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The Wave Equation with Computable Initial Data Such That Its Unique Solution Is Not Computable

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We consider the three-dimensional wave equation. It is well known that the solution $u(x, y, z, t)$ is uniquely determined by two initial conditions: the values of u and $\partial u/\partial t$ at time $t = 0$. Our question is, can computable initial data give rise to noncomputable solutions? The answer is "yes," and two quite different types of noncomputability can occur. Theorem 1 below gives an example in which the solution $u(x, y, z, t)$ takes a noncomputable real value at a computable point in space-time. By contrast, Theorem 2 provides an example in which the solution maps each computable sequence of points in space-time into a computable sequence: nevertheless $u(x, y, z, t)$ is not a computable function. [We note in passing that "computability," as used in this paper, is a technical term familiar to mathematical logicians: the precise definitions are spelled out in the next section.] The proof of Theorem 1 is quite short. That for Theorem 2 is considerably more intricate.

It should be mentioned that these noncomputable solutions of the wave equation are of the type commonly referred to as "weak solutions"—i.e., although continuous, they are not twice differentiable at all points. Weak solutions describe creases, cusps and other non-differentiable patterns which frequently appear in models of wave phenomena. The use of weak solutions is inevitable; we will prove that no C^2 solutions of the desired kind exist. For the convenience of readers unfamiliar with weak solutions, we have put them into a coherent framework in an Addendum, basing our presentation on the "energy integral" associated with the wave. (Just as the initial conditions are computable, the energy integral in our examples is also a computable real.) Finally, the proof that all noncomputable solutions must be of "weak" type is sketched at the end of this addendum. We note, however, that the main part of the paper may be read without reference to the addendum.

The results in this paper are related to comments of Kreisel. In [4] Kreisel asks whether existing physical theories—e.g., classical mechanics or quantum mechanics—can predict theoretically the existence of a physical constant which is not a recursive real. Previous work of the authors in this area [9] was concerned with ordinary differential equations: it was proved that there exists a computable—and hence continuous—function F such that

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$dy/dx = F(x, y)$ has no computable solution in any rectangle however small within its domain. In the present paper, by passing to partial differential equations, we obtain similar results with an equation which is more familiar.

The plan of the paper is as follows. In Section 1 we present some preliminaries—including the standard, commonly accepted definitions of computability for real numbers and functions. In Section 2 we give the proofs of the main theorems—Theorems 1 and 2. Associated with the proof of Theorem 2 is the "Effective Modulus Lemma," a result which may be of independent interest. The generalization of these results to spaces of dimension other than three is presented in Section 3. The paper concludes with the aforementioned addendum on weak solutions.

Since our results overlap two areas—recursion theory and analysis—we have attempted to make our paper self-contained, and have presented our arguments with great attention to detail. The few exceptions, where we merely give "sketches," are clearly indicated.

1. Preliminaries

In this section we summarize some facts of recursive analysis which will be used in the paper. The reader is advised to glance briefly at this section and return to it when necessary.

We take as known the idea of a recursive function from the set of natural numbers $\mathbb{N}$ into itself, or from $\mathbb{N}^q \to \mathbb{N}$. Then a sequence $\{r_n\}$ of rational numbers is called "recursive" or *computable* if there exist three recursive functions a, b, s from $\mathbb{N}$ to $\mathbb{N}$ such that

$$r_n = (-1)^{s(n)} \frac{a(n)}{b(n)}.$$

Similarly we can define a computable double or triple sequence of rationals.

A real number x is called *computable* if there exists a computable sequence of rationals $\{r_n\}$ which converges *effectively* to x; this means that:

There is a recursive function $e(n)$ such that

$$k \geqq e(n) \qquad \text{implies} \qquad |x - r_k| \leqq 10^{-n}.$$

Then we can use the function $e(n)$ to construct a computable subsequence $\{r'_n\} = \{r_{e(n)}\}$ such that

$$|x - r'_n| \leqq 10^{-n}.$$

In the same vein, a sequence of real numbers $\{x_k\}$ is called *computable* if there is a computable double sequence of rationals $\{r_{kn}\}$ such that

$$|x_k - r_{kn}| \leqq 10^{-n} \qquad \text{for all } k, n.$$

A vector $(x_1, \ldots, x_q) \in \mathbb{R}^q$ or a sequence of vectors in $\mathbb{R}^q$ is called *computable* if each of its components is a computable real number or sequence of real numbers, respectively.

Now we come to the notion of a computable function $f: \mathbb{R}^q \to \mathbb{R}^1$. For our purposes, we can restrict attention to the case where f is defined on a closed bounded rectangle $I^q = \{a_i \leqq x_i \leqq b_i, 1 \leqq i \leqq q\}$ in $\mathbb{R}^q$; we assume that the endpoints a_i, b_i are computable reals. Here several equivalent definitions have been given. Perhaps the best, from a foundational point of view, is the "recursive functional" definition of Grzegorczyk [1, 2]. We also have the "effective polynomial approximation" approach of Pour-El and Caldwell [8]. A definition which is equivalent to these, and which is useful in applications, is the following:

A function f from a computable closed bounded rectangle I^q in $\mathbb{R}^q$ into $\mathbb{R}^1$ is called *computable* if:

(a) f is *sequentially computable*, i.e. for every computable sequence $\{\mathbf{x}_k\}$ of points in I^q, the sequence of values $\{f(\mathbf{x}_k)\}$ is computable; and

(b) f is *effectively uniformly continuous*, i.e., there exists a recursive function $d(n)$ such that, for all points $\mathbf{x}, \mathbf{y} \in I^q$:

$$|\mathbf{x} - \mathbf{y}| \leqq 1/d(n) \qquad \text{implies} \qquad |f(\mathbf{x}) - f(\mathbf{y})| \leqq 10^{-n}.$$

In an obviously analogous way, we define the notion of a *computable sequence* of functions $\{f_k\}$. Previously we defined "effective" convergence for a sequence of rationals converging to a computable real. The same definition applies to sequences of reals, and—with the obvious modifications—to a sequence of functions $\{f_k\}$ which converges *effectively and uniformly* to a limit function f. It is a basic result that:

(*) If a sequence of functions $\{f_k\}$ is computable, and converges effectively and uniformly to a limit f, then f is computable (cf. [1, 2]).

We will also use the obvious fact that Riemann integration, applied to functions satisfying (a) and (b) above, is an effective process.

So far we have defined computability only for functions defined on compact rectangles $I^q \subsetneqq \mathbb{R}^q$. There is a natural extension of this definition to functions defined on all of $\mathbb{R}^q$ (cf. [1, 2]). However, we do not need it here, because all of the functions we construct will have compact support, and solutions of the wave equation propagate with a finite velocity. So we omit these complications.

We conclude by stating a couple of well-known facts from recursion theory.

A set A of natural numbers is called *recursively enumerable* if there is a recursive function $a(n)$ which enumerates A; we can always assume that the function $a(n)$ is one to one. The set is called *recursive* if both it and its complement in $\mathbb{N}$ are recursively enumerable. It is a standard fact of recursion theory that there are recursively enumerable sets which are not recursive. Also there exist recursively inseparable pairs of sets A, B: this means that the sets A and B are recursively enumerable and disjoint, and there is no recursive set C such that $A \subsetneq C$ and $B \subsetneq \bar{C}$.

2. Main Theorems

We consider the three-dimensional wave equation

$$\frac{\partial^2 u}{dx^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = 0 \tag{1a}$$

with the initial conditions

$$\begin{aligned} u(x, y, z, 0) &= f(x, y, z), \\ \frac{\partial u}{\partial t}(x, y, z, 0) &= 0. \end{aligned} \tag{1b}$$

It will be the case that f is of class C^1. This implies, as will become clear below, that the solution $u(x, y, z, t)$ is continuous. [We note that u need not be C^2; i.e., u may be a "weak solution" of (1a, b). For a summary of the properties of weak solutions, see the Addendum below. In particular, we show there that the solution is unique.]

Our problem is: If the function f is assumed "computable" (in the precise sense of recursive analysis described in Section 1), is the solution u then also computable? The answer turns out to be "no." Furthermore, as noted in the Introduction, two quite different kinds of noncomputability can occur. Namely, we have:

Theorem 1. *There exists a computable—and hence continuous—function $f(x, y, z)$ such that the solution $u(x, y, z, t)$ of (1) is continuous but not computable, and furthermore the value $u(0, 0, 0, 1)$ is a noncomputable real number.*

Theorem 2. *There exists a computable function $f(x, y, z)$ such that the solution u of (1) is continuous, and*

(a) *$u(x, y, z, t)$ is sequentially computable, but*

(b) *$u(x, y, z, 1)$ is not a computable function of (x, y, z).*

Proof of Theorem 1. The initial function $f(x, y, z)$ will consist of an infinite sequence of "spherical waves" superimposed on one another. All of these waves are symmetrical about the origin. Their supports are a sequence of spherical shells $\{1-\varepsilon_k \leqq \rho \leqq 1+\varepsilon_k\}$, $\rho=(x^2+y^2+z^2)^{1/2}$, lying close to the unit sphere $\{\rho=1\}$. As we shall see, at time $t=1$ these waves are "focused" at the origin. Now, for any C^1 initial function f, the equation (1a, b) has a unique solution given by Kirchhoff's formula (cf. [7, p. 104]):

For brevity we write $\mathbf{x}=(x, y, z)$, and denote by "grad" the gradient in terms of the space variables (x, y, z); we let $\mathbf{n}$ be an arbitrary unit vector ranging over the unit sphere in $\mathbb{R}^3$, and denote by $d\sigma(\mathbf{n})$ the area measure on this sphere, normalized so that the total area equals 1. Then Kirchhoff's formula reads:

$$u(\mathbf{x}, t)=\iint_{\text{unit sphere}} [f(\mathbf{x}+t\mathbf{n})+t(\operatorname{grad} f)(\mathbf{x}+t\mathbf{n})\cdot\mathbf{n}]\, d\sigma(\mathbf{n}). \tag{2}$$

[In our proof it will be convenient to view Eq. (2) geometrically. Because of the variable $(\mathbf{x}+t\mathbf{n})$ which occurs in both terms, we are really averaging f and $(\operatorname{grad} f)\cdot\mathbf{n}$ over a sphere of radius t centered on the point $\mathbf{x}$. We note that the second term is multiplied by t.]

In our construction, the wave $f(\mathbf{x})$ will be a C^1 function which is spherically symmetric about the origin, $f(\mathbf{x})=f(\rho)$, where $\rho=(x^2+y^2+z^2)^{1/2}$. The gradient of such a function $f(\rho)$ at any point $\mathbf{x}=\rho\mathbf{n}$ ($\rho>0$, $\mathbf{n}=$ unit vector) is just

$$(\operatorname{grad} f)(\rho\mathbf{n})=f'(\rho)\,\mathbf{n}.$$

Hence, setting $\mathbf{x}=\mathbf{0}$ in Kirchhoff's formula (2) we obtain:

$$u(0,0,0,t)=f(t)+tf'(t),$$

since $f=f(t)$ is constant on the sphere $\{\rho=t\}$, and the term $t(\operatorname{grad} f)(t\mathbf{n})\cdot\mathbf{n}=tf'(t)\,\mathbf{n}\cdot\mathbf{n}=tf'(t)$ is also constant on this sphere. In particular, for $t=1$ we have

$$u(0,0,0,1)=f(1)+f'(1).$$

Thus to prove Theorem 1, all we need is to construct a computable function $f(\rho)$ such that $f'(\rho)$ is continuous but not computable, and f' takes a noncomputable value at $\rho=1$. An example of this type—but not fitting the conditions we need—was given by Myhill [6].

We construct a canonical "pulse function" $\varphi(x)$ which is C^∞ with support on $[-\frac{1}{2}, \frac{1}{2}]$, and such that $\varphi(x)\geqq 0$ for all x and $\varphi'(0)=1$. An example of such a function, which is computable together with all of its derivatives, is

$$\varphi(x)=(1+x)\,e^{-[x^2/(1-4x^2)]} \qquad \text{for } -\tfrac{1}{2}<x<\tfrac{1}{2},$$
$$=0 \qquad \text{elsewhere.}$$

Now we come to the heart of the construction. We take a one to one recursive function $a(k)$ which enumerates a nonrecursive set A (cf. Section 1). Then the sum of the series

$$\sigma = \sum_{k=0}^{\infty} 10^{-a(k)} = \sum_{a \in A} 10^{-a}$$

is a noncomputable real number. [This well-known fact is easy to see: if we had an effective method for computing σ, then we could tell exactly which numbers a belong to A, and the set A would be recursive.]

We define $f(\rho)$ as the superposition of an infinite sequence of waves $w_k(\rho)$:

$$f(\rho) = \sum_{k=0}^{\infty} w_k(\rho),$$

where (3)

$$w_k(\rho) = 10^{-(k+a(k))}\varphi[10^k(\rho - 1)].$$

The kth wave $w_k(\rho)$ is supported on the spherical shell $\{1 - 10^{-k} \leqslant \rho \leqslant 1 + 10^{-k}\}$, of mean radius 1 and half-width 10^{-k}, centered on the origin. (Actually, because φ has support on $[-\frac{1}{2}, \frac{1}{2}]$, the half-width is $\frac{1}{2} \cdot 10^{-k}$; this is significant only for $k = 0$, since we want $w_k(\rho)$ to vanish in a neighborhood of $\rho = 0$, where the spherical coordinates are singular.)

The sequence of functions $\{w_k\}$ is computable, and the series for f converges effectively and uniformly by comparison with $\sum 10^{-k}$. Hence by (*) in Section 1, $f(x, y, z) = f(\rho)$ is computable. Now for $f'(\rho)$ we have

$$\begin{aligned} f'(\rho) &= \sum_{k=0}^{\infty} w_k'(\rho) \\ &= \sum_{k=0}^{\infty} 10^{-a(k)}\varphi'[10^k(\rho - 1)], \end{aligned}$$

where the series converges uniformly (but not effectively) by comparison with $\sum 10^{-a(k)}$ (which must converge since the function $a(k)$ is one to one). Hence the function f is of class C^1.

Now a glance at Kirchhoff's formula shows that, if f is C^1, then the solution u is continuous. However, as we have seen, the value $u(0, 0, 0, 1) = f(1) + f'(1)$. Since f is computable, $f(1)$ is computable. But for $f'(1)$ we have

$$f'(1) = \sum_{k=0}^{\infty} 10^{-a(k)}\varphi'(0) = \sum_{k=0}^{\infty} 10^{-a(k)},$$

since $\varphi'(0) = 1$. Thus $f'(1)$ is equal to the noncomputable real number σ, and so $u(0, 0, 0, 1)$ is not computable. This proves Theorem 1.

Proof of Theorem 2. As in the proof of Theorem 1, we use an infinite sequence of spherical waves supported on thin spherical shells whose inner and outer radii approach one. However, here the *centers* of these spherical shells are not at the origin. They are situated on the x-axis at a sequence of points $\mathbf{r}_k = (r_k, 0, 0)$ which converge, but not effectively, to a noncomputable real point $\boldsymbol{\alpha} = (\alpha, 0, 0)$. At time $t = 1$, the kth wave is "focused" at the point r_k. The resulting solution is not effectively uniformly continuous, and hence not computable. However, it turns out that this solution is *sequentially* computable as a function of all four variables. Roughly speaking, sequential computability is possible because of the fact that the singularities $\mathbf{r}_k$ "pile up" on a noncomputable point $\boldsymbol{\alpha}$.

Despite the similarities, the proof of Theorem 2 is considerably more delicate than that of Theorem 1. We divide it into five parts. In the first part, we prove a result which we call "The Effective Modulus Lemma." This lemma embodies the key idea behind our construction. In it, we show that there exists a *noncomputable* real number α with the following property: for any computable sequence of real numbers $\{\gamma_k\}$, there exists a recursive function $d(k)$ such that the moduli $|\gamma_k - \alpha| > 1/d(k)$ for all k. The striking thing about this lemma is that there is no effective way to determine the *signs* of the numbers $(\gamma_k - \alpha)$. Indeed, if there were, then α would be computable. (To see this, take for $\{\gamma_k\}$ any recursive enumeration of the rationals.) Thus the differences $(\gamma_k - \alpha)$ can be effectively bounded away from zero, but we are uncertain about their sign.

In the second part of our proof, we construct the initial function $f(x, y, z)$. In the third part, we prove some properties of the corresponding solution $u(x, y, z, t)$. The fourth step is to show that $u(x, y, z, t)$ is sequentially computable. Finally, we show that $u(x, y, z, t)$ is not computable, even when we restrict t to the value $t = 1$.

EFFECTIVE MODULUS LEMMA. *There exists a computable sequence of rational numbers $\{r_m\}$ which converges (noneffectively) to a noncomputable real number α, and in addition has the following property: For any computable sequence of reals $\{\gamma_k\}$, there exist recursive functions $d(k)$ and $e(k)$ such that*

$$\textit{for all} \qquad m \geqq e(k),$$

$$|\gamma_k - r_m| > 1/d(k).$$

Proof. We begin with a recursively inseparable pair of sets of natural numbers A, B: i.e., the sets A, B are recursively enumerable and disjoint, but there is no recursive set C such that $A \subsetneq C$ and $B \subsetneq \bar{C}$. We can assume that $0 \notin A \cup B$. Let $a(n)$ and $b(n)$ be recursive functions which enumerate A and B in a one to one manner. Now we give the construction of the sequence $\{r_m\}$ and its limit α.

$$r_m = \tfrac{5}{9} + \sum_{n=0}^{m} 10^{-a(n)},$$

and

$$\alpha = \tfrac{5}{9} + \sum_{n=0}^{\infty} 10^{-a(n)}.$$

We observe that the decimal expansion of α is a sequence of 5's and 6's, with a 6 in the sth place if and only if $s \in A$.

The reason we choose for α a decimal built up only of 5's and 6's is quite simple. We wish to avoid the ambiguity between terminating decimals and decimals ending in an infinite string of 9's. Unfortunately, we still have to face this ambiguity for the numbers γ_k.

The number α is not a computable real; for if it were, there would be an effective test to determine where in its decimal expansion the 6's occur (cf. the preceding paragraph)—and then the set A would be recursive, a contradiction. Hence the sequence $\{r_m\}$ cannot converge effectively.

Now we must show that, given any preassigned computable sequence of reals $\{\gamma_k\}$, recursive functions $d(k)$ and $e(k)$ as described in the lemma exist. It is here that the other set B in our recursively inseparable pair is used. We must develop a procedure which is effective *uniformly in k*. However, we shall describe this procedure for a fixed one of the γ_k, but in such a way that the uniformity is obvious. Here is the procedure:

To show the main idea without tedious details, we will first postulate that none of the numbers γ_k has a terminating decimal expansion. (Unfortunately there is no effective way to check this.) We will treat the general case later. By our assumption, the $000 \cdots$ versus $999 \cdots$ ambiguity in decimal expansions does not occur. Hence the decimals $\gamma_k = N_0 \cdot N_1 N_2 N_3 \cdots$, $N_s = N_s(k)$, are effectively determined, uniformly in k: i.e., $N_s(k)$ is a recursive function of s and k. [As Mostowski showed [5], this would not be the case for an arbitrary computable sequence of reals γ_k. However, when it is known a-priori that none of the decimals terminate, then it is easy to see that $N_s(k)$ is recursive in both variables.]

Now we fix our attention on a particular γ_k. We begin listing the sets A and B in turn, using the recursive functions $a(n)$ and $b(n)$, until we come to an integer $s \in A \cup B$ such that either

(a) $s \in A$ and $N_s \neq 6$, or

(b) $s \in B$ and $N_s = 6$.

One of the situations (a) or (b) must eventually occur. For suppose it did not. Let C denote the set of integers $\{s \mid N_s = 6\}$. Since the function $N_s(k)$ is recursive, C is recursive, uniformly in k. Hence, since the sets A and B are recursively inseparable, we cannot have $A \subseteq C$ and $B \subseteq \bar{C}$. If $A \nsubseteq C$ then we have situation (a), and if $B \nsubseteq \bar{C}$ then we have (b). Furthermore, the above procedure can be carried out uniformly for the entire sequence $\{\gamma_k\}$, by an effective procedure which returns to each γ_k infinitely often.

Recall that the sth decimal digit for α is a 6 if and only if $s \in A$. Also A and B are disjoint. Hence in case (a) we have: the s-th decimal digit for α is a 6 and that for γ_k is not; in case (b) the situation is reversed. In either case, the sth decimal digits for α and γ_k differ. Since the decimal for α has only 5's and 6's, it follows that

$$|\gamma_k - \alpha| > 10^{-s-1}.$$

So we define

$$d(k) = 10^{s+1},$$

where $s = s(k)$ is the integer determined by the above process. To define the cut-off function $e(k)$, we note that $s \in A \cup B$, and hence either $s = a(n)$ or $s = b(n)$ for some $n = n(k)$, which is also effectively determined by the above construction. We set

$$e(k) = n.$$

Now instead of α, consider the sequence of partial sums $\{r_m\}$. Since $a(n)$ gives a one to one enumeration of A, and $A \cap B = \varnothing$, the sth decimal digit of r_m is determined as soon as either some $a(n)$ or some $b(n)$ equals s. (In the first case it is 6 and in the second case 5.) This coincides with the sth digit for α. Thus, as soon as $m \geqq n$, all of the above arguments for α apply just as well to r_m. We conclude that

$$m \geqq n = e(k) \qquad \text{implies} \qquad |\gamma_k - r_m| > 10^{-s-1} = 1/d(k).$$

This completes the proof for the case where no γ_k has a terminating decimal.

For the general case, instead of the "true" or exact decimal expansion $\gamma_k = N_0 \cdot N_1 N_2 \cdots$, we must use a sequence of finite decimal approximations. These approximations may have a different appearance than the true decimal (e.g., if the approximation is 0.500 ± 10^{-3}, the true decimal might be $0.499 \cdots$). Now we show that there exists for each k a sequence of decimal approximations

$$N_{q,0} \cdot N_{q,1} N_{q,2} \cdots N_{q,q+1} \qquad \text{to} \quad \gamma_k,$$

such that the digits $N_{q,s} = N_{q,s}(k)$ (where $s \leqq q + 1$) are recursive in q, s, and k, and the error

$$|(q\text{th decimal for } \gamma_k) - \gamma_k| \leqq 10^{-(q+1)}.$$

To show this: Since $\{\gamma_k\}$ is computable (cf. Section 1), there exists a computable double sequence of rationals $\{R_{k,q}\}$ such that $|R_{k,q} - \gamma_k| \leqq 10^{-(q+2)}$. Of course, the decimal expansion of a rational number is computable. We define the (qth decimal for γ_k) to be that decimal of length $q + 1$ which most closely approximates $R_{k,q}$; in case of ties, we take the smaller one. Then $|(q\text{th decimal for } \gamma_k) - R_{k,q}| \leqq (1/2)\, 10^{-(q+1)}$ and $|R_{k,q} - \gamma_k| \leqq (1/10)\, 10^{-(q+1)}$, so $|(q\text{th decimal for } \gamma_k) - \gamma_k| \leqq 10^{-(q+1)}$.

Now the previous construction (for the case of nonterminating γ_k) is modified by adding to the two cases (a) and (b) above, a third alternative. As before, we list the sets A and B in turn, using the recursive functions $a(n)$ and $b(n)$. We stop when we come to an integer $s \in A \cup B$ such that either:

(a) $s \in A$ and $N_{s,s} \neq 6$, $N_{s,s+1} \neq 0$ or 9, or

(b) $s \in B$ and $N_{s,s} = 6$, $N_{s,s+1} \neq 0$ or 9, or

(c) $s \in A \cup B$ and $N_{s,s+1} = 0$ or 9.

Again we can show that this process terminates. For if alternative (c) never occurs, then we are back in the previous situation, and our proof that either (a) or (b) must occur goes through as above.

As before, we define $d(k) = 10^{s+1}$ and $e(k) = n$, where $s = a(n)$ or $s = b(n)$. To prove that $m \geqq e(k)$ implies $|\gamma_k - r_m| > 1/d(k)$: When our procedure terminates in case (a) or case (b), the proof goes as before. [Since the $(s + 1)$st digit of (sth decimal for γ_k) is not a 0 or 9, the sth digit is "true"; and since the decimal for r_m contains only 5's and 6's, we get a difference $|\gamma_k - r_m| \geqq 10^{-(s+1)}$.] When our procedure terminates in case (c), then there is a significant discrepancy in the $(s + 1)$st digit: for γ_k it is 8, 9, 0, or 1 (allowing for errors in the decimal approximation), whereas for r_m it is 5 or 6. This proves the lemma.

Construction of the Initial Function $f(x, y, z)$

As in the proof of Theorem 1, $f(x, y, z)$ will be the sum of an infinite sequence of thin spherical waves. The kth wave will be supported on a spherical shell centered on the point $\mathbf{r}_k = (r_k, 0, 0)$, where r_k is given by the Effective Modulus Lemma above. We let $a(k)$ be the recursive function used in the proof of the modulus lemma, and set

$$c(k) = 2k + 4 \sum_{j=0}^{k+1} a(j).$$

Also, we use the same C^∞ function φ as in the proof of Theorem 1: recall that φ has support on $[-\frac{1}{2}, \frac{1}{2}]$, $\varphi(x) \geqq 0$, and $\varphi'(0) = 1$. Then, following the scheme of (3) above, we set:

$$f(x, y, z) = f(\mathbf{x}) = \sum_{k=0}^{\infty} w_k(\mathbf{x}),$$

where (4)

$$w_k(\mathbf{x}) = 10^{-(c(k)+a(k))}\varphi[10^{c(k)}(|\mathbf{x} - \mathbf{r}_k| - 1)].$$

We note some properties of the kth wave $w_k(\mathbf{x})$. Since φ is supported on the interval $[-\frac{1}{2}, \frac{1}{2}]$, the wave $w_k(\mathbf{x})$ is supported on the spherical shell $H_k = \{\mathbf{x} \mid 1 - \varepsilon_k \leqq |\mathbf{x} - \mathbf{r}_k| \leqq 1 + \varepsilon_k\}$, where $\varepsilon_k = 10^{-c(k)}$. Thus we have a shell of mean radius 1 and half-width $10^{-c(k)}$, centered about the point $\mathbf{r}_k$. The amplitude of w_k is

$$\max_{\mathbf{x}} |w_k(\mathbf{x})| = 10^{-(c(k)+a(k))} \max_{x} |\varphi(x)|,$$

whereas for the term $\text{grad}(w_k)$ in Kirchhoff's formula (2) we have

$$\max_{\mathbf{x}} |\text{grad } w_k(\mathbf{x})| = 10^{-a(k)} \max_{x} |\varphi'(x)|.$$

As in the proof of Theorem 1, the difference between the exponents $-(c(k) + a(k))$ and $-a(k)$ for w_k and grad w_k will prove crucial. The exponents for w_k give an effectively convergent series, since $c(k) \geqslant 2k$. The exponents for grad w_k give a series which converges, but not effectively. Actually, the function $c(k)$ was constructed with two goals in mind. Firstly, we have $c(k) \geqq 2k$, which gives the effective convergence mentioned above. Secondly, in the definition of $c(k)$ we have a finite sum of $a(j)$, $j \leqq k + 1$. The reason for this is the following: Later on in our proof, we will use a sequence of disks D_k of radius $10^{-c(k)/2}$ about the points $(r_k, 0, 0)$, where $r_k = (5/9) + \sum_{j=0}^{k} 10^{-a(j)}$. As is easily verified, our definition of $c(k)$ ensures that these disks do not overlap.

Since the series $f = \sum w_k$ is effectively uniformly convergent (being dominated by $\sum 10^{-c(k)} \leqq \sum 10^{-2k}$), and since $\{w_k\}$ is a computable sequence of functions, the limit f is computable (cf. (*) in Section 1). Since the series for grad f converges uniformly, though not effectively (being dominated by $\sum 10^{-a(k)}$), the function f is of class C^1. Then Kirchhoff's formula implies that the solution $u(x, y, z, t)$ is continuous.

Properties of the Solution

We collect here some facts which will be needed in our proof. Since the wave equation is linear, and since $f = \sum_{k=0}^{\infty} w_k$, the solution is given by

$$u(x, y, z, t) = u(\mathbf{x}, t) = \sum_{k=0}^{\infty} u_k(\mathbf{x}, t), \tag{5}$$

where $u_k(\mathbf{x}, t)$ = the solution of the wave equation with initial conditions

$$u_k(\mathbf{x}, 0) = w_k(\mathbf{x}), \ (\partial u_k/\partial t)(\mathbf{x}, 0) = 0.$$

Of course, the functions $u_k(\mathbf{x}, t)$ can be computed by substituting w_k for f in Kirchhoff's formula (2). We now state three facts about the terms $u_k(\mathbf{x}, t)$ in the series for $u(x, y, z, t)$. One of these involves a separation condition:

About each point $\mathbf{r}_k = (r_k, 0, 0)$ we construct an open disk D_k of radius $10^{-c(k)/2}$. Thus

$$D_k = \{\mathbf{x} \mid |\mathbf{x} - \mathbf{r}_k| < 10^{-c(k)/2}\},$$

where we recall

$$c(k)/2 = k + 2 \sum_{j=0}^{k+1} a(j).$$

These disks do not overlap.

(I) The functions $u_k(\mathbf{x}, t)$ form a computable sequence of functions of (x, y, z, t).

(II) At the point $\mathbf{x} = \mathbf{r}_k$ and time $t = 1$ we have

$$u_k(\mathbf{r}_k, 1) \geqq 10^{-a(k)}.$$

(III) At all points $\mathbf{x}$ outside the kth disk D_k and all times t we have

$$|u_k(\mathbf{x}, t)| \leqq \text{Const} \cdot 10^{-c(k)/2} \leqslant \text{Const} \cdot 10^{-k},$$

where

$$\text{Const} = \max_x |\varphi(x)| + \max_x |\varphi'(x)|.$$

Proof of (I). This is obvious from Kirchhoff's formula (2) (with $w_k(\mathbf{x})$ in place of $f(\mathbf{x})$) and the fact that the sequence $\{w_k(\mathbf{x})\}$ in (4) is a computable sequence of C^∞ functions.

Proof of (II). Again we apply Kirchhoff's formula (2) to the function $w_k(\mathbf{x})$ in (4), setting $\mathbf{x} = \mathbf{r}_k$ and $t = 1$. First consider the "grad" term in Kirchhoff's formula. The kth wave $w_k(\mathbf{x})$ is spherically symmetric about the point $\mathbf{r}_k$. Let us (here only) write $\rho = |\mathbf{x} - \mathbf{r}_k|$, so that $w_k(\mathbf{x})$ becomes a function of ρ. Then by the same reasoning used in Theorem 1 we get

$$\begin{aligned}
&(\text{Effect of "grad" term}) \\
&\quad = w_k'(\rho) \qquad (\text{at } \rho = 1) \\
&\quad = (d/d\rho)[10^{-c(k)-a(k)}\varphi(10^{c(k)}(\rho - 1))] \qquad (\text{at } \rho = 1) \\
&\quad = 10^{-(c(k)+a(k))} \cdot 10^{c(k)} \cdot \varphi'(0) \\
&\quad = 10^{-a(k)}, \qquad \text{since } \varphi'(0) = 1.
\end{aligned}$$

Similarly for the non-gradient term in (2) we have, again by the spherical symmetry of $w_k(\mathbf{x})$ about $\mathbf{r}_k$

$$\begin{aligned}(\text{Effect of "non-grad" term}) &= w_k(\rho) \qquad (\text{at } \rho = 1)\\ &= 10^{-(c(k)+a(k))}\varphi(0).\end{aligned}$$

Since $\varphi(x) \geqq 0$, this term is nonnegative. This proves (II).

Proof of (III). Again we use the solution formula (2) applied to $w_k(\mathbf{x})$. Recall that the definition of $w_k(\mathbf{x})$ involves the coefficient $10^{-(c(k)+a(k))}$, whereas $\operatorname{grad} w_k(\mathbf{x})$ has coefficient $10^{-a(k)}$. Here we will make no use of $a(k)$. Instead, a geometric argument based on the separation condition $\mathbf{x} \notin D_k$ will form the core of our proof. First, some trivial estimates:

$$10^{-(c(k)+a(k))} \leqq 10^{-c(k)/2},$$

and so

$$|w_k(\mathbf{x})| \leqq 10^{-c(k)/2} \cdot \max_x |\varphi(x)|.$$

For the "grad" term we have, as in (II) above:

$$\begin{aligned}|\operatorname{grad} w_k(\mathbf{x})| &\leqq 10^{-a(k)} \cdot \max_x |\varphi'(x)|\\ &\leqq \max_x |\varphi'(x)|.\end{aligned}$$

Now consider separately the two terms in Kirchhoff's formula (2). The term involving f itself (as opposed to $\operatorname{grad} f$) causes no difficulty. We obtain immediately for this term, replacing f by w_k:

$$\begin{aligned}|\text{Effect of non-gradient term}| &\leqq \max_x |w_k(\mathbf{x})|\\ &\leqq 10^{-c(k)/2} \cdot \max_x |\varphi(x)|.\end{aligned}$$

The gradient term is harder to handle, since we do not have good enough bounds on $\operatorname{grad} w_k(\mathbf{x})$. Consequently we reason geometrically.

Formula (2) involves integration over a sphere S of radius t *centered on the point* $\mathbf{x}$ (where the measure is normalized so that the whole sphere has measure 1). The spherical wave $w_k(\mathbf{y})$ is supported on a spherical shell $H = \{\mathbf{y} \mid 1 - 10^{-c(k)} \leqq |\mathbf{y} - \mathbf{r}_k| \leqq 1 + 10^{-c(k)}\}$ having mean radius 1 and half-width $10^{-c(k)}$ and *centered at* $\mathbf{r}_k$. Since $\mathbf{x} \notin D_k$, the distance d between $\mathbf{x}$ and $\mathbf{r}_k$ exceeds the radius of D_k which is $10^{-c(k)/2}$. These facts set the geometric situation.

Now (III) can be read out of the following lemma. We consider the "grad" term in Kirchhoff's formula (2) (the other term having already been dealt with). The factor $1/t$ occurring in the lemma is cancelled by the factor of t which appears in Kirchhoff's formula. For the other parameters in the lemma we substitute: $\varepsilon = 10^{-c(k)}$ and $d \geqq 10^{-c(k)/2}$. As noted above, we can dominate $|\text{grad } w_k|$ by $\max_x |\varphi'(x)|$. We get

$$|\text{Effect of "grad" term}| \leqq (10^{-c(k)}/10^{-c(k)/2}) \cdot \max_x |\varphi'(x)|$$
$$\leqq 10^{-c(k)/2} \cdot \max_x |\varphi'(x)|,$$

which suffices to prove (III).

LEMMA (spherical cross sections). *Let S be a sphere in $\mathbb{R}^3$ of radius t centered at a point P, $S = \{\mathbf{x} \mid |\mathbf{x} - P| = t\}$. Let H be a spherical shell of unit mean radius and half-width ε centered at a point Q, i.e., $H = \{\mathbf{x} \mid 1 - \varepsilon \leqq |\mathbf{x} - Q| \leqq 1 + \varepsilon\}$. Let the distance between P and Q be d. Then the area of the region in S where S intersects H is $\leqslant(\varepsilon/td)$ (area of S).*

Proof. For the sake of completeness, we include the elementary proof. Without loss of generality, assume that P and Q lie on the x-axis. We use the following well-known cross section principle for spheres in $\mathbb{R}^3$. Take an interval I of length L on the x-axis situated inside the sphere S: then the portion of S whose x-coordinates lie in I has area $= 2\pi tL$.

Now consider a triangle with vertices P, Q, and an arbitrary point $X \in S \cap H$. Its sides have lengths $PQ = d$, $PX = t$ (since $X \in S$), and for QX we know only that $1 - \varepsilon \leqq QX \leqq 1 + \varepsilon$ (since $X \in H$). Write $\rho = QX$. Apply the law of cosines to this triangle, where θ is the angle between the t and d sides (and recall that the d side lies on the x-axis). We have

$$\cos\theta = [t^2 + d^2 - \rho^2]/2td.$$

Now $\cos\theta$ is just $1/t$ times the projection of the side of length t on the x-axis. Thus the x-interval spanned when ρ varies between $1 - \varepsilon$ and $1 + \varepsilon$ has length

$$\leqq [(1+\varepsilon)^2 - (1-\varepsilon)^2]/2d = 2\varepsilon/d.$$

(The "$\leqq$" occurs because some of the values ascribed to $\cos\theta$ may fall outside of $[-1, 1]$ and thus represent a geometrically impossible triangle.) Combining this with the spherical cross section principle above, we see that the intersection of S with H has area $\leqq (2\pi t)(2\varepsilon/d) = (\varepsilon/td)$ (area of S). This proves the lemma, and completes the proof of (III).

Proof That $u(x, y, z, t)$ Is Sequentially Computable

We use the Effective Modulus Lemma, together with (I) and (III) above. Let $\{(x_k, y_k, z_k, t_k)\}$ be an arbitrary computable sequence of points in $\mathbb{R}^4$. We need to show that the sequence of values $\{u(x_k, y_k, z_k, t_k)\}$ is also computable.

Recall that the solution u is the sum of an infinite series $u = \sum_{m=0}^{\infty} u_m$, and that the result (III) applies to the individual terms u_m. As before, we use the abbreviations $\mathbf{x}_k = (x_k, y_k, z_k)$ and $\mathbf{r}_m = (r_m, 0, 0)$. Now we apply the Effective Modulus Lemma, with $\gamma_k = x_k$: there exist recursive functions $d(k)$ and $e(k)$ such that

$$|x_k - r_m| > 1/d(k) \qquad \text{for} \quad m \geqq e(k).$$

Of course, $|\mathbf{x}_k - \mathbf{r}_m| \geqq |x_k - r_m|$. Since the mth disk D_m in (III) has radius $10^{-c(m)/2} \leqq 10^{-m}$ about r_m, we have

$$\mathbf{x}_k \notin D_m \qquad \text{if} \quad m \geqq e(k) \qquad \text{and} \qquad 10^{-m} < 1/d(k).$$

Let $m_0(k)$ be the least integer m with $m \geqq e(k)$ and $10^{-m} < 1/d(k)$. Since $e(k)$ and $d(k)$ are integer-valued recursive functions, $m_0(k)$ is recursive. Now the sequence of values $\{u(x_k, y_k, z_k, t_k)\}$ is given by the infinite series

$$u(x_k, y_k, z_k, t_k) = \sum_{m=0}^{\infty} u_m(\mathbf{x}_k, t_k). \tag{6}$$

By virtue of (I), the sequence of functions $\{u_m\}$ is computable. For $m \geqq m_0(k)$ we have by (III), since $\mathbf{x}_k \notin D_m$

$$|u_m(\mathbf{x}_k, t_k)| \leqq \text{Const} \cdot 10^{-m},$$

and so

$$\sum_{m=m_0(k)}^{\infty} u_m(\mathbf{x}_k, t_k)$$

converges effectively and uniformly by comparison with $\sum 10^{-m}$. Since $m_0(k)$ is recursive, this implies that the original series (6) is computable, uniformly in k. Thus $u(x, y, z, t)$ is sequentially computable.

Proof That $u(x, y, z, 1)$ Is Not Computable

We refer to the definition of a "computable function," as given in Section 1. If $u(x, y, z, 1)$ is computable, then it must be effectively uniformly continuous. We will use (I), (II), and (III) above to show that this is not the case. Since the details are somewhat complicated, we begin with a rough sketch. Suppose that $u(x, y, z, 1)$ is effectively uniformly continuous. Recall that we have a sequence of disks D_m about the points $\mathbf{r}_m$, of radii $10^{-c(m)/2}$

which approach zero effectively. For each m, let $\mathbf{r}_m^*$ be a point on the boundary of D_m. Because of effective uniform continuity, the difference $[u(\mathbf{r}_m, 1) - u(\mathbf{r}_m^*, 1)]$ approaches zero effectively as $m \to \infty$.

Now the solution u is the sum of an infinite series $\sum u_i$. The results (I), (II), and (III) above apply to the individual terms u_i. By (I), the sequence $\{u_i\}$ is computable, which means that any initial segment of the series for u can be estimated effectively. Let this initial segment consist of the terms u_i with $i \leqq k$, where we assume $k < m$ (the precise relationship between k and m will be spelled out below). Then we consider the infinite series

(a) $u(\mathbf{r}_m, 1) - u(\mathbf{r}_m^*, 1) = \sum_{i=0}^{\infty} [u_i(\mathbf{r}_m, 1) - u_i(\mathbf{r}_m^*, 1)]$,

and break this series into three parts:

(b) = the sum of the ith terms for $0 \leqq i \leqq k$,

(c) = the sum of the ith terms for $i > k$, $i \neq m$, and

(d) = the single term with $i = m$.

Thus the original series (a) = (b) + (c) + (d). We have seen above that, since u is effectively uniformly continuous, (a) approaches zero effectively as $m \to \infty$. Using (I), we show that (b) approaches zero effectively, and using (III) we can prove the same thing for (c). [The use of (III) hinges on the fact that the disks D_i do not intersect, and hence since $i \neq m$, the points $\mathbf{r}_m$ and $\mathbf{r}_m^*$ lie outside of D_i.] This leaves the single term (d). Now it will follow from (II) that (d) is "large," specifically on the order of $10^{-a(m)}$, a quantity which does not approach zero effectively. Since

$$(\mathrm{d}) = (\mathrm{a}) - (\mathrm{b}) - (\mathrm{c}),$$

and the terms (a), (b), and (c) approach zero effectively, whereas (d) does not, we have a contradiction.

Now for the details. Suppose that $u(x, y, z, 1)$ is effectively uniformly continuous. By definition, this means that there exists a recursive function $s(k)$ such that

$$|\mathbf{x} - \mathbf{y}| \leqq 10^{-s(k)} \qquad \text{implies} \qquad |u(\mathbf{x}, 1) - u(\mathbf{y}, 1)| \leqq 10^{-k}.$$

We know by (I) that the sequence $\{u_k(\mathbf{x})\}$ is computable, uniformly in k. Hence there exists a recursive function $s'(k)$ such that

$$|\mathbf{x} - \mathbf{y}| \leqq 10^{-s'(k)} \qquad \text{implies} \qquad |u_i(\mathbf{x}, 1) - u_i(\mathbf{y}, 1)| \leqq 10^{-k}/(k+1) \quad \text{for} \quad 0 \leqq i \leqq k.$$

For convenience, we assume that $s'(k) \geqq k + 1$. Now consider the recursive function $a(m)$ which generates the non-recursive set A. Our goal is to show that, *except for a finite set of values of k*

$$a(m) > k/2 \qquad \text{for} \quad m \geqq s(k) + s'(k). \tag{7}$$

The remainder of this proof will be devoted to establishing (7). For (7) implies that $a(m)$ is effectively bounded below. This implies in turn that the set A is recursive, giving a contradiction.

Let $\{\mathbf{r}_m\} = \{r_m, 0, 0\}$ be the sequence of points we have used throughout, and let

$$r_m^* = r_m + 10^{-c(m)/2}, \qquad \mathbf{r}_m^* = (r_m^*, 0, 0).$$

Thus $\mathbf{r}_m^*$ is a point on the boundary of the disk D_m (cf. (III)). Also

$$|\mathbf{r}_m - \mathbf{r}_m^*| = 10^{-c(m)/2} \leqq 10^{-m}.$$

We examine the difference (which we label (a)):

(a) $u(\mathbf{r}_m, 1) - u(\mathbf{r}_m^*, 1)$.

Since $u = \sum u_i$, the difference (a) is equal to

$$\sum_{i=0}^{\infty} [u_i(\mathbf{r}_m, 1) - u_i(\mathbf{r}_m^*, 1)].$$

We will see later that $m > k$. Now we break this sum into three parts (b), (c), and (d), where

(b) = the sum of the ith terms for $0 \leqq i \leqq k$;

(c) = the sum of the ith terms for $i > k$, $i \neq m$;

(d) = the single term for which $i = m$.

Thus (a) = (b) + (c) + (d). We will see that (d) is the dominant term: we give upper bounds for |(a)|, |(b)|, and |(c)|, and a lower bound for (d). Since our aim is to prove (7), we can assume that $m \geqq s(k) + s'(k)$. (This implies, in particular, that $m \geqq s'(k) \geqq k + 1$, as promised above.)

Bound for (a). Since $m \geqq s(k)$ and $|\mathbf{r}_m - \mathbf{r}_m^*| \leqq 10^{-m}$, we have by the definition of $s(k)$

$$|(a)| = |u(\mathbf{r}_m, 1) - u(\mathbf{r}_m^*, 1)| \leqq 10^{-k}.$$

Now a detail: some of our estimates will involve the constant, Const = $\max_x |\varphi(x)| + \max_x |\varphi'(x)|$ in (III). Since $\varphi'(0) = 1$, this constant is $\geqq 1$. To achieve a standard format, we use the weaker inequality:

$$|(a)| \leqq \text{Const} \cdot 10^{-k}.$$

Bound for (b). For $0 \leqq i \leqq k$ we have, since $m \geqq s'(k)$ and $|\mathbf{r}_m - \mathbf{r}_m^*| \leqq 10^{-m}$

$$|u_i(\mathbf{r}_m, 1) - u_i(\mathbf{r}_m^*, 1)| \leqq 10^{-k}/(k+1), \qquad \text{and so}$$
$$|(\mathrm{b})| = |\text{sum of first } (k+1) \text{ terms in (a)}| \leqq 10^{-k},$$

and hence as before,

$$|(\mathrm{b})| \leqq \mathrm{Const} \cdot 10^{-k}.$$

Bound for (c). Recall that if $i \neq m$, the disks D_i and D_m do not overlap. Hence $\mathbf{r}_m$ and $\mathbf{r}_m^*$ are outside of D_i for $i \neq m$. Thus from (III) we have

$$|u_i(\mathbf{r}_m, 1)| \quad \text{and} \quad |u_i(\mathbf{r}_m^*, 1) \leqq \mathrm{Const} \cdot 10^{-i} \quad \text{for} \quad i \neq m.$$

Hence $|u_i(\mathbf{r}_m, 1) - u_i(\mathbf{r}_m^*, 1)| \leqq 2 \cdot \mathrm{Const} \cdot 10^{-i}$ for $i \neq m$, and summing over all $i > k$, $i \neq m$, we obtain

$$\begin{aligned} |(\mathrm{c})| &\leqq \sum_{i=k+1}^{\infty} 2 \cdot \mathrm{Const} \cdot 10^{-i} = \tfrac{2}{9} \cdot \mathrm{Const} \cdot 10^{-k} \\ &\leqq \mathrm{Const} \cdot 10^{-k}. \end{aligned}$$

Bound for (d). From (II) we have

$$u_m(\mathbf{r}_m, 1) \geqq 10^{-a(m)}.$$

Also, since $\mathbf{r}_m^*$ lies outside of the open disk D_m, we have by (III)

$$|u_m(\mathbf{r}_m^*, 1)| \leqq \mathrm{Const} \cdot 10^{-m} \leqq \mathrm{Const} \cdot 10^{-k},$$

since $m \geqq s'(k) \geqq k+1$. Thus $(\mathrm{d}) = u_m(\mathbf{r}_m, 1) - u_m(\mathbf{r}_m^*, 1)$ satisfies

$$(\mathrm{d}) \geqq 10^{-a(m)} - \mathrm{Const} \cdot 10^{-k}.$$

Now recall that $(\mathrm{a}) = (\mathrm{b}) + (\mathrm{c}) + (\mathrm{d})$, and examine the inequalities which we have proved for these four terms. We have

$$(\mathrm{a}) = (\mathrm{b}) + (\mathrm{c}) + (\mathrm{d}),$$

where

$$|(\mathrm{a})| \leqq \mathrm{Const} \cdot 10^{-k},$$
$$|(\mathrm{b})| \leqq \mathrm{Const} \cdot 10^{-k},$$
$$|(\mathrm{c})| \leqq \mathrm{Const} \cdot 10^{-k},$$

and

$$(\mathrm{d}) \geqq 10^{-a(m)} - \mathrm{Const} \cdot 10^{-k}.$$

Putting this together, we obtain

$$10^{-a(m)} \leqq 4 \cdot \mathrm{Const} \cdot 10^{-k},$$

whence

$$a(m) \geqq k - \log_{10}(4 \cdot \mathrm{Const}),$$

and

$$a(m) > k/2 \text{ provided } k > 2 \cdot \log_{10}(4 \cdot \mathrm{Const}),$$

(where in deriving these inequalities, we have assumed that $m \geqq s(k) + s'(k)$).

This proves (7). As we have noted, (7) implies that $a(m)$ is effectively bounded below, and hence that the set A is recursive, a contradiction. Thus our initial assumption, that $u(x, y, z, 1)$ is computable, must be false. Previously we showed that $u(x, y, z, t)$ is sequentially computable. This completes the proof of Theorem 2.

3. The Wave Equation in n Dimensions

Our results can be extended to the wave equation in any number of space dimensions $n > 1$. We have emphasized the three-dimensional case because of its physical significance and to avoid complications.

For $n = 1$, the situation is reversed: the solution $u(x, t)$ is computable whenever $f(x)$ is. This follows from d'Alembert's formula: $u(x, t) = \frac{1}{2}[f(x - t) + f(x + t)]$. (The same thing holds if, instead of the initial conditions $u = f$, $\partial u/\partial t = 0$ at $t = 0$, we put $\partial u/\partial t = g$ with a computable g. The d'Alembert formula for these initial conditions shows this also.)

For $n \geqq 3$, we can extend Theorems 1 and 2 by the "method of descent." This means that we take the three-dimensional solution $u(x_1, x_2, x_3, t)$ from Theorem 1 or 2, and simply set $u(x_1, ..., x_n, t) = u(x_1, x_2, x_3, t)$, so that the extended solution is constant in the variables $x_4, ..., x_n$.

Remark. In our three-dimensional examples, the initial function f is C^1 and the solution u is continuous. Now the analog of Kirchhoff's formula for higher dimensions involves higher derivatives. Consequently, by using it we could build examples where f was C^2, C^3, or differentiable to any finite degree, by going to sufficiently high space dimension. However, the solution u would still be merely continuous. Thus the structure of these noncomputable solutions seems to be largely independent of the dimension, as soon as $n > 1$.

The Case of dimension $n = 2$

We will consider only Theorem 1. Theorem 2 can also be extended, but for reasons of space we will not do so here. We obtain the same results as before. (One minor change: here f is not C^1, although the solution u is continuous.) We give a brief sketch, indicating the modifications in the proof of Theorem 1 which are needed.

Again we use the method of descent, this time from dimension 3 to 2. We will construct an initial function $f(x, y, z) = f(x, y)$ which is independent of z. Then we apply Kirchhoff's formula for dimension 3. Previously in Theorem 1 we used functions which were spherically symmetric about the origin. Here we use functions which are circularly symmetric, i.e., functions of $r = (x^2 + y^2)^{1/2}$ only. The function f has the form

$$f(x, y) = \sum_{k=0}^{\infty} w_k(x, y),$$

where the waves $w_k(x, y) = w_k(x, y, z)$ are supported on thin *cylindrical* shells $\{1 - \varepsilon_k \leqq r \leqq 1 + \varepsilon_k\}$ in $\mathbb{R}^3$. Now Kirchhoff's formula still involves integration over a *spherical* surface. The crucial question is: as a function of ε_k, what percentage of the unit sphere is contained within the cylindrical shell $\{1 - \varepsilon_k \leqq r \leqq 1 + \varepsilon_k\}$? (Compare the "sphere cross section lemma" used to prove (III) in Theorem 2.) It is easy to verify that the intersection of the unit sphere with the cylindrical shell has an area asymptotic to const $\cdot\ \varepsilon_k^{1/2}$ as $\varepsilon_k \to 0$. (We do not bother about the constant.) The key point is that we have $\varepsilon_k^{1/2}$.

Now we describe the functions $w_k(x, y) = w_k(r)$, $r = (x^2 + y^2)^{1/2}$. Using the same recursive function $a(n)$ as in Theorem 1, we define

$$C_k = 10^{-(k + a(k))},$$
$$\varepsilon_k = 10^{-2k}.$$

Let $w_k(r)$ be the piecewise linear ("tent shaped") function which takes the values: $w_k(1) = C_k$, $w_k(1 \pm \varepsilon_k) = 0$, and such that $w_k(r)$ is linear except at $r = 1$, $1 \pm \varepsilon_k$, and $w_k(r)$ vanishes for $|r - 1| \geqq \varepsilon_k$.

The unit sphere in Kirchhoff's formula (for time $t = 1$, $(x, y, z) = (0, 0, 0)$) intersects the cylindrical shell only in the part where $1 - \varepsilon_k \leqq r \leqq 1$, and does not touch the region where $r > 1$. Assuming that ε_k is small, the outward normal to this sphere is *nearly* parallel to the xy-plane; we shall ignore the discrepancy. As we have seen, the area of intersection of sphere with cylindrical shell is $\cong \varepsilon_k^{1/2}$ (where in this sketch the symbol "$\cong$" means that a constant factor has been dropped).

Now for $1 - \varepsilon_k < r < 1$, the derivative $(d/dr)\, w_k(r) = C_k/\varepsilon_k$. Consequently the effect of the "grad" term in Kirchhoff's formula is on the order of (ignoring constants)

$$\begin{aligned}&(\text{size of } dw_k/dr)\ (\text{area of intersection})\\&\cong (C_k/\varepsilon_k)\cdot \varepsilon_k^{1/2} = C_k/\varepsilon_k^{1/2}.\end{aligned}$$

As before, the "non-grad" term in Kirchhoff's formula can be neglected. Now we have

$$f = \sum_{k=0}^{\infty} w_k,$$

$$u = \sum_{k=0}^{\infty} u_k \qquad \text{(where } u_k \text{ is the solution corresponding to } w_k \text{ in Kirchhoff's formula);}$$

and so by the above estimates

the series for f is dominated by $\sum C_k$,

the series for u is dominated by $\sum C_k/\varepsilon_k^{1/2}$ + (non-grad term).

Recall that $C_k = 10^{-(k+a(k))}$ and $\varepsilon_k = 10^{-2k}$. One verifies that: the series for f converges effectively by comparison with $\sum 10^{-k}$; and the series for u converges, but not effectively, by comparison with $\sum 10^{-a(k)}$. Since the series for f converges effectively and is built up from computable components, f is computable. Since the series for u does not converge effectively we can (copying the proof of Theorem 1) make u continuous but not computable. This completes our sketch for the case $n = 2$.

ADDENDUM: WEAK SOLUTIONS

Our theorems lead to "weak solutions" of the wave equation, i.e., solutions which are not twice differentiable. (As we shall show, this is not an accident arising from our proofs; it is inevitable.) For the convenience of readers who may be unfamiliar with weak solutions, we summarize the essential facts about them here— thus putting our results into a coherent framework.

As in Theorems 1 and 2, we consider the three-dimensional wave equation (1a), except that now we take the general initial conditions: $u = f$ and $\partial u/\partial t = g$ at $t = 0$, instead of assuming that g vanishes. Since our proofs used Kirchhoff's formula, we could simply decree that a "weak solution" means one given by that formula—an obviously ad-hoc approach. A physically reasonable framework is provided by the "finite energy theory." In describing this, we will use the language of Schwartz distributions. Despite the generality of distribution theory, there are several things which must be verified before a treatment of the wave equation based on distributions makes any sense. These are: (a) that solutions exist; (b) that they are unique; and (c) that the operation of restricting a solution to the hyperplane $\{t=0\}$ is well defined, so that we can discuss initial conditions. Of special importance is the uniqueness: this means, of course, that the solution to the wave equation predicted by the general theory is the same as the solution given explicitly in our paper. In what follows, we will state only those facts which are necessary to substantiate (a), (b), and (c).

First we remark that solutions of the wave equation propagate with a finite velocity. Therefore, in studying these solutions over a finite time period, there is no harm in assuming that the initial functions f and g have compact support.

Secondly, associated with any solution $u(x, y, z, t)$ of (1a) and any particular time $t=t_0$, there is the "energy integral" defined by

$$\iiint_{\mathbb{R}^3} [|(\text{grad } u)(x, y, z, t_0)|^2 + |(\partial u/\partial t)(x, y, z, t_0)|^2] \, dx \, dy \, dz. \tag{8}$$

It is a fundamental property of the wave equation that this integral has a value independent of t_0 ("conservation of energy," cf. [3]).

Now we can describe the space X^* of finite energy solutions of the wave equation. We start with the space X of all solutions u which are C^∞, and have compact support for each finite time interval $t_0 \leqq t \leqq t_1$. [Since the elements of X are C^∞, any of the standard existence or uniqueness theorems applies to X, and we have: for every pair of C^∞, compact support functions f, g, there exists a unique solution $u \in X$.] We define the "energy norm" on X to be the square-root of the energy integral (8). Then we take the completion X^* of X with respect to its norm. This is the space of "finite energy" solutions.

To finish this outline, we need some information about the structure of X^*. Since the energy norm topology is stronger than any of the standard topologies used in distribution theory, the points in X^* are representable as Schwartz distributions. Also, differentiation is a continuous operation on distribution space; hence the distributions $u \in X^*$ still satisfy the wave equation.

Regarding the initial conditions, we must now verify property (c). In general, distributions on $\mathbb{R}^4$ do not admit restriction to hyperplanes $\{t=t_0\}$ within $\mathbb{R}^4$. However, in this case they do. *Since the energy norm involves integration over* (x, y, z) *only*, the preceding arguments about the structure of $u \in X^*$ apply mutatis mutandis to give well-defined distributions $u[t_0]$ and $(\partial u/\partial t)[t_0]$ on each hyperplane $\{t=t_0\}$. There is one more point we must verify: in terms of the energy norm on u (which implies the L^2 norm for $\partial u/\partial t$), $u[t_0]$ and $(\partial u/\partial t)[t_0]$ are continuous functions of t_0. This is clear for the C^∞ functions $u \in X$, and since the energy norm is independent of t_0, the general case $u \in X^*$ follows from the fact that a uniform limit of continuous functions is continuous. (Continuity means that the restriction of u or $\partial u/\partial t$ to $\{t = t_0\}$ is an "intrinsic" operation, i.e., one that depends only on $u(x, y, z, t)$, and not on the manner by which u was constructed.) This establishes (a) and (c).

Now we come to the question of uniqueness. We must show that if the initial functions $f = g = 0$, then $u = 0$. Recall that f and g are the restrictions of u and $\partial u/\partial t$ to the hyperplane $\{t = 0\}$. Now we get uniqueness immediately from the fact that the energy integral is independent of t_0: if $f = g = 0$, then this integral vanishes for $t_0 = 0$ and hence for all t_0, so that $u(x, y, z, t_0)$ is identically zero. This proves (b).

The "finite energy theory" outlined above is more than sufficient for our purposes. For a pair of initial conditions f, g corresponds to some $u \in X^*$ if and only if:

$$\operatorname{grad} f \in L^2(\mathbb{R}^3) \qquad \text{and} \qquad g \in L^2(\mathbb{R}^3).$$

In our main theorems, we have $f \in C^1(\mathbb{R}^3)$ (where f has compact support) and $g = 0$. Clearly this fits the above conditions.

We remark that all of the functional-analytic results listed in this addendum become transparent if one uses the Fourier transform (with respect to the space variables x, y, z) and studies the equation dual to (1a) in Fourier transform space. However, although the Fourier transform approach in neater, our aim has been to reach our goal by the shortest path. We have obtained (a), (b), and (c) using only the invariance of the energy integral.

Remark (computability of the energy integral). It is natural to ask whether in Theorems 1 and 2 above, the energy integral is a computable real number. The answer is "yes." Recall that in these theorems the initial conditions are $u = f$, $\partial u/\partial t = 0$ at $t = 0$. Hence the energy integral is just the integral of $|\operatorname{grad} f|^2$ over $\mathbb{R}^3$. We sketch a proof that this is computable: There is a slight difficulty because the series $\operatorname{grad} f = \sum \operatorname{grad} w_k$ in Theorem 1 or 2 does not converge effectively at all points (see (3) and (4) above). However, when we *integrate* $|\operatorname{grad} f|^2$, the difficulty disappears. For the *supports* of the functions w_k in the series for f have volumes which approach zero effectively as $k \to \infty$. This in turn gives an effectively convergent series for the L^2-norms. We omit the details.

Now we show that the appearance of weak solutions in Theorems 1 and 2 is unavoidable.

PROPOSITION. *Let u be a noncomputable solution of the wave equation*

(1a, b) *with computable initial data f. Then u is not effectively uniformly continuous. (Hence u cannot be of class C^2 or even C^1, and thus u is a weak solution.*

Proof (sketch). Let $\varphi(\mathbf{x})$ be any function on $\mathbb{R}^3$ which is C^∞ with compact support and is computable together with all of its derivatives. We first show: (i) the convolution $(u * \varphi)(\mathbf{x}, t) = \iiint_{\mathbb{R}^3} u(\mathbf{x} - \mathbf{y}, t)\, \varphi(\mathbf{y})\, d\mathbf{y}$ is computable.

To prove (i). A trivial computation shows that (since differentiation commutes with convolution), $u * \varphi$ is a solution of the wave equation for any $u \in X$. Then, passing to the closure in energy norm, the same thing holds for all $u \in X^*$. Setting $t = 0$, we see that the solution $u * \varphi$ satisfies the initial conditions: $(u * \varphi)(\mathbf{x}, 0) = (f * \varphi)(\mathbf{x})$ and $(\partial/\partial t)(u * \varphi) = (\partial u/\partial t) * \varphi = 0$. Thus we can use Kirchhoff's formula (2) to deduce $(u * \varphi)(\mathbf{x}, t)$ from $(f * \varphi)(\mathbf{x})$. Now, since f is computable, and φ is computable together with all of its derivatives, the convolution $(f * \varphi)$ is *computable together with all of its derivatives*. Then Kirchhoff's formula immediately shows that $(u * \varphi)$ is computable.

The same result applies to a sequence $\{\varphi_k\}$ of C^∞-compact support functions which are computable together with their derivatives, uniformly in k. We have: (i′) the sequence $\{u * \varphi_k\}$ is computable.

Now we show that (i′) implies the proposition. Suppose that u is effectively uniformly continuous. Take any function φ as above with $\varphi \geqq 0$ and whose integral over all of $\mathbb{R}^3$ is equal to 1. Let $\varphi_k(\mathbf{x}) = k^3 \varphi(k\mathbf{x})$ (so that the integral of φ_k remains equal to 1, while support $(\varphi_k) = (1/k) \cdot$ support (φ); i.e., $\{\varphi_k\}$ is an "approximate identity"). Since u is effectively uniformly continuous, it is easy to verify that $(u * \varphi_k) \to u$ effectively and uniformly as $k \to \infty$. By (i′), the sequence $\{u * \varphi_k\}$ is computable. Hence by (*) in Section 1, u is computable, a contradiction.

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Note added in Proof. In answer to several series, we state the following. Examples of the type presented in this paper can not be given for the standard elliptic or parabolic partial differential equations. Such equations include the Dirichlet problem with a suitable regular boundary, and the heat equation with initial data having compact support. The proof of this fact depends on the maximum principle for elliptic and parabolic equations.

REFERENCES

1. A. GRZEGORCZYK, Computable functionals, *Fund. Math.* **42** (1955), 168–202.
2. A. GRZEGORCZYK, On the definitions of computable real continuous functions, *Fund. Math.* **44** (1957), 61–71.
3. G. HELLWIG, "Partial Differential Equations," Blaisdell, New York, 1964.
4. G. KREISEL, A notion of mechanistic theory, *Synthese* **29** (1974), 11–16.
5. A. MOSTOWSKI, On computable sequences, *Fund. Math.* **44** (1957), 37–51.
6. J. MYHILL, A recursive function defined on a compact interval and having a continuous derivative that is not recursive, *Michigan Math. J.* **18** (1971), 97–98.
7. I. G. PETROVSKII, "Partial Differential Equations," Saunders, Philadelphia, 1967.
8. M. B. POUR-EL AND J. CALDWELL, On a simple definitions of computable function of a real variable—with applications to functions of a complex variable. *Z. Math. Logik. Grundlagen Math.* **21** (1975), 1–19.
9. M. B. POUR-EL AND I. RICHARDS, A computable ordinary differential equation which possesses no computable solution, *Ann. of Math. Logic* **17** (1979), 61–90.