Available online at www.sciencedirect.com

SCIENCE DIRECT®

DISCRETE MATHEMATICS

Discrete Mathematics 300 (2005) 16–29

www.elsevier.com/locate/disc

Chromatic-index-critical graphs of orders 13 and 14

Drago Bokal[a], Gunnar Brinkmann[b], Stefan Grünewald[c]

[a] *Inštitut za matematiko, fiziko in mehaniko, Univerza v Ljubljani, Jadranska 19, SI-1000 Ljubljana, Slovenia*
[b] *Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany*
[c] *The Linnaeus Centre for Bioinformatics, Uppsala Universitet, BMC, Box 598, SE-75124 Uppsala, Sweden*

Received 11 April 2003; received in revised form 23 June 2005; accepted 28 June 2005
Available online 9 August 2005

Abstract

A graph is chromatic-index-critical if it cannot be edge-coloured with Δ colours (with Δ the maximal degree of the graph), and if the removal of any edge decreases its chromatic index. The Critical Graph Conjecture stated that any such graph has odd order. It has been proved false and the smallest known counterexample has order [[18] A.J.W. Hilton, R.J. Wilson, Edge-colorings of graphs: a progress report, in: M.F. Cabobianco, et al. (Eds.), Graph Theory and its Applications: East and West, New York, 1989, pp. 241–249; [31] H.P. Yap, Some topics in graph theory, London Mathematical Society, Lecture Note Series, vol. 108, Cambridge University Press, Cambridge, 1986].

In this paper we show that there are no chromatic-index-critical graphs of order 14. Our result extends that of [[5] G. Brinkmann, E. Steffen, Chromatic-index-critical graphs of orders 11 and 12, European J. Combin. 19 (1998) 889–900] and leaves order 16 as the only case to be checked in order to decide on the minimality of the counterexamples given by Chetwynd and Fiol. In addition we list all *nontrivial* critical graphs of order 13.

Keywords: Critical graph; Edge-colouring; Graph generation; Chromatic index; Structure generation

1. Introduction

The class of simple graphs can be divided into two subclasses by Vizing's well-known theorem—into the class of *colourable graphs* (or *class 1 graphs*), whose chromatic index is equal to the maximum degree Δ of the graph, and into the class of *noncolourable graphs*

E-mail addresses: drago.bokal@fmf.uni-lj.si (D. Bokal), gunnar@mathematik.uni-bielefeld.de (G. Brinkmann), Stefan.Grunewald@lcb.uu.se (S. Grünewald).

0012-365X/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2005.06.010

(or *class 2 graphs*), whose chromatic index is $\Delta + 1$. A noncolourable graph G is said to be *Δ-critical*, or simply critical, if the removal of any edge from G decreases its chromatic index.

In this paper we consider only simple graphs. Also, all our graph colourings are edge-colourings. Furthermore we assume that the reader is familiar with basic graph theory notions, taken from [11,17,30] for instance.

In 1973, Beineke and Wilson [1] presented several methods to construct critical graphs. Clearly, each graph with more than $\Delta\lfloor\frac{n}{2}\rfloor$ edges is class 2. Their constructions yield many critical graphs with $\Delta\lfloor\frac{n}{2}\rfloor + 1$ edges. Graphs with at least this number of edges are called *overfull*. They conjectured that there are no critical graphs of even order. This Critical Graph Conjecture was later independently stated by Jakobsen [20].

The critical graphs with $\Delta\lfloor\frac{n}{2}\rfloor + 1$ edges are called *trivial critical graphs*. Critical graphs with at most $\Delta\lfloor\frac{n}{2}\rfloor$ edges are critical for *structural reasons*—they are called *nontrivial critical graphs*.

The Critical Graph Conjecture was disproved by Goldberg [13]: he found an infinite family of critical graphs of even order, the smallest of them having 22 vertices. Chetwynd and Fiol independently found a critical graph on 18 vertices. In order to check the minimality of this counterexample, Yap [31] asked whether critical graphs of orders 12, 14 or 16 do exist (cf. [9,31]). First lists of critical graphs of order less than 10 are given in [10,12].

In 1997 Brinkmann and Steffen [4] established that the graphs of Fiol and Chetwynd are the smallest 4-critical graphs of even order, and that the Goldberg graph is the smallest 3-critical graph of even order. They did this by a combination of computational and theoretical results. Later they also partially answered Yap's question by showing that there are no critical graphs of order 12, and found the only two nontrivial critical graphs of order 11 (see [5]). In order to avoid possible computational errors, this was done by two independent approaches—one of them discussing a large number of cases (possible degree sequences) by hand and checking only the remaining cases with a computer and one a more or less straightforward computer search. Another result on the topic was found by Steffen and Grünewald [16], who constructed Δ-critical graphs of even order for any $\Delta \geqslant 3$. A more detailed description of the techniques described here can be found in [2].

2. Results

We prove the following two theorems:

Theorem 1. *No graph of order* 14 *is critical*.

Theorem 2. *There are exactly* 14 *nontrivial critical graphs of order* 13. *They are listed in Fig.* 1.

All nontrivial critical graphs of order at most 13 have maximum degree 3. They can be obtained from the critical subgraph P_c of the Petersen graph—that is the Petersen graph minus one vertex—by three well-known constructions. Additional information on these methods can be found in [12,16].

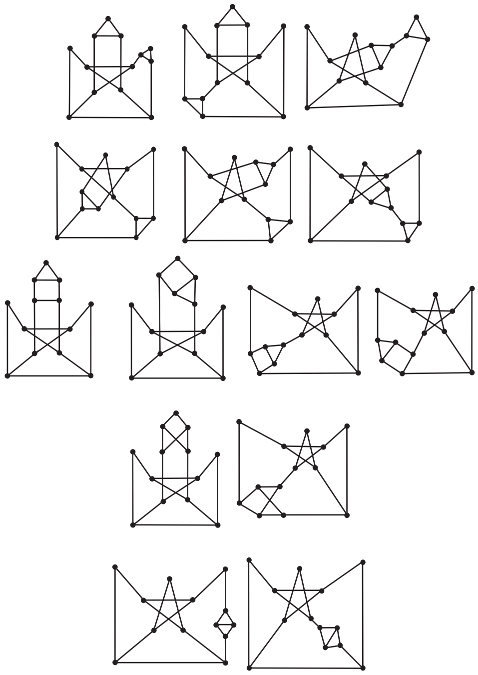

Fig. 1. All nontrivial critical graphs of order 13.

Lemma 3. *Let G be a 3-critical graph and G' a graph obtained from G by replacing a vertex v of G by a triangle and connecting the two or three edges formerly containing v with different vertices of the triangle. Then G' is also 3-critical.*

Lemma 4. *Let G be a graph with $\Delta(G) \geqslant 2$, and $v_1, \ldots, v_d$ be the neighbours of $v \in V(G)$. Let $u_1, \ldots, u_\Delta$ be the vertices of degree $\Delta - 1$ in the complete bipartite graph $K_{\Delta,\Delta-1}$, and G' be the graph obtained from $G - v$ and $K_{\Delta,\Delta-1}$ by adding edges $v_i u_i$ for $i = 1, \ldots, d$.*

Then G is Δ-critical, if and only if G' is Δ-critical.

A graph H is said to be obtained from G and G' by a *Hajós-union* where v and v' are identified, if $v \in V(G)$, $v' \in V(G')$, and H is constructed from G and G' as follows:

1. One edge $uv \in E(G)$ and one edge $u'v' \in E(G')$ are removed.
2. The vertices v and v' are identified.
3. A new edge uu' is inserted.

Lemma 5 (*Jakobsen [19]*). *Let G and G' be two Δ-critical graphs and $v \in V(G)$, $v' \in V(G')$ two vertices such that $\deg_G(v) + \deg_{G'}(v') \leqslant \Delta + 2$. Then any graph obtained from G and G' by a Hajós-union where v and v' are identified is Δ-critical.*

The two nontrivial critical graphs of order 11 can be constructed from P_c by applying Lemma 3 either to a vertex of degree 2 or to a vertex of degree 3. By applying Lemma 3 to

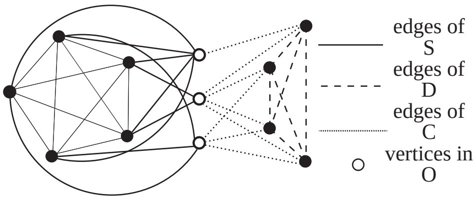

Fig. 2. A 6-maximal graph with a deficit clique of size 4 and 3 open vertices.

one of those graphs we obtain 10 nontrivial critical graphs of order 13 (graphs in first three rows of Fig. 1), by applying Lemma 4 to P_c we find two nontrivial critical graphs of order 13 without triangles (fourth row of Fig. 1), and two additional nontrivial critical graphs of order 13 are obtained from P_c and the trivial 3-critical graph of order 5 by a Hajós-union where the vertex of degree 2 in the latter graph is one of the identified vertices (fifth row).

Theorems 2 and 1 are proved using two (almost) independent computer programs to do an exhaustive computer search for nontrivial critical graphs. We will only describe the algorithm on which one of the programs is based in detail and sketch the other. The main strategy of the approaches is that instead of generating and testing all possibly critical graphs, we generated a *relatively small* set of graphs (called *Δ-maximal candidates*) so that each critical graph is contained in at least one of these supergraphs of the same order. The search for critical subgraphs was performed like in [5].

To prove Theorem 2, all Δ-maximal candidates of order 13 were generated for $\Delta \in \{3, 4, \ldots, 9\}$. For $\Delta \in \{10, 11, 12\}$ the fact that no critical graphs exist follows from [7,8,28]. To prove Theorem 1 only $\Delta \in \{5, \ldots, 10\}$ had to be considered. The results of [4] imply that there are no Δ-critical graphs of order 14 for $\Delta \in \{3, 4\}$, and the results of [6,28,29] imply this for $\Delta \in \{11, 12, 13\}$.

The remainder of this paper is structured as follows: Section 3 describes how Δ-maximal candidates can be efficiently generated, Section 4 discusses how we searched the candidates for critical subgraphs and Section 5 discusses the results obtained and sketches the second approach.

3. Δ-maximal graphs, Δ-spanned graphs and Δ-maximal candidates

Let G be a graph of maximal degree Δ. Vertices v with $\deg_G(v) = \Delta$ are called *saturated vertices* and vertices of degree less than Δ are called *deficit vertices*. We call the subgraph $S(G)$ that is induced by the saturated vertices the *saturated subgraph* of G and the subgraph $D(G)$ induced by the deficit vertices the *deficit subgraph*. The subgraph $C(G)$ consisting of all edges between vertices of $S(G)$ and $D(G)$ and their endpoints is called the *connecting subgraph*. Obviously $C(G)$ is bipartite. In the saturated subgraph S the vertices v with $\deg_S(v) < \Delta$ are of considerable importance. These are G-saturated vertices that are adjacent to G-deficit vertices. They are called *open vertices* of G and the set of open vertices is denoted as $\mathrm{O}(G)$. Fig. 2 illustrates these concepts.

The order of a graph G is denoted by $|G|$.

A *Δ-maximal graph G* is a graph with maximal degree Δ that is not a proper subgraph of any graph with the same number of vertices and maximal degree—or equivalently: a graph in which $D(G)$ is a clique or empty.

A Δ-maximal graph with a spanning Δ-critical subgraph is called a *Δ-spanned graph*.

Lemma 6. *Every critical graph is* 2*-connected.*

An easy proof of Lemma 6 can be found in, for example, [31].

Let a critical graph *G* be given. By Lemma 6, *G* is 2-connected. Since this property as well as the property of being class 2 are preserved when edges are added to *G* in a way that the maximum degree is preserved, each Δ-critical graph is contained in a Δ-spanned graph of the same order.

Lemma 7 (*Vizing's Adjacency Lemma*). *Let G be a Δ-critical graph, $uv \in E(G)$. Then v is adjacent to at least $\Delta - \deg(u) + 1$ saturated vertices different from u.*

The proof is rather technical and can be found in [12]. This lemma implies the following proposition:

Proposition 8. *Every vertex of a Δ-spanned graph G is adjacent to at least two saturated vertices.*

If a Δ-maximal graph has the property that every vertex has at least two saturated neighbours, then it is called a *Δ-maximal candidate*. By Proposition 8 every Δ-spanned graph is a Δ-maximal candidate. Thus, it is sufficient to generate all Δ-maximal candidates, apply a filtering procedure that removes those that are colourable or not 2-connected and finally search the remaining graphs for spanning critical subgraphs.

The decomposition into induced subgraphs *S* and *D* and the subgraph *C* leads to a natural way of constructing all Δ-maximal candidates of a given order: first construct all saturated graphs *S*, then connect each of them with the deficit clique *D* of appropriate size in a way that every deficit vertex has at least two saturated neighbours. Isomorphism rejection methods [3] must be applied in order to construct only one copy of each graph up to isomorphisms. For the construction of the saturated subgraphs existing programs like described in [14,15,23,26] can be used.

In the approach we will present in detail, the saturated subgraphs are generated from their degree sequence, that is the sequence $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_{|S|}$ of vertex degrees of the saturated subgraph. Obviously, all isomorphic graphs have the same degree sequence of $S(G)$.

The following propositions give necessary criteria for degree sequences that belong to saturated subgraphs of Δ-maximal candidates of a given order.

Proposition 9. *For a Δ-maximal candidate G we have $|D(G)| \leqslant \Delta - 2$.*

Proof. The maximal degree of a deficit vertex is $\Delta - 1$. By definition this is at least $|D| + 1$. Together we have $|D| \leqslant \Delta - 2$. $\square$

Proposition 10. *For a Δ-maximal candidate G we have* $|S|>|D|$ *and* $2|D|<|G|$.

Proof. If there are no deficit vertices, $0=2|D|<|G|$. For $|D|\geqslant 1$, let $\Delta_2<\Delta$ be the second largest degree in G. Let $l=|E(C)|$. Obviously $|D||S|\geqslant l$ and equality holds if and only if each vertex in S is adjacent to each vertex in D.

For given $|D|$, $|S|$ and l, we construct an upper bound for Δ and a lower bound for Δ_2. Summing up the G-degrees of vertices in S and counting edges we get $|S|\Delta\leqslant 2(|S|(|S|-1)/2)+l$ yielding $\Delta\leqslant|S|-1+l/|S|$. A similar argument for vertices in D leads to $|D|\Delta_2\geqslant 2(|D|(|D|-1)/2)+l$, $\Delta_2\geqslant|D|-1+l/|D|$.

Since $\Delta>\Delta_2$, we get $(|S|-|D|)(|D||S|-l)>0$. As $|D||S|-l\geqslant 0$, we have $|S|>|D|$ and also $|G|=|S|+|D|>2|D|$. $\square$

Combining Propositions 9 and 10 we have the following:

Corollary 11. *For a Δ-maximal candidate G we have* $0\leqslant|D|\leqslant \min\left\{\Delta-2, \lfloor\frac{|G|-1}{2}\rfloor\right\}$.

Note that, as $|S|=|G|-|D|$, Corollary 11 gives necessary conditions for the lengths of degree sequences of saturated subgraphs that can occur.

Proposition 12. *Let $d_1, d_2, \ldots, d_{|S|}$ be the degree sequence of the saturated subgraph S of a Δ-maximal candidate. Then $\Delta-|D|\leqslant d_i\leqslant\Delta$ for $1\leqslant i\leqslant|S|$.*

Proof. Since a saturated vertex can be adjacent to at most $|D|$ deficit vertices, we have $\Delta-|D|\leqslant d_i$. $\square$

For a graph G let $m\geqslant\Delta(G)$. The *m-deficiency* of a vertex v is defined as $\Theta_m(v):=m-\deg(v)$. The *m-deficiency* of a graph G is

$$\Theta_m(G):=\sum_{v\in V}\Theta_m(v)=m|G|-\sum_{v\in V}\deg(v).$$

Obviously, if we add more than $\Theta_m(G)/2$ edges to G, then the new graph will have at least one vertex of degree larger than m. Note that for a Δ-maximal graph G, $\Theta_\Delta(S)$ equals the number of edges in the connecting subgraph C.

Proposition 13. *For a Δ-maximal candidate G we have* $2|D|\leqslant\Theta_\Delta(S)\leqslant(\Delta-|D|)|D|$.

Proof. By definition, each vertex in D has at least two neighbours in S, yielding $2|D|\leqslant\Theta_\Delta(S)$.

For an arbitrary deficit vertex u we have $\deg(u)\leqslant\Delta-1$. Since u has $|D|-1$ G-deficit neighbours there are at most $\Delta-|D|$ neighbours in S. Summing over all deficit vertices, we obtain $\Theta_\Delta(S)\leqslant(\Delta-|D|)|D|$. $\square$

Proposition 14. *Let o be the number of open vertices in a Δ-maximal candidate G. Then* $o\geqslant\left\lceil\frac{\Theta_\Delta(S)}{|D|}\right\rceil$.

Proof. There are $\Theta_\Delta(S)$ edges in the bipartite graph C. By the pigeon-hole principle, there is a vertex in D that has C-degree at least $\left\lceil \frac{\Theta_\Delta(S)}{|D|} \right\rceil$, which gives a lower bound for the number of open vertices. $\square$

In the following let $V(D) = \{u_1, \ldots, u_{|D|}\}$ and let O_{u_i} denote the set of neighbours of u_i in C.

For a Δ-maximal candidate G let $\mathcal{O}$ be the multiset $\{O_{u_i} \mid 1 \leqslant i \leqslant |D|\}$. Each u_i in D is neighbouring exactly all $v \in O_{u_i}$ in S, so the graph G is uniquely determined by S, D, $\mathcal{O}$ and a function f mapping the vertices of D to the element of $\mathcal{O}$ containing its neighbours. As we are only interested in graphs up to isomorphisms, and since D is a clique and therefore has the full permutation group of its vertices as the automorphism group and since the order of D equals the cardinality of $\mathcal{O}$, D and f are redundant information. We can take any clique with a vertex set of the given cardinality and assign the elements of $\mathcal{O}$ in any order to get an isomorphic graph. For given S and $\mathcal{O}$ let us denote a corresponding graph by $G_S(\mathcal{O})$.

Proposition 15. *For a Δ-maximal candidate $G_S(\mathcal{O})$ we have*

- $O_{u_i} \subseteq \mathrm{O}(G)$,
- $\sum_i |O_{u_i}| = \Theta_\Delta(S)$,
- $2 \leqslant |O_{u_i}| \leqslant \Delta - |\mathcal{O}|$,
- *Each $v \in \mathrm{O}(G)$ appears in exactly $\Delta - \deg_S(v)$ sets O_{u_i}.*

The question we will discuss now is under which conditions $G_S(\mathcal{O})$ and $G_{S'}(\mathcal{O}')$ are isomorphic. As any isomorphism must map S onto S', the fact that S and S' are isomorphic is a necessary condition. Since every saturated subgraph will be constructed only once up to isomorphisms, in the construction this would mean $S = S'$. Assuming $S = S'$, how must $\mathcal{O}$ and $\mathcal{O}'$ be related?

Proposition 16. *Let ψ be an automorphism of S and let $\mathcal{O} = \{O_{u_i} \mid 1 \leqslant i \leqslant |D|\}$ and $\mathcal{O}' = \{O'_{u_i} \mid 1 \leqslant i \leqslant |D|\}$ be representations of connecting subgraphs. If there exists an automorphism π of D such that $O'_{\pi(u_i)} = \psi(O_{u_i})$, then $G_S(\mathcal{O})$ and $G_S(\mathcal{O}')$ are isomorphic.*

Proof. Let $G = G_S(\mathcal{O})$, $G' = G_S(\mathcal{O}')$. The isomorphism $\Phi : G \to G'$ is given by the following bijection of $V(G)$ onto $V(G')$: $\Phi|_D = \pi$ and $\Phi|_S = \psi$. Let $uv \in E(G)$. If $u, v \in V(S)$ or $u, v \in V(D)$ then clearly $\Phi(u)\Phi(v) \in E(G')$. Suppose $u \in V(D)$, $v \in V(S)$. Then $\Phi(u)\Phi(v) = \pi(u)\psi(v)$ is an edge in G', since $O'_{\pi(u)} = \psi(O_u)$ and therefore $v \in O_u$ implies $\psi(v) \in \psi(O_u) = O'_{\pi(u)}$. In the same way it can be shown that Φ^{-1} is a homomorphism. $\square$

Proposition 17. *Let $G = G_S(\mathcal{O})$ and $G' = G_S(\mathcal{O}')$ be isomorphic graphs and $\Phi : G \to G'$ an isomorphism. Let $\psi = \Phi|_S$, $\pi = \Phi|_D$. Then ψ is an automorphism of S and π is an automorphism of D such that $O'_{\pi(u_i)} = \psi(O_{u_i})$.*

Proof. Φ maps saturated vertices of G to saturated vertices of G' and therefore ψ is an automorphism of S. Similarly, π must be an automorphism of D. Let $u_i \in D(G)$. Since Φ is an isomorphism, the saturated neighbours of $\Phi(u_i)$ are in $\Phi(O_{u_i})$, that is $O'_{\Phi(u_i)} = \Phi(O_{u_i})$, and therefore $O'_{\pi(u_i)} = \psi(O_{u_i})$. $\square$

For a given graph *S* and cardinality *d* of the deficit subgraph *D*, let $\mathbb{O}$ denote the set of all those multisets $\mathcal{O}$ of cardinality *d* that fulfill the requirements of Proposition 15. Then the automorphism group of *S* acts in a natural way on $\mathbb{O}$ and Propositions 16 and 17 imply the following theorem:

Theorem 18. *Two graphs $G_S(\mathcal{O})$ and $G_S(\mathcal{O}')$ are isomorphic if and only if $|\mathcal{O}| = |\mathcal{O}'|$ and $\mathcal{O}$ and $\mathcal{O}'$ are in the same orbit of the automorphism group of S on $\mathbb{O}$.*

Note that the number of times that an open vertex w_i occurs in the adjacency lists of a given representation $\mathcal{O}$ of a connecting subgraph *C* depends only on $\deg_S(w_i)$ and not on the structure of the graph. Since the program we use for generating saturated subgraphs (see [14]) lists the graph in a way that the vertex labels are sorted with respect to their degree, so that the degree of the vertex does only depend on the degree sequence and not on the graph itself, the set $\mathbb{O}$ is identical for all these graphs, so it must be computed only once for every degree sequence. This is used to speed up the generation of the set of Δ-maximal candidates.

The Δ-maximal candidates were generated in the following way: first, all degree sequences, satisfying the conditions of Corollary 11 and Propositions 12–14 were generated. Then, the set $\mathbb{O}$ of all possible $\mathcal{O}$ for a given degree sequence are generated and stored. All the nonisomorphic graphs with this degree sequence, that is all the *G*-saturated subgraphs *S*, are generated by a program of Grund [14]. For each *S*, the automorphism group and the orbits of the automorphism group on $\mathbb{O}$ are computed. From each orbit one representative is chosen to connect the given *S* to the deficit clique *D*, yielding a Δ-maximal candidate. The automorphism group is computed by McKay's program *nauty* [21,22,24].

4. Searching for the critical subgraphs

This section describes how the Δ-maximal candidates were searched for spanning Δ-critical subgraphs.

In the first step, all colourable candidates were discarded. The main idea for checking colourability is the following proposition, which leads to a recursive algorithm. The proof of this proposition is trivial and is omitted.

Proposition 19. *Given a graph G and $e \in E(G)$. G is of class* 1 *if and only if there exists an inclusion-maximal matching $M \subseteq E(G)$ (i.e., M is not properly contained in any other matching), such that $e \in M$, all vertices of degree $\Delta(G)$ are contained in edges of M and $G - M$ is of class* 1.

The way the initial edge *e* is chosen has an effect on the efficiency of the program. We have chosen *e* as an edge with endpoints of the smallest possible degree.

Furthermore a test on whether G, resp. G minus a vertex of minimum degree is overfull is performed. This allowed fast detection of many class 2 graphs.

In the second step we discard the candidates that are not 2-connected. Since the graphs to be tested were small, this was done by a straightforward algorithm that merely checks the definition.

The following lemma by Miao and Liu will be used to show that Δ-maximal candidates with some special degree sequences cannot contain spanning Δ-critical subgraphs. Graphs with these degree sequences were discarded without further checks.

Lemma 20 (*Miao and Liu [27]*). *Let G be a Δ-critical graph containing vertices u and x such that the distance between u and x is 3. Then $\deg(u) = 2$ implies $\deg(x) \geqslant \Delta - 1$.*

Corollary 21. *Let $\Delta > 3$ and G be a Δ-critical graph containing at least $k \geqslant 1$ vertices $w_1, \ldots, w_k$ of degree 2 and an additional vertex $v_d \notin \{w_1, \ldots, w_k\}$ of degree $2 \leqslant d \leqslant \Delta - 2$. Then the order of G is at least $3k + \Delta + d - 1$.*

Proof. By Vizing's Adjacency Lemma, the sets of neighbours of $w_1, \ldots, w_k, v_d$ and the set $\{w_1, \ldots, w_k, v_d\}$ must be pairwise disjoint, thus their union U contains $3k + d + 1$ vertices. Let x_1, x_2 be the neighbours of w_1. Then $\deg_G(x_1) = \Delta$, and the neighbours of x_1 different from w_1 and x_2 cannot be in U (Lemma 20). Hence, G contains at least $\Delta - 2$ vertices which are not in U, and therefore the order of G is at least $3k + \Delta + d - 1$. $\square$

From this result we can deduce that for $|G| = n \leqslant 14$ there is no Δ-spanned graph G with degree sequence $8^{n-3}444$, $9^{n-3}544$, $10^{n-3}444$, $10^{n-3}554$, or $10^{n-3}644$: a spanning critical subgraph cannot contain one of the edges connecting the deficit vertices of G in view of Vizing's adjacency Lemma, thus Corollary 21 is violated. Hence, the degree sequences mentioned above do not have to be considered.

All the remaining graphs were checked for colourability and searched for spanning Δ-critical subgraphs as described below:

Let us call an edge *e VAL-removable in G*, if e does not fulfill the necessary condition for being in a critical subgraph of G stated in the following Corollary of Lemma 7.

Corollary 22. *Let G be a graph of maximum degree Δ, $uv \in E(G)$. For a vertex w let s_w denote the number of saturated neighbours of w.*

If $s_u < \Delta - \deg(v) + 1$ or $s_v < \Delta - \deg(u) + 1$, then uv is not contained in any spanning Δ-critical subgraph of G.

We denote with $\mathtt{val}_0(G)$ the graph obtained from G after removing all VAL-removable edges from G. Let $\mathtt{val}_n(G) := \mathtt{val}_0(\mathtt{val}_{n-1}(G))$. As the number of edges in G is finite, there exists $N \in \mathbb{N}$, such that $\mathtt{val}_N(G) = \mathtt{val}_{N+1}(G)$. Let us define $\mathtt{val}(G) := \mathtt{val}_N(G)$.

In order to compute $\mathtt{val}(G)$ we did not delete all edges that are VAL-removable in G in parallel, but having detected an edge that can be removed, this edge is removed at once and all parameters that might be affected by this operation (the numbers of saturated neighbours and the valencies) are updated at once. An edge that could be removed with the old set of parameters can also be removed with this updated sets of parameters, but not necessarily the other way around. So, in general, at least some tests on removability with negative results are avoided which leads to a faster algorithm. The proof that this leads to the same graph $\mathtt{val}(G)$ is easy and is left to the reader.

When searching for a spanning critical subgraph in a 2-connected noncolourable graph G, first $\mathtt{val}(G)$ is computed, and then for every edge we check whether its removal yields a colourable graph. If this is the case for all edges, then the graph is critical. If after the removal of an edge e the graph is still non-colourable, we recursively apply the algorithm to $G - e$. For a (possibly empty) sequence $e_1, e_2, \dots, e_k$ of edges in $E(G)$, we define the graph $G(e_1, \dots, e_k)$ recursively as $G(\emptyset) := \mathtt{val}(G)$, and $G(e_1, \dots, e_j) := \mathtt{val}(G(e_1, \dots, e_{j-1}) - e_j)$ for $j \geqslant 1$. Let $\preceq$ be a linear (e.g. lexicographic) ordering of the edges of G. We say that $e_1, e_2, \dots, e_k$ is a *regular pruning sequence*, if $e_i \prec e_{i+1}$ for all $i \in \{1, \dots, k-1\}$ and if for $i \in \{1, \dots, k\}$ $e_i \preceq e$ for all $e \in E(G(e_1, \dots, e_{i-1}) \setminus E(G(e_1, \dots, e_i))$.

Proposition 23. *Let H be a spanning Δ-critical subgraph of G. Then there exists a unique regular pruning sequence $e_1, \dots, e_k$ of edges in G, such that $H = G(e_1, \dots, e_k)$.*

Proof. Let $F_1 = E(G(\emptyset)) \setminus E(H)$, and define recursively for $F_i \neq \emptyset$: $e_i = \min_{\preceq} F_i$ and for $i > 1$ let $F_i := E(G(e_1, \dots, e_{i-1})) \setminus E(H)$. Note that H is a spanning critical subgraph of $G(\emptyset)$ and therefore none of the edges of H is VAL-removable in any $G(e_1, ..., e_j)$. As F_1 is finite, and $F_{i+1} \subsetneq F_i$, there is some k such that $F_{k+1} = \emptyset$. Then $H = G(e_1, \dots, e_k)$. As $F_i \subseteq F_{i+1}$, we have $e_i \prec e_{i+1}$ and since $E(G(e_1, \dots, e_{i-1})) \setminus E(G(e_1, \dots, e_i)) \subseteq F_i$ also the second requirement for a regular pruning sequence is fulfilled.

Now assume we have another regular pruning sequence $e'_1, e'_2, \dots, e'_{k'}$ for the graph H and adopt the notation above. Let $j \leqslant \min\{k, k'\}$ be the smallest index so that $e'_j \neq e_j$. Such an index must exist since otherwise not all edges would be removed and due to the minimality of j the set F_j is the same for both sequences. But since in order to form H, e_j must occur in $e'_{j+1}, \dots, e'_{k'}$ or in $E(G(e'_1, \dots, e'_{l-1})) \setminus E(G(e'_1, \dots, e'_l))$ for some $l \geqslant j$, the conditions for regular pruning sequences will be violated. $\square$

We searched for spanning critical subgraphs by removing edges in a way corresponding to regular pruning sequences and backtracking in case the computation of $\mathtt{val}(G)$ notices that an edge must be removed that would violate the condition of a regular pruning sequence. The set of critical subgraphs is finally filtered for pairwise nonisomorphic copies by a program using the canonical labelling routine in *nauty* [21,22,24].

5. The second approach and discussion

A second program implemented an algorithm very similar to the one described above. The main difference is that it did not construct saturated subgraphs S for given degree sequences, but for given δ and Δ and given $|S|$ and $\Theta_\Delta(S)$ (or equivalently: $|S|$ and $|E(S)|$). We used the following lemma to detect some cases where no Δ-maximal candidates exist:

Lemma 24. *For a Δ-maximal graph G we have*

$$\left\lceil \frac{\Delta(|S|-|D|)+|D|^2}{2} \right\rceil \leqslant |E(S)| \leqslant \left\lfloor \frac{\Delta|S|-2|D|}{2} \right\rfloor$$

Proof. Every vertex in D has valency at most $\Delta-1$, so at most $\Delta-1-(|D|-1)=\Delta-|D|$ edges starting at a vertex in D can end in vertices of S. So the total sum of degrees in S must be at least $\Delta|S|-|D|(\Delta-|D|)$ proving the first part of the inequality.

Since every vertex in D has at least two neighbours in S, the sum of degrees in S can be at most $\Delta|S|-2|D|$ which gives the second part of the inequality. □

This lemma excluded, for example, cases like $|G|=14$, $\Delta=5$ and $|D|=3$ because of the lower bound for the edges exceeding the upper bound or cases like $|G|=13$, $\Delta=10$ and $|D|=6$, since the lower bound for the number of edges exceeds the number of edges in a complete graph on $|S|$ vertices.

This second approach had a smaller generation rate but the advantage of being able to use a generation program for these graphs that is independent of Grund's program. We used Brendan McKay's *geng* [23,24] for the graphs with $|D|>0$ and also Markus Meringers *genreg* [25,26] for graphs with $|D|=0$. The only remaining overlap of the two implementations is (except for the platform and the compiler) the program *nauty*.

Our approach was designed to work fast for graphs of even order. As can be seen from the following tables, for graphs of odd order, almost all Δ-maximal candidates are 2-connected class 2 graphs. This can be easily deduced from the numbers of edges in the graphs:

Lemma 25. *Let G be a Δ-maximal graph of odd order. If* $|D| \in \{0, 1, 2, \Delta-2\}$ *then G contains an overfull subgraph with maximal degree Δ.*

Proof. In case $\Theta_\Delta(G) < \Delta$ the graph G is overfull. This is the case for $|D| \in \{0, 1\}$.

In case $|D|=2$ and G is not overfull, with v, w the deficit vertices, we have $\deg(v)+\deg(w) \leqslant \Delta$. This gives $\Theta_\Delta(S)=|E(C)| \leqslant \Delta-2$ proving that S is overfull.

In case $|D|=\Delta-2$, every vertex in D must have degree $\Delta-1$, so $\Theta_\Delta(G)=\Delta-2$. □

In other cases (like $|D|=3$) it can be easily seen that only those cases where the deficit vertices have very few neighbours in S do not lead to overfull graphs, so that the ratio of class 2 graphs is very large.

Searching these graphs for critical subgraphs is very expensive, since for each of these graphs a large number of subgraphs has to be tested. The advantage of our approach compared to the straightforward approach of generating all graphs with a suitable number of edges and testing them directly for being critical can be seen for even order, where a lot of graphs could be filtered out.

Compared to the number of all graphs on 14 vertices only a fraction of 1/250 000 was generated and filtered for being class 2 and 2-connected. Including the subgraphs that occurred while searching for spanning critical subgraphs, only a fraction of 1/100 000 was tested for being critical (not counting the colourability tests for class 1 graphs, since this is

Table 1
Results for $|G| = 13$

$\|G\|$, Δ, $\|D\|$	Δ-maximal candidates	2-connected class 2 Δ-maximal candidates
13,3,1	872	777
13,4,0	10 786	10 768
13,4,1	35 689	35 647
13,4,2	57 016	56 933
13,5,1	1 696 704	1 696 697
13,5,2	1 323 139	1 323 137
13,5,3	161 919	161 915
13,6,0	367 860	367 860
13,6,1	2 979 292	2 979 292
13,6,2	11 642 407	11 642 407
13,6,3	1 848 811	1 766 352
13,6,4	24 643	24 643
13,7,1	2 749 744	2 749 744
13,7,2	8 222 853	8 222 853
13,7,3	4 840 355	4 481 705
13,7,4	153 418	137 061
13,7,5	323	323
13,8,0	10 786	10 786
13,8,1	165 358	165 358
13,8,2	1 334 020	1 334 020
13,8,3	1 236 313	1 129 237
13,8,4	157 392	133 781
13,8,5	842	712
13,8,6	0	0
13,9,1	4103	4103
13,9,2	29 374	29 374
13,9,3	57 743	51 862
13,9,4	15 413	12 825
13,9,5	352	314
13,9,6	0	0
13,9,7	0	0

part of the test for being critical of the class 2 graph from which the graph was constructed by deleting an edge).

The programs were run on a cluster of 100 Linux machines with 133 to 450 MHz at the Universität Bielefeld. For the approach described in detail, generating the saturated subgraphs took approximately 490 h of CPU time, assembling the Δ-maximal candidates took additional 230 h. The most time consuming task was the examination of the candidates, which took 3770 h, yielding altogether approximately 4500 h of pure CPU time.

The results of both implementations are given in Theorems 1 and 2 and in Tables 1 and 2 .

Even with much more and much faster computers it is not possible to use the same approach for order 16. New ideas and new insight into the structure of critical graphs are needed in order to finally answer Yap's question.

Table 2
Results for $|G| = 14$

$\|G\|$, Δ, $\|D\|$	Δ-maximal candidates	2-connected class 2 Δ-maximal candidates[a]
14,5,0	3 459 386	22
14,5,1	17 526 403	17 526 384
14,5,2	43 353 428	552
14,5,3	0	0
14,6,0	21 609 301	7
14,6,1	171 046 398	171 046 398
14,6,2	648 221 257	132 292 661
14,6,3	88 127 504	257
14,6,4	919 510	9
14,7,0	21 609 301	0
14,7,1	239 967 643	239 967 643
14,7,2	1 304 849 058	379 522 893
14,7,3	472 124 665	0
14,7,4	15 994 671	0
14,7,5	0	0
14,8,0	3 459 386	0
14,8,1	53 889 268	53 889 268
14,8,2	413 311 923	154 615 911
14,8,3	324 131 831	19 622 620[a]
14,8,4	32 727 669	0
14,8,5	141 360	0
14,8,6	24	0
14,9,0	88 193	0
14,9,1	1 850 802	1 850 802
14,9,2	19 871 394	8 789 828
14,9,3	29 738 464	2 836 593[a]
14,9,4	7 206 269	0
14,9,5	129 315	0
14,9,6	66	0
14,9,7	0	0
14,10,0	540	0
14,10,1	11 400	11 400
14,10,2	157 783	80 827
14,10,3	408 485	54 235[a]
14,10,4	204 932	2394
14,10,5	10 152	0
14,10,6	19	0
14,10,7	0	0
14,10,8	0	0

[a] For 14,8,3, 14,9,3 and 14,10,3 only those 2-connected class 2 Δ-maximal candidates are listed that do not have one of the degree sequences $8^{11}444$, $9^{11}544$, $10^{11}444$, $10^{11}554$, or $10^{11}644$.

References

[1] L.W. Beineke, R.J. Wilson, On the edge-chromatic number of a graph, Discrete Math. 5 (1973) 15–20.
[2] D. Bokal, Grafi, kritični za barvanje povezav, diplomsko delo, Fakulteta za matematiko in fiziko, Univerza v Ljubljani, Ljubljana, 2002.

[3] G. Brinkmann, Isomorphism rejection in structure generation programs, published in P.Hansen, P.W. Fowler, M. Zheng (Eds.), Discrete Mathematical Chemistry, DIMACS Series on Discrete Mathematics and Theoretical Computer Science, vol. 51, 2000, pp. 25–38.
[4] G. Brinkmann, E. Steffen, 3- and 4- critical graphs of small even order, Discrete Math. 169 (1997) 193–197.
[5] G. Brinkmann, E. Steffen, Chromatic-index-critical graphs of orders 11 and 12, European J. Combin. 19 (1998) 889–900.
[6] A.G. Chetwynd, A.J.W. Hilton, The chromatic index of graphs of even order with many edges, J. Graph Theory 8 (1984) 463–470.
[7] A.G. Chetwynd, A.J.W. Hilton, Partial edge-colourings of complete graphs or of graphs which are nearly complete, Graph Theory and Combinatorics (Cambridge, 1983), Academic Press, London, 1984 pp. 81–97.
[8] A.G. Chetwynd, A.J.W. Hilton, The edge-chromatic class of graphs with maximum degree at least $|V| - 3$, Graph Theory in memory of G.A. Dirac (Sandbjerg, 1985), Annals of Discrete Mathematics, vol. 41, 1989, pp. 91–110.
[9] A.G. Chetwynd, R.G. Wilson, The rise and fall of the critical graph conjecture, J. Graph Theory 7 (1983) 153–157.
[10] A.G. Chetwynd, H.P. Yap, Chromatic index critical graphs of order 9, Discrete Math. 47 (1983) 23–33.
[11] R. Diestel, Graph Theory, Springer, Berlin, 1997.
[12] S. Fiorini, R.J. Wilson, Edge-colorings of graphs, Research Notes in Mathematics, vol. 16, Pitman Publishing Limited, London, 1977.
[13] M.K. Goldberg, Construction of class 2 graphs with maximum vertex degree 3, J. Combin. Theory Ser. B 31 (1981) 282–291.
[14] R. Grund, Konstruktion schlichter Graphen mit gegebener Gradpartition, Bayreuth. Math. Schr. 44 (1993) 73–104.
[15] T. Grüner, Ein neuer Ansatz zur rekursiven Erzeugung von schlichten Graphen, Diplomarbeit, Universität Bayreuth, 1995.
[16] S. Grünewald, E. Steffen, Chromatic-index-critical graphs of even order, J. Graph Theory 30 (1999) 27–36.
[17] F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1969.
[18] A.J.W. Hilton, R.J. Wilson, Edge-colorings of graphs: a progress report, in: M.F. Cabobianco et al. (Eds.), Graph Theory and its Applications: East and West, New York, 1989, pp. 241–249.
[19] I.T. Jakobsen, Some remarks on the chromatic index of a graph, Arch. Math. (Basel) 24 (1973) 440–448.
[20] I.T. Jakobsen, On critical graphs with chromatic index 4, Discrete Math. 9 (1974) 265–276.
[21] B.D. McKay, Practical graph isomorphism, 10th Manitoba Conference on Numerical Mathematics and Computing (Winnipeg, 1980); Congressus Numerantium, vol. 30, 1981, pp. 45–87.
[22] B.D. McKay, Nauty Users Guide (version 1.5), Technical Reports TR-CS-90-02, Department of Computer Science, Australian National University, 1990.
[23] B.D. McKay, Isomorph-free exhaustive generation, J. Algorithms 26 (1998) 306–324.
[24] B.D. McKay, Nauty, see http://cs.anu.edu.au/~bdm/nauty/, Copyright (1994–2000) Brendan McKay.
[25] M. Meringer, Erzeugung regulärer Graphen, Diplomarbeit, Universität Bayreuth, 1996.
[26] M. Meringer, Fast generation of regular graphs and construction of cages, J. Graph Theory 30 (2) (1999) 137–146.
[27] L. Miao, G. Liu, An extension of one theorem of critical edge-chromatic graphs, Math. Appl. 12 (3) (1999) 69–71.
[28] M. Plantholt, The chromatic index of graphs with a spanning star, J. Graph Theory 5 (1981) 45–53.
[29] M. Plantholt, The chromatic index of graphs with large maximum degree, Discrete Math. 47 (1983) 91–96.
[30] D.B. West, Introduction to Graph Theory, Prentice-Hall, Englewood Cliffs, NJ, 1996.
[31] H.P. Yap, Some topics in graph theory, London Mathematical Society, Lecture Note Series, vol. 108, Cambridge University Press, Cambridge, 1986.