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Regularity and decay of solutions of nonlinear harmonic oscillators

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Abstract

We prove sharp analytic regularity and decay at infinity of solutions of variable coefficients nonlinear harmonic oscillators. Namely, we show holomorphic extension to a sector in the complex domain, with a corresponding Gaussian decay, according to the basic properties of the Hermite functions in $\mathbb{R}^d$. Our results apply, in particular, to nonlinear eigenvalue problems for the harmonic oscillator associated to a real-analytic scattering, or asymptotically conic, metric in $\mathbb{R}^d$, as well as to certain perturbations of the classical harmonic oscillator.

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1. Introduction

The harmonic oscillator $H = -\Delta + |x|^2$ in $\mathbb{R}^d$ represents one of the simplest and yet more useful models for several physical phenomena, and its relevance both in Mathematical Analysis and Physics is well known. Its eigenfunctions, namely the Hermite functions $h_\alpha(x)$, are given by the formulae $h_\alpha(x) = p_\alpha(x)e^{-|x|^2/2}$, $\alpha \in \mathbb{N}^d$, where p_α is a polynomial of degree $|\alpha|$ (see

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e.g. [29]). Two remarkable features of the Hermite functions are their Gaussian decay at infinity, and their very high regularity. In fact, we have

$$\left|h_\alpha(x)\right| \lesssim e^{-c|x|^2} \quad \text{for } x \in \mathbb{R}^d, \qquad \left|\widehat{h_\alpha}(\xi)\right| \lesssim e^{-c|\xi|^2} \quad \text{for } \xi \in \mathbb{R}^d \tag{1.1}$$

for every $c < 1/2$, where $\widehat{h_\alpha}(\xi)$ denotes the Fourier transform of h_α. The functions h_α in fact extend to entire functions $h_\alpha(x+iy)$ in the complex space $\mathbb{C}^d$ and, for every $0 < \varepsilon < 1$, we have the estimates

$$\left|h_\alpha(x+iy)\right| \lesssim e^{-c|x|^2} \quad \text{in the sector } |y| < \varepsilon\bigl(1+|x|\bigr), \tag{1.2}$$

for some $c > 0$.

In this paper we wonder to what extent these properties continue to hold for *nonlinear* perturbations of the harmonic oscillator, possibly with variable coefficients. Relevant models are equations of the type

$$-\Delta u + |x|^2 u - \lambda u = F[u], \quad \lambda \in \mathbb{C}, \tag{1.3}$$

with a nonlinearity of the form $F[u] = \sum_{|\alpha|+|\beta|\leqslant 1} c_{\alpha\beta} x^\beta \partial^\alpha u^k$, $k \geqslant 2$. Cappiello, Gramchev and Rodino in [8] showed by a counterexample that generally, even in dimension $d = 1$, there can exist Schwartz solutions of (1.3) which do not extend to entire functions in $\mathbb{C}$. In fact, a refinement of their argument (see Section 5 below) shows that a sequence of complex singularities may occur, approaching a straight line at infinity. On the other hand, as a positive result, it was proved in [8] that every solution $u \in H^s(\mathbb{R}^d)$, $s > d/2 + 1$, of (1.3) extends to a holomorphic function $u(x+iy)$ on the strip $\{z \in \mathbb{C}^d\colon |\operatorname{Im} z| < T\}$ and satisfies there an estimate of the type $|u(x+iy)| \leqslant Ce^{-c|x|^2}$, for some $c, C, T > 0$. Similar results, namely, holomorphic extension to a *strip* and super-exponential decay, were proved in [8,11] for more general classes of elliptic operators with polynomial coefficients.

The above mentioned negative result as well as the estimates (1.2), valid in a sector in the linear case, suggest the possibility, even in the presence of certain nonlinear perturbations, of a holomorphic extension of the solutions to a *sector*, rather than only a strip, with a corresponding Gaussian decay estimate. In this paper we show, for a large class of equations including (1.3), even with *non-polynomial* coefficients, that this is in fact the case. The techniques developed here actually will apply to much more general differential (and pseudodifferential) operators. To motivate the class of operators we will consider, we first discuss a special yet important example.

Consider the equation $Pu = F[u]$, with

$$P = \sum_{j,k=1}^{d} g^{jk}(x)\partial_j\partial_k + \sum_{k=1}^{d} b_k(x)\partial_k + V(x), \tag{1.4}$$

where the functions g^{jk}, b_k, and the potential V are real-analytic in $\mathbb{R}^d$, and satisfy the following conditions.

We suppose that the matrix (g^{jk}) is real and symmetric and that there exists a constant $C > 0$ such that

$$\sum_{j,k=1}^{d} g^{jk}(x)\xi_j\xi_k \geqslant C^{-1}|\xi|^2 \quad \forall x, \xi \in \mathbb{R}^d, \tag{1.5}$$

as well as

$$\left|\partial^\alpha g^{jk}(x)\right| + \left|\partial^\alpha b_k(x)\right| \leqslant C^{|\alpha|+1}\alpha!\langle x\rangle^{-|\alpha|} \quad \forall x \in \mathbb{R}^d,\ \alpha \in \mathbb{N}^d, \tag{1.6}$$

where $\langle x\rangle = (1+|x|^2)^{1/2}$. Moreover we assume that

$$\begin{cases} \operatorname{Re} V(x) \geqslant C^{-1}|x|^2 & \text{for } |x| > C, \\ \left|\partial^\alpha V(x)\right| \leqslant C^{|\alpha|+1}\alpha!\langle x\rangle^{2-|\alpha|} & \forall x \in \mathbb{R}^d,\ \alpha \in \mathbb{N}^d. \end{cases} \tag{1.7}$$

We consider a nonlinearity of the form

$$F[u] = \sum_{\substack{2\leqslant h+l\leqslant N \\ 1\leqslant j\leqslant d}} F_{jhl}(x)u^h(\partial_j u)^l, \tag{1.8}$$

for some $N \in \mathbb{N}$, where

$$\left|\partial^\alpha F_{jhl}(x)\right| \leqslant C^{|\alpha|+1}\alpha!\langle x\rangle^{1-\min\{1,l\}-|\alpha|} \quad \forall x \in \mathbb{R}^d,\ \alpha \in \mathbb{N}^d. \tag{1.9}$$

Then, we claim that:

Under these assumptions, every solution $u \in H^s(\mathbb{R}^d)$, $s > d/2 + \min\{1,l\}$, of the equation $Pu = F[u]$, extends to a holomorphic function $u(x+iy)$ in the sector $\{z = x+iy \in \mathbb{C}^d\colon |y| < \varepsilon(1+|x|)\}$ of $\mathbb{C}^d$ for some $\varepsilon > 0$, satisfying there the estimates $|u(x+iy)| \leqslant Ce^{-c|x|^2}$, for some constants $C > 0$, $c > 0$.

Notice that if $V(x)$ satisfies (1.7) then also $V(x) - \lambda$, $\lambda \in \mathbb{C}$, satisfies it, so that the above result applies to the corresponding eigenvalue problem as well. In the linear case ($F[u] = 0$) this result intersects the wide literature on the decay and regularity of eigenfunctions of Schrödinger operators, cf. Agmon [1], Nakamura [23], Sordoni [28], Rabinovich [25], Rabinovich and Roch [26] and many others.

We also remark that suitable perturbations of the standard harmonic oscillator fall in this class of equations, as well as the harmonic oscillator associated to a real-analytic scattering, or asymptotically conic, Riemannian metric in $\mathbb{R}^d$ (see Section 5 below). For a detailed analysis of these metrics and their important role in geometric scattering theory we refer to Melrose [20,21], Melrose and Zworski [22].

Let us now state our main result in full generality. We consider nonlinear equations whose linear part is a differential or even pseudodifferential operator

$$Pu(x) = p(x,D)u(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\xi}p(x,\xi)\hat{u}(\xi)\,d\xi, \tag{1.10}$$

with symbol p in the class $\Gamma_a^m(\mathbb{R}^d)$, $m > 0$, defined as the space of all functions $p \in C^\infty(\mathbb{R}^{2d})$ satisfying the estimates

$$\left|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)\right| \leqslant C^{|\alpha|+|\beta|+1}\alpha!\beta!\big(1+|x|+|\xi|\big)^{m-|\alpha|}\langle x\rangle^{-|\beta|} \tag{1.11}$$

for all $(x,\xi) \in \mathbb{R}^{2d}$, $\alpha, \beta \in \mathbb{N}^d$, and for some positive constant C independent of α, β. This class is particularly suited to study harmonic oscillators with variable analytic coefficients and it is inspired by the class considered by Shubin [27] and Helffer [16] which was in fact modeled on the harmonic oscillator and its real powers. However, a *differential* operator belongs to that class only if its coefficients are polynomial, which is an unpleasant limitation. With respect to [16,27], we overcome this restriction by assuming the less demanding estimates (1.11), which at the same time imply that the symbol p is real-analytic. For example, a function

$$p(x,\xi)=\sum_{|\alpha|\leqslant m}c_\alpha(x)\xi^\alpha$$

belongs to $\Gamma_a^m(\mathbb{R}^d)$ if the coefficients c_α satisfy $|\partial^\beta c_\alpha(x)|\leqslant C^{|\beta|+1}\beta!\langle x\rangle^{m-|\alpha|-|\beta|}$.

We shall assume moreover the symbol p of our operator to be Γ-elliptic, in the sense that, for some constant $R>0$,

$$\inf_{|x|+|\xi|\geqslant R}\bigl(1+|x|+|\xi|\bigr)^{-m}\bigl|p(x,\xi)\bigr|>0. \tag{1.12}$$

This is clearly a global version of the classical notion of ellipticity. For example, by (1.5)–(1.7), the symbol of the operator in (1.4) belongs to $\Gamma_a^2(\mathbb{R}^d)$ and satisfies (1.12) with $m=2$.

We moreover consider a nonlinearity of the form

$$F[u]=\sum_{h,l,\rho_1,\dots,\rho_l}F_{h,l,\rho_1\dots\rho_l}(x)\prod_{k=1}^{l}\partial^{\rho_k}u, \tag{1.13}$$

where the above sum is finite and $h,l\in\mathbb{N}$, $l\geqslant 2$, $\rho_1,\dots,\rho_l\in\mathbb{N}^d$ satisfy the condition $h+\max\{|\rho_k|\}\leqslant\max\{m-1,0\}$. We assume that the functions $F_{h,l,\rho_1\dots\rho_l}(x)$ satisfy the following estimates

$$\bigl|\partial^\beta F_{h,l,\rho_1\dots\rho_l}(x)\bigr|\leqslant C^{|\beta|+1}\beta!\langle x\rangle^{h-|\beta|}, \tag{1.14}$$

for some positive constant C depending on $h,l,\rho_1,\dots,\rho_l$ and independent of β. Our main result is the following.

Theorem 1.1. *Let $p\in\Gamma_a^m(\mathbb{R}^d)$, $m>0$, satisfy* (1.12) *and let $F[u]$ be of the form* (1.13), (1.14) *(possibly with some factors in the product replaced by their conjugates). Assume that $u\in H^s(\mathbb{R}^d)$, $s>d/2+\max_k\{|\rho_k|\}$, is a solution of the equation $Pu=F[u]$. Then u extends to a holomorphic function $u(x+iy)$ in the sector*

$$\bigl\{z=x+iy\in\mathbb{C}^d:\ |y|<\varepsilon\bigl(1+|x|\bigr)\bigr\} \tag{1.15}$$

of $\mathbb{C}^d$, for some $\varepsilon>0$, satisfying there the estimates

$$\bigl|u(x+iy)\bigr|\leqslant Ce^{-c|x|^2}, \tag{1.16}$$

for some constants $C>0$, $c>0$. The same holds for all the derivatives of u.

The linear case $F[u]=0$ deserves a special interest and will be treated in detail in Section 5. We emphasize the fact that the form of the domain of holomorphic extension as a sector is, in a sense, completely sharp, even for the model (1.3) (see Section 5).

Let us briefly compare our result with those in the existent literature. Several papers were devoted to the problem of holomorphic extension to a strip and exponential decay of solutions of certain semilinear elliptic equations arising in the theory of solitary waves or bound states, whose model is $-\Delta u + u = |u|^{p-1}u$, cf. Berestycki and Lions [2], Bona and Grujic' [4], Bona and Li [5,6], Biagioni and Gramchev [3], Gramchev [12], Cappiello, Gramchev and Rodino [9,10] and the references therein; see also our recent paper [7] for the extension to a sector. However, as it is clear from our model (1.3), we consider a different class of equations here, and in fact we deal with Gaussian, rather than exponential decay. Instead, as already mentioned, a class similar to the present one was considered in [8], where the problem of the extension to a strip, combined with super-exponential decay, was addressed. The main novelties of the present work are the possibility of treating non-polynomial coefficients and nonlinearities, and the achievement of the optimal extension result, namely to a sector. Finally, we stress the fact that the machinery developed in the present paper should hopefully apply to evolution counterparts of the above equations, in the spirit of Hayashi et al., see [14,15]. It seems interesting, in particular, to find lower bounds estimates on the width of the above sector, depending on time. We do not consider these topics here, but we plan to devote a future paper to them.

The paper is organized as follows. In Section 2 we list some known factorial and binomial estimates and we collect some basic properties of the pseudodifferential operators introduced before, which will be instrumental in the proofs of our results. In Section 3 we introduce a suitable space of analytic functions which exploits the two properties (1.15) and (1.16). Section 4 is devoted to the proof of Theorem 1.1 which is based on an iterative scheme on the space defined in Section 3. Finally, in Section 5 we give some concluding remarks. In particular, we read our results on the models introduced above and treat in detail their application to the Schrödinger operator in $\mathbb{R}^d$ with a scattering metric. Finally, we discuss the sharpness of our results for what concerns the shape of the domain of the holomorphic extension as a sector of $\mathbb{C}^d$.

2. Notation and preliminary results

2.1. Factorial and binomial coefficients

We use the usual multi-index notation for factorial and binomial coefficients. Hence, for $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ we set $\alpha! = \alpha_1! \dots \alpha_d!$ and for $\beta, \alpha \in \mathbb{N}^d$, $\beta \leqslant \alpha$, we set $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$.

The following inequality is standard and used often in the sequel:

$$\binom{\alpha}{\beta} \leqslant 2^{|\alpha|}. \tag{2.1}$$

Also, we recall the identity

$$\sum_{\substack{|\alpha'|=j \\ \alpha' \leqslant \alpha}} \binom{\alpha}{\alpha'} = \binom{|\alpha|}{j}, \quad j = 0, 1, \dots, |\alpha|,$$

which follows from $\prod_{i=1}^{d}(1+t)^{\alpha_i}=(1+t)^{|\alpha|}$, and gives in particular

$$\binom{\alpha}{\beta}\leqslant\binom{|\alpha|}{|\beta|},\quad \alpha,\beta\in\mathbb{N}^d,\ \beta\leqslant\alpha. \tag{2.2}$$

The last estimate implies in turn, by induction,

$$\frac{\alpha!}{\delta_1!\cdots\delta_j!}\leqslant\frac{|\alpha|!}{|\delta_1|!\cdots|\delta_j|!},\quad \alpha=\delta_1+\cdots+\delta_j, \tag{2.3}$$

as well as

$$\frac{\alpha!}{(\alpha-\beta)!}\leqslant\frac{|\alpha|!}{|\alpha-\beta|!},\quad \beta\leqslant\alpha. \tag{2.4}$$

Finally we recall the so-called inverse Leibniz' formula:

$$x^\beta\partial^\alpha u(x)=\sum_{\gamma\leqslant\beta,\ \gamma\leqslant\alpha}\frac{(-1)^{|\gamma|}\beta!}{(\beta-\gamma)!}\binom{\alpha}{\gamma}\partial^{\alpha-\gamma}\big(x^{\beta-\gamma}u(x)\big). \tag{2.5}$$

2.2. Pseudodifferential operators

We collect here some basic properties of the class $\Gamma_a^m(\mathbb{R}^d)$ defined by the estimates (1.11) and of the corresponding operators (1.10). Actually, for our purposes it is not necessary to develop a specific calculus for the analytic symbols of $\Gamma_a^m(\mathbb{R}^d)$. We shall deduce the properties we need from those of the larger class $\Gamma^m(\mathbb{R}^d)$ defined as the space of all functions $p\in C^\infty(\mathbb{R}^{2d})$ satisfying the following estimates: for every $\alpha,\beta\in\mathbb{N}^d$ there exists a constant $C_{\alpha,\beta}>0$ such that

$$\big|\partial_x^\beta\partial_\xi^\alpha p(x,\xi)\big|\leqslant C_{\alpha,\beta}\big(1+|x|+|\xi|\big)^{m-|\alpha|}\langle x\rangle^{-|\beta|} \tag{2.6}$$

for every $x,\xi\in\mathbb{R}^d$. Clearly $\Gamma_a^m(\mathbb{R}^d)\subset\Gamma^m(\mathbb{R}^d)$. We shall denote by $\mathrm{OP}\Gamma^m(\mathbb{R}^d)$ (respectively $\mathrm{OP}\Gamma_a^m(\mathbb{R}^d)$) the class of pseudodifferential operators with symbol in $\Gamma^m(\mathbb{R}^d)$ (respectively in $\Gamma_a^m(\mathbb{R}^d)$). We endow $\Gamma^m(\mathbb{R}^d)$ with the topology defined by the seminorms

$$\|p\|_N^{(\Gamma^m)}=\sup_{|\alpha|+|\beta|\leqslant N}\ \sup_{(x,\xi)\in\mathbb{R}^{2d}}\big\{\big|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)\big|\big(1+|x|+|\xi|\big)^{-m+|\alpha|}\langle x\rangle^{|\beta|}\big\},\quad N\in\mathbb{N}.$$

The properties of $\Gamma^m(\mathbb{R}^d)$ follow from the general Weyl–Hörmander's calculus in [18, Chapter XVIII]; with the notation used there, $\Gamma^m(\mathbb{R}^d)=S(M,g)$, with the weight $M(x,\xi)=(1+|x|^2+|\xi|^2)^{m/2}$ and the metric

$$g_{x,\xi}(y,\eta)=\frac{|dy|^2}{1+|x|^2}+\frac{|d\eta|^2}{1+|x|^2+|\xi|^2}.$$

We also refer the reader to [24, Chapter 1] for an elementary and self-contained presentation; with the notation in [24, Definition 1.1.1] we have $\Gamma^m(\mathbb{R}^d)=S(M;\Phi,\Psi)$, with $M(x,\xi)$ as above and

$$\Phi(x,\xi)=\langle x\rangle,\qquad \Psi(x,\xi)=\big(1+|x|^2+|\xi|^2\big)^{1/2}.$$

Now, if $p \in \Gamma^m(\mathbb{R}^d)$ then $p(x,D)$ defines a continuous map $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ which extends to a continuous map $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. The composition of two such operators is therefore well defined in $\mathcal{S}(\mathbb{R}^d)$ and in $\mathcal{S}'(\mathbb{R}^d)$; more precisely, if $p_1 \in \Gamma^{m_1}(\mathbb{R}^d)$ and $p_2 \in \Gamma^{m_2}(\mathbb{R}^d)$, then $p_1(x,D)p_2(x,D) = p_3(x,D)$ with $p_3 \in \Gamma^{m_1+m_2}(\mathbb{R}^d)$ and the map $(p_1,p_2) \mapsto p_3$ is continuous $\Gamma^{m_1}(\mathbb{R}^d) \times \Gamma^{m_2}(\mathbb{R}^d) \to \Gamma^{m_3}(\mathbb{R}^d)$. Moreover we have that $\bigcap_{m\in\mathbb{R}} \Gamma^m(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^{2d})$. In particular, operators with Schwartz symbols are globally regularizing, i.e. they map continuously $\mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$.

Operators in $\mathrm{OP}\Gamma^m(\mathbb{R}^d)$ are also bounded on certain weighted Sobolev spaces. We consider, for simplicity, the case of integer positive exponents (we will only need this case, see [27] for the general case). For $s \in \mathbb{N}$, we define

$$Q^s(\mathbb{R}^d) = \Big\{ u \in L^2(\mathbb{R}^d):\ \|u\|_{Q^s} := \sum_{|\alpha|+|\beta|\leqslant s} \big\| x^\beta \partial^\alpha u \big\|_{L^2} < \infty \Big\}. \tag{2.7}$$

We recall that $\bigcap_{s\in\mathbb{N}} Q^s(\mathbb{R}^d)\mathcal{S}(\mathbb{R}^d)$. Now, if $P \in \mathrm{OP}\Gamma^m(\mathbb{R}^d)$, $m \in \mathbb{Z}$, $m \leqslant s$, we have $P: Q^s(\mathbb{R}^d) \to Q^{s-m}(\mathbb{R}^d)$ continuously with

$$\big\| p(x,D) \big\|_{\mathcal{B}(Q^s, Q^{s-m})} \leqslant C \|p\|_N^{(\Gamma^m)}$$

for suitable $C > 0$, $N \in \mathbb{N}$ depending only on s, m and on the dimension d; see [24, Proposition 1.5.5, Theorem 2.1.12]. Moreover, for $m \in \mathbb{Z}$, $m \leqslant s$, there exists an operator $T \in \mathrm{OP}\Gamma^{-m}(\mathbb{R}^d)$ which gives an isomorphism $Q^{s-m}(\mathbb{R}^d) \to Q^s(\mathbb{R}^d)$.

We will also need the following Schauder's estimates for the weighted Sobolev spaces $Q^s(\mathbb{R}^d)$ in (2.7).

Proposition 2.1. *Let $s \in \mathbb{N}$, $s > d/2$. There exists $C_s > 0$ such that*

$$\|uv\|_{Q^s} \leqslant C_s \|u\|_{Q^s} \|v\|_{Q^s} \quad \forall u, v \in Q^s(\mathbb{R}^d).$$

Proof. We have

$$\begin{aligned} \|uv\|_{Q^s} &= \sum_{|\alpha|+|\beta|\leqslant s} \big\| x^\beta \partial^\alpha (uv) \big\|_{L^2} = \sum_{|\alpha|+|\beta|\leqslant s} \sum_{\gamma\leqslant\alpha} \binom{\alpha}{\gamma} \big\| x^\beta \partial^{\alpha-\gamma} u \cdot \partial^\gamma v \big\|_{L^2} \\ &\leqslant 2^s \sum_{|\alpha|+|\beta|\leqslant s} \sum_{\gamma\leqslant\alpha} \big\| x^\beta \partial^{\alpha-\gamma} u \big\|_{L^p} \big\| \partial^\gamma v \big\|_{L^q}, \end{aligned}$$

where $1 \leqslant p, q \leqslant \infty$ are chosen to satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, and $\frac{1}{2} < \frac{1}{p} + \frac{|\gamma|}{d}$, $\frac{1}{2} < \frac{1}{q} + \frac{s-|\gamma|}{d}$. This is possible because $s > \frac{d}{2}$. Then, by the Sobolev embeddings we have

$$\|uv\|_{Q^s} \leqslant C_s \sum_{|\alpha|+|\beta|\leqslant s} \sum_{\gamma\leqslant\alpha} \big\| x^\beta \partial^{\alpha-\gamma} u \big\|_{H^{|\gamma|}} \big\| \partial^\gamma v \big\|_{H^{s-|\gamma|}}. \tag{2.8}$$

On the other hand,

$$\big\| x^\beta \partial^{\alpha-\gamma} u \big\|_{H^{|\gamma|}} \asymp \sum_{|\mu|\leqslant|\gamma|} \big\| \partial^\mu \big(x^\beta \partial^{\alpha-\gamma} u \big) \big\|_{L^2} \leqslant C_s' \|u\|_{Q^s}. \tag{2.9}$$

Similarly,

$$\left\| \partial^\gamma v \right\|_{H^{s-|\gamma|}} \leqslant C_s'' \|u\|_{Q^s}. \tag{2.10}$$

Combining (2.8), (2.9) and (2.10) we get the desired result. □

As a technical tool, we will also use the scale of weighted Sobolev spaces

$$H^{s_1,s_2}(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d) \colon \|u\|_{H^{s_1,s_2}} := \left\| \langle x \rangle^{s_2} u \right\|_{s_1} < \infty \right\}, \tag{2.11}$$

defined for $s_1, s_2 \in \mathbb{R}$. In particular, we need the following result (see e.g. [24, Definition 3.1.1 and Theorem 3.1.5]).

Proposition 2.2. *Consider a symbol $p(x,\xi)$ satisfying the estimates*

$$\left| \partial_x^\beta \partial_\xi^\alpha p(x,\xi) \right| \leqslant C_{\alpha,\beta} \langle x \rangle^{n-|\beta|} \langle \xi \rangle^{m-|\alpha|}, \quad \forall \alpha, \beta \in \mathbb{N}^d,\ x \in \mathbb{R}^d. \tag{2.12}$$

Then the corresponding operator $p(x,D)$ is bounded $H^{s_1,s_2}(\mathbb{R}^d) \to H^{s_1-m,s_2-n}(\mathbb{R}^d)$ for every $s_1, s_2 \in \mathbb{R}$, with operator norm estimated by an upper bound of a finite number of the constants $C_{\alpha,\beta}$ appearing in (2.12).

By using Schauder's estimates in the standard Sobolev spaces and the inclusion $H^{s_1,s_2}(\mathbb{R}^d) \hookrightarrow H^{s_1}(\mathbb{R}^d)$, valid if $s_2 \geqslant 0$, one also gets

$$\|uv\|_{H^{s_1,s_2}} \leqslant C_{s_1,s_2} \|u\|_{H^{s_1,s_2}} \|v\|_{H^{s_1,s_2}}, \quad s_1 > \frac{d}{2},\ s_2 \geqslant 0. \tag{2.13}$$

A symbol $p \in \Gamma^m(\mathbb{R}^d)$ (and the corresponding operator) is said to be *Γ-elliptic* if it satisfies the condition (1.12).

The notion of Γ-ellipticity for an operator in $\mathrm{OP}\Gamma^m(\mathbb{R}^d)$ will be crucial in the subsequent arguments because it guaranties the existence of a parametrix $E \in \mathrm{OP}\Gamma^{-m}(\mathbb{R}^d)$. Namely we have the following result, see [24, Theorem 1.3.6] for the proof.

Proposition 2.3. *Let $p \in \Gamma^m(\mathbb{R}^d)$ be Γ-elliptic. Then there exists an operator $E \in \mathrm{OP}\Gamma^{-m}(\mathbb{R}^d)$ such that $EP = I + R$ and $PE = I + R'$, where R, R' are* globally *regularizing pseudodifferential operators, i.e. R and R' are continuous maps $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$. The operator E is said to be a parametrix for P.*

Finally we point out for further reference the following formulae, which can be verified by a direct computation: for $\alpha, \beta \in \mathbb{N}^d$, $u \in \mathcal{S}(\mathbb{R}^d)$:

$$x^\beta p(x,D)u = \sum_{\gamma \leqslant \beta} (-1)^{|\gamma|} \binom{\beta}{\gamma} \left(D_\xi^\gamma p \right)(x,D) \left(x^{\beta-\gamma} u \right), \tag{2.14}$$

$$\partial^\alpha p(x,D)u = \sum_{\delta \leqslant \alpha} \binom{\alpha}{\delta} \left(\partial_x^\delta p \right)(x,D) \partial^{\alpha-\delta} u. \tag{2.15}$$

3. A space of analytic functions

We introduce a space of real-analytic functions in $\mathbb{R}^d$, which extend holomorphically on a sector in $\mathbb{C}^d$ and display there a Gaussian decay.

Definition 3.1. We denote by $\mathcal{H}_{sect}(\mathbb{R}^d)$ the space of all functions $f \in C^\infty(\mathbb{R}^d)$ satisfying the following condition: there exists a constant $C > 0$ such that

$$\left|x^\beta \partial^\alpha f(x)\right| \leqslant C^{|\alpha|+|\beta|+1} M(\alpha, \beta), \quad \text{for all } \alpha, \beta \in \mathbb{N}^d, \tag{3.1}$$

where

$$M(\alpha, \beta) = |\alpha|!^{1/2} \max\bigl\{|\alpha|, |\beta|\bigr\}!^{1/2}. \tag{3.2}$$

It is easy to verify that the space $\mathcal{H}_{sect}(\mathbb{R}^d)$ is closed under differentiation.

Theorem 3.2. *Let $f \in \mathcal{H}_{sect}(\mathbb{R}^d)$. Then f extends to a holomorphic function $f(x+iy)$ in the sector*

$$\mathcal{C}_\varepsilon = \bigl\{z = x + iy \in \mathbb{C}^d\colon\ |y| < \varepsilon\bigl(1 + |x|\bigr)\bigr\} \tag{3.3}$$

of $\mathbb{C}^d$ for some $\varepsilon > 0$, satisfying there the estimates

$$\bigl|f(x+iy)\bigr| \leqslant C e^{-c|x|^2}, \tag{3.4}$$

for some constants $C > 0$, $c > 0$.

Proof. First we show the estimates

$$\left|x^\beta \partial^\alpha f(x)\right| \leqslant C^{|\alpha|+1} |\alpha|! e^{-c|x|^2}, \quad \text{for } |\beta| \leqslant |\alpha|. \tag{3.5}$$

Indeed, since $|x|^{2n} \leqslant k^n \sum_{|\gamma|=n} |x^{2\gamma}|$ for a constant $k > 0$ depending only on the dimension d, by (3.1) we have (assuming $C \geqslant 1$ in (3.1))

$$\begin{aligned}
e^{c|x|^2}\left|x^\beta \partial^\alpha f(x)\right| &= \sum_{n=0}^{\infty} \frac{(c|x|^2)^n}{n!} \left|x^\beta \partial^\alpha f(x)\right| \\
&\leqslant \sum_{n=0}^{\infty} (ck)^n \sum_{|\gamma|=n} \frac{1}{|\gamma|!} \left|x^{\beta+2\gamma} \partial^\alpha f(x)\right| \\
&\leqslant \sum_{n=0}^{\infty} (ck)^n \sum_{|\gamma|=n} C^{2|\alpha|+2|\gamma|+1} \frac{|\alpha|!^{1/2}(|\alpha|+2|\gamma|)!^{1/2}}{|\gamma|!} \\
&\leqslant \sum_{n=0}^{\infty} (ck)^n \sum_{|\gamma|=n} (2C)^{2|\alpha|+2|\gamma|+1} |\alpha|!,
\end{aligned}$$

where in the last step we used the inequality $(|\alpha|+2|\gamma|)! \leqslant 2^{|\alpha|+4|\gamma|}|\alpha|!|\gamma|!^2$, which follows by applying twice (2.1). Since the number of multi-indices γ satisfying $|\gamma| = n$ does not exceed 2^{d+n-1}, we get (3.5) for a new constant C, if c is small enough. Now, (3.5) and the estimate $|\alpha|! \leqslant d^{|\alpha|}\alpha!$ give

$$\left|\partial^\alpha f(x)\right| \leqslant C^{|\alpha|+1}\alpha!\langle x\rangle^{-|\alpha|}e^{-c|x|^2}, \tag{3.6}$$

for a new constant $C > 0$. By considering the Taylor expansion of the function f centered in any $x \in \mathbb{R}^d$ and using the estimates in (3.6) we obtain the desired extension property in a sector of the type (3.3) together with the estimates (3.4). □

In the sequel we will use the following characterization of the space $\mathcal{H}_{sect}(\mathbb{R}^d)$ in terms of Q^s-based norms.

Set, for $f \in \mathcal{S}'(\mathbb{R}^d)$,

$$S_\infty^{s,\varepsilon}[f] = \sum_{\alpha,\beta\in\mathbb{N}^d} \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}\left\|x^\beta\partial^\alpha f\right\|_{Q^s}, \tag{3.7}$$

where $M(\alpha,\beta)$ is defined in (3.2).

Proposition 3.3. *Let $f \in \mathcal{H}_{sect}(\mathbb{R}^d)$. Then for every $s \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $S_\infty^{s,\varepsilon}[f] < \infty$.*

In the opposite direction, if for some $s \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $S_\infty^{s,\varepsilon}[f] < \infty$, then $f \in \mathcal{H}_{sect}(\mathbb{R}^d)$.

Proof. Assume $f \in \mathcal{H}_{sect}(\mathbb{R}^d)$. We have

$$\left\|x^\beta\partial^\alpha f\right\|_{Q^s} = \sum_{|\delta|+|\gamma|\leqslant s}\left\|x^\delta\partial^\gamma\left(x^\beta\partial^\alpha f\right)\right\|_{L^2}.$$

Now, if $M \in \mathbb{N}$ satisfies $M > d/4$ we have

$$\left\|x^\delta\partial^\gamma\left(x^\beta\partial^\alpha f\right)\right\|_{L^2} \leqslant C'\left\|\left(1+|x|^2\right)^M x^\delta\partial^\gamma\left(x^\beta\partial^\alpha f\right)\right\|_{L^\infty}. \tag{3.8}$$

By Leibniz' formula, (3.1) and (2.1) we get

$$\left\|x^\beta\partial^\alpha f\right\|_{Q^s} \leqslant C_s^{|\alpha|+|\beta|+1}M(\alpha,\beta)$$

for some constant $C_s > 0$. Hence $S_\infty^{s,\varepsilon}[f] < \infty$ if $\varepsilon < C_s^{-1}$.

In the opposite direction, we may take $s = 0$; hence assume $S_\infty^{0,\varepsilon}[f] < \infty$ for some $\varepsilon > 0$. Then $\|x^\beta\partial^\alpha f(x)\|_{L^2} \leqslant C^{|\alpha|+|\beta|+1}M(\alpha,\beta)$ for all $\alpha,\beta \in \mathbb{N}^d$. If M is an integer, $M > d/2$, we have

$$\left\|x^\beta\partial^\alpha f\right\|_{L^\infty} \leqslant C\sum_{|\gamma|\leqslant M}\left\|\partial^\gamma\left(x^\beta\partial^\alpha f\right)\right\|_{L^2},$$

and similarly one gets that $f \in \mathcal{H}_{sect}(\mathbb{R}^d)$. □

4. Proof of the main result (Theorem 1.1)

In this section we prove Theorem 1.1. In fact we shall state and prove this result for the more general non-homogeneous equation

$$Pu = f + F[u], \tag{4.1}$$

where P and $F[u]$ satisfy the assumptions of Theorem 1.1 and f is a function in the space $\mathcal{H}_{sect}(\mathbb{R}^d)$ defined in Section 3. Moreover we can restate our result in terms of estimates in $\mathcal{H}_{sect}(\mathbb{R}^d)$. Namely, in view of Theorem 3.2, it will be sufficient to prove the following theorem.

Theorem 4.1. *Let $P = p(x, D) \in \mathrm{OP}\Gamma_a^m(\mathbb{R}^d)$, $m > 0$, be Γ-elliptic, that is* (1.12) *is satisfied. Let $F[u]$ be of the form* (1.13) *(possibly with some factors in the product replaced by their conjugates) and $f \in \mathcal{H}_{sect}(\mathbb{R}^d)$. Assume moreover that $u \in H^s(\mathbb{R}^d)$, $s > d/2 + \max_k\{|\rho_k|\}$, is a solution of* (4.1). *Then $u \in \mathcal{H}_{sect}(\mathbb{R}^d)$.*

In fact we always assume that $F[u]$ has the form in (1.13), and we leave to the reader the easy changes when some factors of the product in (1.13) are replaced by their conjugates.

The first step is to show that, under the assumptions of Theorem 4.1, the sum $S_N^{s,\varepsilon}[u]$ is finite for every $N \in \mathbb{N}$ and for some $\varepsilon > 0$. In particular, we prove the following preliminary result.

Lemma 4.2. *Under the assumptions of Theorem* 4.1*, we have $u \in \mathcal{S}(\mathbb{R}^d)$.*

Proof. The proof is based on a bootstrap argument in the scale of weighted Sobolev spaces defined in (2.11). Notice that $H^s(\mathbb{R}^d) = H^{s,0}(\mathbb{R}^d)$ and that $\bigcap_{s_1,s_2\in\mathbb{R}} H^{s_1,s_2}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d)$. To prove the lemma it is then sufficient to show that if $u \in H^{s_1,s_2}(\mathbb{R}^d)$, $s_1 > d/2 + M$, with $M = \max_k\{|\rho_k|\}$, $s_2 \geqslant 0$, then $u \in H^{s_1+\tau,s_2+\tau}$, with $\tau = \min\{m/2, 1/2\}$, and to iterate this argument. We first consider the case $m \geqslant 1$. Let $E \in \mathrm{OP}\Gamma^{-m}(\mathbb{R}^d)$ be a parametrix of P (Proposition 2.3). Applying E to both sides of Eq. (4.1), we get

$$u = -Ru + Ef + EF[u],$$

where R is globally regularizing. In particular, we have $Ru \in \mathcal{S}(\mathbb{R}^d)$ and $Ef \in \mathcal{S}(\mathbb{R}^d)$ because $f \in \mathcal{S}(\mathbb{R}^d)$. Concerning the nonlinear term, we observe that, since $-m \leqslant -M - h - 1$, then the symbol $e(x, \xi)$ of E satisfies the following estimates

$$\begin{aligned}\left|\partial_\xi^\alpha \partial_x^\beta e(x, \xi)\right| &\leqslant C_{\alpha\beta}\big(1 + |x| + |\xi|\big)^{-M-h-1-|\alpha|}\langle x\rangle^{-|\beta|} \\ &\leqslant C_{\alpha\beta}\langle\xi\rangle^{-M-1/2-|\alpha|}\langle x\rangle^{-h-1/2-|\beta|}.\end{aligned}$$

It follows from Proposition 2.2 that $E : H^{s_1-M,s_2-h}(\mathbb{R}^d) \to H^{s_1+1/2,s_2+1/2}(\mathbb{R}^d)$ continuously for every $s_1, s_2 \in \mathbb{R}$. In particular, in view of (1.14), we have

$$\big\|EF[u]\big\|_{H^{s_1+1/2,s_2+1/2}} = \left\| E \sum_{h,l,\rho_1,\ldots,\rho_l} F_{h,l,\rho_1,\ldots,\rho_l}(x) \prod_{k=1}^{l} \partial^{\rho_k} u \right\|_{H^{s_1+1/2,s_2+1/2}}$$

$$\leqslant C \sum_{h,l,\rho_1,\dots,\rho_l} \left\| F_{h,l,\rho_1,\dots,\rho_l}(x) \prod_{k=1}^{l} \partial^{\rho_k} u \right\|_{H^{s_1-M,s_2-h}}$$

$$\leqslant C' \left\| \prod_{k=1}^{l} \partial^{\rho_k} u \right\|_{H^{s_1-M,s_2}} \leqslant C'' \|u\|^l_{H^{s_1,s_2}} < \infty$$

by Schauder's estimates (2.13), because $s_1 - M > d/2$. The case $0 < m < 1$ is completely similar, considering that in this case $h = M = 0$ and the symbol $e(x,\xi)$ of E satisfies the estimates

$$\left|\partial_\xi^\alpha \partial_x^\beta e(x,\xi)\right| \leqslant C_{\alpha\beta} \langle\xi\rangle^{-m/2-|\alpha|} \langle x\rangle^{-m/2-|\beta|},$$

so that E maps continuously $H^{s_1,s_2}(\mathbb{R}^d)$ into $H^{s_1+m/2,s_2+m/2}(\mathbb{R}^d)$. □

In order to prove Theorem 4.1 it suffices to verify that $S_\infty^{s,\varepsilon}[u] < \infty$ for some $s \geqslant 0, \varepsilon > 0$, in view of Proposition 3.3. This will be achieved by an iteration argument involving the partial sums of the series in (3.7), that are

$$S_N^{s,\varepsilon}[f] = \sum_{|\alpha|+|\beta| \leqslant N} \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)} \left\| x^\beta \partial^\alpha f \right\|_{Q^s}, \tag{4.2}$$

where $M(\alpha,\beta)$ is defined in (3.2).

4.1. Proof of Theorem 4.1

We need several estimates to which we address now.

Proposition 4.3. *Let $R \in \mathrm{OP}\Gamma^{-1}(\mathbb{R}^d)$. Then for every $s \in \mathbb{R}$ there exists a constant $C_s > 0$ such that, for every $\varepsilon > 0$, $N \geqslant 1$ and $u \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$\sum_{0<|\alpha|+|\beta| \leqslant N} \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)} \left\| R\left(x^\beta \partial^\alpha u\right) \right\|_{Q^s} \leqslant C_s \varepsilon S_{N-1}^{s,\varepsilon}[u].$$

Proof. We first estimate the terms with $\beta = 0$, hence $\alpha \neq 0$. Let $k \in \{1,\dots,d\}$ such that $\alpha_k \neq 0$. Since $R \circ \partial_k \in \mathrm{OP}\Gamma^0(\mathbb{R}^d)$ is bounded on $Q^s(\mathbb{R}^d)$ we have[1]

$$\frac{\varepsilon^{|\alpha|}}{|\alpha|!} \left\| R\left(\partial^\alpha u\right) \right\|_{Q^s} \leqslant C_s \varepsilon \frac{\varepsilon^{|\alpha|-1}}{|\alpha|!} \left\| \partial^{\alpha-e_k} u \right\|_{Q^s}.$$

On the other hand, when $\beta \neq 0$, hence $\beta_j \neq 0$ for some $j \in \{1,\dots,d\}$, we use the fact that $R \circ x_j \in \mathrm{OP}\Gamma^0(\mathbb{R}^d)$ is bounded on $Q^s(\mathbb{R}^d)$. We get

$$\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)} \left\| R\left(x^\beta \partial^\alpha u\right) \right\|_{Q^s} \leqslant C_s \varepsilon \frac{\varepsilon^{|\alpha|+|\beta|-1}}{M(\alpha,\beta)} \left\| x^{\beta-e_j} \partial^\alpha u \right\|_{Q^s}.$$

Since $M(\alpha,\beta) \geqslant M(\alpha,\beta-e_j)$, this gives the desired result. □

[1] We denote by e_k the kth vector of the standard basis of $\mathbb{R}^d$.

Proposition 4.4. *Let $P = p(x, D)$ be a pseudodifferential with symbol $p(x, \xi)$ satisfying the estimates* (1.11)*, with $m \geqslant 0$. Let $E \in \mathrm{OP}\Gamma^{-m}(\mathbb{R}^d)$. Then for every $s \in \mathbb{R}$ there exists a constant $C_s > 0$ such that, for every ε small enough, $N \geqslant 1$ and $u \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$\sum_{0<|\alpha|+|\beta|\leqslant N} \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha, \beta)} \big\| E\big[P, x^\beta \partial^\alpha\big]u\big\|_{Q^s} \leqslant C_s \varepsilon S^{s,\varepsilon}_{N-1}[u]. \tag{4.3}$$

Proof. We estimate separately each term arising in the sum (4.3). We write

$$\big[P, x^\beta \partial^\alpha\big] = \big[P, x^\beta\big]\partial^\alpha + x^\beta\big[P, \partial^\alpha\big].$$

Hence, using (2.14), (2.15), we get

$$\begin{aligned}\big[P, x^\beta \partial^\alpha\big]u = &\sum_{0\neq\gamma_0\leqslant\beta} (-1)^{|\gamma_0|+1}\binom{\beta}{\gamma_0}\big(D_\xi^{\gamma_0} p\big)(x, D)\big(x^{\beta-\gamma_0}\partial^\alpha u\big) \\ &- \sum_{0\neq\delta\leqslant\alpha}\binom{\alpha}{\delta} x^\beta \partial_x^\delta p(x, D)\partial^{\alpha-\delta}u.\end{aligned} \tag{4.4}$$

Given β, δ, let $\tilde\delta$ be a multi-index of maximal length among those satisfying $|\tilde\delta| \leqslant |\delta|$ and $\tilde\delta \leqslant \beta$ (hence, if $|\tilde\delta| < |\delta|$ then $\beta - \tilde\delta = 0$). Writing $x^\beta = x^{\tilde\delta}x^{\beta-\tilde\delta}$ in the last term of (4.4) and using again (2.14) we get

$$\big[P, x^\beta \partial^\alpha\big]u = \sum_{\delta\leqslant\alpha} \sum_{\substack{\gamma_0\leqslant\beta-\tilde\delta \\ (\delta,\gamma_0)\neq(0,0)}} (-1)^{|\gamma_0|+1}\binom{\beta-\tilde\delta}{\gamma_0}\binom{\alpha}{\delta} x^{\tilde\delta}\big(D_\xi^{\gamma_0}\partial_x^\delta p\big)(x, D)\big(x^{\beta-\tilde\delta-\gamma_0}\partial^{\alpha-\delta}u\big). \tag{4.5}$$

We now work out this formula to obtain a useful representation of the commutator $[P, x^\beta\partial^\alpha]$. Namely, we look at the operator $x^{\beta-\tilde\delta-\gamma_0}\partial^{\alpha-\delta}$. Given γ_0, α, δ, let $\tilde\gamma_0$ be a multi-index, to be chosen later on, satisfying $|\tilde\gamma_0| \leqslant |\gamma_0|$ and $\tilde\gamma_0 \leqslant \alpha - \delta$. We write, by the inverse Leibniz formula (2.5),

$$\begin{aligned}x^{\beta-\tilde\delta-\gamma_0}\partial^{\alpha-\delta} &= x^{\beta-\tilde\delta-\gamma_0}\partial^{\tilde\gamma_0}\partial^{\alpha-\delta-\tilde\gamma_0} = \partial^{\tilde\gamma_0}\circ x^{\beta-\tilde\delta-\gamma_0}\partial^{\alpha-\delta-\tilde\gamma_0} \\ &+ \sum_{\substack{0\neq\gamma_1\leqslant\beta-\tilde\delta-\gamma_0 \\ \gamma_1\leqslant\tilde\gamma_0}} \frac{(-1)^{|\gamma_1|}(\beta-\tilde\delta-\gamma_0)!}{(\beta-\tilde\delta-\gamma_0-\gamma_1)!}\binom{\tilde\gamma_0}{\gamma_1}\partial^{\tilde\gamma_0-\gamma_1}\circ x^{\beta-\tilde\delta-\gamma_0-\gamma_1}\partial^{\alpha-\delta-\tilde\gamma_0}.\end{aligned} \tag{4.6}$$

We now look at the operator $x^{\beta-\tilde\delta-\gamma_0-\gamma_1}\partial^{\alpha-\delta-\tilde\gamma_0}$. We denote by $\tilde\gamma_1$ a multi-index, to be chosen later on, satisfying $|\tilde\gamma_1| \leqslant |\gamma_1|$, $\tilde\gamma_1 \leqslant \alpha - \delta - \tilde\gamma_0$. Applying again the inverse Leibniz formula we have

$$x^{\beta-\tilde\delta-\gamma_0-\gamma_1}\partial^{\alpha-\delta-\tilde\gamma_0} = x^{\beta-\tilde\delta-\gamma_0-\gamma_1}\partial^{\tilde\gamma_1}\partial^{\alpha-\delta-\tilde\gamma_0-\tilde\gamma_1} = \partial^{\tilde\gamma_1}\circ x^{\beta-\tilde\delta-\gamma_0-\gamma_1}\partial^{\alpha-\delta-\tilde\gamma_0-\tilde\gamma_1}$$
$$+ \sum_{\substack{0\neq\gamma_2\leqslant\beta-\tilde\delta-\gamma_0-\gamma_1\\ \gamma_2\leqslant\tilde\gamma_1}} \frac{(-1)^{|\gamma_2|}(\beta-\tilde\delta-\gamma_0-\gamma_1)!}{(\beta-\tilde\delta-\gamma_0-\gamma_1-\gamma_2)!}$$
$$\times \binom{\tilde\gamma_1}{\gamma_2}\partial^{\tilde\gamma_1-\gamma_2}\circ x^{\beta-\tilde\delta-\gamma_0-\gamma_1-\gamma_2}\partial^{\alpha-\delta-\tilde\gamma_0-\tilde\gamma_1}. \tag{4.7}$$

Continuing in this way and substituting all in (4.5) we get

$$[P, x^\beta\partial^\alpha]u = \sum_{\delta\leqslant\alpha}\sum_{j=0}^{r}\sum_{\substack{\gamma_0\leqslant\beta-\tilde\delta\\ (\delta,\gamma_0)\neq(0,0)}}\sum_{\substack{0\neq\gamma_1\leqslant\beta-\tilde\delta-\gamma_0\\ \gamma_1\leqslant\tilde\gamma_0}}\cdots\sum_{\substack{0\neq\gamma_j\leqslant\beta-\tilde\delta-\gamma_0-\cdots-\gamma_{j-1}\\ \gamma_j\leqslant\tilde\gamma_{j-1}}} C_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}$$
$$\times p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,D)\big(x^{\beta-\tilde\delta-\gamma_0-\cdots-\gamma_j}\partial^{\alpha-\delta-\tilde\gamma_0-\cdots-\tilde\gamma_j}u\big), \tag{4.8}$$

where $\tilde\gamma_j$ satisfy $|\tilde\gamma_j|\leqslant|\gamma_j|$ and $\tilde\gamma_j\leqslant\alpha-\delta-\tilde\gamma_0-\cdots-\tilde\gamma_{j-1}$,

$$p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,\xi) = x^{\tilde\delta}\big(D_\xi^{\gamma_0}\partial_x^\delta p\big)(x,\xi)\xi^{\tilde\gamma_0-\gamma_1+\tilde\gamma_1-\cdots-\gamma_j+\tilde\gamma_j}, \quad j\geqslant 0, \tag{4.9}$$

and

$$|C_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}| = \frac{\alpha!(\beta-\tilde\delta)!}{(\alpha-\delta)!\delta!\gamma_0!(\beta-\tilde\delta-\gamma_0-\cdots-\gamma_j)!}\prod_{k=1}^{j}\binom{\tilde\gamma_{k-1}}{\gamma_k}$$
$$\leqslant \frac{|\alpha|!|\beta-\tilde\delta|!}{|\alpha-\delta|!\delta!\gamma_0!|\beta-\tilde\delta-\gamma_0-\cdots-\gamma_j|!}2^{|\tilde\gamma_0+\cdots+\tilde\gamma_{j-1}|}, \tag{4.10}$$

cf. (2.4) and (2.1). (If $j=0$ in (4.10) we mean that there are not the binomial factors, nor the power of 2.) Observe that, since we have $\gamma_j\neq 0$ for every $j\geqslant 1$, this procedure in fact stops after a finite number r of steps.

We now study separately the terms with $|\alpha|\geqslant|\beta|$ and $|\beta|>|\alpha|$.

Case $|\alpha|\geqslant|\beta|$. We use the formula (4.8), where now we choose γ_0 satisfying, in addition, $|\tilde\gamma_0|=|\gamma_0|$. Such a multi-index exists, because $|\alpha|\geqslant|\beta|$. Similarly, at each subsequent step we can choose $\tilde\gamma_j$, $j\geqslant 1$, satisfying in addition $|\tilde\gamma_j|=|\gamma_j|$.

Now we observe that, by (1.11), (2.1), and Leibniz' formula, for every $\theta,\sigma\in\mathbb{N}^d$ we have

$$\big|\partial_\xi^\theta\partial_x^\sigma p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,\xi)\big| \leqslant C^{|\gamma_0|+|\delta|+1}\gamma_0!\delta!\big(1+|x|+|\xi|\big)^{m-|\theta|}\langle x\rangle^{-|\sigma|}, \tag{4.11}$$

for some constant C depending only on θ and σ. In fact, $|\tilde\delta|\leqslant|\delta|$, $|\tilde\gamma_0-\gamma_1+\tilde\gamma_1-\cdots-\gamma_j+\tilde\gamma_j| = |\tilde\gamma_0|=|\gamma_0|$, and the powers of $|\delta|$ and $|\gamma_0|$ which arise can be estimated by $C^{|\gamma_0|+|\delta|+1}$ for some $C>0$.

We now use these last bounds to estimate $E\circ p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,D)$. To this end, observe that this operator belongs to $\mathrm{OP}\Gamma^0(\mathbb{R}^d)$, and therefore its norm as a bounded operator on $Q^s(\mathbb{R}^d)$ is estimated by a seminorm of its symbol in $\Gamma^0(\mathbb{R}^d)$, depending only on s and d. Such a seminorm is in turn estimated by the product of a seminorm of the symbol of E in $\Gamma^{-m}(\mathbb{R}^d)$ and a seminorm of $p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}$ in $\Gamma^m(\mathbb{R}^d)$, again depending only on s, d. Hence, from (4.11) we get

$$\left\| E \circ p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,D)\right\|_{\mathcal{B}(Q^s)} \leqslant C_s^{|\gamma_0|+|\delta|+1}\gamma_0!\delta!. \tag{4.12}$$

Since $|\tilde{\gamma}_k| = |\gamma_k|$, $0 \leqslant k \leqslant j$, we have

$$\frac{|\beta-\tilde{\delta}|!|\alpha-\delta-\tilde{\gamma}_0-\cdots-\tilde{\gamma}_j|!}{|\alpha-\delta|!|\beta-\tilde{\delta}-\gamma_0-\cdots-\gamma_j|!} \leqslant 1 \tag{4.13}$$

(recall that if $|\tilde{\delta}| < |\delta|$ then $\beta-\tilde{\delta} = \gamma_0 = \cdots = \gamma_j = \tilde{\gamma}_0 = \cdots = \tilde{\gamma}_j = 0$).

By (4.10), (4.12) (4.13), we get in this case

$$\begin{aligned} &\frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}|C_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}|\left\| E\circ p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,D)\left(x^{\beta-\tilde{\delta}-\gamma_0-\cdots-\gamma_j}\partial^{\alpha-\delta-\tilde{\gamma}_0-\cdots-\tilde{\gamma}_j}u\right)\right\|_{Q^s} \\ &\quad \leqslant C_s(C_s\varepsilon)^{|\delta|+|\tilde{\delta}|+|\gamma_0+\cdots+\gamma_j|+|\tilde{\gamma}_0+\cdots+\tilde{\gamma}_j|}\frac{\varepsilon^{|\alpha|+|\beta|-|\delta|-|\tilde{\delta}|-|\gamma_0+\cdots+\gamma_j|-|\tilde{\gamma}_0+\cdots+\tilde{\gamma}_j|}}{|\alpha-\delta-\tilde{\gamma}_0-\cdots-\tilde{\gamma}_j|!} \\ &\quad\quad \times \left\| x^{\beta-\tilde{\delta}-\gamma_0-\cdots-\gamma_j}\partial^{\alpha-\delta-\tilde{\gamma}_0-\cdots-\tilde{\gamma}_j}u\right\|_{Q^s}. \end{aligned} \tag{4.14}$$

Now, the assumption $|\alpha| \geqslant |\beta|$ and the choice of $\tilde{\delta}$ and $\tilde{\gamma}_j$, $j \geqslant 0$, imply $M(\alpha,\beta) = |\alpha|!$ and $|\beta-\tilde{\delta}-\gamma_0-\cdots-\gamma_j| \leqslant |\alpha-\delta-\tilde{\gamma}_0-\cdots-\tilde{\gamma}_j|$. Hence

$$M(\alpha-\delta-\tilde{\gamma}_0-\cdots-\tilde{\gamma}_j, \beta-\tilde{\delta}-\gamma_0-\cdots-\gamma_j) = |\alpha-\delta-\tilde{\gamma}_0-\cdots-\tilde{\gamma}_j|!.$$

We can therefore rewrite (4.14) as

$$\begin{aligned} &\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}|C_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}|\left\| E\circ p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,D)\left(x^{\beta-\tilde{\delta}-\gamma_0-\cdots-\gamma_j}\partial^{\alpha-\delta-\tilde{\gamma}_0-\cdots-\tilde{\gamma}_j}u\right)\right\|_{Q^s} \\ &\quad \leqslant C_s(C_s\varepsilon)^{|\delta|+|\tilde{\delta}|+|\gamma_0+\cdots+\gamma_j|+|\tilde{\gamma}_0+\cdots+\tilde{\gamma}_j|} \\ &\quad\quad \times \frac{\varepsilon^{|\alpha|+|\beta|-|\delta|-|\tilde{\delta}|-|\gamma_0+\cdots+\gamma_j|-|\tilde{\gamma}_0+\cdots+\tilde{\gamma}_j|}}{M(\alpha-\delta-\tilde{\gamma}_0-\cdots-\tilde{\gamma}_j,\beta-\tilde{\delta}-\gamma_0-\cdots-\gamma_j)}\left\| x^{\beta-\tilde{\delta}-\gamma_0-\cdots-\gamma_j}\partial^{\alpha-\delta-\tilde{\gamma}_0-\cdots-\tilde{\gamma}_j}u\right\|_{Q^s}. \end{aligned} \tag{4.15}$$

We now perform the change of variables $\tilde{\alpha} = \alpha-\delta-\tilde{\gamma}_0-\cdots-\tilde{\gamma}_j$, $\tilde{\beta} = \beta-\tilde{\delta}-\gamma_0-\cdots-\gamma_j$. In fact, the map $(\alpha,\beta,\delta,j,\gamma_0,\gamma_1,\dots,\gamma_j) \to (\tilde{\alpha},\tilde{\beta},\delta,j,\gamma_0,\gamma_1,\dots,\gamma_j)$ defined in this way is not injective, because of the presence of $\tilde{\delta}$, $\tilde{\gamma}_0,\dots,\tilde{\gamma}_j$ (of course, one should think of $\tilde{\delta}$ as a function of α,β,δ, and to every $\tilde{\gamma}_j$, $j \geqslant 0$, as a function of $\alpha,\beta,\delta,\gamma_k$, $k \leqslant j$). Anyhow, since $|\tilde{\delta}| \leqslant |\delta|$ and $|\tilde{\gamma}_j| = |\gamma_j|$, the number of pre-images of a given point is at most $2^{|\delta|+|\gamma_0|+\cdots+|\gamma_j|+d(j+2)}$. Hence we deduce from (4.8) and (4.15) that, if ε is small enough,

$$\sum_{\substack{|\alpha|+|\beta|\leqslant N\\|\alpha|\geqslant|\beta|}}\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}\big\|E\big[P,x^\beta\partial^\alpha\big]u\big\|_{Q^s}$$

$$\leqslant 2^dC_s\sum_{|\tilde\alpha|+|\tilde\beta|\leqslant N-1}\frac{\varepsilon^{|\tilde\alpha|+|\tilde\beta|}}{M(\tilde\alpha,\tilde\beta)}\big\|x^{\tilde\beta}\partial^{\tilde\alpha}u\big\|_{Q^s}\sum_{j=0}^r 2^{d(j+1)}\sum_\delta\sum_{\substack{\gamma_1\neq0,\dots,\gamma_j\neq0\\\gamma_0:\ (\delta,\gamma_0)\neq(0,0)}}(2C_s\varepsilon)^{|\delta|+|\gamma_0+\gamma_1+\cdots+\gamma_j|}$$

$$\leqslant S^{s,\varepsilon}_{N-1}[u]\sum_{j=0}^r\big(C_s'\varepsilon\big)^{j+1}\leqslant C_s''\varepsilon S^{s,\varepsilon}_{N-1}[u]. \tag{4.16}$$

Case $|\beta|>|\alpha|$. Here it is convenient to get separate estimates when $|x|$ is large or small at the scale $|\beta|^{1/2}$. To make this precise, consider a function $\varphi\in C_0^\infty(\mathbb{R}^d)$, $\varphi(x)=1$ for $|x|\leqslant1$ and $\varphi(x)=0$ for $|x|\geqslant2$. Let then

$$\varphi_\beta(x)=\varphi\left(\frac{x}{|\beta|^{1/2}}\right)$$

(notice that $\beta\neq0$, because of our hypothesis $|\beta|>|\alpha|$). Hence $\varphi_\beta(x)=1$ for $|x|\leqslant|\beta|^{1/2}$ and $\varphi_\beta(x)=0$ for $|x|\geqslant2|\beta|^{1/2}$. Moreover, we have

$$\big|\partial^\gamma\varphi_\beta(x)\big|\leqslant C_\gamma|\beta|^{-|\gamma|/2},\quad \gamma\in\mathbb{N}^d,\ x\in\mathbb{R}^d, \tag{4.17}$$

for constants $C_\gamma>0$.

We write

$$\big[P,x^\beta\partial^\alpha\big]=\varphi_\beta(x)\big[P,x^\beta\partial^\alpha\big]+\big(1-\varphi_\beta(x)\big)\big[P,x^\beta\partial^\alpha\big],$$

and we split consequently the terms in (4.3).

Estimate of $\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}\|E((1-\varphi_\beta(x))[P,x^\beta\partial^\alpha]u)\|_{Q^s}$. We use the expression in (4.5) for $[P,x^\beta\partial^\alpha]$, and we split the second sum, by considering separately the terms with $|\beta-\tilde\delta-\gamma_0|\leqslant|\alpha-\delta|$ or $|\beta-\tilde\delta-\gamma_0|>|\alpha-\delta|$. This is equivalent to saying $|\beta-\gamma_0|\leqslant|\alpha|$ or $|\beta-\gamma_0|>|\alpha|$ because $|\beta|>|\alpha|$ implies $|\tilde\delta|=|\delta|$. Moreover we apply the iterative argument at the beginning of the present proof to the terms with $|\beta-\gamma_0|\leqslant|\alpha|$. We obtain

$$\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}\big\|E\big(\big(1-\varphi_\beta(x)\big)\big[P,x^\beta\partial^\alpha\big]u\big)\big\|_{Q^s}\leqslant(I)+(II) \tag{4.18}$$

where

$$(I)=\sum_{\delta\leqslant\alpha}\sum_{\substack{\gamma_0\leqslant\beta-\tilde\delta:\\(\delta,\gamma_0)\neq(0,0),\ |\beta-\gamma_0|>|\alpha|}}\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}\binom{\beta-\tilde\delta}{\gamma_0}\binom{\alpha}{\delta}$$

$$\times\big\|E\big(x^{\tilde\delta}\big(1-\varphi_\beta(x)\big)\big(D_\xi^{\gamma_0}\partial_x^\delta p\big)(x,D)\big(x^{\beta-\tilde\delta-\gamma_0}\partial^{\alpha-\delta}u\big)\big)\big\|_{Q^s}, \tag{4.19}$$

whereas

$$(II) = \sum_{\delta\leqslant\alpha}\sum_{j=0}^{r} \sum_{\substack{\gamma_0\leqslant\beta-\tilde{\delta}:\\ (\delta,\gamma_0)\neq(0,0),\ |\beta-\gamma_0|\leqslant|\alpha|}} \sum_{\substack{0\neq\gamma_1\leqslant\beta-\tilde{\delta}-\gamma_0\\ \gamma_1\leqslant\tilde{\gamma}_0}} \cdots \sum_{\substack{0\neq\gamma_j\leqslant\beta-\tilde{\delta}-\gamma_0-\cdots-\gamma_{j-1}\\ \gamma_j\leqslant\tilde{\gamma}_{j-1}}} \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}$$
$$\times |C_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}| \big\| E\big(\big(1-\varphi_\beta(x)\big) p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,D)$$
$$\times \big(x^{\beta-\tilde{\delta}-\gamma_0-\cdots-\gamma_j}\partial^{\alpha-\delta-\tilde{\gamma}_0-\cdots-\tilde{\gamma}_j}u\big)\big)\big\|_{Q^s}, \tag{4.20}$$

where the multi-indices $\tilde{\gamma}_j$ will be chosen later on, satisfying $|\tilde{\gamma}_j|\leqslant|\gamma_j|$ and $\tilde{\gamma}_j\leqslant\alpha-\delta-\tilde{\gamma}_0-\cdots-\tilde{\gamma}_{j-1}$; the constants $C_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}$ satisfy (4.10), and (4.9) holds.

Estimate of the terms in (I) (hence $|\beta-\gamma_0|>|\alpha|$). Since $|\tilde{\delta}|=|\delta|$, by Leibniz' formula, (1.11) and (2.1), for every $\theta,\sigma\in\mathbb{N}^d$ we have

$$\big|\partial_\xi^\theta\partial_x^\sigma\big(x^{\tilde{\delta}}\big(D_\xi^{\gamma_0}\partial_x^\delta p\big)(x,\xi)\big)\big| \leqslant C^{|\gamma_0|+|\delta|+1}\gamma_0!\delta!\big(1+|x|+|\xi|\big)^{m-|\gamma_0|-|\theta|}\langle x\rangle^{-|\sigma|} \tag{4.21}$$

for some constant C depending only on θ and σ.

Since $1-\varphi_\beta(x)$ is supported where $|x|\geqslant|\beta|^{1/2}$, using Leibniz' formula again and (4.17) we get

$$\big|\partial_\xi^\theta\partial_x^\sigma\big(x^{\tilde{\delta}}\big(1-\varphi_\beta(x)\big)\big(D_\xi^{\gamma_0}\partial_x^\delta p\big)(x,\xi)\big)\big|$$
$$\leqslant C^{|\gamma_0|+|\delta|+1}\gamma_0!\delta!|\beta|^{-\frac{|\gamma_0|}{2}}\big(1+|x|+|\xi|\big)^{m-|\theta|}\langle x\rangle^{-|\sigma|}$$

for some constant C depending only on θ and σ. As a consequence,

$$\big\|E\circ\big(x^{\tilde{\delta}}\big(1-\varphi_\beta(x)\big)\big(D_\xi^{\gamma_0}\partial_x^\delta p\big)(x,D)\big)\big\|_{\mathcal{B}(Q^s)} \leqslant C_s^{|\gamma_0|+|\delta|+1}\gamma_0!\delta!|\beta|^{-\frac{|\gamma_0|}{2}}. \tag{4.22}$$

On the other hand, we have

$$\binom{\beta-\tilde{\delta}}{\gamma_0}\binom{\alpha}{\delta}\leqslant\frac{|\beta-\tilde{\delta}|!|\alpha|!}{|\beta-\tilde{\delta}-\gamma_0|!|\alpha-\delta|!\gamma_0!\delta!} \tag{4.23}$$

as well as

$$\frac{1}{|\beta|!^{1/2}|\alpha|!^{1/2}}\frac{|\beta-\tilde{\delta}|!|\alpha|!}{|\beta-\tilde{\delta}-\gamma_0|!|\alpha-\delta|!}|\beta-\tilde{\delta}-\gamma_0|!^{1/2}|\alpha-\delta|!^{1/2}|\beta|^{-|\gamma_0|/2}$$
$$=\underbrace{\left(\frac{|\beta-\tilde{\delta}|!|\alpha|!}{|\alpha-\delta|!|\beta|!}\right)^{1/2}}_{\leqslant1}\underbrace{\left(\frac{|\beta-\tilde{\delta}|!}{|\beta-\tilde{\delta}-\gamma_0|!}|\beta|^{-|\gamma_0|}\right)^{1/2}}_{\leqslant1}\leqslant1. \tag{4.24}$$

By (4.22), (4.23) and (4.24) we obtain

$$\frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!^{1/2}|\beta|!^{1/2}}\binom{\beta-\tilde{\delta}}{\gamma_0}\binom{\alpha}{\delta}\left\| E\left(x^{\tilde{\delta}}\left(1-\varphi_\beta(x)\right)\left(D_\xi^{\gamma_0}\partial_x^\delta p\right)(x,D)u\right)\right\|_{Q^s}$$
$$\leqslant C_s(C_s\varepsilon)^{2|\delta|+|\gamma_0|}\frac{\varepsilon^{|\alpha|+|\beta|-2|\delta|-|\gamma_0|}}{|\alpha-\delta|!^{1/2}|\beta-\tilde{\delta}-\gamma_0|!^{1/2}}\left\|x^{\beta-\tilde{\delta}-\gamma_0}\partial^{\alpha-\delta}u\right\|_{Q^s}. \tag{4.25}$$

Since $|\beta|>|\alpha|$ and $|\beta-\gamma_0|>|\alpha|$, then we have $M(\alpha,\beta)=|\alpha|!^{1/2}|\beta|!^{1/2}$ and $M(\alpha-\delta,\beta-\tilde{\delta}-\gamma_0)|\alpha-\delta|!^{1/2}|\beta-\tilde{\delta}-\gamma_0|!^{1/2}$, so that (4.25) can be rephrased as

$$\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}\binom{\beta-\tilde{\delta}}{\gamma_0}\binom{\alpha}{\delta}\left\| E\left(x^{\tilde{\delta}}\left(1-\varphi_\beta(x)\right)\left(D_\xi^{\gamma_0}\partial_x^\delta p\right)(x,D)u\right)\right\|_{Q^s}$$
$$\leqslant C_s(C_s\varepsilon)^{2|\delta|+|\gamma_0|}\frac{\varepsilon^{|\alpha|+|\beta|-2|\delta|-|\gamma_0|}}{M(\alpha-\delta,\beta-\tilde{\delta}-\gamma_0)}\left\|x^{\beta-\tilde{\delta}-\gamma_0}\partial^{\alpha-\delta}u\right\|_{Q^s}. \tag{4.26}$$

Estimate of the terms in (*II*) (hence $|\beta-\gamma_0|\leqslant|\alpha|$). In the iterative argument which led to (4.20), we choose the multi-indices $\tilde{\gamma}_j$, $j\geqslant 0$, in the following way: $\tilde{\gamma}_0$ is a multi-index satisfying, in addition, $|\beta-\tilde{\delta}-\gamma_0|=|\alpha-\delta-\tilde{\gamma}_0|$. Such a multi-index exists, because $|\tilde{\delta}|=|\delta|$, $|\beta|>|\alpha|$ and $|\beta-\gamma_0|\leqslant|\alpha|$; moreover $|\gamma_0|-|\tilde{\gamma}_0|=|\beta|-|\alpha|$. Similarly, we can choose $\tilde{\gamma}_1$ satisfying $|\beta-\tilde{\delta}-\gamma_0-\gamma_1|=|\alpha-\delta-\tilde{\gamma}_0-\tilde{\gamma}_1|$; in particular $|\tilde{\gamma}_1|=|\gamma_1|$. In general we can choose $\tilde{\gamma}_j$ such that

$$|\beta-\tilde{\delta}-\gamma_0-\cdots-\gamma_j|=|\alpha-\delta-\tilde{\gamma}_0-\cdots-\tilde{\gamma}_j|; \tag{4.27}$$

hence $|\tilde{\gamma}_j|=|\gamma_j|$ if $j\geqslant 1$.

Notice that now in (4.9) we have $|\tilde{\delta}|=|\delta|$ and $|\tilde{\gamma}_0-\gamma_1+\tilde{\gamma}_1-\cdots-\gamma_j+\tilde{\gamma}_j|=|\tilde{\gamma}_0|$. Hence, since $|\gamma_0|-|\tilde{\gamma}_0|=|\beta|-|\alpha|$, by (1.11), (2.1), and Leibniz' formula, for every $\theta,\sigma\in\mathbb{N}^d$ we have

$$\left|\partial_\xi^\theta\partial_x^\sigma p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,\xi)\right|\leqslant C^{|\gamma_0|+|\delta|+1}\gamma_0!\delta!\left(1+|x|+|\xi|\right)^{m-|\beta|+|\alpha|-|\theta|}\langle x\rangle^{-|\sigma|}, \tag{4.28}$$

for some constant C depending only on θ and σ.

Since $1-\varphi_\beta(x)$ is supported where $|x|\geqslant|\beta|^{1/2}$, using Leibniz' formula again and (4.17) we get

$$\left|\partial_\xi^\theta\partial_x^\sigma\left(\left(1-\varphi_\beta(x)\right)p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,\xi)\right)\right|$$
$$\leqslant C^{|\gamma_0|+|\delta|+1}|\beta|^{-\frac{|\beta|-|\alpha|}{2}}\gamma_0!\delta!\left(1+|x|+|\xi|\right)^{m-|\theta|}\langle x\rangle^{-|\sigma|}$$

for some new constant C depending only on θ and σ. We obtain

$$\left\|E\circ\left(\left(1-\varphi_\beta(x)\right)p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,D)\right)\right\|_{\mathcal{B}(Q^s)}\leqslant C_s^{|\gamma_0|+|\delta|+1}\gamma_0!\delta!|\beta|^{-\frac{|\beta|-|\alpha|}{2}}. \tag{4.29}$$

Moreover we have

$$\frac{1}{|\alpha|!^{1/2}|\beta|!^{1/2}}\frac{|\alpha|!|\beta-\tilde{\delta}|!}{|\alpha-\delta|!}|\beta|^{-\frac{|\beta|-|\alpha|}{2}}=\underbrace{\left(\frac{|\alpha|!|\beta-\tilde{\delta}|!}{|\alpha-\delta|!|\beta|!}\right)^{1/2}}_{\leqslant 1}\underbrace{\left(\frac{|\beta-\tilde{\delta}|!}{|\alpha-\delta|!}|\beta|^{-|\beta|+|\alpha|}\right)^{1/2}}_{\leqslant 1}\leqslant 1. \tag{4.30}$$

By (4.10), (4.29), (4.30), we get

$$\begin{aligned}&\frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!^{1/2}|\beta|!^{1/2}}|C_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}|\\&\quad\times\left\|E\big(\big(1-\varphi_\beta(x)\big)p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,D)\big(x^{\beta-\tilde\delta-\gamma_0-\dots-\gamma_j}\partial^{\alpha-\delta-\tilde\gamma_0-\dots-\tilde\gamma_j}u\big)\big)\right\|_{Q^s}\\&\quad\leqslant C_s(C_s\varepsilon)^{|\delta|+|\tilde\delta|+|\gamma_0+\dots+\gamma_j|+|\tilde\gamma_0+\dots+\tilde\gamma_j|}\frac{\varepsilon^{|\alpha|+|\beta|-|\delta|-|\tilde\delta|-|\gamma_0+\dots+\gamma_j|-|\tilde\gamma_0+\dots+\tilde\gamma_j|}}{|\beta-\tilde\delta-\gamma_0-\dots-\gamma_j|!}\\&\quad\times\left\|x^{\beta-\tilde\delta-\gamma_0-\dots-\gamma_j}\partial^{\alpha-\delta-\tilde\gamma_0-\dots-\tilde\gamma_j}u\right\|_{Q^s}.\end{aligned}\tag{4.31}$$

Since $|\beta|>|\alpha|$, we have $M(\alpha,\beta)=|\alpha|!^{1/2}|\beta|!^{1/2}$, whereas from (4.27) we see that $M(\alpha-\delta-\tilde\gamma_0-\dots-\tilde\gamma_j,\beta-\tilde\delta-\gamma_0-\dots-\gamma_j)=|\beta-\tilde\delta-\gamma_0-\dots-\gamma_j|!$. Hence we deduce from (4.31) that

$$\begin{aligned}&\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}|C_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}|\\&\quad\times\left\|E\big(\big(1-\varphi_\beta(x)\big)p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,D)\big(x^{\beta-\tilde\delta-\gamma_0-\dots-\gamma_j}\partial^{\alpha-\delta-\tilde\gamma_0-\dots-\tilde\gamma_j}u\big)\big)\right\|_{Q^s}\\&\quad\leqslant C_s(C_s\varepsilon)^{|\delta|+|\tilde\delta|+|\gamma_0+\dots+\gamma_j|+|\tilde\gamma_0+\dots+\tilde\gamma_j|}\frac{\varepsilon^{|\alpha|+|\beta|-|\delta|-|\tilde\delta|-|\gamma_0+\dots+\gamma_j|-|\tilde\gamma_0+\dots+\tilde\gamma_j|}}{M(\alpha-\delta-\tilde\gamma_0-\dots-\tilde\gamma_j,\beta-\tilde\delta-\gamma_0-\dots-\gamma_j)}\\&\quad\times\left\|x^{\beta-\tilde\delta-\gamma_0-\dots-\gamma_j}\partial^{\alpha-\delta-\tilde\gamma_0-\dots-\tilde\gamma_j}u\right\|_{Q^s}.\end{aligned}\tag{4.32}$$

We now use (4.18)–(4.20), (4.26), (4.32) to conclude, by the same arguments as in the case $|\alpha|\geqslant|\beta|$, that

$$\sum_{\substack{|\alpha|+|\beta|\leqslant N\\|\beta|>|\alpha|}}\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}\left\|E\big(\big(1-\varphi_\beta(x)\big)\big[P,x^\beta\partial^\alpha\big]u\big)\right\|_{Q^s}\leqslant C'_s\varepsilon S^{s,\varepsilon}_{N-1}[u].\tag{4.33}$$

Estimate of $\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}\|E(\varphi_\beta(x)[P,x^\beta\partial^\alpha]u)\|_{Q^s}$. We now start from the formula (4.4). For any fixed α, β, δ, we choose $\tilde\delta\leqslant\beta$ such that $|\beta-\tilde\delta|=|\alpha-\delta|$, which is possible because here $|\beta|>|\alpha|$. Writing $x^\beta=x^{\tilde\delta}x^{\beta-\tilde\delta}$ in (4.4) and using (2.14) we still get the formula (4.5) (notice however the choice of $\tilde\delta$ is different from the one we made there). We now apply the iterative argument detailed at the beginning of the present proof, which led to (4.8), where the coefficients $C_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}$ satisfy (4.10) and (4.9) holds. The multi-indices $\tilde\gamma_j$, $j\geqslant0$, are chosen here to satisfy, in addition, $|\tilde\gamma_j|=|\gamma_j|$, which is possible because $|\beta-\tilde\delta|=|\alpha-\delta|$. Hence, we can rewrite (4.10) as

$$|C_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}|\leqslant\frac{|\alpha|!}{\delta!\gamma_0!|\beta-\tilde\delta-\gamma_0-\dots-\gamma_j|!}2^{|\tilde\gamma_0+\dots+\tilde\gamma_{j-1}|}.\tag{4.34}$$

Now, on the support of φ_β we have $|x| \leqslant 2|\beta|^{1/2}$; moreover we have $|\tilde{\delta}| = |\beta| - |\alpha| + |\delta|$. Hence it follows from (1.11) and (4.9) that

$$\left|\partial_\xi^\theta \partial_x^\sigma \left(\varphi_\beta(x) p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,\xi)\right)\right| \leqslant C^{|\gamma_0|+|\tilde{\delta}|+1} \gamma_0!\delta!|\beta|^{\frac{|\beta|-|\alpha|}{2}} \left(1+|x|+|\xi|\right)^{m-|\theta|} \langle x\rangle^{-|\sigma|}$$

for some constant C depending on σ, θ. As a consequence,

$$\left\| E \circ \left(\varphi_\beta(x) p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,D)\right)\right\|_{\mathcal{B}(Q^s)} \leqslant C_s^{|\gamma_0|+|\tilde{\delta}|+1} \gamma_0!\delta!|\beta|^{\frac{|\beta|-|\alpha|}{2}}. \tag{4.35}$$

Now we see from Stirling's formula that, for some $C > 1$,

$$\frac{|\alpha|!^{1/2}|\beta|^{\frac{|\beta|-|\alpha|}{2}}}{|\beta|!^{1/2}} \leqslant C^{|\beta|-|\alpha|} \leqslant C^{|\tilde{\delta}|}. \tag{4.36}$$

By applying (4.34), (4.35) and (4.36) we obtain

$$\begin{aligned}
&\frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!^{1/2}|\beta|!^{1/2}} |C_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}| \\
&\qquad \times \left\| E\left(\varphi_\beta(x) p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,D)\left(x^{\beta-\tilde{\delta}-\gamma_0-\dots-\gamma_j} \partial^{\alpha-\delta-\tilde{\gamma}_0-\dots-\tilde{\gamma}_j} u\right)\right)\right\|_{Q^s} \\
&\quad \leqslant C_s (C_s\varepsilon)^{|\delta|+|\tilde{\delta}|+|\gamma_0+\dots+\gamma_j|+|\tilde{\gamma}_0+\dots+\tilde{\gamma}_j|} \frac{\varepsilon^{|\alpha|+|\beta|-|\delta|-|\tilde{\delta}|-|\gamma_0+\dots+\gamma_j|-|\tilde{\gamma}_0+\dots+\tilde{\gamma}_j|}}{|\beta-\tilde{\delta}-\gamma_0-\dots-\gamma_j|!} \\
&\qquad \times \left\| x^{\beta-\tilde{\delta}-\gamma_0-\dots-\gamma_j} \partial^{\alpha-\delta-\tilde{\gamma}_0-\dots-\tilde{\gamma}_j} u\right\|_{Q^s}.
\end{aligned} \tag{4.37}$$

Since $|\beta| > |\alpha|$ we have $M(\alpha,\beta) = |\alpha|!^{1/2}|\beta|!^{1/2}$, whereas our choice of $\tilde{\delta}$ and $\tilde{\gamma}_j$ implies that $|\alpha-\delta-\tilde{\gamma}_0-\dots-\tilde{\gamma}_j| = |\beta-\tilde{\delta}-\gamma_0-\dots-\gamma_j|$. Then we have $M(\alpha-\delta-\tilde{\gamma}_0-\dots-\tilde{\gamma}_j, \beta-\tilde{\delta}-\gamma_0-\dots-\gamma_j) = |\beta-\tilde{\delta}-\gamma_0-\dots-\gamma_j|!$ and we can rewrite (4.37) as

$$\begin{aligned}
&\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)} |C_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}| \\
&\qquad \times \left\| E\left(\varphi_\beta(x) p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,D)\left(x^{\beta-\tilde{\delta}-\gamma_0-\dots-\gamma_j} \partial^{\alpha-\delta-\tilde{\gamma}_0-\dots-\tilde{\gamma}_j} u\right)\right)\right\|_{Q^s} \\
&\quad \leqslant C_s (C_s\varepsilon)^{|\delta|+|\tilde{\delta}|+|\gamma_0+\dots+\gamma_j|+|\tilde{\gamma}_0+\dots+\tilde{\gamma}_j|} \frac{\varepsilon^{|\alpha|+|\beta|-|\delta|-|\tilde{\delta}|-|\gamma_0+\dots+\gamma_j|-|\tilde{\gamma}_0+\dots+\tilde{\gamma}_j|}}{M(\alpha-\delta-\tilde{\gamma}_0-\dots-\tilde{\gamma}_j, \beta-\tilde{\delta}-\gamma_0-\dots-\gamma_j)} \\
&\qquad \times \left\| x^{\beta-\tilde{\delta}-\gamma_0-\dots-\gamma_j} \partial^{\alpha-\delta-\tilde{\gamma}_0-\dots-\tilde{\gamma}_j} u\right\|_{Q^s}.
\end{aligned} \tag{4.38}$$

It follows from (4.8) and (4.38), by the same arguments[2] as in the case $|\alpha| \geqslant |\beta|$, that

$$\sum_{\substack{|\alpha|+|\beta|\leqslant N\\ |\beta|>|\alpha|}} \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)} \left\| E\left(\varphi_\beta(x)\left[P, x^\beta\partial^\alpha\right]u\right)\right\|_{Q^s} \leqslant C_s'\varepsilon S_{N-1}^{s,\varepsilon}[u]. \tag{4.39}$$

This estimate, together with (4.16) and (4.33), implies (4.3), which concludes the proof. □

We now turn the attention to the nonlinear term. We first treat the case when $m \geqslant 1$.

Proposition 4.5. *Let $E \in \mathrm{OP}\Gamma^{-m}(\mathbb{R}^d)$, $m \geqslant 1$, $h \in \mathbb{N}$, $\rho_1, \dots, \rho_l \in \mathbb{N}^d$, with $h + \max\{|\rho_k|\} \leqslant m-1$. Let g be a real-analytic function on $\mathbb{R}^d$ satisfying the estimates*

$$\left|\partial^\alpha g(x)\right| \leqslant C^{|\alpha|+1}\alpha!\langle x\rangle^{h-|\alpha|}, \quad x \in \mathbb{R}^d,\ \alpha \in \mathbb{N}^d, \tag{4.40}$$

for some $C > 0$ independent of α. Then for every integer $s > d/2 + \max_k\{|\rho_k|\}$ there exists a constant $C_s > 0$ such that, for every ε small enough, $N \geqslant 1$ and $u \in \mathcal{S}(\mathbb{R}^d)$, the following estimates hold:

$$\sum_{0<|\alpha|+|\beta|\leqslant N} \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)} \left\| E\left(x^\beta\partial^\alpha\left(g(x)\prod_{k=1}^{l}\partial^{\rho_k}u\right)\right)\right\|_{Q^s} \leqslant C_s\varepsilon\left(S_{N-1}^{s,\varepsilon}[u]\right)^l. \tag{4.41}$$

Proof. We first treat the terms with $\beta \neq 0$ in the sum (4.41). Let $j \in \{1,\dots,d\}$ such that $\beta_j \neq 0$. By Leibniz' formula, we have

$$x^\beta\partial^\alpha\left(g(x)\prod_{k=1}^{l}\partial^{\rho_k}u\right) = x_j\sum_{\delta_0\leqslant\alpha-e_j}\ \sum_{\delta_1+\dots+\delta_l=\alpha-\delta_0} \frac{\alpha!}{\delta_0!\delta_1!\cdots\delta_l!}\partial^{\delta_0}g(x)x^{\beta-e_j}\prod_{k=1}^{l}\partial^{\delta_k+\rho_k}u.$$

Let now $\tilde\delta_0$ be a multi-index of maximal length among those satisfying $|\tilde\delta_0| \leqslant |\delta_0|$ and $\tilde\delta_0 \leqslant \beta - e_j$. Write $x^{\beta-e_j} = x^{\tilde\delta_0}x^{\beta-e_j-\tilde\delta_0}$ and observe that the symbol $x_jx^{\tilde\delta_0}\partial^{\delta_0}g(x)$ belongs to $\Gamma^{h+1}(\mathbb{R}^d)$, with every seminorm estimated by $A^{|\delta_0|+1}\delta_0!$ for some positive constant A independent of δ_0. Then $E \circ x_jx^{\tilde\delta_0}\partial^{\delta_0}g(x)$ belongs to $\mathrm{OP}\Gamma^{-m+1+h}(\mathbb{R}^d)$. Consequently, it is continuous $Q^{s-M}(\mathbb{R}^d) \to Q^s(\mathbb{R}^d)$, with $M = \max\{|\rho_k|\}$, since $-m+1+h \leqslant -M$ and its operator norm is bounded by $A^{|\delta_0|+1}\delta_0!$ for a new constant A independent of δ_0. Then we have

$$\begin{aligned}
&\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)} \left\| E\left(x^\beta\partial^\alpha\left(g(x)\prod_{k=1}^{l}\partial^{\rho_k}u\right)\right)\right\|_{Q^s}\\
&\quad\leqslant C_s\sum_{\delta_0\leqslant\alpha} A^{|\delta_0|+1}\sum_{\delta_1+\dots+\delta_l=\alpha-\delta_0} \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}\frac{\alpha!}{\delta_1!\cdots\delta_l!}\left\| x^{\beta-e_j-\tilde\delta_0}\prod_{k=1}^{l}\partial^{\delta_k+\rho_k}u\right\|_{Q^{s-M}}.
\end{aligned}$$

[2] To be precise, here we do not have longer $|\tilde\delta| \leqslant |\delta|$, but rather $|\tilde\delta| = |\beta| - |\alpha| + |\delta|$, so that the number of multi-indices $\tilde\delta$ which may arise is estimated by $2^{|\beta|-|\alpha|+|\delta|+d-1}$, and this factor is absorbed by the power $\varepsilon^{|\tilde\delta|} = \varepsilon^{|\beta|-|\alpha|+|\delta|}$ in (4.38).

We can now write

$$x^{\beta-e_j-\tilde{\delta}_0}\prod_{k=1}^{l}\partial^{\delta_k+\rho_k}u=\prod_{k=1}^{l}x^{\gamma_k}\partial^{\delta_k+\rho_k}u,$$

where $\gamma_1+\dots+\gamma_l=\beta-e_j-\tilde{\delta}_0$ and, if $|\beta|\leqslant|\alpha|$, with $|\gamma_k|\leqslant|\delta_k|$ for $1\leqslant k\leqslant l$ (which is possible because in that case $|\beta-e_j-\tilde{\delta}_0|\leqslant|\alpha-\delta_0|$; observe that if $|\tilde{\delta}_0|<|\delta_0|$ then $\beta-e_j-\tilde{\delta}_0=0$), whereas, if $|\beta|\geqslant|\alpha|+1$, with $|\gamma_k|\geqslant|\delta_k|$ for $1\leqslant k\leqslant l$ (which is possible because in that case $|\tilde{\delta}_0|=|\delta_0|$ and $|\beta-e_j-\tilde{\delta}_0|\geqslant|\alpha-\delta_0|$). Moreover, if $|\beta|\leqslant|\alpha|$ (then $M(\alpha,\beta)=|\alpha|!$), we have by (2.3) the following inequality

$$\frac{1}{|\alpha|!}\cdot\frac{\alpha!}{\delta_1!\cdots\delta_l!}\leqslant\frac{1}{|\delta_1|!\cdots|\delta_l|!},\tag{4.42}$$

whereas, for $|\beta|\geqslant|\alpha|+1$ (then $M(\alpha,\beta)=|\alpha|!^{1/2}|\beta|!^{1/2}$), we have

$$\frac{1}{|\alpha|!^{1/2}|\beta|!^{1/2}}\cdot\frac{\alpha!}{\delta_1!\cdots\delta_l!}\leqslant\frac{1}{(|\delta_1|!\cdots|\delta_l|!|\gamma_1|!\cdots|\gamma_l|!)^{1/2}},\tag{4.43}$$

which also follows at once from (2.3). Hence by Proposition 2.1 we get

$$\begin{aligned}&\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}\left\|E\left(x^\beta\partial^\alpha\left(g(x)\prod_{k=1}^{l}\partial^{\rho_k}u\right)\right)\right\|_{Q^s}\\&\quad\leqslant C_s\varepsilon\sum_{\delta_0\leqslant\alpha}(A\varepsilon)^{|\delta_0|}\varepsilon^{|\tilde{\delta}_0|}\sum_{\delta_1+\dots+\delta_l=\alpha-\delta_0}\prod_{k=1}^{l}\frac{\varepsilon^{|\gamma_k|+|\delta_k|}}{M(\gamma_k,\delta_k)}\left\|x^{\gamma_k}\partial^{\delta_k+\rho_k}u\right\|_{Q^{s-M}}.\end{aligned}\tag{4.44}$$

Let now $T\in\mathrm{OP}\Gamma^{-M}(\mathbb{R}^d)$ be any operator which gives an isomorphism $Q^{s-M}\to Q^s$, and write $x^{\gamma_k}\partial^{\delta_k+\rho_k}u=\partial^{\rho_k}(x^{\gamma_k}\partial^{\delta_k}u)+[x^{\gamma_k}\partial^{\delta_k},\partial^{\rho_k}]u$ in the last term of (4.44). We get

$$\left\|x^{\gamma_k}\partial^{\delta_k+\rho_k}u\right\|_{Q^{s-M}}\leqslant\left\|x^{\gamma_k}\partial^{\delta_k}u\right\|_{Q^s}+\left\|T\left[x^{\gamma_k}\partial^{\delta_k},\partial^{\rho_k}\right]u\right\|_{Q^s},$$

where we used the fact that ∂^{ρ_k} is bounded $Q^s(\mathbb{R}^d)\to Q^{s-M}(\mathbb{R}^d)$.

Using this last estimate we obtain

$$\begin{aligned}&\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}\left\|E\left(x^\beta\partial^\alpha\left(g(x)\prod_{k=1}^{l}\partial^{\rho_k}u\right)\right)\right\|_{Q^s}\\&\quad\leqslant C_s\varepsilon\sum_{\delta_0\leqslant\alpha}(A\varepsilon)^{|\delta_0|}\sum_{\tilde{\delta}_0\leqslant\beta-e_j}\varepsilon^{|\tilde{\delta}_0|}\sum_{\delta_1+\dots+\delta_l=\alpha-\delta_0}\prod_{k=1}^{l}\frac{\varepsilon^{|\gamma_k|+|\delta_k|}}{M(\gamma_k,\delta_k)}\\&\quad\quad\times\left\{\left\|x^{\gamma_k}\partial^{\delta_k}u\right\|_{Q^s}+\sum_{|\gamma|\leqslant m-1}\left\|T\left[x^{\gamma_k}\partial^{\delta_k},\partial^{\gamma}\right]u\right\|_{Q^s}\right\}\end{aligned}$$

(recall that the γ_k's depend on $\alpha, \beta, \tilde{\delta}_0, \delta_1, \dots, \delta_l$ and the choice of e_j). We now sum the above expression over $|\alpha| + |\beta| \leqslant N$, $\alpha \neq 0$. When α and β vary but δ and $\tilde{\delta}_0$ are fixed, every term in the above sum also appears in the development of

$$\left\{ \sum_{|\tilde{\alpha}|+|\tilde{\beta}| \leqslant N-1} \frac{\varepsilon^{|\tilde{\alpha}|+|\tilde{\beta}|}}{M(\tilde{\alpha}, \tilde{\beta})} \left\{ \left\| x^{\tilde{\beta}} \partial^{\tilde{\alpha}} u \right\|_{Q^s} + \sum_{|\gamma| \leqslant m-1} \left\| T\left[x^{\tilde{\beta}} \partial^{\tilde{\alpha}}, \partial^{\gamma}\right] u \right\|_{Q^s} \right\} \right\}^l$$

and is repeated at most d times (corresponding to the possible choices of e_j). Hence, taking ε sufficiently small, we obtain

$$\begin{aligned} &\sum_{\substack{0<|\alpha|+|\beta| \leqslant N \\ \beta \neq 0}} \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha, \beta)} \left\| E\left(x^{\beta} \partial^{\alpha} \left(g(x) \prod_{k=1}^{l} \partial^{\rho_k} u \right) \right) \right\|_{Q^s} \\ &\quad \leqslant C_s'' \varepsilon \left\{ \sum_{|\tilde{\alpha}|+|\tilde{\beta}| \leqslant N-1} \frac{\varepsilon^{|\tilde{\alpha}|+|\tilde{\beta}|}}{M(\tilde{\alpha}, \tilde{\beta})} \left\{ \left\| x^{\tilde{\beta}} \partial^{\tilde{\alpha}} u \right\|_{Q^s} + \sum_{|\gamma| \leqslant m-1} \left\| T\left[x^{\tilde{\beta}} \partial^{\tilde{\alpha}}, \partial^{\gamma}\right] u \right\|_{Q^s} \right\} \right\}^l \\ &\quad \leqslant C_s'' \varepsilon \left\{ S_{N-1}^{s,\varepsilon}[u] + C_s''' \varepsilon S_{N-2}^{s,\varepsilon}[u] \right\}^l \leqslant C_s'''' \varepsilon \left(S_{N-1}^{s,\varepsilon}[u] \right)^l, \end{aligned}$$

where we used Proposition 4.4 applied with ∂^{γ} and T in place of P and E respectively, and we understand $S_{-1}^{s,\varepsilon}[u] = 0$.

We now treat the terms with $\beta = 0$ in the sum (4.41) (recall, $M(\alpha, 0) = |\alpha|!$). Let $\alpha \neq 0$ and $j \in \{1, \dots, d\}$ such that $\alpha_j \neq 0$. By writing $\partial^{\alpha} = \partial_j \partial^{\alpha - e_j}$ and using Leibniz' formula we have

$$\partial^{\alpha} \left(g(x) \prod_{k=1}^{l} \partial^{\rho_k} u \right) = \partial_j \sum_{\delta_0 \leqslant \alpha - e_j} \sum_{\delta_1 + \cdots + \delta_l = \alpha - e_j - \delta_0} \frac{(\alpha - e_j)!}{\delta_0! \delta_1! \cdots \delta_l!} \partial^{\delta_0} g(x) \prod_{k=1}^{l} \partial^{\delta_k + \rho_k} u.$$

Observe that $E \partial_j \circ \partial^{\delta_0} g(x) \in \mathrm{OP}\Gamma^{-m+1+h}(\mathbb{R}^d)$ is bounded $Q^{s-M}(\mathbb{R}^d) \to Q^s(\mathbb{R}^d)$, with $M = \max\{|\rho_k|\}$, because $-m + 1 + h \leqslant -M$, and its operator norm is estimated by $A^{|\delta_0|+1} \delta_0!$ for some positive constant A independent of δ_0. Hence

$$\begin{aligned} &\frac{\varepsilon^{|\alpha|}}{|\alpha|!} \left\| E\left(\partial^{\alpha} \left(g(x) \prod_{k=1}^{l} \partial^{\rho_k} u \right) \right) \right\|_{Q^s} \\ &\quad \leqslant C_s \sum_{\delta_0 \leqslant \alpha - e_j} A^{|\delta_0|+1} \sum_{\delta_1 + \cdots + \delta_l = \alpha - e_j - \delta_0} \frac{\varepsilon^{|\alpha|}}{|\alpha|!} \frac{(\alpha - e_j)!}{\delta_1! \cdots \delta_l!} \left\| \prod_{k=1}^{l} \partial^{\delta_k + \rho_k} u \right\|_{Q^{s-M}}. \end{aligned}$$

Using the inequality

$$\frac{1}{|\alpha|!} \cdot \frac{(\alpha - e_j)!}{\delta_1! \cdots \delta_l!} \leqslant \frac{1}{|\delta_1|! \cdots |\delta_l|!},$$

Proposition 2.1 and the boundedness of $\partial^{\rho_k} : Q^s \to Q^{s-M}$ we get

$$\frac{\varepsilon^{|\alpha|}}{|\alpha|!}\left\| E\left(\partial^\alpha\left(g(x)\prod_{k=1}^{l}\partial^{\rho_k}u\right)\right)\right\|_{Q^s} \leqslant C_s\varepsilon \sum_{\delta_0\leqslant\alpha-e_j}(A\varepsilon)^{|\delta_0|}\sum_{\delta_1+\cdots+\delta_l=\alpha-e_j-\delta_0}\prod_{k=1}^{l}\frac{\varepsilon^{|\delta_k|}}{|\delta_k|!}\left\|\partial^{\delta_k}u\right\|_{Q^s}.$$

By the same arguments as above we obtain

$$\sum_{0<|\alpha|\leqslant N}\frac{\varepsilon^{|\alpha|}}{|\alpha|!}\left\| E\left(x^\beta\partial^\alpha\left(g(x)\prod_{k=1}^{l}\partial^{\rho_k}u\right)\right)\right\|_{Q^s} \leqslant C_s\varepsilon\left(S^{s,\varepsilon}_{N-1}[u]\right)^l,$$

which concludes the proof. □

We are now ready to conclude the proof of Theorem 4.1.

End of the proof of Theorem 4.1 *(the case $m \geqslant 1$).* From (4.1) we have, for $\alpha, \beta \in \mathbb{N}^d$, $\varepsilon > 0$,

$$\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}x^\beta\partial^\alpha Pu = \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}x^\beta\partial^\alpha f + \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}x^\beta\partial^\alpha F[u],$$

so that

$$\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}P\left(x^\beta\partial^\alpha u\right) = \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}\left[P, x^\beta\partial^\alpha\right]u + \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}x^\beta\partial^\alpha f + \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}x^\beta\partial^\alpha F[u].$$

We now apply to both sides the parametrix E of P. With $R = EP - I \in \mathrm{OP}\Gamma^{-1}(\mathbb{R}^d)$ we obtain

$$\begin{aligned}\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}x^\beta\partial^\alpha u = {}& -\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}R\left(x^\beta\partial^\alpha u\right) + \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}E\left[P, x^\beta\partial^\alpha\right]u \\ & + \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}E\left(x^\beta\partial^\alpha f\right) + \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}E\left(x^\beta\partial^\alpha F[u]\right).\end{aligned}$$

Taking the Q^s norms and summing over $|\alpha| + |\beta| \leqslant N$ give

$$\begin{aligned}S^{s,\varepsilon}_N[u] \leqslant {}& \|u\|_{Q^s} + \sum_{0<|\alpha|+|\beta|\leqslant N}\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}\left\|R\left(x^\beta\partial^\alpha u\right)\right\|_{Q^s} \\ & + \sum_{0<|\alpha|+|\beta|\leqslant N}\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}\left\|E\left[P, x^\beta\partial^\alpha\right]u\right\|_{Q^s} \\ & + \sum_{0<|\alpha|+|\beta|\leqslant N}\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}\left\|E\left(x^\beta\partial^\alpha f\right)\right\|_{Q^s} \\ & + \sum_{0<|\alpha|+|\beta|\leqslant N}\frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)}\left\|E\left(x^\beta\partial^\alpha F[u]\right)\right\|_{Q^s}.\end{aligned} \tag{4.45}$$

The second and the third term in the right-hand side of (4.45) can be estimated using Propositions 4.3 and 4.4 while the term containing f is obviously dominated by $S_\infty^{s,\varepsilon}[f]$. For the last term we can apply Proposition 4.5. Hence, we have that, for ε small enough,

$$S_N^{s,\varepsilon}[u] \leqslant \|u\|_{Q^s} + C_s S_\infty^{s,\varepsilon}[f] + C_s \varepsilon \left(S_{N-1}^{s,\varepsilon}[u] + \sum_l \left(S_{N-1}^{s,\varepsilon}[u]\right)^l \right).$$

Iterating the last estimate and possibly shrinking ε, we obtain that $S_\infty^{s,\varepsilon}[u] < \infty$, which implies $u \in \mathcal{H}_{sect}(\mathbb{R}^d)$ by Proposition 3.3.

4.2. Proof of Theorem 4.1: the case $0 < m < 1$

In this case the nonlinearity (1.13), due to the restriction $h + \max\{|\rho_k|\} \leqslant \max\{m-1, 0\}$ reduces to the following form

$$F[u] = \sum_l F_l(x) u^l, \tag{4.46}$$

the above sum being finite, with $l \in \mathbb{N}$, $l \geqslant 2$, and $F_l(x)$ real-analytic functions satisfying the following estimates

$$\left|\partial^\alpha F_l(x)\right| \leqslant C^{|\alpha|+1} \alpha! \langle x\rangle^{-|\alpha|}, \quad x \in \mathbb{R}^d,\ \alpha \in \mathbb{N}^d, \tag{4.47}$$

for some $C > 0$ independent of α.

We follow the same argument used for the case $m \geqslant 1$, so that we only sketch the proof. For technical reasons which will be clear in the sequel, here it is convenient to work in the framework of the usual Sobolev spaces, i.e. by defining

$$\widetilde{S}_N^{s,\varepsilon}[f] = \sum_{|\alpha|+|\beta| \leqslant N} \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)} \left\|x^\beta \partial^\alpha f\right\|_{H^s}, \qquad \widetilde{S}_\infty^{s,\varepsilon}[f] = \sum_{\alpha,\beta \in \mathbb{N}^d} \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)} \left\|x^\beta \partial^\alpha f\right\|_{H^s}. \tag{4.48}$$

It is easy to see that the results in Propositions 3.3, 4.3 and 4.4 continue to hold with $S_N^{s,\varepsilon}[u]$ and $S_\infty^{s,\varepsilon}[u]$ replaced by $\widetilde{S}_N^{s,\varepsilon}[u]$ and $\widetilde{S}_\infty^{s,\varepsilon}[u]$ and with the spaces Q^s replaced by H^s everywhere (operators in $\mathrm{OP}\Gamma^0(\mathbb{R}^d)$ are bounded on every H^s by Proposition 2.2 with $m = n = 0$). It remains to estimate the nonlinear term. On this point we observe that although this term is more elementary than before, the action of the parametrix gives a lower "gain", since $0 < m < 1$. Then we have to modify slightly our technique. We have the following result.

Proposition 4.6. *Let $E \in \mathrm{OP}\Gamma^{-m}(\mathbb{R}^d)$, $0 < m < 1$, and let $l \in \mathbb{N}$, $l \geqslant 2$ and g be a real-analytic function on $\mathbb{R}^d$ satisfying the same estimates as in* (4.47). *Then, for every integer $s > d/2$ there exists a constant $C_s' > 0$ and, for every $\tau > 0$, there exists $C_\tau > 0$ such that, for every ε small enough, $N \geqslant 1$ and $u \in \mathcal{S}(\mathbb{R}^d)$ we have*

$$\sum_{0<|\alpha+\beta|\leqslant N} \frac{\varepsilon^{|\alpha|+|\beta|}}{M(\alpha,\beta)} \left\| E\left(x^\beta \partial^\alpha \left(g(x)u^l\right)\right)\right\|_{H^s}$$
$$\leqslant \tau C_s' \|u\|_{H^s}^{l-1} S_N^{s,\varepsilon}[u] + C_s'(\varepsilon C_\tau + \tau + \varepsilon)\left(S_{N-1}^{s,\varepsilon}[u]\right)^l. \tag{4.49}$$

Proof. We first consider the terms with $\beta \neq 0$. We can write

$$x^\beta \partial^\alpha\left(g(x)u^l\right) = g(x)x^\beta \partial^\alpha u^l + \sum_{\substack{\delta_0+\delta_1+\cdots+\delta_k=\alpha \\ \delta_0\neq 0}} \frac{\alpha!}{\delta_0!\delta_1!\cdots\delta_l!} \partial^{\delta_0} g(x) x^\beta \prod_{k=1}^{l} \partial^{\delta_k} u.$$

Let $\tilde{\delta}_0$ be a multi-index of maximal length among those satisfying $|\tilde{\delta}_0| \leqslant |\delta_0|$, $\tilde{\delta}_0 \leqslant \beta$. The operators $E \circ g(x)$ and $E \circ x^{\tilde{\delta}_0}\partial^{\delta_0} g(x)$ belong to $\mathrm{OP}\Gamma^{-m}(\mathbb{R}^d)$, and the symbol of the second one has each seminorm estimated by $A^{|\delta_0|+1}\delta_0!$, for some positive constant A independent of δ_0. Hence, by the continuity properties on weighted Sobolev spaces (Proposition 2.2 with $n = 0$) we have

$$\left\| E\left(x^\beta \partial^\alpha\left(g(x)u^l\right)\right)\right\|_{H^s}$$
$$\leqslant C_s \left\| \langle x\rangle^{-m} x_j x^{\beta-e_j} \partial^\alpha\left(u^l\right)\right\|_{H^s} + \sum_{\substack{\delta_0+\delta_1+\cdots+\delta_k=\alpha \\ \delta_0\neq 0}} C_s^{|\delta_0|+1} \frac{\alpha!}{\delta_1!\cdots\delta_l!} \left\| x^{\beta-\tilde{\delta}_0} \prod_{k=1}^{l} \partial^{\delta_k} u \right\|_{H^s}.$$

Now, since $\beta_j \neq 0$ for some $j \in \{1,\dots,d\}$, we have

$$\begin{aligned}
\left\| \langle x\rangle^{-m} x_j x^{\beta-e_j} \partial^\alpha\left(u^l\right)\right\|_{H^s} &= \sum_{|\gamma|\leqslant s} \left\| \partial^\gamma\left(\langle x\rangle^{-m} x_j x^{\beta-e_j} \partial^\alpha\left(u^l\right)\right)\right\|_{L^2} \\
&\leqslant \sum_{|\gamma|\leqslant s} \left\| \langle x\rangle^{-m} x_j \partial^\gamma\left(x^{\beta-e_j} \partial^\alpha\left(u^l\right)\right)\right\|_{L^2} \\
&\quad + \sum_{|\gamma|\leqslant s} \sum_{0\neq\gamma'\leqslant\gamma} \binom{\gamma}{\gamma'} \left\| \partial^{\gamma'}\left(\langle x\rangle^{-m} x_j\right) \partial^{\gamma-\gamma'}\left(x^{\beta-e_j} \partial^\alpha\left(u^l\right)\right)\right\|_{L^2}.
\end{aligned}$$

Now, for every $\tau > 0$ there exists $C_\tau' > 0$ such that

$$\langle x\rangle^{-m}|x_j| \leqslant \tau|x_j| + C_\tau'. \tag{4.50}$$

Using this inequality and commuting x_j with ∂^γ we get

$$\begin{aligned}
&\sum_{|\gamma|\leqslant s} \left\| \langle x\rangle^{-m} x_j \partial^\gamma\left(x^\beta \partial^\alpha\left(u^l\right)\right)\right\|_{L^2} \\
&\quad\leqslant \tau \sum_{|\gamma|\leqslant s} \left\| x_j \partial^\gamma\left(x^{\beta-e_j} \partial^\alpha u^l\right)\right\|_{L^2} + C_{s,\tau} \left\| x^{\beta-e_j} \partial^\alpha\left(u^l\right)\right\|_{H^s} \\
&\quad\leqslant \tau C_s \left\| x^\beta \partial^\alpha\left(u^l\right)\right\|_{H^s} + C_{s,\tau}' \left\| x^{\beta-e_j} \partial^\alpha\left(u^l\right)\right\|_{H^s}.
\end{aligned}$$

We notice moreover that for $\gamma' \neq 0$ we have $\partial^{\gamma'}(\langle x\rangle^{-m}x_j) \in L^\infty(\mathbb{R}^d)$, so that

$$\sum_{|\gamma|\leqslant s}\sum_{0\neq\gamma'\leqslant\gamma}\binom{\gamma}{\gamma'}\big\|\partial^{\gamma'}\big(\langle x\rangle^{-m}x_j\big)\partial^{\gamma-\gamma'}\big(x^{\beta-e_j}\partial^\alpha u^l\big)\big\|_{L^2} \leqslant C_s\big\|x^{\beta-e_j}\partial^\alpha u^l\big\|_{H^s}.$$

Hence we have obtained that

$$\begin{aligned}\big\|E\big(x^\beta\partial^\alpha\big(g(x)u^l\big)\big)\big\|_{H^s} &\leqslant \tau C_s\big\|x^\beta\partial^\alpha\big(u^l\big)\big\|_{H^s} + C'_{s,\tau}\big\|x^{\beta-e_j}\partial^\alpha\big(u^l\big)\big\|_{H^s} \\ &\quad + \sum_{\substack{\delta_0+\delta_1+\cdots+\delta_k=\alpha\\ \delta_0\neq 0}} C_s^{|\delta_0|+1}\frac{\alpha!}{\delta_1!\cdots\delta_l!}\left\|x^{\beta-\tilde\delta_0}\prod_{k=1}^{l}\partial^{\delta_k}u\right\|_{H^s}. \qquad (4.51)\end{aligned}$$

Let us estimate the three terms in the right-hand side of (4.51). To treat the first one we observe that

$$x^\beta\partial^\alpha\big(u^l\big) = lu^{l-1}x^\beta\partial^\alpha u + \sum_{\substack{\delta_1+\cdots+\delta_l=\alpha\\ \delta_k\neq\alpha\ \forall k}}\frac{\alpha!}{\delta_1!\cdots\delta_l!}\prod_{k=1}^{l}x^{\gamma_k}\partial^{\delta_k}u,$$

where, as before, we can choose $\gamma_1,\ldots,\gamma_l \in \mathbb{N}^d$ such that $\gamma_1+\cdots+\gamma_l = \beta$ and $|\gamma_j| \leqslant |\delta_j|$ (respectively $|\gamma_j| \geqslant |\delta_j|$) if $|\beta| \leqslant |\alpha|$ (respectively if $|\beta| \geqslant |\alpha|$). Then, using the same arguments as in the case $m \geqslant 1$, we obtain

$$\tau C_s\sum_{\substack{|\alpha+\beta|\leqslant N\\ \beta\neq 0}}\frac{\varepsilon^{|\alpha+\beta|}}{M(\alpha,\beta)}\big\|x^\beta\partial^\alpha\big(u^l\big)\big\|_{H^s} \leqslant \tau l C_s\|u\|_{H^s}^{l-1}\widetilde{S}_N^{s,\varepsilon}[u] + \tau C_s\big(\widetilde{S}_{N-1}^{s,\varepsilon}[u]\big)^l. \qquad (4.52)$$

Similarly, we easily prove that

$$\sum_{\substack{|\alpha+\beta|\leqslant N\\ \beta\neq 0}}\frac{\varepsilon^{|\alpha+\beta|}}{M(\alpha,\beta)}\big\|x^{\beta-e_j}\partial^\alpha\big(u^l\big)\big\|_{H^s} \leqslant C_s\varepsilon\big(\widetilde{S}_{N-1}^{s,\varepsilon}[u]\big)^l. \qquad (4.53)$$

Concerning the third term in (4.51) we can write $x^{\beta-\tilde\delta_0} = \prod_{k=1}^{l}x^{\gamma_k}$, where $\gamma_1,\ldots,\gamma_l$ satisfy $\gamma_1+\cdots+\gamma_l = \beta-\tilde\delta_0$ and $|\gamma_j| \leqslant |\delta_j|$ (respectively $|\gamma_j| \geqslant |\delta_j|$) if $|\beta| \leqslant |\alpha|$ (respectively if $|\beta| \geqslant |\alpha|$). Then, the same arguments as in the case $m \geqslant 1$ yield

$$\sum_{\substack{|\alpha+\beta|\leqslant N\\ \beta\neq 0}}\frac{\varepsilon^{|\alpha+\beta|}}{M(\alpha,\beta)}\sum_{\substack{\delta_0+\delta_1+\cdots+\delta_k=\alpha\\ \delta_0\neq 0}}C^{|\delta_0|+1}\frac{\alpha!}{\delta_1!\cdots\delta_l!}\left\|x^{\beta-\tilde\delta_0}\prod_{k=1}^{l}\partial^{\delta_k}u\right\|_{H^s} \leqslant C_s\varepsilon\big(\widetilde{S}_{N-1}^{s,\varepsilon}[u]\big)^l \qquad (4.54)$$

for $\varepsilon > 0$ sufficiently small. The estimate of the terms in (4.49) with $\beta = 0$ (hence $\alpha \neq 0$) is very similar but easier, relying on the inequality

$$\langle\xi\rangle^{-m}|\xi_j| \leqslant \tau|\xi_j| + C'_\tau \qquad (4.55)$$

in place of (4.50). We omit the details for the sake of brevity. □

End of the proof of Theorem 4.1 *(the case* $0 < m < 1$*)*. Using the same argument as in the case $m \geqslant 1$, by the variants of Propositions 3.3, 4.3 and 4.4 with $\widetilde{S}_N^{s,\varepsilon}[f]$ and $\widetilde{S}_\infty^{s,\varepsilon}[f]$ defined in (4.48) in place of $S_N^{s,\varepsilon}[f]$ and $S_\infty^{s,\varepsilon}[f]$, and with the spaces Q^s replaced by H^s, and by Proposition 4.6 we obtain

$$\widetilde{S}_N^{s,\varepsilon}[u] \leqslant \|u\|_{H^s} + C_s' \widetilde{S}_\infty^{s,\varepsilon}[f] + C_s' \varepsilon \widetilde{S}_{N-1}^{s,\varepsilon}[u] + \sum_l \left(\tau C_s' \|u\|_{H^s}^{l-1} \widetilde{S}_N^{s,\varepsilon}[u] + C_s'(\varepsilon C_\tau + \tau + \varepsilon)\left(\widetilde{S}_{N-1}^{s,\varepsilon}[u]\right)^l\right)$$

for every $N \geqslant 1$ and ε small enough. Now, choosing $\tau < (2\sum_l C_s' \|u\|_s^{l-1})^{-1}$ we obtain

$$\widetilde{S}_N^{s,\varepsilon}[u] \leqslant 2\|u\|_{H^s} + 2C_s' \widetilde{S}_\infty^{s,\varepsilon}[f] + 2C_s' \varepsilon \widetilde{S}_{N-1}^{s,\varepsilon}[u] + \sum_l \left(2C_s'(\varepsilon C_\tau + \tau + \varepsilon)\left(\widetilde{S}_{N-1}^{s,\varepsilon}[u]\right)^l\right).$$

Then we can iterate the last estimate observing that, shrinking τ and then ε, the quantity $\varepsilon C_\tau + \tau + \varepsilon$ can be taken arbitrarily small. This gives $\widetilde{S}_\infty^{s,\varepsilon}[u] < \infty$ and therefore $u \in \mathcal{H}_{sect}(\mathbb{R}^d)$.

5. Examples and concluding remarks

5.1. Some remarks on the analyticity estimates

Let us say a few words on the estimates

$$\left|\partial^\alpha f(x)\right| \leqslant C^{|\alpha|+1} \alpha! \langle x\rangle^{-|\alpha|} \quad \forall \alpha \in \mathbb{N}^d,\ x \in \mathbb{R}^d, \tag{5.1}$$

which are assumed for the coefficients of the metric in (1.6) (cf. also the nonlinearity in (1.8), (1.9), and (1.13), (1.14)).

Locally they are exactly the usual estimates of real-analyticity. To better understand the meaning of the decay for $|x| \to +\infty$, let us consider the following important class of examples. Consider a real-analytic function f in $\mathbb{R}^d$ satisfying, in polar coordinates r, ω, $r > 0$, $\omega \in \mathbb{S}^{d-1}$,

$$f(r\omega) = h\left(r^{-1}, \omega\right), \quad \text{for } r > r_0,\ \omega \in \mathbb{S}^{d-1},$$

for some $r_0 > 0$, where h is an analytic function on $[0, r_0^{-1}) \times \mathbb{S}^{d-1}$, hence *analytic up to* 0 *in the first variable*. Let us verify that then f satisfies the estimates (5.1).

Clearly, it is sufficient to check the estimates (5.1) for large $|x|$. Now, by assumption we have

$$f(x) = f(r\omega) = \sum_{k=0}^{\infty} \frac{1}{k!} \varphi_k(\omega) r^{-k} = \sum_{k=0}^{\infty} \tilde{\varphi}_k(x), \quad |x| > r_0,$$

where φ_k are analytic functions on $\mathbb{S}^{d-1}$, and $\tilde{\varphi}_k(x) \frac{1}{k!} \varphi_k(\omega) r^{-k}$. Observe that the functions $\tilde{\varphi}_k(x)$ are real-analytic functions for $x \neq 0$ and positively homogeneous of degree $-k$. Moreover, by the very definition of φ_k, for every $\bar{x}$, $|\bar{x}| > r_0$, we have the estimates

$$\left|\partial^\alpha \tilde{\varphi}_k(x)\right| \leqslant C^{|\alpha|+k+1} \alpha!, \tag{5.2}$$

for some constant $C > 0$, in some neighborhood of $\bar{x}$ (this is easily verified in polar coordinates and then one uses the analyticity of the change of variables). Hence, by compactness, the estimates (5.2) hold, say, for $|x| = 2r_0$. By homogeneity we deduce that

$$\left|\partial^\alpha \tilde{\varphi}_k(x)\right| \leqslant C^{|\alpha|+k+1}\alpha!\left|\frac{x}{2r_0}\right|^{-k-|\alpha|}, \quad x \neq 0. \tag{5.3}$$

Hence we obtain

$$\left|\partial^\alpha f(x)\right| \leqslant \sum_{k=0}^{\infty}\left|\partial^\alpha \tilde{\varphi}_k(x)\right| \leqslant C\alpha!\sum_{k=0}^{\infty}\left|\frac{x}{2r_0 C}\right|^{-k-|\alpha|} \leqslant C(2r_0 C)^{|\alpha|}\alpha!|x|^{-|\alpha|},$$

if $|x| > 4r_0 C$. This concludes the proof of (5.1).

As another remark, we observe that the estimates (5.1) are in fact equivalent to requiring that $f(x)$ extends to a bounded holomorphic function $f(x+iy)$ in a sector of the type (1.15) (see e.g. [7, Proposition 5.1]). This is very useful to check the estimates (5.1) in concrete situations, as we will see below.

5.2. *Metric Laplacians*

Consider a smooth Riemannian metric $g_{jk}(x)$ in $\mathbb{R}^d$. The corresponding Laplace–Beltrami operator has the form

$$\begin{aligned}\mathcal{L}u &= \sum_{j,k=1}^{d}\frac{1}{\sqrt{g(x)}}\partial_j\left(\sqrt{g(x)}g^{jk}(x)\partial_k u\right)\\ &= \sum_{j,k=1}^{d}\left(g^{jk}(x)\partial_j\partial_k u - g^{jk}(x)\sum_{l=1}^{d}\Gamma_{jk}^l(x)\partial_l u\right),\end{aligned}$$

where g^{jk} is the inverse matrix of g_{jk}, $g = \det(g_{jk})$, and the Christoffel symbols are defined by

$$\Gamma_{ij}^l = \frac{1}{2}\sum_{k=1}^{d} g^{kl}(\partial_i g_{kj} + \partial_j g_{ik} - \partial_k g_{ij}).$$

Let us assume that the metric is real-analytic and satisfies the estimates

$$\left|\partial^\alpha g_{jk}(x)\right| \leqslant C^{|\alpha|+1}\alpha!\langle x\rangle^{-|\alpha|}, \qquad g(x) > C^{-1}, \tag{5.4}$$

for some $C > 0$, and every $\alpha \in \mathbb{N}^d$, $x \in \mathbb{R}^d$. Then the matrix g^{jk} satisfies (1.5) and the estimates in (1.6). If in addition we consider $V(x)$ and $F[u]$ as in (1.7), (1.8), then the equation

$$-\mathcal{L}u + V(x)u - \lambda u = F[u], \quad \lambda \in \mathbb{C}, \tag{5.5}$$

is a special case of (1.4). Hence, by Theorem 1.1, every solution $u \in H^s(\mathbb{R}^d)$, $s > d/2 + 1$, of (5.5), extends to a holomorphic function $u(x+iy)$ in the sector of $\mathbb{C}^d$ in (1.15), satisfying there the estimates in (1.16) for some constants $C > 0$, $c > 0$.

As a model for the above type of metrics, one may consider the hyperboloid $\mathbb{R}^{d+1}$:

$$S = \{(x,t) \in \mathbb{R}^d \times \mathbb{R}\colon\ t = \sqrt{1+|x|^2}\},$$

parametrized by $x \in \mathbb{R}^d$. The Riemannian metric induced on S by the Euclidean one then satisfies the estimates (5.4), as one can easily verify.

More generally, we can consider real-analytic scattering metrics in $\mathbb{R}^d$. They play an important role in geometric scattering theory, see [20,22], [21, Chapter 6], and have received particular attention in the last years (see e.g. [13] and the references therein). Indeed, natural perturbations of the Euclidean metric fall in that category.

A real-analytic metric in $\mathbb{R}^d$ is of scattering type if for any coordinate chart $V \subset \mathbb{S}^{d-1}$ and for some $r_0 > 0$ it has the form

$$h(r^{-1}, \eta; dr, r\,d\eta), \quad \text{for } r > r_0, \tag{5.6}$$

where $r = |x|$, $\eta = (\eta_1, \eta_2, \ldots, \eta_{d-1})$ are real-analytic coordinates on V, and h is a positive definite quadratic form in the last couple of variables, whose coefficients are analytic functions on $[0, r_0^{-1}) \times V$. Moreover one requires that $h(0, \eta; dr, d\eta)$ is positive definite.[3]

Notice that this metric approaches the conic metric $h(0, \eta; dr, r\,d\eta)$ as $r \to +\infty$, which explains the terminology, sometimes used in the literature, of *asymptotically conic* metric. Notice, by comparison, that the Euclidean metric $|dx|^2$ in polar coordinates reads in fact $dr^2 + r^2 h'$, where h' is the usual metric on $\mathbb{S}^{d-1}$.

Now, using the remark in Section 5.1 one sees that, in Euclidean coordinates, such a metric satisfies the estimates (5.4). In particular, the bound from below in (5.4) is satisfied because $h(0, \omega; dr, r\,d\omega)$ is positive definite, and this last metric in Euclidean coordinates has coefficients which are homogeneous functions of degree 0 (in fact, each η_j is a real-analytic function of x on $\mathbb{R}_+ \times V$, positively homogeneous of degree 0).

5.3. The linear case

This above result for Eq. (5.4) seems interesting even in the linear case ($F[u] = 0$), namely for the eigenfunctions of $-\mathcal{L} + V(x)$. That equation appears naturally, for example, when looking for standing wave solutions (i.e. solutions of the type $v(t,x) = e^{i\lambda t}u(x)$) of the Schrödinger equation $i\partial_t v - \mathcal{L}v + V(x)v = 0$ for scattering metrics (cf. [13]).

In the linear case we can even assume $u \in \mathcal{S}'(\mathbb{R}^d)$. In fact, the existence of a parametrix for $-\mathcal{L} + V(x)$ (Proposition 2.3), shows that such a solution is automatically in $\mathcal{S}(\mathbb{R}^d)$. Moreover, if $V(x)$ is in addition real-valued, we know e.g. from [17] (see also [24, Theorem 4.2.9]) that the operator $-\mathcal{L} + V(x)$, regarded as a symmetric operator in $L^2(\mathbb{R}^d, \sqrt{g}\,dx)$ with domain $\mathcal{S}(\mathbb{R}^d)$, is essentially self-adjoint and $L^2(\mathbb{R}^d, \sqrt{g}\,dx)$ has an orthonormal basis made of eigenfunctions.[4] Also, $\dim \operatorname{Ker}(-\mathcal{L} + V(x) - \lambda) < \infty$, which implies that the width ε of the sector in (1.15) can then be chosen uniformly with respect to the solutions.

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[3] Of course, this is equivalent to saying that $h(0, \eta; dr, r\,d\eta)$ is positive definite for every $r > 0$.

[4] Since the eigenvalues of the metric are bounded from below and from above, $L^2(\mathbb{R}^d, dx) = L^2(\mathbb{R}^d, \sqrt{g}\,dx)$ as normed space.

5.4. Sharpness of Theorem 1.1

We recall that in the case of a differential operator with polynomial coefficients the solutions $u \in \mathcal{S}'(\mathbb{R}^d)$ of the equation $Pu = 0$ extend to entire functions on $\mathbb{C}^d$ satisfying estimates (1.16) in a sector of the form (1.15), cf. [8, Theorem 1.1]. Very simple examples show that, even in the linear case, in Theorem 1.1 we cannot expect an entire extension for the solution u. For example, consider, in dimension $d = 1$, the equation

$$-u'' + V(x)u = 0, \tag{5.7}$$

where $V(x) = x^2 + 3 + \frac{2x^2-6}{(x^2+1)^2}$. A solution is given by $u(x) = \frac{1}{x^2+1}e^{-x^2/2}$, which does not extend to an entire function on $\mathbb{C}$.

The following example shows that, in fact, infinitely many singularities can occur along any fixed ray in the complex domain. Let $\theta \in (-\pi/2, \pi/2)$, and consider again Eq. (5.7), with

$$V(x) = x^2 - 1 - 2e^{-2i\theta} + 4e^{-i\theta}x\tanh\bigl(e^{-i\theta}x\bigr) + 6e^{-2i\theta}x\tanh^2\bigl(e^{-i\theta}x\bigr).$$

By applying the remark at the end of Section 5.1 to the function $\tanh(e^{-i\theta}x)$ it is immediate to check that $V(x)$ satisfies the estimates in (1.7). On the other hand, the function

$$u(x) = \cosh^{-2}\bigl(e^{-i\theta}x\bigr)e^{-x^2/2}$$

is a solution of (5.7) and extends to a meromorphic function in the complex plane with poles at $z = e^{i(\theta+\pi/2)}(2k+1)\pi$, $k \in \mathbb{Z}$.

This shows that in Theorem 1.1, even in the linear case, the form of the domain of holomorphic extension as a sector is sharp in general. The following example shows a similar phenomenon in the nonlinear case, even for the standard harmonic oscillator.

Consider the following nonlinear perturbation of the harmonic oscillator, in dimension $d = 1$, at the first eigenvalue $\lambda = 1$:

$$\begin{cases} u'' - x^2u + u = \left(\dfrac{d}{dx} - x\right)u^k, & k \geqslant 2, \\ u(0) = u_0 > 0. \end{cases} \tag{5.8}$$

It was shown in [8] that the solution of (5.8) is given by

$$u(x) = e^{-\frac{x^2}{2}}\left[\lambda + \sqrt{2k-2}\,\mathrm{Erfc}\left(\sqrt{\frac{k-1}{2}}x\right)\right]^{\frac{1}{1-k}} \tag{5.9}$$

with $\lambda = u_0^{1-k} - \sqrt{\frac{\pi(k-1)}{2}}$, where we used the complementary error function defined by

$$\mathrm{Erfc}(t) = \int_t^{+\infty} e^{-v^2}\,dv.$$

Here and in the following, roots are defined to be positive for positive numbers, with continuous extension to the complex domain, i.e., we take principal branches. Suppose now $\lambda > 0$, that is $0 < u_0 < (\frac{\pi(k-1)}{2})^{\frac{1}{2-2k}}$. In this case, since

$$0 < \lambda < \lambda + \sqrt{2k-2}\,\mathrm{Erfc}\left(\sqrt{\frac{k-1}{2}}x\right),$$

the solution $u(x)$ in (5.9) is well defined analytic in $\mathbb{R}$ and

$$0 < u(x) < \lambda^{\frac{1}{1-k}} e^{-\frac{x^2}{2}}.$$

Similar estimates are valid for $u'(x)$, $u''(x)$. Hence we have $u \in H^2(\mathbb{R})$, and Theorem 1.1 applies, implying the desired holomorphic extension $u(z)$ to a sector. However, as observed in [8], $u(z)$ is not entire, but has a singularity at $z_0 \in \mathbb{C}$ when

$$\lambda + \sqrt{2k-2}\,\mathrm{Erfc}\left(\sqrt{\frac{k-1}{2}}z_0\right) = 0, \tag{5.10}$$

where $\mathrm{Erfc}(z)$ is the entire extension of $\mathrm{Erfc}(x)$. Such singularities in fact occur, because the great Picard theorem in the complex domain grants the existence of infinitely many solutions z_0 of (5.10) for all $\lambda \in \mathbb{C}$, but for a possible exceptional value, see [30].

Indeed, we now prove the following more precise result.

Proposition 5.1. *For every $\lambda > 0$, but for a possible exceptional value, and every $\varepsilon > 0$, $u(z)$ has a sequence of singularities which tends to infinity in the sector $\pi/4 < \arg z < \pi/4 + \varepsilon$ or in $3\pi/4 - \varepsilon < \arg z < 3\pi/4$.*

Proof. Using the great Picard theorem as above and the reflection properties

$$\mathrm{Erfc}(\bar{z}) = \overline{\mathrm{Erfc}(z)}, \qquad \mathrm{Erfc}(-\bar{z}) = \sqrt{\pi} - \mathrm{Erfc}(\bar{z}) = \sqrt{\pi} - \overline{\mathrm{Erfc}(z)},$$

which can be verified directly from the definition, it is sufficient to prove that

$$\mathrm{Erfc}(z) \to 0 \quad \text{as } z \to \infty \text{ in the sector } |\arg z| \leqslant \pi/4, \tag{5.11}$$

and that, for every $\varepsilon > 0$,

$$\left|\mathrm{Erfc}(z)\right| \to +\infty \quad \text{as } z \to \infty \text{ in the sector } \pi/4 + \varepsilon < \arg z \leqslant \pi/2. \tag{5.12}$$

Now, (5.11) follows at once from the expansion

$$\mathrm{Erfc}(z) = \frac{e^{-z^2}}{2z}\bigl(1 + R(z)\bigr) \quad \text{with } \left|R(z)\right| \leqslant \frac{1}{\sqrt{2}|z|^2},$$

valid when $|\arg z| \leqslant \pi/4$; see e.g. [19, pp. 18–20].

The property (5.12) can be verified directly as follows. Observe that, for $z = x + iy$, $x > 0$, $y > 0$, we can write

$$\operatorname{Erfc}(z) = -\int_{\gamma} e^{-u^2}\,du,$$

where the path γ is given by the hyperbola through $z = x + iy$ with parametrization $u = \gamma(t) = \frac{xy}{t} + it$, $t \in (0, y]$. Then

$$\begin{aligned}\left|\operatorname{Erfc}(x+iy)\right| &= \left|\int_0^y e^{-\frac{x^2y^2}{t^2}+t^2}\left(-\frac{xy}{t^2}+i\right)dt\right| \\ &\geqslant \int_0^y e^{-\frac{x^2y^2}{t^2}+t^2}\,dt.\end{aligned}$$

Let $0 < \mu < 1$ be a number to be chosen later. We have

$$\left|\operatorname{Erfc}(x+iy)\right| \geqslant \int_{\mu y}^{y} e^{-\frac{x^2y^2}{t^2}+t^2}\,dt \geqslant (1-\mu)ye^{-\mu^{-2}x^2+\mu^2y^2}.$$

Now, if z belongs in addition to the sector in (5.12), we have $0 < x < \tilde{\varepsilon}y$, for some $\tilde{\varepsilon} < 1$. We obtain then

$$\left|\operatorname{Erfc}(x+iy)\right| \geqslant (1-\mu)ye^{(\mu^2-\tilde{\varepsilon}^2\mu^{-2})y^2}.$$

If we choose $\mu > \sqrt{\tilde{\varepsilon}}$, we get $|\operatorname{Erfc}(x+iy)| \to +\infty$ as $y \to +\infty$, which gives the desired conclusion when $x = \operatorname{Re} z > 0$. The case when $x = 0$ is immediate, because

$$\operatorname{Erfc}(iy) = -\int_0^y e^{t^2}\,dt + \frac{\sqrt{\pi}}{2}.$$

Property (5.12) is then proved. □

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