nonmonotonic modal logics into nonmonotonic rule systems. Then in Section 4, we shall give the basic definitions of annotated nonmonotonic rule systems. The entire theory of nonmonotonic rule systems as developed by Marek et al. [15–19] can then be lifted to the setting of annotated nonmonotonic rule systems. This fact is not too surprising because, as we shall indicate when we formally define annotated nonmonotonic rule systems, for each annotated nonmonotonic rule system $\mathcal{AS}$, one can define an essentially equivalent nonmonotonic rule system $\mathcal{S}$. The difference between $\mathcal{AS}$ and $\mathcal{S}$ is that we have to add potentially infinitely many new rules to the system to produce effect of the annotation directly in a nonmonotonic rule system. Adding many additional rules is undesirable however since the complexity of most algorithms in nonmonotonic rule systems such as algorithms for finding extensions, determining whether an element lies in an extension, etc., is directly dependent on the number of rules. Thus just from complexity considerations, developing a proper theory of annotated rules systems is essential. However, while it not surprising that the theory of nonmonontontic rule systems can be lifted to the annotated setting, it requires a fair amount of care to seamlessly incorporate annotation into nonmonotonic rule systems so that the entire theory can be lifted. Such a program is much too long to be carried out in a single paper. However we will carry out one part of this program in this paper. Namely, we shall show how to extend FC-normal nonmonotonic rule systems to the setting of annotated nonmonotonic rule systems. Thus in Section 5, we shall review the basic properties of normal default theories which were proved by Reiter in [28] . Then in Section 6, we shall give the definitions and basic properties of annotated FC-normal rule systems. Essentially, all the desirable properties that are possessed by normal default theories are possessed by annotated FC-normal rule systems. The proofs of the basic properties of annotated FC-normal rule systems will be given in Section 7.

2. Nonmonotonic rule systems

In this section we shall very briefly review the key definitions of nonmonotonic rule systems of Marek, Nerode, and Remmel.

Inspired by Reiter [28] and Apt [1], Marek, Nerode, and Remmel introduced the notion of a nonmonotonic rule system in [15, 16]. A *nonmonotonic rule of inference* is a triple $\langle P, G, \varphi \rangle$, where $P = \{\alpha_1, \ldots, \alpha_n\}$, $G = \{\beta_1, \ldots, \beta_m\}$ are finite lists of objects from a set U and $\varphi \in U$. Each such rule is written in the form

$$r = \frac{\alpha_1, \ldots, \alpha_n : \beta_1, \ldots, \beta_m}{\varphi} \tag{1}$$

Here $\{\alpha_1, \ldots, \alpha_n\}$ are called the *premises* of rule r, $\{\beta_1, \ldots, \beta_m\}$ are called the *constraints* of rule r, and φ is called the *conclusion* of r and will be denoted by $prem(r)$, $cons(r)$, and $cln(r)$, respectively. A rule $r \in N$ as in (1) is called *monotonic* if $cons(r) = \emptyset$ and r is called *strictly nonmonotonic* otherwise. We say that r is an *axiom* if $prem(r) = \emptyset$.

A *nonmonotonic formal system* $\mathcal{S}$ is a pair $\langle U, N \rangle$, where U is a nonempty set and N is a set of nonmonotonic rules over U. We let $mon(\mathcal{S})$ denote the set of monotonic rules of $\mathcal{S}$ and let $nmon(\mathcal{S})$ denote the set of strictly nonmonotonic rules of $\mathcal{S}$.

A subset $S \subseteq U$ is called *deductively closed* if for every rule r of N, if all the premises of r are in S and all the constraints of r are not in S, i.e. $prem(r) \subseteq S$ and $cons(r) \bigcap S = \emptyset$, then the conclusion of r belongs to S. Similarly, a subset $S \subseteq U$ is called *monotonically closed* if for every monotonic rule r of N, if all the premises of r are in S, i.e. $prem(r) \subseteq S$, then the conclusion of r belongs to S.

Next we introduce two closure operators for $\mathcal{S}$. The first is the monotonic closure operator which is defined in the usual way by restricting our attention to the monotonic rules of $\mathcal{S}$. That is, for every $I \subseteq U$ there is the least set $cl_{\text{mon}}(I)$, called the monotonic closure of I, such that $I \subseteq cl_{\text{mon}}(I)$ and $cl_{\text{mon}}(I)$ is monotonically closed. Our second closure operator is a nonmonotonic operator which depends on a subset S of U and which we call the *S-consequence operator*, $C_S(I)$. Given a set S and an $I \subseteq U$, an *S-deduction* of φ from I in $\langle U, N \rangle$ is a finite sequence $\langle \varphi_1, \ldots, \varphi_k \rangle$ such that $\varphi_k = \varphi$

and, for all $i \leqslant k$, either

(i) φ_i is in I, or

(ii) φ_i is the conclusion of an axiom r such that $cons(r) \cap S = \emptyset$, or

(iii) φ_i is the conclusion of a rule $r \in N$ such that all the premises of r are included in $\{\varphi_1, \dots, \varphi_{i-1}\}$ and $cons(r) \cap S = \emptyset$.

An *S-consequence* of I is an element of U occurring in some S-deduction from I. Let $C_S(I)$ denote the set of all S-consequences of I in $\langle U, N \rangle$. Note that S enters solely as a restraint on the use of the rules that can be used in an S-deduction. A single constraint in a rule in N may be in S and therefore prevent the rule from ever being applied in an S-deduction from I, even though all the premises of that rule occur earlier in the deduction. Thus S contributes no members directly to $C_S(I)$, although members of S may turn up in $C_S(I)$ by an application of a rule which happens to have its conclusion in S. For a fixed S, the operator $C_S(\cdot)$ is monotonic. That is, if $I \subseteq J$, then $C_S(I) \subseteq C_S(J)$. Also, $C_S(C_S(I)) = C_S(I)$. However $C_S(I)$ is antimonotonic as a function of S, i.e. if $S \subseteq S'$, then $C_{S'}(I) \subseteq C_S(I)$.

Generally, $C_S(I)$ is not deductively closed in $\langle U, N \rangle$. It is perfectly possible that all the premises of a rule be in $C_S(I)$, the constraints of that rule are outside $C_S(I)$, but a constraint of that rule be in S, preventing the conclusion from being put into $C_S(I)$.

Example 1. Let $U = \{a,b,c\}$, $N = \{\frac{a:b}{c}\}$, and $S = \{b\}$. Then it easy to see that $C_S(a) = \{a\}$ since the rule $\frac{a:b}{c}$ cannot be used in an S-deduction. However, clearly $\{a\}$ is not deductively closed.

Next we come to a fundamental notion in nonmonotonic rule systems which is the notion of extension. We say that $S \subseteq U$ is *grounded* in I if $S \subseteq C_S(I)$. We say that $S \subseteq U$ is an *extension* of I if $C_S(I) = S$. The notion of groundedness is related to the phenomenon of "reconstruction". S is grounded in I if all elements of S are S-deducible from I (remember that S influences only the negative sides of rules). S is an extension of I if two things happen. First, every element of S is S-deducible from I, that is, S is grounded in I (this is an analogue of the adequacy property in logical calculi). Second, the converse holds, i.e. all the S-consequences of I belong to S (this is the analogue of completeness). Thus extensions are analogues for a nonmonotonic rule system of the set of all consequences of a monotonic rule system. The best way to think of extensions is to consider the model of common sense reasoning. We can then think of the rules of our nonmonotonic rule system as certain rules of thumb that we believe. Then a reasonable or justifiable set of beliefs S is a set where we can justify all our beliefs by appealing to the rules, i.e. S should be grounded, and we can derive no other conclusions from rules that are consistent with our set of beliefs. Moreover, the concept of an extension is a generalization of stable models of logic programs, extensions of default logics, and extensions of truth maintenance systems.

The notion of an extension is also related to that of a minimal deductively closed set. The following propositions were proved in [15].

Proposition 1. *If S is an extension of I, then:*
(1) *S is a minimal deductively closed superset of I.*
(2) *For every I' such that $I \subseteq I' \subseteq S$, $C_S(I') = S$.*

Proposition 2. *The set of extensions of I forms an antichain. That is, if S_1, S_2 are extensions of I and $S_1 \subseteq S_2$, then $S_1 = S_2$.*

With each rule r of form (1), we associate a monotonic rule

$$r' = \frac{\alpha_1, \ldots, \alpha_n :}{\varphi} \tag{2}$$

obtained from r by dropping all the constraints. Rule r' is called the *projection* of rule r. Let $NG(S, \mathscr{S})$ be the collection of all S-applicable rules. That is, a rule r belongs to $NG(S, \mathscr{S})$ if all the premises of r belong to S and all constraints of r are outside of S. We write $M(S)$ for the collection of all projections of all rules from $NG(S, \mathscr{S})$. The projection $\langle U, N\rangle|_S$ is the monotonic system $\langle U, M(S)\rangle$. Thus $\langle U, N\rangle|_S$ is obtained as follows: First, non-S-applicable rules are eliminated. Then, the constraints are dropped altogether. The following characterization theorem is proven in [15]:

Theorem 1. *A subset $S \subseteq U$ is an extension of I in $\langle U, N\rangle$ if and only if S is the deductive closure of I in $\langle U, N\rangle|_S$.*

There is yet another characterization of extensions which will be useful for our purposes. For this we need the concept of a proof scheme. A *proof scheme* for φ is a finite sequence

$$p = \langle\langle \varphi_0, r_0, G_0\rangle, \ldots, \langle \varphi_m, r_m, G_m\rangle\rangle \tag{3}$$

such that $\varphi_m = \varphi$ and
(1) If $m = 0$ then r_0 is an axiom with conclusion φ_0, that is,

$$r_0 = \frac{: b_1, \ldots, b_r}{\varphi}$$

and $G_0 = cons(r)$.
(2) If $m > 0$, $\langle\langle \varphi_i, r_i, G_i\rangle\rangle_{i=0}^{m-1}$ is a proof scheme of length m and φ_m is a conclusion of r_m, that is, $r_m = \varphi_{i_0}, \ldots, \varphi_{i_s} : b_1, \ldots, b_r / \varphi_m$, where $i_0, \ldots, i_s < m$ and $G_m = G_{m-1} \cup cons(r)$.

The formula φ_m is called the *conclusion* of p and is written $cln(p)$. The set G_m is called the *support* of p and is written $supp(p)$.

The idea behind this concept is as follows. An S-deduction in the system $\langle U, N\rangle$, say D, uses some negative information about S to ensure that the constraints of rules that were used are outside of S. But this negative information is finite, that is, it involves a finite subset of the complement of S. Thus, there exists a finite subset G of the complement of S such that as long as $G \cap S_1 = \emptyset$, D is an S_1-deduction as well. Our notion of proof scheme captures this finitary character of S-deduction.

We can then characterize extensions of $\langle U,N\rangle$ as follows.

Theorem 2. *Let $\mathscr{S} = \langle U,N\rangle$ be a nonmonotonic rule system and let $S \subset U$. Then S is an extension of $\mathscr{S}$ if and only if*

(i) *for each $\varphi \in S$, there is a proof scheme p such that $cln(p) = \varphi$ and $supp(p) \cap S = \emptyset$ and*

(ii) *for each $\varphi \notin S$, there is a no proof scheme p such that $cln(p) = \varphi$ and $supp(p) \cap S = \emptyset$.*

We close this section by defining another useful concept in the theory of nonmonotonic rule systems which is closely related with the fixpoints of the operator T_P in logic programming and Clark's completion, see [1]. Given collection of rules $R \subseteq N$, let $c(R) = \{cln(r)\colon r \in R\}$. Then we say that S is a *weak extension* of I iff $S = C_S(I \cup c(NG(S,\mathscr{S})))$. The idea behind the concept of weak extension is the following. In the process of constructing $C_S(I)$, S is used only negatively as a restraint. But we can relax our requirements and allow deductions that use S also on the positive side. However for weak extensions, the elements of S are not treated as "axioms", but are used to generate objects from U by also testing the positive side of a rule for membership in S.

For the rest of this paper, we shall only consider extensions of $\emptyset$ unless explicitly stated otherwise. We say that T is an *extension* of $\mathscr{S}$ if T is an extension of $\emptyset$ in $\mathscr{S}$.

3. Nonmonotonic rule systems and other nonmonotonic reasoning formalisms

In this section, we shall briefly describe the translations of general logic programs, default logic, truth maintenance systems, and nonmonotonic modal logics into nonmonotonic rule systems. This will show not only how nonmonotonic rule systems generalize these types of nonmonotonic reasoning formalisms but it will also show how our annotated nonmonotonic rules systems, once they are defined, can easily be translated back into these types of formalisms.

3.1. Logic programming, general case

A *general program clause* is an expression of the form

$$C = p \leftarrow q_1,\ldots,q_n,\neg r_1,\ldots,\neg r_m \tag{4}$$

where $p,q_1,\ldots,q_n,r_1,\ldots,r_m$ are atomic formulas possibly with variables in some first order language $\mathscr{L}$. A *program* is a set of clauses of the form (4). A clause C is called a *Horn clause* if $m = 0$. We let $H(P)$ denote the set of all Horn clauses of P. $\mathscr{H}_P$ is the Herbrand base of P, that is, the set of all ground atomic formulas of the language of P. Let *ground*(P) be the set of ground Herbrand substitutions of clauses in P. Given a set $M \subseteq \mathscr{H}_P$, the Gelfond–Lifschitz [6] reduct of P, P^M is the set of ground Horn clauses $p \leftarrow q_1,\ldots,q_n$ such that for some $r_1,\ldots,r_m \notin M$, the clause

$p \leftarrow q_1,\ldots,q_n,\neg r_1,\ldots,\neg r_m \in ground(P)$. M is called a *stable model* of P if M coincides with the least model of P^M.

Assign to a ground clause $p \leftarrow q_1,\ldots,q_n,\neg r_1,\ldots,\neg r_m \in ground(P)$ the rule

$$r(C) = \frac{q_1,\ldots,q_n : r_1,\ldots,r_m}{p}. \tag{5}$$

Let $r(P) = \langle \mathscr{H}_P, \{r(C)\colon\ C \in ground(P)\}\rangle$. The following proposition was proved ([15]).

Proposition 3. *M is a stable model of P if and only if M is an extension of $r(P)$.*

3.2. *Default logic*

Let U be the collection of all formulas of a propositional logic $\mathscr{L}$. Recall that a default theory $\langle D, W\rangle$ is a pair where D is a collection of default rules, i.e. rules of the form

$$\frac{\alpha\colon M\beta_1,\ldots,M\beta_m}{\psi} \tag{6}$$

where $\alpha,\beta_1,\ldots,\beta_m$, and ψ are formulas of $\mathscr{L}$ and W a set of formulas of $\mathscr{L}$. Let $2^{\mathscr{L}}$ denote the set of all subsets of formulas of $\mathscr{L}$. We associate an operator, Γ mapping $2^{\mathscr{L}}$ into $2^{\mathscr{L}}$ by stipulating:
$\Gamma(S) = T$ if T is the least theory in $\mathscr{L}$ such that $W \subseteq T$, T is closed under propositional consequence and T satisfies the following condition:

$$\text{whenever } d = \frac{\alpha\colon M\beta_1,\ldots,M\beta_m}{\psi} \in D, \alpha \in T,$$
$$\neg\beta_1 \notin S,\ldots,\neg\beta_m \notin S, \text{then} \psi \in T.$$

Then a theory $S \subseteq \mathscr{L}$ is called an extension of $\langle D, W\rangle$ if $\Gamma(S) = S$.

We can represent a default theory as a nonmonotonic rule system $\langle U, S\rangle$ where U is the set of all formulas of $\mathscr{L}$ and S consists of all rules of the following three types:

(i) Elements $\gamma \in W$ are represented as rules: $:/\gamma$.

(ii) Rules of form (6) are represented as $\alpha\colon\neg\beta_1,\ldots,\neg\beta_m/\gamma$.

(That is, the constraints of the rule representing a default rule r have an additional negation in front).

(iii) Processing rules of logic. That is, all the monotonic rules of the system of classical logic, e.g. modus ponens would be a set of rules of the form $a \rightarrow b, a : /b$.

We then have the following proposition from [15]:

Proposition 4. *A collection $S \subseteq U$ is an extension of a system consisting of rules of type* (i)–(iii) *if and only if S is a default extension of $\langle D, W\rangle$.*

3.3. *Truth maintenance systems*

Our description takes care of both truth maintenance systems as defined by Doyle [4] and De Kleer [3], with subsequent contributions of Reinfrank et al. [29].

Let At be a collection of atoms. By a rule over At we mean a object of the form $r = \langle A|B\rangle \to c$ where $A, B \subseteq At$, $c \in At$. A truth maintenance system (*TMS*) is a collection of rules.

Let S be a TMS. Given $M \subseteq At$, an M-derivation of an atom $a \in At$ is a finite sequence $\langle a_1, \ldots, a_n\rangle$ satisfying the conditions:

(1) $a_n = a$.

(2) For every $j \leqslant n$, either a rule $\langle\emptyset|\emptyset\rangle \to a_j$ belongs to S or there is a rule $\langle A|B\rangle \to a_j$ in S such that $A \subseteq \{a_1, \ldots, a_{j-1}\}$, $B \cap M = \emptyset$.

We call M a *TMS*-extension of S if and only if M has the property that M consists of precisely these atoms that possess an M-derivation.

We translate a rule $r = \langle A|B\rangle \to c$ as the rule

$$t(r) = \frac{a_1, \ldots, a_m : b_1, \ldots, b_n}{c}$$

where $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$. Then $M \subset At$ is an extension of the truth maintenance system iff M is an extension of the nonmonotonic rule system $\langle At, N\rangle$ where N the set of translations of rules of the truth maintenance system. In this fashion, TMS-extensions become extensions of corresponding nonmonotonic rule systems.

3.4. *Logic programming with classical negation*

We now discuss the so-called "logic programming with classical negation" of [7] as a chapter in the theory of nonmonotonic rule systems.

Recall the basic notions introduced in [7]. The collection of objects appearing in heads or bodies of clauses is the set of all literals, that is, atoms or negated atoms. In particular, a negated atom may appear in the head of a clause. Consider first "general Horn" clauses in which literals may appear in arbitrary places. To each set P of such clauses assign its *answer set*, the least collection A of literals satisfying the following two conditions:

(1) If $a \leftarrow b_1, \ldots, b_m$ is in P and $b_1, \ldots, b_m \in A$, then $a \in A$.

(2) If for some atom p, p and $\neg p$ are both in A, then A is the whole collection Lit of all literals.

Introduce a collection Str of *structural processing* rules over the set $U = Lit$. These are all monotonic rules of the form:

$$\frac{p, \neg p :}{a}$$

for all atoms p and literals a.

Translate the clause: $a \leftarrow b_1, \ldots, b_n$ as rule:

$$\frac{b_1, \ldots, b_n :}{a}$$

and let tr(P) be the collection of translations of clauses in P plus the structural rules *Str*. Then we have

Proposition 5. *A subset $A \subseteq Lit$ is an answer set for P if and only if A is an extension of $tr(P)$. Since $tr(P)$ is a set of monotonic rules, such an answer set is the least fixpoint of the (monotonic) operator associated with the translation.*

Gelfond and Lifschitz also introduced general rules in this setting. Since, the negation used in literals is not the "negation-as-failure" of general logic programming, Gelfond and Lifschitz introduce another negation symbol "*not*" and a general logic clause with classical negation in the form:

$$a \leftarrow b_1, \ldots, b_n, not(c_1), \ldots, not(c_m)$$

They define the answer set for a program with classical negation as follows:
Let $M \subseteq Lit$ and P be a general program. Define P/M as a collection of clauses lacking *not* obtained as follows:

(1) If a clause C contains a substring $not(a)$ where $a \in M$, then eliminate C altogether.

(2) In remaining clauses eliminate all substrings of the form $not(a)$.

The resulting program P/M lacks the symbol *not*, so the answer set is well-defined. Let M' be the answer set for P/M. We call M an answer set for P precisely when $M' = M$.

Gelfond and Lifschitz gave a computational procedure for finding such answer sets, and subsequently reduced computing them to computing default logic extensions. In [15], Marek, Nerode, and Remmel gave a general result showing that the construction of Gelfond and Lifschitz can also be faithfully represented within nonmonotonic rule systems; Define U to be *Lit*, and translate the clause:

$$a \leftarrow b_1, \ldots, b_n, not(c_1), \ldots, not(c_m)$$

as the rule:

$$\frac{b_1, \ldots, b_n : c_1, \ldots, c_m}{a}$$

The translation of the program P then consists of the translations of individual clauses C of P, incremented by the structural rules *Str*. The following result was proved in [15].

Proposition 6. *Let P be a general logic program with classical negation and N_P be the translation described above. Then a collection M is an answer set for P if and only if M is an extension for the rule system $\langle U, N_P \rangle$.*

3.5. McDermott and Doyle systems

McDermott and Doyle [23] and McDermott [22] investigated another system of nonmonotonic reasoning. This system is based on modal logic. We now give a brief description of the approach of McDermott and Doyle and a description of how it fits into nonmonotonic rule systems. Let $\mathscr{L}_L$ be the propositional modal language based on one modal operator L (expressing the necessity operator). We consider a strong notion of proof based on the application of the necessitation rule to all formulas, not just all theorems, of the logic under consideration. That is, this notion of proof from a set of formulas I allows one to apply necessitation to all formulas previously proved.

Let $\mathscr{S}$ be a modal logic. Examples of such a logic includes the familiar $S4$, $S5$, K, or even a logic that does not includes the schemes of K. We associate with $\mathscr{S}$ its consequence operation based on the above strong notion of proof. We denote it by $Cn_{\mathscr{S}}(\cdot)$. We now introduce the notion of $\mathscr{S}$-**expansion**. Given a set of formulas $I \subseteq \mathscr{L}_L$, we say that a theory $T \subseteq \mathscr{L}_L$ is a $\mathscr{S}$-expansion of I if

$$T = Cn_{\mathscr{S}}(I \cup \{\neg L\varphi\colon\ \varphi \notin T\}) \tag{7}$$

Notice that the role of the logic $\mathscr{S}$ here is slightly different than in the usual applications of modal logic. $\mathscr{S}$ serves as means of *reconstruction* of T from the initial assumptions I and the *negative introspection* with respect to T. It should be clear that regardless of what $\mathscr{S}$ is (it does not even need to be included in $S5$) that an expansion of any theory is closed under $S5$-consequence. It is the discipline of reconstruction that makes the difference. Note the weaker the logic, the more difficult it is to reconstruct.

Marek et al. [18] showed how this formalism can be faithfully represented as a nonmonotonic rule system. Let $\mathscr{S}$ be a fixed modal logic, axiomatized by a set of axioms AX. We define a rule system $\langle U_{\mathscr{S}}, N_{\mathscr{S}}\rangle$ as follows. The universe $U_{\mathscr{S}}$ of our system is $\mathscr{L}_L$. The set $N_{\mathscr{S}}$ consists of the following five groups of rules:

1. $:/\varphi$, where φ ranges over all the axioms of propositional logic in the language $\mathscr{L}_L$, treating every formula of the form $L\psi$ as an atom.
2. $:/\varphi$, where φ ranges over all the axioms of the logic $\mathscr{S}$.
3. $\varphi :/L\varphi$ for all the formulas $\varphi \in \mathscr{L}_L$.
4. $\varphi, \varphi \supset \psi\colon/\psi$ for all the formulas $\varphi, \psi \in \mathscr{L}_L$.
5. $:\varphi/\neg L\varphi$ for all $\varphi \in \mathscr{L}_L$.

Notice that the groups (1)–(4) of rules are monotonic, only the group (5) consists of nonmonotonic rules.

Theorem 3. *Let $\mathscr{S}$ be a modal logic. Let $I \subseteq \mathscr{L}_L$. Then T is an $\mathscr{S}$-expansion of I if and only if T is an extension of I in the nonmonotonic rule system $\langle U_{\mathscr{S}}, N_{\mathscr{S}}\rangle$.*

4. Annotated nonmonotonic rule systems

We now come to the main results of the paper. In this section, we describe our theory of annotated nonmonotonic rule systems which can be viewed as a natural amalgamation of the theory of nonmonotonic rule systems and Kifer's and Subrahmanian's work on annotated logic programs. We let $\mathscr{P} = (P, \leqslant_{\mathscr{P}})$ be any partially ordered set or preordered set. The kind of statements that we will consider will be of the form (ϕ, p) where ϕ is an element of some universe U and $p \in P$. Here are some examples of the sorts of things one might express by picking the appropriate ordering $\mathscr{P}$.

1. $\mathscr{P} = ([0,1], \leqslant)$. Then $(\phi, 1)$ would assert that ϕ is true for certain, $(\phi, 0.9)$ would assert that ϕ is true with at least 90% confidence or at least 90% probability and $(\phi, 0)$ would assert that we do not know anything about the truth of ϕ.

2. One can reverse the ordering in example (1) so that $(\phi, 0.2)$ would now mean that our confidence in the truth of ϕ is less than equal to 0.2.

3. We can achieve confidence intervals by letting P be the set of all closed intervals contained in $[0,1]$ and by defining $\leqslant_{\mathscr{P}}$ by declaring that $[a,b] \leqslant_{\mathscr{P}} [c,d]$ iff $a \leqslant c$ and $b \leqslant d$. Thus a statement $(\phi, [0.2, 0.8])$ would mean that our confidence in the truth of ϕ is between 0.2 and 0.8.

4. $\mathscr{P} = ([0,1] \times 2^T, \leqslant \times \subseteq)$ where T is a set of times and 2^T denotes the set of all subsets of T. Here $\leqslant_{\mathscr{P}}$ is just the usual product ordering of $\leqslant$ and $\subseteq$. Thus a statement $(\phi, (p, S))$ would mean that ϕ is true with confidence at least p at all times $t \in S$.

5. One can extend example (4) to have more coordinates. For example, we can let $\mathscr{P} = ([0,1] \times 2^T \times 2^P, \leqslant \times \subseteq \times \subseteq)$ where P is a set of places. In this case we could have statements like (rains, (0.9, {Friday, Saturday}, {Ithaca, Cortland})) which would mean that there is a 90% probability that it will rain in Ithaca and Cortland on Friday and Saturday.

In this setting our rules will be of the form

$$r = \frac{(\alpha_1, a_1), \ldots, (\alpha_n, a_n) : (\beta_1, b_1), \ldots, (\beta_m, b_m)}{(\varphi, c)}. \tag{8}$$

If we let $\mathscr{P}$ be as in example 1 above, then the rule means that if α_i has been established with confidence $\geqslant a_i$ for $i = 1, \ldots, n$ and no β_j can be established with confidence $\geqslant b_j$ for $j = 1, \ldots, m$, then we may conclude that φ holds with confidence at least c.

A *annotated nonmonotonic rule system* $\mathscr{S}$ consists of a triple $\langle U, \mathscr{P}, N \rangle$ where U is a set called the universe of $\mathscr{S}$, $\mathscr{P} = (P, \leqslant_{\mathscr{P}})$ is preordered set, and N is a set of rules as in (8) where α_i, β_j and φ are in U for all i and j and a_i, b_j, and c are in P for all i and j. Here $\{(\alpha_1, a_1), \ldots, (\alpha_n, a_n)\}$ are called the *premises* of rule r, $\{(\beta_1, b_1), \ldots, (\beta_m, b_n)\}$ are called the *constraints* of rule r, and (φ, c) is called the *conclusion* of r and will be denoted by $prem(r)$, $cons(r)$, and $cln(r)$, respectively. A rule $r \in N$ as in (8) is called *monotonic* if $cons(r) = \emptyset$ and r is called *strictly nonmonotonic* otherwise. We say that r is an *axiom* if $prem(r) = \emptyset$. We let $mon(\mathscr{S})$ denote the set of monotonic rules of $\mathscr{S}$ and we let $nmon(\mathscr{S})$ denote the set of strictly nonmonotonic rules of $\mathscr{S}$

We note that a nonmonotonic rule system $\langle U, N \rangle$ can be thought of as an annotated nonmonotonic rule system $\langle U, \mathscr{P}, N \rangle$ where $\mathscr{P}$ is the one element poset $(\{1\}, \leqslant)$.

For the rest of this paper, we shall assume that $\mathscr{P} = (P, \leqslant_{\mathscr{P}})$. However we shall normally write $\leqslant$ for $\leqslant_{\mathscr{P}}$ when there is no possibility for confusion.

Definition 1. (1) A subset $W \subseteq U \times P$ is called *deductively closed* if for every rule

$$r = \frac{(\alpha_1, a_1), \ldots, (\alpha_n, a_n) \colon (\beta_1, b_1), \ldots, (\beta_m, b_m)}{(\varphi, c)}$$

of N, whenever there exists $(\alpha_1, d_1), \ldots, (\alpha_n, d_n)$ in W such that $d_i \geqslant a_i$ for all i and there is no (β_j, e_j) in W with $e_j \geqslant b_j$ for any j, then $(\varphi, c) \in W$.

(2) A subset $S \subseteq U \times P$ is called *monotonically closed* if for every monotonic rule

$$r = \frac{(\alpha_1, a_1), \ldots, (\alpha_n, a_n) :}{(\varphi, c)}$$

of N, whenever there exists $(\alpha_1, d_1), \ldots, (\alpha_n, d_n)$ in S such that $d_i \geqslant a_i$ for all i, then $(\varphi, c) \in S$.

Now it is easy to see that if $W_i \subseteq U \times P$ are monotonically closed for all i in some index set I, then $\bigcap_{i \in I} W_i$ is monotonically closed. Thus for any set $V \subseteq U \times P$, we can define the monotonic closure of V, denoted $cl_{\mathrm{mon}}(V)$, by

$$cl_{\mathrm{mon}}(V) = \bigcap \{W \subseteq U \times P\colon\ V \subseteq W \ \&\ W \text{ is monotonically closed}\}.$$

Next we define the *S-consequences* operator C_S for any set $S \subseteq U \times P$. Given a set $I \subseteq U \times P$, an *S-deduction* of (φ, c) from I in $\langle U, \mathscr{P}, N\rangle$ is a finite sequence $\langle(\varphi_1, c_1), \ldots, (\varphi_k, c_k)\rangle$ such that $(\varphi_k, c_k) = (\varphi, c)$ and, for all $i \leqslant k$, either

(i) there is some $(\varphi_i, e_i) \in I$ such that $e_i \geqslant c_i$, or

(ii) there is an axiom

$$r = \frac{: (\beta_1, b_1), \ldots, (\beta_m, b_m)}{(\varphi_i, f_i)}$$

of N such that there is no $(\beta_j, d_j) \in S$ with $d_j \geqslant b_j$ for any $j = 1, \ldots, m$ and $f_i \geqslant c_i$, or

(iii) there is a rule

$$r = \frac{(\varphi_{j_1}, a_1), \ldots, (\varphi_{j_n}, a_n) \colon (\beta_1, b_1), \ldots, (\beta_m, b_m)}{(\varphi_i, f_i)}$$

of N, where $j_1, \ldots, j_n < i$, $a_{j_s} \leqslant c_{j_s}$ for all $s = 1, \ldots, n$, there is no $(\beta_j, d_j) \in S$ with $d_j \geqslant b_j$ for any $j = 1, \ldots, m$, and $f_i \geqslant c_i$.

An *S-consequence* of I is an element of $U \times P$ occurring in some S-deduction from I. Let $C_S(I)$ denote the set of all S-consequences of I in $\langle U, \mathscr{P}, N\rangle$. It is then easy to see that $C_S(I)$ is monotonic in I, i.e.,

$$I \subseteq I' \Rightarrow C_S(I) \subseteq C_S(I') \tag{9}$$

and that $C_S(I)$ is antimonotonic in S, i.e.,

$$S \subseteq S' \Rightarrow C_{S'}(I) \subseteq C_S(I). \tag{10}$$

We note that clauses (i)–(iii) in our definition of S-deduction ensure that when we have derived (φ_i, c_i), then we can also derive (φ_i, d_i) for any $d_i \leqslant c_i$. Now we could also ensure this property by explicitly adding rules of the form

$$r = \frac{(\gamma,c):}{(\gamma,d)} \tag{11}$$

into N whenever $d \leqslant c$. That is, given the annotated nonmonotonic rule system $\mathscr{S} = \langle U,\mathscr{P},N\rangle$, we could form a new nonmonotonic rule system $\overline{\mathscr{S}} = \langle U \times P,\overline{N}\rangle$ where $\overline{N}$ consists of N plus all rules of the form (11). Then one can show that for any sets $I,S \subseteq U \times P$, the set S-consequences of I relative to $\mathscr{S}$ is equal to the set of S-consequences of I relative to $\overline{\mathscr{S}}$. However as mentioned in the introduction, the extra rules of $\overline{\mathscr{S}}$ has many undesirable consequences with regard to the complexity of the algorithms needed to reason in such systems since the complexity of almost all reasoning tasks in the system are directly dependent on the sum of lengths of the rules. Moreover, if we go back to our first example where $\mathscr{P} = ([0,1],\leqslant)$ and $[0,1]$ is the real interval, there would be uncountably many rules of the form (11) so that the number of rules would become uncountable even if U is finite. However in the case where $\mathscr{P} = ([0,1],\leqslant)$ and U is finite, we can often specify the set of S-consequences of I relative to $\mathscr{S}$ by simply giving the set $\{(u,p_u)\colon u \in U$ and $p_u = max(\{p\colon (u,p) \in C_S(I)\}\}$ so that we can still carry out reasoning tasks based on finitary information. Our formulation of annotated nonmonotonic rule systems is designed explicity to allow us to carry out such reasoning tasks in a finitary manner whenever possible with ease.

We say that $S \subseteq U \times P$ is *downward closed* is whenever $(\gamma,c) \in S$, then $(\gamma,d) \in S$ for all $d \leqslant c$. In fact, it will be useful to define another closure operator, called the downward closure operator, defined for each $A \subseteq U$ by

$$cl_{\text{down}}(A) = \{(\alpha,q) \in U\colon \exists(\alpha,p) \in A(p \geqslant_{\mathscr{P}} q)\}$$

We note that our definitions of the closure operators, cl_{mon} and cl_{down} imply the following propostion.

Proposition 7. *Let $\langle U,\mathscr{P},N\rangle$ be an annotated nonmonotonic rule system. Then for any $S \subseteq U \times P$, $cl_{\text{down}}(cl_{\text{mon}}(cl_{\text{down}}(S))) = cl_{\text{down}}(cl_{\text{mon}}(S))$*

Proof. Clearly, the operators cl_{down} and cl_{mon} are monotonic so that $cl_{\text{down}}(cl_{\text{mon}}(cl_{\text{down}}(S))) \supseteq cl_{\text{down}}(cl_{\text{mon}}(S))$.

For the reverse inclusion note that any monotonic rule $r = (\alpha_1,a_1),\ldots,(\alpha_n,a_n):/(\varphi,c)$ which is $cl_{\text{down}}(S)$-applicable is also S-applicable. That is, if there exists $(\alpha_1,b_1),\ldots,(\alpha_n,b_n)$ in $cl_{\text{down}}(S)$ with $b_i \geqslant a_i$ for $i = 1,\ldots,n$, then there must exist $(\alpha_1,c_1),\ldots,(\alpha_n,c_n)$ in S with $c_i \geqslant b_i \geqslant a_i$ for $i = 1,\ldots,n$. It is then easy to prove by induction of the length of a deduction that $cl_{\text{mon}}(cl_{\text{down}}(S)) - cl_{\text{down}}(S) = cl_{\text{mon}}(S) - cl_{\text{down}}(S)$ so that $cl_{\text{down}}(cl_{\text{mon}}(cl_{\text{down}}(S))) \subseteq cl_{\text{down}}(cl_{\text{mon}}(S))$. □

This given, we then have the following.

Proposition 8. *For all $S,I \subseteq U \times P$, $C_S(I)$ is downward closed.*

Because nonmonotonic rule systems can be viewed as special cases of annotated nonmonotonic rule systems when $\mathscr{P} = (\{1\}, \leqslant)$, $C_S(I)$ will not, in general, be deductively closed in $\langle U, \mathscr{P}, N \rangle$. Of course, it is easy to construct examples where $C_S(I)$ is not dedecutively closed when $\mathscr{P}$ is nontrivial as well. That is, say a rule

$$r = \frac{(\alpha_1, a_1), \ldots, (\alpha_n, a_n) \colon (\beta_1, b_1), \ldots, (\beta_m, b_m)}{(\varphi, c)}$$

of N is *S-applicable* if there exists $(\alpha_1, d_1), \ldots, (\alpha_n, d_n)$ in S such that $d_i \geqslant a_i$ for all i and there is no (β_j, e_j) in S with $e_j \geqslant b_j$ for any j. Then it is perfectly possible, that a rule r is $C_S(I)$-applicable but $cons(r) \cap S \neq \emptyset$ so that it cannot be used to put elements in $C_S(I)$. We also note that while C_S is downward closed, it is not closed under limits in $\mathscr{P}$.

Example 2. Suppose that $\mathscr{P}$ is just the rational interval $[0, 1]$ under the usual ordering. Let $\langle U, \mathscr{P}, N \rangle$ be the following rule system. $U = \{a, b, c\}$ and N consists of the following set of rules.

1. $\dfrac{(a, 1 - \frac{1}{n}) :}{(a, 1 - \frac{1}{n+1})}$ for $n = 2, 3, \ldots$.
2. $\dfrac{:}{(a, \frac{1}{2})}$
3. $\dfrac{(b, 1) : (c, 1)}{a : 1}$

Then let $I = \{(b, 1)\}$ and $S = \{(c, 1)\}$. Then it is easy to see that $C_S(I) = \{(a, q) : q < 1\} \cup \{(b, 1)\}$. Note that rule (3) is $C_S(I)$-applicable but it is not S-applicable so that $C_S(I)$ is not deductively closed.

We note that the failure of $C_S(I)$ being closed under limits in $\mathscr{P}$ is not something that can be easily remedied in our setting because requiring that $C_S(I)$ is closed under limits in $\mathscr{P}$ would require an infinitary type rule of the form

$$r = \frac{\{(a, q) \colon q \in T\};}{(a, p)}$$

where $p = sup(T)$ if $\mathscr{P}$ is an infinite poset like the rational interval $[0, 1]$ under the usual ordering. However in any real applications, we never actually deal with infinite posets since we can use only finitely many rules and hence we can restrict ourselves to the poset consisting of those elements which are actually mentioned in the rules.

The analogue of extension is now straightforward. Namely, we say that $S \subseteq U \times P$ is an *extension* of I if $C_S(I) = S$. We also can prove the obvious analogues of Propositions 1 and 2.

Proposition 9. *If S is an extension of I, then:*

(1) *S is a minimal deductively and downward closed superset of I.*

(2) *For every I' such that $I \subseteq I' \subseteq S$, $C_S(I') = S$.*

Proof. Suppose $S = C_S(I)$. Clearly, $I \subset C_S(I)$ by clause (i) of the definition of S-deduction. $C_S(I)$ is always downward closed by Proposition 8. To see that $S = C_S(I)$ is deductively closed suppose we have a rule

$$r = \frac{(\alpha_1, a_1), \ldots, (\alpha_n, a_n) : (\beta_1, b_1), \ldots, (\beta_m, b_m)}{(\varphi, c)}$$

of N such that there exists $(\alpha_1, d_1), \ldots, (\alpha_n, d_n)$ in S such that $d_i \geqslant a_i$ for all i and there is no (β_j, e_j) in S with $e_j \geqslant b_j$ for any j. Then there are S-deductions D_i of (α_i, d_i) from I for $i = 1, \ldots, n$. It is then easy to see that we can construct a S-deduction of (φ, c) by concatenating the sequences $D_1, \ldots, D_n, \langle(\varphi, c)\rangle$ so that $(\varphi, c) \in C_S(I) = S$. Thus, S is a deductively and downward closed superset of I.

Next suppose that T is a deductively and downward closed superset of I such that $T \subset S$. Then let $(\varphi_k, c_k) \in S - T$ and $\langle(\varphi_1, c_1), \ldots, (\varphi_k, c_k)\rangle$ be a S-deduction of (φ_k, c_k). Then let i be the least j such that $(\varphi_j, c_j) \notin T$. But then either:

(i) there is some $(\varphi_i, e_i) \in I$ such that $e_i \geqslant c_i$ and hence $(\varphi_i, c_i) \in T$ since $I \subseteq T$ and T is downward closed.

(ii) there is an axiom

$$r = \frac{: (\beta_1, b_1), \ldots, (\beta_m, b_m)}{(\varphi_i, f_i)}$$

of N such that there is no $(\beta_j, d_j) \in S$ with $d_j \geqslant b_j$ for any $j = 1, \ldots, m$ and $f_i \geqslant c_i$ in which case there is certainly no $(\beta_j, d_j) \in T$ with $d_j \geqslant b_j$ for any $j = 1, \ldots, m$ and hence $(\varphi_i, c_i) \in T$ since T is deductively closed.

(iii) there is a rule

$$r = \frac{(\varphi_{j_1}, a_1), \ldots, (\varphi_{j_n}, a_n) : (\beta_1, b_1), \ldots, (\beta_m, b_m)}{(\varphi_i, f_i)}$$

of N where $j_1, \ldots, j_n < i$, $a_{j_s} \leqslant c_{j_s}$ for all $s = 1, \ldots, n$, there is no $(\beta_j, d_j) \in S$ with $d_j \geqslant b_j$ for any $j = 1, \ldots, m$, and $f_i \geqslant c_i$. But then $\{(\varphi_{j_1}, a_1), \ldots, (\varphi_{j_n}, a_n)\} \subseteq T$ by our choice of i and there is no $(\beta_j, d_j) \in T$ with $d_j \geqslant b_j$ for any $j = 1, \ldots, m$. Hence, $(\varphi_i, c_i) \in T$ since T is deductively closed.

Thus there can be no such i and hence there can be no such $T \subset S$. Thus, S is a minimally deductively and downward closed superset of I.

For part (2), observe that every S-deduction from I' can be expanded to an S-deduction from I. That is, if $D = \langle(\varphi_1, c_1), \ldots, (\varphi_k, c_k)\rangle$ be an S-deduction of (φ_k, c_k) from I', then the only reason that D is not an S-deduction of (φ_k, c_k) from I comes from the fact that in an application of clause (i) in the definition of an S-deduction from I', it may be the case that for some i, there is some $(\varphi_i, e_i) \in I'$ such that $e_i \geqslant c_i$ rather than there is some $(\varphi_i, e_i) \in I$ such that $e_i \geqslant c_i$. But since $(\varphi_i, e_i) \in I' \subseteq S$, there is an S-deduction of (φ_i, e_i) from I, $D_i = \langle(\theta_1, e_1), \ldots, (\theta_l, e_l)\rangle$ where $(\theta_l, e_l) = (\varphi_i, e_i)$. If whenever there is such an i, we replace (φ_i, e_i) in D by the sequence D_i, then we can easily expand D to an S-deduction of (φ_k, c_k) from I. Thus, $C_S(I') \subseteq S$. On the other hand, since $C_S(J)$ is monotonic in J, $S = C_S(I) \subseteq C_S(I')$ and hence $C_S(I') = C_S(I)$. □

Proposition 10. *The set of extensions of I forms an antichain. That is, if S_1, S_2 are extensions of I and $S_1 \subseteq S_2$, then $S_1 = S_2$.*

Proof. Suppose S_1, S_2 are extensions of I and $S_1 \subseteq S_2$. Since $C_S(I)$ is antimonotonic in S, we $S_2 = C_{S_2}(I) \subseteq C_{S_1}(I) = S_1$. Thus $S_1 = S_2$. □

Next we shall extend the notion of proof scheme to annotated nonmonotonic rule systems which will allow us to give another characterization of extensions. A *proof scheme* for (φ, c) is a finite sequence

$$p = \langle\langle(\varphi_0, c_0), r_0, G_0\rangle, \ldots, \langle(\varphi_m, c_m), r_m, G_m\rangle\rangle \tag{12}$$

such that $(\varphi_m, c_m) = (\varphi, c)$ and

(1) If $m = 0$, then

$$r_0 = \frac{:(\beta_1, b_1), \ldots, (\beta_m, b_m)}{(\varphi_0, f_0)}$$

where $f_0 \geqslant c_0$ and $G_0 = cons(r)$.

(2) If $m > 0$, then $\langle\langle\varphi_i, r_i, G_i\rangle\rangle_{i=0}^{m-1}$ is a proof scheme of length m and

$$r_m = \frac{(\varphi_{j_1}, a_1), \ldots, (\varphi_{j_n}, a_n) : (\beta_1, b_1), \ldots, (\beta_m, b_m)}{(\varphi_m, f_m)}$$

where $j_1, \ldots, j_n < m$, $a_{j_s} \leqslant c_{j_s}$ for all $s = 1, \ldots, n$, $f_m \geqslant c_m$, and $G_m = \{(\beta, d) \in G_{m-1} \cup cons(r) : \neg\exists(\beta, e) \in G_{m-1} \cup cons(r) \text{ with } e \leqslant d\}$. The pair (φ_m, c_m) is called the *conclusion* of p and is written $cln(p)$. The set G_m is called the *support* of p and is written $supp(p)$.

The idea behind this concept is as follows. An S-deduction in the system $\langle U, \mathcal{P}, N, \rangle$, say D, uses some negative information about $cl_{\text{down}}(S)$ to ensure that the constraints of rules that were used are outside of $cl_{\text{down}}(S)$. But this negative information is finite, that is, it involves a finite subset of the complement of $cl_{\text{down}}(S)$. Thus, there exists a finite subset G of the complement of $cl_{\text{down}}(S)$ such that as long as $G \cap cl_{\text{down}}(S_1) = \emptyset$, D is an S_1-deduction as well. Our notion of proof scheme captures this finitary character of S-deduction.

We can then characterize extensions of $\langle U, \mathcal{P}, N\rangle$ as follows.

Theorem 4. *Let $\mathcal{S} = \langle U, \mathcal{P}, N\rangle$ be an annotated nonmonotonic rule system and let $S \subset U \times P$. Then S is an extension of $\mathcal{S}$ if and only if*

(i) *S is downward closed,*

(ii) *for each $(\varphi, c) \in S$, there is a proof scheme p such that $cln(p) = (\varphi, c)$ and $supp(p) \cap S = \emptyset$ and*

(iii) *for each $(\varphi, c) \notin S$, there is a no proof scheme p such that $cln(p) = (\varphi, c)$ and $supp(p) \cap S = \emptyset$.*

Proof. It is straightforward to prove by induction on the length of a proof scheme p that if S is a downward closed set, then $cln(p) \in C_S(\emptyset)$ if $S \cap supp(p) = \emptyset$. Similarly, it is straightforward to take an S-deduction of (φ, c) from $\emptyset$ and turn it into a proof scheme p such that $cln(p) = (\varphi, c)$ and $supp(p) \cap S = \emptyset$. Thus for downward closed sets S, $C_S(\emptyset)$ equals the set of all $(\varphi, c) \in U \times P$ such that there is a proof scheme p such that $cln(p) = (\varphi, c)$ and $supp(p) \cap S = \emptyset$. Hence for downward closed S, conditions (ii) and (iii) are equivalent to saying that $C_S(\emptyset) = S$, i.e. that S is an extension. □

We can also characterize extensions in $\langle U, \mathscr{P}, N \rangle$ via an analogue of the Gelfond–Lifschitz transform. Recall that a rule

$$r = \frac{(\alpha_1, a_1), \ldots, (\alpha_n, a_n) : (\beta_1, b_1), \ldots, (\beta_m, b_m)}{(\varphi, c)}$$

of N is *S-applicable* if there exists $(\alpha_1, d_1), \ldots, (\alpha_n, d_n)$ in S such that $d_i \geqslant a_i$ for all i and there is no (β_j, e_j) in S with $e_j \geqslant b_j$ for any j. Let $NG(S, \mathscr{S})$ denote the set of all S-applicable rules in N. Then with each rule r of form (8), we associate a monotonic rule

$$r' = \frac{(\alpha_1, a_1), \ldots, (\alpha_n, a_n) :}{(\varphi, c)}. \tag{13}$$

obtained from r by dropping all the constraints. Rule r' is called the *projection* of rule r. $M(S)$ for the collection of all projections of all rules from $NG(S, \mathscr{S})$. The projection $\langle U, \mathscr{P}, N \rangle|_S$ is the monotonic system $\langle U, \mathscr{P}, M(S) \rangle$. Thus, $\langle U, \mathscr{P}, N \rangle|_S$ is obtained as follows: First, non-S-applicable rules are eliminated. Then, the constraints are dropped altogether. We then have the following analogue of Theorem 1.

Theorem 5. *A subset $S \subseteq U$ is an extension of I in $\langle U, \mathscr{P}, N \rangle$ if and only if $S = cl_{\text{down}}(cl_{\text{mon}}(I))$ in $\langle U, \mathscr{P}, N \rangle|_S$.*

Proof. It is straightforward to prove by induction on a length of a sequence, that a sequence $\langle (\alpha_0, a_0), \ldots, (\alpha_n, a_n) \rangle$ is an S-deduction from I in $\langle U, \mathscr{P}, N \rangle$ iff the same sequence represents a deduction which shows that $(\alpha_n, a_n) \in cl_{\text{down}}(cl_{\text{mon}}(I))$ in $\langle U, N \rangle|_S$. That is, $C_S(I)$ in $\langle U, \mathscr{P}, N \rangle$ equals $cl_{\text{down}}(cl_{\text{mon}}(I))$ in $\langle U, \mathscr{P}, N \rangle|_S$. Thus, S is an extension of I in $\langle U, \mathscr{P}, N \rangle$ if and only if $S = C_S(I)$ in $\langle U, \mathscr{P}, N \rangle$ if and only if $S = cl_{\text{down}}(cl_{\text{mon}}(I))$ in $\langle U, \mathscr{P}, N \rangle|_S$. □

We can also define the analogue of weak extensions for annotated nonmontonic rule systems $\langle U, \mathscr{P}, N \rangle$. Given a collection of rules $R \subseteq N$, let

$$c(R) = \{cln(r) : r \in R\}.$$

Then we say that S is a *weak extension* of I iff $S = C_S(I \cup c(NG(S, \mathscr{S})))$.

Once again for the rest of this paper, we shall only consider extensions of $\emptyset$ in an annotated nonmontonic rule systems $\mathscr{S} = \langle U, \mathscr{P}, N \rangle$ unless explicitly stated otherwise. Thus, we shall say that S is an *extension* of $\mathscr{S}$ if S is an extension of $\emptyset$ in $\mathscr{S}$.

As pointed out in the introduction, it is possible to adapt the entire machinery of extensions, proof schemes, forward chaining rule processing, and FC-normal rule systems as developed by Marek, Nerode, and Remmel for nonmonotonic rule systems to the setting of annotated nonmonotonic rule systems. In particular, we shall show how one can extend Reiter's normal default theories to annotated nonmonotonic rule systems by providing an analogue of *FC*-normal rule systems as in [18]. To motivate our results, we shall recall the main properties of normal default theories in our next section.

5. Normal default theories

In this section we shall recall Reiter's definitions of normal default theories and state some of the basic theorems about normal default theories as proved in [28].

Recall a *default rule* is a rule of proof of the form

$$\frac{\varphi:\ M\psi_1,\ldots,M\psi_m}{\gamma} \tag{14}$$

where $\varphi,\psi_1,\ldots,\psi_m,\gamma$ are formulas of a propositional language $\mathscr{L}$. A *default theory* is a pair $\langle D,W\rangle$, where D is a set of default rules and $W\subseteq\mathscr{L}$. For any subset of formulas $S\subseteq\mathscr{L}$, we let $Cn(S)$ denote the set of all logical consequences of S. Also if D is a set of default rules, let

$$c(D)=\left\{\gamma:\ \frac{\varphi:\ M\psi_1,\ldots,M\psi_m}{\gamma}\in D\right\}.$$

Given a subset $S\subseteq\mathscr{L}$, define $\Gamma(S)$ as the least set T (under inclusion) satisfying these conditions:

1. $W\subseteq T$;
2. $Cn(T)=T$;
3. Whenever $r\in D$ is a default rule of the form (14) and $\varphi\in T$ and for all $j\leqslant m$, $\neg\psi_j\notin S$, then $\gamma\in T$.

It is easy to see that $\Gamma(S)$ always exists. We say that $S\subseteq\mathscr{L}$ is an *extension* of $\langle D,W\rangle$ if $\Gamma(S)=S$. A default rule of the form (14) is called *generating* for S if $\varphi\in S$, $\neg\psi_1,\ldots,\neg\psi_m\notin S$. Let $S\subseteq\mathscr{L}$. Then we define $NG(D,S)$ to be the set of all generating rules for S in D and $c(NG(D,S))$ to be the set of their conclusions. S is called a *weak extension* of $\langle D,W\rangle$ if $S=Cn(W\cup c(NG(D,S)))$.

A rule r is *normal* if it is of the form

$$\frac{\varphi:\ M\psi}{\psi}. \tag{15}$$

A default theory $\langle D,W\rangle$ is *normal* if every $r\in D$ is a normal default rule. Reiter [28] proved the following theorems about normal default theories.

Theorem 6. *Every normal default theory possesses an extension.*

Theorem 7 (Semi-monotonicity). *Suppose that D and D' are sets of normal defaults with $D'\subseteq D$. Let E' be an extension of the normal default theory $\Delta'=\langle D',W\rangle$ and*

let $\Delta = \langle D, W \rangle$. Then Δ has an extension E such that
1. $E' \subseteq E$ *and*
2. $NG(E', \Delta') \subseteq NG(E, \Delta)$.

Theorem 8 (Orthogonality of extensions). *If a normal default theory $\langle D, W \rangle$ has distinct extensions E and F, then $E \cup F$ is inconsistent.*

Theorem 9. *Suppose that $\Delta = \langle D, W \rangle$ is a normal default theory and $W \cup c(D)$ is consistent. Then Δ has a unique extension.*

Theorem 10. *Suppose that $\Delta = \langle D, W \rangle$ is a normal default theory and that $D' \subseteq D$. Suppose further that E_1' and E_2' are distinct extensions of $\langle D', W \rangle$. Then Δ has distinct extensions E_1 and E_2 such that $E_1' \subseteq E_1$ and $E_2' \subseteq E_2$.*

6. FC-normal nonmonotonic rule systems

In [18], Marek, Remmel, and Nerode defined a generalization of Reiter's normal default theories [28] for nonmonotonic rule systems called FC-normal default theories. This allows one to define analogues of normal default theories in all the nonmonotonic reasoning formalisms mentioned in Section 3. Indeed, when FC-normal rule systems are translated back to default logic, one obtains a larger class of default theories, called extended normal default theories, which strictly contains the class of normal default theories but still has all the desirable properties of default theories.

In this section we shall define FC-normal annotated nonmonotonic rule systems and state the analogues of the results about such systems proved in [18]. We shall delay the proofs of our results until Section 7.

Definition 2. Let $\langle U, \mathscr{P}, N \rangle$ be a nonmonotonic rule system. We say that a subset $Con \subseteq \mathscr{P}(U \times P)$ (where $\mathscr{P}(U \times P)$ is the power set of $U \times P$) is a *consistency property* over $\mathscr{S} = \langle U, \mathscr{P}, N \rangle$ if
1. $\emptyset \in Con$,
2. $(\forall A, B \subseteq U)(A \subseteq B \ \&\ Con(B) \Rightarrow Con(A))$,
3. $(\forall A \subseteq U)(Con(A) \Rightarrow Con(cl_{\text{down}}(A)))$
4. $(\forall A \subseteq U)(Con(A) \Rightarrow Con(cl_{\text{mon}}(A)))$, and
5. whenever $\mathscr{A} \subseteq Con$ has the property that $(\forall A, B \in \mathscr{A})\ (\exists C \in \mathscr{A})(A \subseteq C \wedge B \subseteq C)$, then $Con(\bigcup \mathscr{A})$.

Condition (1) says that the empty set is consistent. Condition (2) requires that a subset of a consistent set is also consistent. Condition (3) says that the downward closure of a consistent set is consistent. Condition (4) postulates that the closure of a consistent set under monotonic rules is consistent. Finally, the last condition says that the union of a *directed* family of consistent sets is also consistent. We note that conditions (1), (2) and (5) are Scott's conditions for information systems. Condition (4) connects "consistent" sets to the monotonic part of the rule system; if A is consistent, then adding elements derivable from A via monotonic rules preserves "consistency".

Definition 3. Let $\mathscr{S} = \langle U, \mathscr{P}, N \rangle$ be an annotated nonmonotonic rule system and let *Con* be a consistency property over $\langle U, \mathscr{P}, N \rangle$.

1. A rule

$$r = \frac{(a_1, p_1), \ldots, (a_n, p_n) \colon (b_1, q_1), \ldots, (b_k, q_k)}{(c, r)} \in nmon(\mathscr{S})$$

is *FC-normal* (with respect to *Con*) if $Con(V \cup \{(c,r)\})$ and not $Con(V \cup \{(c,r), (b_i, q_i)\})$ for all $i \leqslant k$ whenever $V \subseteq U \times P$ is such that

(a) $Con(V)$, (b) $cl_{\text{down}}(V) = V$, (c) $cl_{\text{mon}}(V) = V$,

(d) $(a_1, p_1), \ldots, (a_n, p_n) \in V$, and

(e) $(c,r), (b_1, q_1), \ldots, (b_k, q_k) \notin V$.

2. $\mathscr{S} = \langle U, \mathscr{P}, N \rangle$ is an *FC-normal* (with respect to *Con*) nonmonotonic rule system if all $r \in nmon(\mathscr{S})$ are *FC*-normal with respect to *Con*.

3. $\langle U, \mathscr{P}, N \rangle$ is an *FC-normal annotated nonmonotonic rule system* if for some consistency property $Con \subseteq \mathscr{P}(U)$, $\langle U, \mathscr{P}, N \rangle$ is *FC*-normal with respect to *Con*.

Example 3. Let $\mathscr{P}$ consists of $P = \{0, 0.1, 0.2, \ldots, 0.9, 1\}$ under the usual ordering and let $U \times P = \{(x, p) \colon x \in \{a, b, c, d, e, f\} \ \& \ p \in P\}$. Let the consistency property be defined by the following condition:
$A \notin Con$ if and only if either $\{(c, 0.7), (d, 0.8)\} \subseteq cl_{\text{down}}(A)$ or $\{(e, 0.7), (f, 0.8)\} \subseteq cl_{\text{down}}(A)$.

Now consider the following set of rules, N:

(1) $\dfrac{:}{(a, 0.9)}$

(2) $\dfrac{(c, 0.8) :}{(b, 0.9)}$

(3) $\dfrac{(b, 0.8) :}{(c, 0.9)}$

(4) $\dfrac{(a, 0.8) : (d, 0.8)}{(c, 0.9)}$

(5) $\dfrac{(c, 0.8) : (f, 0.8)}{(e, 0.7)}$

Then for the annotated nonmonotonic rule system $\mathscr{S} = \langle U, \mathscr{P}, N \rangle$, rules (1)–(3) form the monotonic part of $\mathscr{S}$ and rules (4) and (5) form the nonmonotonic part of $\mathscr{S}$. First it is easy to check that *Con* is a consistency property over $\mathscr{S}$. One can also easily check that rules (4) and (5) are *FC*-normal with respect to *Con*. For example, for rule (4), suppose that $V \subseteq U \times P$ is in *Con* and $cl_{\text{down}}(cl_{\text{mon}}(V)) = V$ and $(a, 0.8) \in V$ but $(d, 0.8)$ and $(c, 0.7)$ are not in V. It is then easy to see that adding $(c, 0.9)$ to V can not cause both $(c, 0.7)$ and $(d, 0.8)$ to appear in $cl_{\text{down}}(V \cup \{(c, 0.9)\})$ unless $(d, 0.8)$ was already contained in V which we are explicitly assuming is not the case.

Similarly adding $(c,0.9)$ to V can not cause both $(e,0.7)$ and $(f,0.8)$ to appear in $cl_{\mathrm{down}}(V \cup \{(c,0.9)\})$ unless they were both in V which is not the case since V is consistent. Thus, $V \cup \{(c,0.9)\}$ is in *Con* but $V \cup \{(c,0.9),(d,0.8)\}$ is not in *Con* so that rule (4) is *FC*-normal with respect to *Con*. Thus, $\mathscr{S}$ is *FC*-normal annotated rule system with respect to *Con*.

One can easily check that $\mathscr{S}$ does have an extension, in fact, it has only one extension M, where

$$M = \{(a,p)\colon\ p \leqslant 0.9\} \cup \{(b,p)\colon\ p \leqslant 0.9\} \cup \{(c,p)\colon\ p \leqslant 0.9\} \cup \{(e,p)\colon\ p \leqslant 0.7\}.$$

If we add to N the rule $(c,0.7)\colon/(d,0.9)$ to get a set of rules N', then *Con* is no longer a consistency property over $\mathscr{S}' = \langle U,\mathscr{P},N'\rangle$ because $\{(c,p)\colon\ p \leqslant 0.7\} \in Con$ but the downward and monotonic closure of $\{(c,p)\colon\ p \leqslant 0.7\}$ relative to $\mathscr{S}' = \langle U,\mathscr{P},N'\rangle$ contains both $(c,0.7)$ and $(d,0.8)$ and hence is not in *Con*.

If we add the rule $(e,0.7)\colon (f,0.9)/(d,0.9)$ to N to form a new NRS $\mathscr{S}'' = \langle U,\mathscr{P},N''\rangle$, *Con* will still be a consistency property over $\mathscr{S}'' = \langle U,\mathscr{P},N''\rangle$ because the property of being a consistency property depends only on the monotonic part of the rule system. However, $\mathscr{S}''$ is not *FC*-normal with respect to *Con* because $r = (e,0.7)\colon (f,0.9)/(d,0.9)$ is not *FC*-normal with respect to *Con*. That is, for the downward and monotonically closed set $C = \{(x,p)\colon\ x \in \{a,b,c,e\}\ \&\ p \leqslant 0.9\}$, we have $prem(r) \subseteq C$, $cons(r) \cap C = \emptyset$, but $cl_{\mathrm{down}}(cl_{\mathrm{mon}}(\{c(r)\} \cup C))$ contains both $(c,0.7)$ and $(d,0.8)$ and hence is not in *Con*.

FC-normal annotated nonmonotonic rule systems have all the desirable properties that are possessed by normal default theories as defined by Reiter in [28]. We next shall state the basic results about *FC*-normal annotated nonmonotonic rule systems.

Theorem 11. *Let $\mathscr{S} = \langle U,\mathscr{P},N\rangle$ be an FC-normal annotated nonmonotonic rule system with respect to consistency property Con, then there exists an extension of $\mathscr{S}$.*

Theorem 12. *Let $\mathscr{S} = \langle U,\mathscr{P},N\rangle$ be an FC-normal annotated nonmonotonic rule system with respect to consistency property Con and let I be a subset of U such that $I \in Con$. Then there exists an extension I' of $\mathscr{S}$ such that $I \subseteq I'$.*

In fact, all extensions of *FC*-normal annotated nonmonotonic rule systems can be constructed via a forward chaining type construction which we shall call the normal forward chaining construction. The more general forward chaining construction of [19] can also be adapted to annotated nonmonotonic rule systems but we do not have the space to give the details in this paper. To this end, let $\mathscr{S} = \langle U,\mathscr{P},N\rangle$ be an *FC*-normal annotated nonmonotonic rule system and fix some well-ordering $\prec$ of $nmon(\mathscr{S})$. That is, the well-ordering $\prec$ determines some listing of the rules of $nmon(\mathscr{S}), \{r_\alpha\colon\ \alpha \in \gamma\}$ where γ is some ordinal. Let Θ_γ be the least cardinal such that $\gamma \leqslant \Theta_\gamma$. In what follows, we shall assume that the ordering among ordinals is given by $\in$. Our normal forward chaining construction will define an increasing sequence of sets $\{E_\alpha^{\prec}\}_{\alpha\in\Theta_\gamma}$. We will then define $E^{\prec}$ to be the downward closure of $\bigcup_{\alpha\in\Theta_\gamma} E_\alpha^{\prec}$.

The normal forward chaining construction of $E^{\prec}$.

Case 0: Let $E_0^{\prec} = cl_{\text{mon}}(\emptyset)$.

Case 1: $\alpha = \eta + 1$ is a successor ordinal.

Given $E_\eta^{\prec}$, let $\ell(\alpha)$ be the least $\lambda \in \gamma$ such that

$$r_\lambda = \frac{(a_1,p_1),\dots,(a_n,p_n):(b_1,q_1),\dots,(b_k,q_k)}{(c,s)}$$

where $(a_1,p_1),\dots,(a_n,p_n) \in cl_{\text{down}}(E_\eta^{\prec})$ and $(b_1,q_1),\dots,(b_k,q_k)$, $(c,s) \notin cl_{\text{down}}(E_\eta^{\prec})$. If there is no such $\ell(\alpha)$, then let $E_{\eta+1}^{\prec} = E_\alpha^{\prec} = E_\eta^{\prec}$. Otherwise, let

$$E_{\eta+1}^{\prec} = E_\alpha^{\prec} = cl_{\text{mon}}(E_\eta^{\prec} \cup \{cln(r_{\ell(\alpha)})\}).$$

Case 2: α is a limit ordinal. Then let $E_\alpha^{\prec} = \bigcup_{\beta\in\alpha} E_\beta^{\prec}$.

This given, we have the following:

Theorem 13. *If $\mathcal{S} = \langle U, \mathcal{P}, N\rangle$ is an FC-normal annotated nonmonotonic rule system and $\prec$ is* any *well-ordering of nmon($\mathcal{S}$), then*

1. *$E^{\prec}$ is an extension of $\mathcal{S}$.*
2. *(Completeness of the construction). Every extension of $\mathcal{S}$ is of the form $E^{\prec}$ for a suitably chosen ordering $\prec$ of nmon($\mathcal{S}$).*

It is quite straightforward to prove by induction that if $\mathcal{S} = \langle U, \mathcal{P}, N\rangle$ is FC-normal with respect to consistency property *Con*, then $E_\alpha^{\prec} \in Con$ for all α and hence $E^{\prec} \in$ *Con*. Thus the following is an immediate consequence of part 2 of Theorem 13.

Corollary 1. *Let $\mathcal{S} = \langle U, \mathcal{P}, N\rangle$ be an FC-normal annotated nonmonotonic rule system with respect to consistency property Con, then every extension of $\mathcal{S}$ is in Con.*

We should also point out that if we restrict ourselves to countable FC-normal annotated nonmonotonic rules systems $\mathcal{S} = \langle U, \mathcal{P}, N\rangle$, i.e. if U and N are countable, and $\mathcal{P}$ is countable, then we can restrict ourselves to orderings of order type ω where ω is the order type of the natural numbers. That is, suppose we fix some well-ordering $\prec$ of *nmon*($\mathcal{S}$) of order type ω. Thus, the well-ordering $\prec$ determines some listing of the rules of *nmon*($\mathcal{S}$), $\{r_n : n \in \omega\}$. Our normal forward chaining construction can be presented in an even more straightforward manner in this case. Our construction again will define an increasing sequence of sets $\{E_n^{\prec}\}_{n\in\omega}$ in stages. This given, we will then define $E^{\prec} = cl_{\text{down}}(\bigcup_{n\in\omega} E_n^{\prec})$.

The countable normal forward chaining construction of $E^{\prec}$.

Stage 0: Let $E_0^{\prec} = cl_{\text{mon}}(\emptyset)$.

Stage $n+1$: Let $\ell(n+1)$ be the least $s \in \omega$ such that

$$r_s = \frac{(a_1,p_1),\ldots,(a_n,p_n) : (b_1,q_1),\ldots,(b_k,q_k)}{(c,t)}$$

where $(a_1,p_1),\ldots,(a_n,p_n) \in cl_{\mathrm{down}}(E_n^{\prec})$ and $(b_1,q_1),\ldots,(b_k,q_k)$, $(c,t) \notin cl_{\mathrm{down}}(E_n^{\prec})$. If there is no such $\ell(n+1)$, then let $E_{n+1}^{\prec} = E_n^{\prec}$. Otherwise, let

$$E_{n+1}^{\prec} = cl_{\mathrm{mon}}(E_n^{\prec} \cup \{cln(r_{\ell(n+1)})\}).$$

This given, we then have the following:

Theorem 14. *If* $\mathcal{S} = \langle U, \mathcal{P}, N\rangle$ *is a countable FC-normal annotated nonmonotonic rule system where* $\mathcal{P}$ *is countable, then*

1. $E^{\prec}$ *is an extension of* $\mathcal{S}$ *if* $E^{\prec}$ *is constructed via the countable normal forward chaining algorithm with respect to* $\prec$, *where* $\prec$ *is* any *well-ordering of* $nmon(\mathcal{S})$ *of order type* ω.

2. (*Completeness of the construction.*) *Every extension of* $\mathcal{S}$ *is of the form* $E^{\prec}$ *for a suitably chosen well-ordering* $\prec$ *of* $nmon(\mathcal{S})$ *of order type* ω *where* $E^{\prec}$ *is constructed via the countable normal forward chaining algorithm.*

Example 4. Consider the *FC*-normal annotated nonmonotonic rule system $\mathcal{S} = \langle U, \mathcal{P}, N\rangle$ of Example 1 where we order $nmon(\mathcal{S})$ by declaring that $(4) \prec (5)$. Then the stages of the countable normal forward chaining algorithm for $\mathcal{S}$ are the following.

Stage 0: $E_0^{\prec} = cl_{\mathrm{mon}}(\emptyset) = \{(a,0.9)\}$.

Stage 1: $r_1 = (4)$ and $E_1^{\prec} = cl_{\mathrm{mon}}(E_0^{\prec} \cup \{(c,0.9)\}) = \{(a,0.9),(c,0.9),(b,0.9)\}$.

Stage 2: $r_2 = (5)$ and $E_2^{\prec} = cl_{\mathrm{mon}}(E_1^{\prec} \cup \{(e,0.7)\}) = \{(a,0.9),(c,0.9),(b,0.9),(e,0.7)\}$.

Stage 3: r_3 is undefined so the construction stops.

Thus $E^{\prec} = cl_{\mathrm{down}}(E_2^{\prec}) = \{(a,p)\colon\ p \leqslant 0.9\} \cup \{(b,p)\colon\ p \leqslant 0.9\} \cup \{(c,p)\colon\ p \leqslant 0.9\} \cup \{(e,p)\colon\ p \leqslant 0.7\}$.

It is easy to check that if $\prec'$ is the ordering where $(5) \prec' (4)$, the countable normal foward chaining algorithm would give the same result. However suppose we form a new annotated rule system $\mathcal{S}' = \langle U, \mathcal{P}, N'\rangle$ by adding the two rules listed below.

(6) $\dfrac{(c,0.7) : (e,0.7)}{(f,0.9)}$

(7) $\dfrac{(a,0.7) : (c,0.7)}{(d,1)}$

Again it is not difficult to check that $\mathcal{S}'$ is *FC*-normal with respect to *Con*. However in this case, there is more than one extension of $\mathcal{S}'$ so that order will make a difference in the foward chaining construction. For example, suppose that we order the rules by

$$(4) \prec (5) \prec (6) \prec (7).$$

Then the stages of the foward chaining construction are exactly the same. That is, at stage (3), rule (6) is blocked because $(e,0.7) \in cl_{\mathrm{down}}(E_2^{\prec})$ and rule (7) is blocked because $(c, 0.7) \in cl_{\text{down}}(E_2^{\prec})$. However if we order the rules by

$$(7) \prec' (6) \prec' (5) \prec' (4),$$

then we have the following stages.

Stage 0: $E_0^{\prec'} = cl_{\text{mon}}(\emptyset) = \{(a, 0.9)\}$.

Stage 1: $r_1 = (7)$ and $E_1^{\prec'} = cl_{\text{mon}}(E_0^{\prec'} \cup \{(d, 1)\}) = \{(a, 0.9), (d, 1)\}$.

Stage 2: r_2 is undefined so the construction stops.

Thus, $E^{\prec'} = cl_{\text{down}}(E_1^{\prec'}) = \{(a, p)\colon\ p \leqslant 0.9\} \cup \{(d, p)\colon\ p \leqslant 1\}$. However if use the ordering

$$(4) \prec'' (6) \prec'' (5) \prec'' (7),$$

then we have the following stages.

Stage 0: $E_0^{\prec''} = cl_{\text{mon}}(\emptyset) = \{(a, 0.9)\}$.

Stage 1: $r_1 = (4)$ and $E_1^{\prec''} = cl_{\text{mon}}(E_0^{\prec''} \cup \{(c, 0.9)\}) = \{(a, 0.9), (c, 0.9), (b, 0.9)\}$.

Stage 2: $r_2 = (6)$ and $E_2^{\prec''} = cl_{\text{mon}}(E_1^{\prec''} \cup \{(f, 0.9)\}) = \{(a, 0.9), (c, 0.9), (b, 0.9), (f, 0.9)\}$.

Stage 3: r_3 is undefined so the construction stops.

Thus $E^{\prec} = cl_{\text{down}}(E_2^{\prec}) = \{(a, p)\colon\ p \leqslant 0.9\} \cup \{(b, p)\colon\ p \leqslant 0.9\} \cup \{(c, p)\colon\ p \leqslant 0.9\} \cup \{(f, p)\colon\ p \leqslant 0.9\}$.

One can check that these are the only three extensions in this case.

We note that in the case where $\mathscr{S}$ is finite, we can show that the Countable Normal Forward Chaining algorithm for $\mathscr{S}$ runs in polynomial time w.r.t. the sum of the lengths of the rules in N where the length of a rule is the length of the string that codes the rule in some finite alphabet.

FC-normal NRS's also possesses what Reiter terms the "semi-monotonicity" property.

Theorem 15. *Let $\mathscr{S}_1 = \langle U, \mathscr{P}, N_1 \rangle$ and $\mathscr{S}_2 = \langle U, \mathscr{P}, N_2 \rangle$ be two FC-normal annotated NRS such that $N_1 \subseteq N_2$ but $mon(\mathscr{S}_1) = mon(\mathscr{S}_2)$. Assume, in addition, that both are FC-normal with respect to the same consistency property. Then for every extension E_1 of $\mathscr{S}_1$, there is an extension E_2 of $\mathscr{S}_2$ such that $E_1 \subseteq E_2$.*

FC-normal NRS's also satisfy the orthogonality of extensions property with respect to their consistency property.

Theorem 16. *Let $\mathscr{S} = \langle U, \mathscr{P}, N \rangle$ be an FC-normal annotated NRS with respect to a consistency property Con. Then if E_1 and E_2 are two distinct extensions of $\mathscr{S}$, $E_1 \cup E_2 \notin Con$.*

Our next theorem gives a sufficient condition for when an *FC*-normal annotated rule systems has a unique extension.

Theorem 17. *Let $\mathscr{S} = \langle U, \mathscr{P}, N\rangle$ be an FC-normal annotated NRS with respect to a consistency property Con. Suppose that*

$$cl_{\text{down}}(cl_{\text{mon}}(\{cln(r)\colon\ r \in nmon(\mathscr{S})\}))$$

is in Con. Then $\mathscr{S}$ has a unique extension.

We end this section with two more results which are also analogues of the results of Reiter's [28]. We say that $(\varphi, c) \in U \times P$ has a consistent proof scheme with respect to a consistency property *Con* over $\mathscr{S} = \langle U, \mathscr{P}, N\rangle$ if and only if there is a proof scheme

$$p = \langle\langle(\varphi_0, a_0), r_0, G_0\rangle, \dots, \langle(\varphi_m, a_m), r_m, G_m\rangle\rangle \tag{16}$$

such that $(\varphi_m, a_m) = (\varphi, c)$ and $\{(\varphi_0, a_0), \dots, (\varphi_m, a_m)\} \in Con$. We then have the following.

Theorem 18. *Let $\mathscr{S} = \langle U, \mathscr{P}, N\rangle$ be an FC-normal annotated NRS with respect to a consistency property Con. Then $(\varphi, c) \in U \times P$ is an element of some extension of $\mathscr{S}$ if and only if (φ, c) has a consistent proof scheme with respect to Con.*

Theorem 19. *Suppose $\mathscr{S} = \langle U, \mathscr{P}, N\rangle$ is an FC-normal annotated NRS and that $D \subseteq nmon(\mathscr{S})$. Suppose further that E_1' and E_2' are distinct extensions of $\langle U, \mathscr{P}, D \cup mon(\mathscr{S})\rangle$. Then $\mathscr{S}$ has distinct extensions E_1 and E_2 such that $E_1' \subseteq E_1$ and $E_2' \subseteq E_2$.*

7. Proofs of general results on *FC*-normal nonmonotonic rule systems

In this section, we shall give the proof of the results stated in Section 6. From now on we assume all *FC*-normal nonmonotonic rule systems have the consistency property given by *Con*.

Theorem 11. *Every FC-normal annotated nonmonotonic rule system has an extension.*

Proof. We shall show that our forward chaining construction will always produce an extension. Thus fix some well-ordering $\prec$ of $nmon(\mathscr{S})$. Our well-ordering $\prec$ determines some listing of the rules of $nmon(\mathscr{S}), \{r_\alpha\colon\ \alpha \in \gamma\}$, where γ is some ordinal. Let Θ_γ be the least cardinal such that $\gamma \leqslant \Theta_\gamma$. In what follows, we shall assume that the ordering among ordinals is given by $\in$. Recall that our forward chaining construction defines an increasing sequence of sets $\{E_\alpha^\prec\}_{\alpha \in \Theta_\gamma}$ as follows.

The normal forward chaining construction of $E^\prec$.

Case 0: Let $E_0^\prec = cl_{\text{mon}}(\emptyset)$.

Case 1: $\alpha = \eta + 1$ is a successor ordinal. Given $E_\eta^\prec$, let $\ell(\alpha)$ be the least $\lambda \in \gamma$ such that

$$r_\lambda = \frac{(a_1, p_1), \ldots, (a_n, p_n) : (b_1, q_1), \ldots, (b_k, q_k)}{(c, s)}$$

where $(a_1, p_1), \ldots, (a_n, p_n) \in cl_{\text{down}}(E_\eta^\prec)$ and $(b_1, q_1), \ldots, (b_k, q_k)$, $(c, s) \notin cl_{\text{down}}(E_\eta^\prec)$. If there is no such $\ell(\alpha)$, then let $E_{\eta+1}^\prec = E_\alpha^\prec = E_\eta^\prec$. Otherwise, let

$$E_{\eta+1}^\prec = E_\alpha^\prec = cl_{\text{mon}}(E_\eta^\prec \cup \{cln(r_{\ell(\alpha)})\}).$$

Case 2: α is a limit ordinal. Then let $E_\alpha^\prec = \bigcup_{\beta \in \alpha} E_\beta^\prec$. Then, we let $E^\prec = cl_{\text{down}}(\bigcup_{\alpha \in \Theta_\gamma} E_\alpha^\prec)$.

It is straightforward to prove by (transfinite) induction that $Con(E_\alpha^\prec)$ holds for all $\alpha \in \Theta_\gamma$ and hence $Con(E^\prec)$ holds. Next we want to prove by (transfinite) induction that $E_\alpha^\prec \subseteq C_{E^\prec}(\emptyset)$ for all $\alpha \in \Theta_\gamma$. If $\alpha = 0$, then clearly $E_0^\prec = cl_{\text{mon}}(\emptyset) \subseteq C_{E^\prec}(\emptyset)$. Suppose α is a successor ordinal and $\eta + 1 = \alpha$. Assume by induction that $E_\eta^\prec \subseteq C_{E^\prec}(\emptyset)$. Then if $E_\eta^\prec \neq E_{\eta+1}^\prec$, there exists one rule

$$r_{\ell(\eta+1)} = \frac{(a_1, p_1), \ldots, (a_n, p_n) : (b_1, q_1), \ldots, (b_k, q_k)}{(c, s)}$$

where $(a_1, p_1), \ldots, (a_n, p_n) \in cl_{\text{down}}(E_\eta^\prec)$ and $(b_1, q_1), \ldots, (b_k, q_k)$, $(c, s) \notin cl_{\text{down}}(E_\eta^\prec)$, and $E_{\eta+1}^\prec = cl_{\text{mon}}(E_\eta \cup \{(c, s)\})$. But since $r_{\ell(\eta+1)}$ is *FC*-normal, we know that $E_\eta^\prec \cup \{(c, s), (b_i, q_i)\}$ is not consistent for all $i \leqslant k$. Since $E^\prec$ is consistent, it must be the case that $E_\eta^\prec \cup \{(c, s), (b_i, q_i)\} \not\subseteq E^\prec$ for all $i \leqslant k$ since subsets of consistent sets are consistent. Thus for all $i \leqslant k$, $(b_i, q_i) \notin E^\prec$. Hence, $r_{\ell(\eta+1)}$ shows that $(c, s) \in C_{E^\prec}(\emptyset)$. But then $E_\eta^\prec \cup \{(c, s)\} \subseteq C_{E^\prec}(\emptyset)$ from which it easily follows that $cl_{\text{mon}}(E_\eta^\prec \cup \{(c, s)\}) = E_{\eta+1}^\prec \subseteq C_{E^\prec}(\emptyset)$. For α a limit, we can assume by induction that $E_\beta^\prec \subseteq C_{E^\prec}(\emptyset)$ for all $\beta \in \alpha$ and so $E_\alpha^\prec = \bigcup_{\beta \in \alpha} E_\beta^\prec \subseteq C_{E^\prec}(\emptyset)$. Thus we have shown $E_\beta^\prec \subseteq C_{E^\prec}(\emptyset)$ for all $\beta \in \Theta_\gamma$ and hence $E^\prec = cl_{\text{down}}(\bigcup_{\beta \in \Theta_\gamma} E_\beta^\prec) \subseteq C_{E^\prec}(\emptyset)$.

To prove that $C_{E^\prec}(\emptyset) \subseteq E^\prec$, we proceed by induction on the length of minimal proof schemes. That is, suppose that if p is a minimal proof scheme of length $\leqslant m$ such that $\text{supp}(p) \cap E^\prec = \emptyset$, then $cln(p) \in E^\prec$. Now suppose $q = \langle\langle(\alpha_0, a_0), r_{\eta_0}, G_0\rangle, \ldots, \langle(\alpha_m, a_m), r_{\eta_m}, G_m\rangle\rangle$ is a minimal proof scheme of length $m+1$ where $G_m \cap E^\prec = \emptyset$. Note that since $E^\prec$ is downward closed it follows that $G_m \cap cl_{\text{down}}(E_\alpha^\prec) = \emptyset$ for all α. Moreover, by induction $(\alpha_0, a_0), \ldots, (\alpha_{m-1}, a_{m-1}) \in E^\prec$ and hence $(\alpha_0, a_0), \ldots, (\alpha_{m-1}, a_{m-1}) \in cl_{\text{down}}(E_\alpha^\prec)$ for some $\alpha \in \Theta_\gamma$. Suppose

$$r_{\eta_m} = \frac{(\alpha_{i_0}, e_0), \ldots, (\alpha_{i_s}, e_s) : (\beta_1, f_1), \ldots, (\beta_k, f_k)}{(\alpha_m, g_{\eta_m})}$$

where $i_0 < \cdots < i_s < m$, for all $j \leqslant s$, $a_j \geqslant e_j$, and $(\beta_1, f_1), \ldots, (\beta_k, f_k) \notin E^\prec$, and $g_{\eta_m} \geqslant a_m$. Now it is easy to see that our construction ensures that if $r_{\ell(\eta+1)}$ is defined, then $cln(r_{\ell(\eta+1)}) \notin E_\eta^\prec$. Hence if $\lambda \neq \eta$ and $r_{\ell(\lambda)}$ and $r_{\ell(\eta)}$ are defined, then $r_{\ell(\lambda)} \neq r_{\ell(\eta)}$. Thus, the function $\ell(.)$ is one-to-one on its domain. Now suppose that $(\alpha_m, g_{\eta_m}) \notin E^\prec$. Then for all $\lambda + 1$ greater than α, r_{η_m} is a candidate to be $r_{\ell(\lambda+1)}$ at stage $\lambda + 1$. Hence, it must be the case that $r_{\ell(\lambda+1)}$ is defined and $\ell(\lambda+1) \in \eta_m$. But this is impossible. That is, if Θ_γ is infinite, then the cardinality of $\{r_{\ell(\lambda+1)}: \eta_m \in \lambda \in \Theta_\gamma\}$ is equal to the cardinality of Θ_γ which is strictly greater than the cardinality of $\{\delta : \delta \in \eta_m\}$. Similary if Θ_γ is finite, then the fact that $r_{\ell(\lambda)}$ is defined for all $\lambda \leqslant \Theta_\gamma$ and $\ell()$ is one-to-one would mean that the cardinality of $\{r_{\ell(\lambda)}: \lambda \in \Theta_\gamma\}$ is equal to the cardinality of $nmon(\mathcal{S})$ so that every rule $r \in nmon(\mathcal{S})$ must be equal to $r_{\ell(\lambda)}$ for some λ. Thus in either case we have shown that if $(\alpha_m, g_{\eta_m}) \notin E^{\prec}$, then for some $\mu \in \Theta_\gamma$, r_{η_μ} is the least rule

$$r = \frac{(\delta_1, d_1), \ldots, (\delta_s, d_s) : (\gamma_1, c_1), \ldots, (\gamma_m, c_m)}{(\psi, f)}$$

such that $(\delta_1, d_1), \ldots, (\delta_s, d_s) \in cl_{\text{down}}(E^{\prec}_\mu)$ and $(\gamma_1, c_1), \ldots, (\gamma_m, c_m), (\psi, f) \notin cl_{\text{down}}(E^{\prec}_\mu)$. But then by construction $(\alpha_m, g_{\eta_m}) \in E^{\prec}_{\mu+1} \subseteq E^{\prec}$. Thus (α_m, g_{η_m}) must be in $E^{\prec}$. But since $E^{\prec}$ is downward closed, then $(\alpha_m, a_m) \in E^{\prec}$. Hence $C_{E^{\prec}}(\emptyset) \subseteq E^{\prec}$ and $E^{\prec}$ is an extension as claimed. □

Note that the proof of Theorem 11, remains unchanged if instead of starting with $cl_{\text{mon}}(\emptyset)$ at stage 0, we start with $cl_{\text{mon}}(I)$, where $I \in Con$. Thus we also have the following.

Theorem 12. *Let $\mathcal{S} = \langle U, \mathcal{P}, N \rangle$ be an FC-normal annotated nonmonotonic rule system with respect to consistency property Con. Let I be a subset of U such that $I \in Con$. Then there exists an extension I' of $\mathcal{S}$ such that $I \subseteq I'$.*

Next we want to show that every extension of an FC-normal NRS $\mathcal{S} = \langle U, \mathcal{P}, N \rangle$ can be constructed by our forward chaining construction relative to an appropriate ordering of the $nmon(\mathcal{S})$.

Theorem 13. *If $\mathcal{S} = \langle U, \mathcal{P}, N \rangle$ is an FC-normal annotated NRS and $\prec$ is* any *well-ordering of $nmon(\mathcal{S})$, then:*

(1) *$E^{\prec}$ is an extension of $\mathcal{S}$.*

(2) (*completeness of the construction*). *Every extension of $\mathcal{S}$ is of the form $E^{\prec}$ for a suitably chosen ordering $\prec$ of $nmon(\mathcal{S})$.*

Proof. (1) Follows from our proof of Theorem 11.

(2) We prove the following fact:

Let F be an extension of an FC-normal annotated NRS $\mathcal{S} = \langle U, \mathcal{P}, N \rangle$. Let $\mu = \text{card}(NG(F, \mathcal{S}))$ and let $\prec$ be some well-ordering of $nmon(\mathcal{S})$ such that the listing of $nmon(\mathcal{S})$ determined by $\prec$, $\{r_\alpha: \alpha \in \gamma\}$, is such that $\mu \leqslant \gamma$ and $NG(F, \mathcal{S}) = \{r_\alpha: \alpha \in \mu\}$. Then

(i) $F = cl_{\text{down}}(cl_{\text{mon}}(\{cln(r): r \in NG(F, \mathcal{S})\}))$ and

(ii) $F = E^{\prec}$ where $E^{\prec}$ is constructed by our forward chaining construction.

For (i), note that for each

$$r = \frac{(\alpha_0, a_0), \ldots, (\alpha_n, a_n) \colon (\beta_1, b_1), \ldots, (\beta_m, b_m)}{(\varphi, c)} \in NG(F, \mathscr{S}),$$

$cln(r) \in F$. Moreover for any set $W \subseteq C_F(\emptyset)$, $cl_{\text{down}}(cl_{\text{mon}}(W)) \subseteq C_F(\emptyset)$ so that $cl_{\text{down}}(cl_{\text{mon}}(\{cln(r) \colon r \in NG(F, \mathscr{S})\})) \subseteq C_F(\emptyset)$. Then a straightforward induction on the length of a minimal proof scheme p will show that if $\text{supp}(p) \cap F = \emptyset$, then $cln(p) \in cl_{\text{down}}(cl_{\text{mon}}(\{cln(r) : r \in NG(F, \mathscr{S})\}))$. It then follows that $C_F(\emptyset) = cl_{\text{down}}(cl_{\text{mon}}(\{cln(r) \colon r \in NG(F, \mathscr{S})\}))$.

For (ii), let $\{E_\alpha^{\prec} : \alpha \in \Theta_\gamma\}$ be constructed by the forward chaining construction relative to the well-ordering of rules $\{r_\alpha \colon \alpha \in \gamma\}$. Then we claim $cl_{\text{down}}(E_\mu^{\prec}) = F$ and $E_\alpha^{\prec} = E_\mu^{\prec}$ for $\alpha > \mu$. First it is easy to show by induction that $cl_{\text{mon}}(E_\alpha^{\prec}) = E_\alpha^{\prec}$ for all α. Next we claim that if $\alpha \in \mu$, then $E_\alpha^{\prec} \subseteq F$ and moreover if $E_\alpha^{\prec} \neq E_{\alpha+1}^{\prec}$, then $\ell(\alpha + 1) \in \mu$. That is, $E_0^{\prec} = cl_{\text{mon}}(\emptyset) \subseteq F$. Next suppose by induction that $E_\beta^{\prec} \subseteq F$ for all $\beta \in \alpha$. Then if α is a limit ordinal, $E_\alpha^{\prec} = \bigcup_{\beta \in \alpha} E_\beta^{\prec} \subseteq F$. If α is a successor ordinal, we can assume by induction that $E_\eta^{\prec} \subseteq F$ where $\eta + 1 = \alpha$. Now consider $E_\eta^{\prec}$. If $cl_{\text{down}}(E_\eta^{\prec}) = F$, then for any rule

$$r = \frac{(\alpha_0, a_0), \ldots, (\alpha_n, a_n) \colon (\beta_1, b_1), \ldots, (\beta_m, b_m)}{(\varphi, c)}$$

in $nmon(\mathscr{S})$, it must be the case that either $(\varphi, c) \in cl_{\text{down}}(E_\eta^{\prec})$, $\{(\beta_1, b_1), \ldots, (\beta_m, b_m)\} \cap cl_{\text{down}}(E_\eta^{\prec}) \neq \emptyset$, or $\{(\alpha_0, a_0), \ldots (\alpha_n, a_n)\} \not\subseteq cl_{\text{down}}(E_\eta^{\prec})$ since F is an extension. That is, if $cl_{\text{down}}(E_\eta^{\prec}) = F$, then $\ell(\eta + 1)$ must be undefined and hence $E_\eta^{\prec} = E_{\eta+1}^{\prec}$. If $cl_{\text{down}}(E_\eta^{\prec}) \neq F$, then consider some $(\varphi, c) \in F - cl_{\text{down}}(E_\eta^{\prec})$. Since $(\varphi, c) \in F$, there is some minimal proof scheme

$$p = \langle \langle (\alpha_0, a_0), \bar{r}_0, G_0 \rangle, \ldots, \langle (\alpha_m, a_m), \bar{r}_m, G_m \rangle \rangle$$

where $(\alpha_m, a_m) = (\varphi, c)$ and $G_m \cap F = \emptyset$ which witnesses that $(\varphi, c) \in F$. Since $(\varphi, c) \notin cl_{\text{down}}(E_\eta^{\prec})$, there must be some $k < m$ such that $(\alpha_0, a_0), \ldots, (\alpha_{k-1}, a_{k-1}) \in cl_{\text{down}}(E_\eta^{\prec})$ and $(\alpha_k, a_k) \notin cl_{\text{down}}(E_\eta^{\prec})$. But then $\bar{r}_k$ must be of the form

$$\bar{r}_k = \frac{(\alpha_{i_0}, e_{i_0}), \ldots, (\alpha_{i_j}, e_{i_j}) : (\beta_1, b_1), \ldots, (\beta_t, b_t)}{(\alpha_k, f_k)}$$

where $i_0 < \cdots < i_j < k$, for all $h \leqslant j$, $e_{i_h} \leqslant a_{i_h}$, and $f_k \geqslant a_k$. Now it cannot be that $\{(\beta_1, b_1), \ldots, (\beta_t, b_t)\} = \emptyset$ since otherwise $(\alpha_k, f_k) \in cl_{\text{mon}}(E_\eta^{\prec}) = E_\eta^{\prec}$ which would imply that $(\alpha_k, a_k) \in cl_{\text{down}}(E_\eta^{\prec})$. Thus $\{(\beta_1, b_1), \ldots, (\beta_t, b_t)\} \neq \emptyset$. But since $\{(\beta_1, b_1), \ldots, (\beta_t, b_t)\} \subseteq G_m$ and $G_m \cap F = \emptyset$, it must be the case that $\{(\beta_1, b_1), \ldots, (\beta_t, b_t)\} \cap cl_{\text{down}}(E_\eta^{\prec}) = \emptyset$ and $\{(\beta_1, b_1), \ldots, (\beta_t, b_t)\} \cap F = \emptyset$. Hence $\bar{r}_k \in NG(F, \mathscr{S})$ and $\bar{r}_k$ is a candidate to be $r_{\ell(\eta+1)}$. But this means that if $\bar{r}_k = r_\beta$ in our ordering of rules in $nmon(\mathscr{S})$, then $\ell(\eta + 1) \leqslant \beta < \mu$. But for any $\delta \in \mu$, $cln(r_\delta) \in F$ by our choice of our well-ordering. Thus $cln(r_{\ell(\eta+1)}) \in F$ so that $E_\eta^{\prec} \cup \{cln(r_{\ell(\eta+1)})\} \subseteq F$ and hence $cl_{\text{mon}}(E_\eta^{\prec} \cup \{cln(r_{\ell(\eta+1)})\}) = E_{\eta+1}^{\prec} = E_\alpha^{\prec} \subseteq F$.

It follows that $E_\mu^{\prec} \subseteq F$ since $E_\mu = \bigcup_{\alpha \in \mu} E_\alpha^{\prec}$ and $E_\alpha^{\prec} \subseteq F$ for all $\alpha \in \mu$. We claim that it must be the case that $cl_{\text{down}}(E_\mu^{\prec}) = F$ for otherwise $cl_{\text{down}}(E_\mu^{\prec}) \subset F$ and hence for all $\alpha \in \mu$, $cl_{\text{down}}(E_\alpha^{\prec}) \subset F$. But our argument above shows that if $cl_{\text{down}}(E_\alpha^{\prec}) \subset F$, then $E_\alpha^{\prec} \subset E_{\alpha+1}^{\prec}$ and $\ell(\alpha+1) \in \mu$. This fact, in turn, will allow us to prove, by induction on the length of a minimal proof scheme, that for all $r \in NG(F, \mathscr{S})$, $cln(r) \in cl_{\text{down}}(E_\mu^{\prec})$. That is, suppose $(\varphi, c) = cln(r)$ for some $r \in NG(F, \mathscr{S})$. Now $(\varphi, c) = cln(p)$ for some minimal proof scheme $p = \langle\langle(\alpha_0, a_0), r_0, G_0\rangle, \ldots, \langle(\alpha_m, a_m), r_m, G_m\rangle\rangle$ where $G_m \cap F = \emptyset$ and $(\alpha_m, a_m) = (\varphi, c)$. Now assume by induction that all (ψ, d) such that $(\psi, d) = cln(r)$ for some $r \in NG(F, \mathscr{S})$ and (ψ, d) is the conclusion of some minimal proof scheme q such that $\text{supp}(q) \cap F = \emptyset$ and length of $q \leqslant m$ are in $E_\eta^{\prec}$. Note that each r_k for $k \leqslant m$ is in $NG(F, \mathscr{S})$ since p shows $(\alpha_0, a_0), \ldots, (\alpha_m, a_m) \in F$ and $cons(r_k) \subseteq G_m$ where $G_m \cap F = \emptyset$. It follows that each (α_i, a_i) for $i < m$ is in $cl_{\text{down}}(E_\mu^{\prec})$ by our induction hypothesis. But then $\{(\alpha_0, a_0), \ldots, (\alpha_{m-1}, a_{m-1})\} \subseteq cl_{\text{down}}(E_\mu^{\prec})$. So consider r_m. Now if $r_m \in mon(\mathscr{S})$, then r_m is of the form

$$\frac{(\alpha_{i_0}, e_{i_0}), \ldots, (\alpha_{i_j}, e_{i_j}) :}{(\varphi, d)}$$

where $i_0 < \cdots i_j < m$, $a_{i_h} \geqslant e_{i_h}$ for $h = 1, \ldots, j$ and $d \geqslant c$. But since $(\alpha_i, a_i) \in cl_{\text{down}}(E_\mu^{\prec})$ for $i < m$, it follows that for each $i < m$ there is a $(\alpha_i, f_i) \in E_\mu^{\prec}$ with $f_i \geqslant a_i$. But then r_m shows that $(\varphi, d) \in cl_{\text{mon}}(E_\mu^{\prec}) = E_\mu^{\prec}$ and hence $(\varphi, c) \in cl_{\text{down}}(E_\mu^{\prec})$. If $r_m \in NG(F, \mathscr{S})$, then $r_m = r_\xi$ in our orderings of rules where $\xi < \mu$ and r_m is of the form

$$\frac{(\alpha_{i_0}, e_{i_0}), \ldots, (\alpha_{i_j}, e_{i_j}) : (\beta_1, b_1), \ldots, (\beta_k, b_k)}{(\varphi, d)}$$

where $i_0 < \cdots i_j < m$, $a_{i_h} \geqslant e_{i_h}$ for $h = 1, \ldots, j$, $(\beta_1, b_1), \ldots, (\beta_k, b_k) \notin F$, and $d \geqslant c$. Thus there is some $\lambda \in \mu$ such that $\{(\alpha_0, a_0), \ldots, (\alpha_{m-1}, a_{m-1})\}$ is contained in $E_\lambda^{\prec}$. But then for any $\lambda \leqslant \delta \leqslant \mu$, if $(\varphi, c) \notin cl_{\text{down}}(E_\delta^{\prec})$, then r_ξ is a possible candidate to be $r_{\ell(\delta+1)}$. Hence it must be that case that $r_{\ell(\delta+1)}$ is defined and $\ell(\delta+1) \in \xi$. But this is impossible. That is, if μ is infinite, then the cardinality of $\{r_{\ell(\delta+1)} : \lambda \in \delta \in \mu\}$ is equal to the cardinality of μ which is strictly greater than the cardinality of $\{\alpha : \alpha \in \xi\}$. Similary if μ is finite, then the fact that $r_{\ell(\lambda)}$ is defined for all $\lambda \leqslant \mu$ and $\ell()$ is one-to-one would mean that the cardinality of $\{r_{\ell(\lambda)} : \lambda \leqslant \mu\}$ is equal to the cardinality of μ so that every rule r_δ with $\delta \leqslant \mu$ must be equal to $r_{\ell(\lambda)}$ for some $\lambda \leqslant \mu$. Thus in either case we have shown that if $(\varphi, c) \notin E_\mu^{\prec}$, then for some $\delta \leqslant \mu$, $r_\xi = r_{\ell(\delta)}$. But then by construction $(\varphi, d) \in E_\delta^{\prec} \subseteq E_\mu^{\prec}$. Thus (φ, c) must be in $cl_{\text{down}}(E_\mu^{\prec})$. Thus we have shown that $\{cln(r) : r \in NG(F, \mathscr{S})\} \subseteq cl_{\text{down}}(E_\mu^{\prec})$ if $E_\mu^{\prec} \subset F$. But this is a contradiction since by (i), $F = cl_{\text{down}}(cl_{\text{mon}}(\{cln(r) : r \in NG(F, \mathscr{S})\})) \subseteq cl_{\text{down}}(cl_{\text{mon}}(cl_{\text{down}}(E_\mu^{\prec}))) = cl_{\text{down}}(cl_{\text{mon}}(E_\mu^{\prec})) = cl_{\text{down}}(E_\mu^{\prec})$. Thus it must be the case that $cl_{\text{down}}(E_\mu^{\prec}) = F$.

Note we have already shown that if $cl_{\text{down}}(E_\alpha^{\prec}) = F$, then $E_\alpha^{\prec} = E_{\alpha+1}^{\prec}$. Thus since $cl_{\text{down}}(E_\mu^{\prec}) = F$, it easily follows that $E_\lambda^{\prec} = E_\mu^{\prec}$ for all $\mu \leqslant \lambda \leqslant \Theta_\gamma$. Hence, $E^{\prec} = cl_{\text{down}}(\bigcup_{\alpha \in \Theta_\gamma} E_\alpha) = cl_{\text{down}}(E_\mu^{\prec}) = F$ as claimed. □

Since every $E^{\prec}$ is in Con, we immediately get the following corollary.

Corollary 1. *Let $\mathscr{S} = \langle U, \mathscr{P}, N\rangle$ be an FC-normal annotated nonmonotonic rule system with respect to consistency property Con, then every extension of $\mathscr{S}$ is in Con.*

Next we show that if our *FC*-normal annotated nonmonotonic rule system $\langle U, \mathscr{P}, N\rangle$ is countable, i.e. if U and $\mathscr{P}$ are countable which automatically implies that N is countable, then every extension of $\langle U, \mathscr{P}, N\rangle$ can be constructed via the countable forward chaining construction relative to some well-ordering $\prec$ of $nmon(\langle U, \mathscr{P}, N\rangle)$ of the order type of ω.

Theorem 14. *If $\mathscr{S} = \langle U, \mathscr{P}, N\rangle$ is a countable FC-normal annotated nonmonotonic rule system, then*

1. *$E^{\prec}$ constructed via the countable forward chaining construction with respect to $\prec$, where $\prec$ is any* any *well-ordering of $nmon(\mathscr{S})$ of order type ω, is an extension of $\mathscr{S}$.*
2. *(completeness of the construction). Every extension of $\mathscr{S}$ is of the form $E^{\prec}$ for a suitably chosen ordering $\prec$ of $nmon(\mathscr{S})$ of order type ω where $E^{\prec}$ is constructed via the countable forward chaining construction.*

Proof. We note that if $\prec$ is a well-ordering of $nmon(\mathscr{S})$ of order type ω, the countable forward chaining algorithm is just the first ω steps of the forward chaining algorithm. Thus to prove (1), we must show that if we construct $E^{\prec}$ with respect to the forward chaining algorithm, then $E^{\prec}_{\omega} = E^{\prec}_{\lambda}$ for all $\lambda \geqslant \omega$. In fact, we need only show that $E^{\prec}_{\omega} = E^{\prec}_{\omega+1}$. Now suppose that

$$r_{\omega+1} = \frac{(a_1, p_1), \ldots, (a_n, p_n) : (b_1, q_1), \ldots, (b_k, q_k)}{(\psi, c)}$$

is defined. Thus $(a_1, p_1), \ldots, (a_n, p_n) \in cl_{\text{down}}(E^{\prec}_{\omega})$ and $(b_1, q_1), \ldots, (b_k, q_k)$, $(c, s) \notin cl_{\text{down}}(E^{\prec}_{\omega})$ Moreover, $r_{\ell(\omega+1)} = r_q$ for some q where $\{r_n\}_{n\in\omega}$ is the ordering of rules determined by $\prec$. But since $E^{\prec}_{\omega} = \bigcup_{n\in\omega} E^{\prec}_n$, there must be some s such that $(\alpha_1, p_0), \ldots, (\alpha_n, p_n) \in cl_{\text{down}}(E^{\prec}_s)$. Hence for all $t \geqslant s$, $(\beta_1, b_1), \ldots, (\beta_k, b_k), (\psi, c) \notin cl_{\text{down}}(E^{\prec}_t)$ so that r_q is candidate to be $r_{\ell(t)}$ for all $t > s$. Since the function $\ell(t)$ is one-to-one, it easily follows that there would have to be some finite t such that $r_q = r_{\ell(t)}$. Thus, $r_{\ell(\omega+1)}$ must not be defined and hence $E^{\prec}_{\omega} = E^{\prec}$.

Next we consider the proof of (2). Note that if we apply the proof of Theorem 13 to F in this case the most natural thing to do is to order the rules of $NG(F, \mathscr{S})$ first, say $NG(F, \mathscr{S}) = \{s_0, s_1, \ldots\}$, and then follow this ordering by listing all the rules of $nmon(\mathscr{S}) - NG(F, \mathscr{S}) = \{t_0, t_1, \ldots\}$. Now if $NG(F, \mathscr{S})$ is finite, then our listing of rules determines a well-ordering $\prec$ of order type ω in which case the proof of Theorem 13 shows that $F = E^{\prec}$. If $NG(F, \mathscr{S})$ is infinite, then our listing of rules determines a well-ordering $\prec$ of order type $\omega + \omega$. It then follows from the proof of Theorem 13 that

$$E^{\prec}_0 \subseteq E^{\prec}_1 \subseteq \cdots \subseteq E^{\prec}_{\omega} = E^{\prec}_{1+\omega} = \ldots.$$

and that $cl_{\text{down}}(E_\omega^\prec) = F$. The key point to note is that for any

$$r = \frac{(a_1, p_1), \ldots, (a_n, p_n) : (b_1, q_1), \ldots, (b_k, q_k)}{(\psi, c)}$$

which is not in $NG(F, \mathcal{S})$, it must be the case that $\{(b_1, q_1), \ldots, (b_k, q_k)\} \cap F \neq \emptyset$ or $\{(a_1, p_1), \ldots, (a_n, p_n)\} \not\subseteq F$. But since $F = cl_{\text{down}}(\bigcup_{i \in \omega} E_i^\prec)$, it also follows that either

(a) for some i, $\{(b_1, q_1), \ldots, (b_k, q_k)\} \cap cl_{\text{down}}(E_i^\prec) \neq \emptyset$ or

(b) for all j, $\{(a_1, p_1), \ldots, (a_n, p_n)\} \not\subseteq cl_{\text{down}}(E_j^\prec)$.

In case (a) if we insert r between $s_{h(i)}$ and $s_{h(i)+1}$ where $h(i) = max(\{\ell(j) \colon j \leqslant i\})$, then this change will have no affect on the construction of the $E_i^\prec$'s. That is, the construction of $E^\prec$ up to stage i can depend only on $s_0, \ldots, s_{h(i)}$ and hence we will get the same sets, $E_j^\prec$ for $j \leqslant i$, for any ordering which starts out $s_0, \ldots, s_{h(i)}$. Thus if we take the ordering $s_0, \ldots, s_{h(i)}, r, s_{1+h(i)}, \ldots$, then because $\{(b_1, q_1), \ldots, (b_k, q_k)\} \cap E_i^\prec \neq \emptyset$, r is not a candidate to be $r_{\ell(k)}$ for any $k > i$ and hence the insertion of r does not effect the rest of the construction of $E^\prec$. In case (b), we can insert r anywhere in the initial ω part of the list and it will have no effect on the construction of the $E_i^\prec$'s for $i \in \omega$ because the premises of r are never contained in any $cl_{\text{down}}(E_i^\prec)$. In this way, we can see that it is possible to interweave all the r's in $nmon(\mathcal{S}) - NG(F, \mathcal{S})$ into the basic ordering $s_0, s_1, \ldots$ so as to create an ordering of order type ω but with out changing the sequence $E_0^\prec, E_1^\prec, \ldots$. Thus it will still be the case that $F = cl_{\text{down}}(E_\omega^\prec)$ so that $E_\omega^\prec = E_{\omega+1}^\prec = \cdots$. Hence it will still be the case that $F = E^\prec$ □

Theorem 15 follows immediately from the following result:

Theorem 15 (Semi-monotonicity). *Suppose $\mathcal{S} = \langle U, \mathcal{P}, N \rangle$ is an FC-normal annotated NRS. Let $D \subseteq nmon(\mathcal{S})$. Then*

1. *$\mathcal{S}' = \langle U, \mathcal{P}, mon(\mathcal{S}) \cup D \rangle$ is FC-normal annotated NRS and*
2. *if E' is an extension of $\mathcal{S}'$, then there is an extension E of $\mathcal{S}$ such that*
   (a) *$E' \subseteq E$ and*
   (b) *$NG(E', \mathcal{S}') \subseteq NG(E, \mathcal{S})$*

Proof. The fact that $\mathcal{S}'$ is an FC-normal annotated NRS is an immediate consequence of our definitions. For part (ii), let μ equal the cardinality of $NG(E', \mathcal{S}')$ and choose a well-ordering of $NG(E', \mathcal{S}')$, $\{r_\alpha \colon \alpha \in \mu\}$. Then extend this well-ordering to a well-ordering $\{r_\alpha \colon \alpha \in \gamma\}$ of nmon(S). It follows that if $E^\prec$ is constructed via our forward chaining algorithm with respect to the well ordering $\prec$ determined by $\{r_\alpha : \alpha \in \gamma\}$, then proof of Theorem 13 shows $E' = cl_{\text{down}}(E_\mu^\prec)$ so that $E' \subseteq E^\prec$.

It remains to prove that $NG(E', \mathcal{S}') \subseteq NG(E^\prec, \mathcal{S})$. Now suppose

$$r = \frac{(a_1, p_1), \ldots, (a_n, p_n) : (b_1, q_1), \ldots, (b_k, q_k)}{(\psi, c)} \in NG(E', \mathcal{S}').$$

Then $\{(a_1, p_1), \ldots, (a_n, p_n)\} \subseteq E' \subseteq E^\prec$ and $\{(b_1, q_1), \ldots, (b_k, q_k)\} \cap E' = \emptyset$. But note that $Con(E')$ holds since $E' = cl_{\text{down}}(E_\mu^\prec)$. By Theorem 13 $E' = cl_{\text{down}}(cl_{\text{mon}}(\{cln(r) \colon r \in NG(E', \mathcal{S}')\}))$. Thus $(\psi, c) \in E'$ and hence by the FC-normality of r, $E' \cup \{(\psi, c),$

$(\beta_i, b_i)\}$ is not consistent for any $i = 1, \ldots, k$. But since $E^{\prec}$ is consistent, $E' \cup \{(\psi, c), (\beta_i, b_i)\} \not\subseteq E^{\prec}$ for any $i = 1, \ldots, k$. Hence $(\beta_i, b_i) \notin E^{\prec}$ for all $i = 1, \ldots, k$ and $r \in NG(E^{\prec}, \mathscr{S})$. □

We prove now the result on the orthogonality of extensions.

Theorem 16 (Orthogonality of extensions). *Let $\mathscr{S} = \langle U, \mathscr{P}, N\rangle$ be an FC-normal annotated NRS with respect to a consistency property Con. Then if E and F are two distinct extensions of $\mathscr{S}$, $E \cup F \notin Con$.*

Proof. By Theorem 13, $E = cl_{\mathrm{down}}(\bigcup_{\alpha\in\Theta_\gamma} E^{\prec}_{\alpha})$ where $\{E^{\prec}_{\alpha}\}_{\alpha\in\Theta_\gamma}$ is the sequence constructed by the forward chaining construction relative to some well ordering $\prec$ of $nmon(S)$. Let α be the least ordinal such that $cl_{\mathrm{down}}(E^{\prec}_{\alpha}) \subset F$ but $cl_{\mathrm{down}}(E^{\prec}_{\alpha+1}) \not\subset F$. Note there must be such an α since otherwise $E \subseteq F$ and then by the minimality of extensions, $E = F$. Thus the rule

$$r_{\ell(\alpha+1)} = \frac{(a_1, p_1), \ldots, (a_n, p_n) : (b_1, q_1), \ldots, (b_k, q_k)}{(\psi, c)}$$

is such that $\{(a_1, p_1), \ldots, (a_n, p_n)\} \subseteq cl_{\mathrm{down}}(E^{\prec}_{\alpha})$, $\emptyset \neq \{(b_1, q_1), \ldots, (b_k, q_k)\}$, $\{(b_1, q_1), \ldots, (b_k, q_k)\} \cap cl_{\mathrm{down}}(E^{\prec}_{\alpha}) = \emptyset$ and $E^{\prec}_{\alpha+1} = cl_{\mathrm{mon}}(E^{\prec}_{\alpha} \cup \{(\psi, c)\})$. Since $cl_{\mathrm{down}}(E^{\prec}_{\alpha+1}) \not\subseteq F$, it must be that $(\psi, c) \notin F$. But this means that $(\beta_i, b_i) \in F$ for some i since otherwise $r_{\ell(\alpha+1)} \in NG(F, \mathscr{S})$ which would imply that $(\psi, c) \in F$ because F is an extension. By the FC-normality of $r_{\ell(\alpha+1)}$, $cl_{\mathrm{down}}(E^{\prec}_{\alpha}) \cup \{(\psi, c), (\beta_i, b_i)\}$ is not consistent. But since $cl_{\mathrm{down}}(E^{\prec}_{\alpha}) \cup \{(\psi, c), (\beta_i, b_i)\} \subseteq E \cup F$, $E \cup F$ is also not consistent. □

Theorem 17. *Suppose $\mathscr{S} = \langle U, \mathscr{P}, N\rangle$ is an FC-normal annotated NRS with respect to consistency property Con such that $cl_{\mathrm{mon}}(\{cln(r)\colon r \in nmon(\mathscr{S})\})$ is in Con. Then $\mathscr{S}$ has a unique extension.*

Proof. For a contradiction, assume $\mathscr{S}$ has two distinct extensions, E_1 and E_2. Then by our proof of Theorem 13 , $E_i = cl_{\mathrm{down}}(cl_{\mathrm{mon}}(\{cln(r)\colon r \in NG(E_i, \mathscr{S})\}))$ for $i = 1, 2$. But then for $i = 1, 2$, $E_i \subseteq cl_{\mathrm{down}}(cl_{\mathrm{mon}}(\{cln(r) : r \in nmon(\mathscr{S})\}))$. Thus $E_1 \cup E_2$ is contained in a consistent set so that $E_1 \cup E_2$ is consistent, contradicting Theorem 16. □

Theorem 18. *Let $\mathscr{S} = \langle U, \mathscr{P}, N\rangle$ be an FC-normal annotated NRS with respect to a consistency property Con. Then $(\varphi, c) \in U \times P$ is an element of some extension of $\mathscr{S}$ if and only if (φ, c) has a consistent proof scheme with respect to Con.*

Proof. Clearly if $(\varphi, c) \in E$ where E is an extension, then $Con(E)$ by Corollary 1. Thus, since $(\varphi, c) \in C_E(\emptyset)$, there is a consistent minimal proof scheme for (φ, c).

Conversely assume that $p = \langle\langle(\varphi_0, a_0), r_0, G_0\rangle, \ldots, \langle(\varphi_m, a_m), r_m, G_m\rangle\rangle$ is a consistent minimal proof scheme for (φ, c). Let $0 \leqslant i_1 < \cdots < i_k \leqslant m$ be set of all $i \leqslant m$ such that $r_i \in nmon(\mathscr{S})$. Now well-order $nmon(\mathscr{S})$ so that $r_{i_1}, \ldots, r_{i_k}$ are the first k elements in the list. Then if we construct an extension via our forward chaining construction, it is easy to show by induction on k that $(\varphi, c) \in E_k^{\prec}$. Hence $(\varphi, c) \in cl_{\text{down}}(E^{\prec})$ which is an extension. □

Theorem 19. *Suppose $\mathscr{S} = \langle U, \mathscr{P}, N \rangle$ is an FC-normal annotated NRS and that $D \subseteq nmon(\mathscr{S})$. Suppose further that E_1' and E_2' are distinct extensions of $\langle U, \mathscr{P}, D \cup mon(\mathscr{S}) \rangle$. Then $\mathscr{S}$ has distinct extensions E_1 and E_2 such that $E_1' \subseteq E_1$ and $E_2' \subseteq E_2$.*

Proof. By Theorem 18, we know that there are extensions of $\mathscr{S}$, E_1 and E_2, such that $E_1' \subseteq E_1$ and $E_2' \subseteq E_2$. But then the orthogonality of extensions for $\langle U, \mathscr{P}, D \cup mon(\mathscr{S}) \rangle$ ensures $E_1' \cup E_2'$ is not consistent. Hence $E_1 \cup E_2$ is not consistent so that $E_1 \neq E_2$. □

8. Conclusion

In this paper, we introduced the theory of annotated nonmonotonic rule systems which forms a common generalization of the nonmonotonic rule systems of Marek [15–19] and annotated logic programming paradigm of Subrahmanian [30, 31]. Annotated nonmontonic rules system provide a general framework for nonmonotonic reasoning systems which include probabilistic reasoning, uncertainty measurements, and time, place, origin or quality dependencies. We also introduced a generalization of Reiter's normal default theories in the setting of annotated nonmontonic rule systems called FC-normal annotated nonmontonic rule systems. We showed that FC-normal annotated nonmontonic rule systems have all the desirable properties that are possessed by normal default theories. Finally we introduced a forward chaining type construction to construct all extensions of an FC-normal annnotated nonmontonic rule system.

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