The recognition of the class of indecomposable digraphs under low hemimorphy

A. Boussaïri, P. Ille

Faculté des Sciences Ain Chock, Département de Mathématiques et Informatique, Km 8 route d’El Jadida, BP 5366 Maarif, Casablanca, Maroc
Institut de Mathématiques de Luminy, CNRS – UMR 6206, 163 avenue de Luminy, Case 907, 13288 Marseille Cedex 09, France

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Given a digraph \( G = (V, A) \), the subdigraph of \( G \) induced by a subset \( X \) of \( V \) is denoted by \( G[X] \). With each digraph \( G = (V, A) \) is associated its dual \( G^* = (V, A^*) \) defined as follows: for any \( x, y \in V \), \( (x, y) \in A^* \) if \( (y, x) \in A \). Two digraphs \( G \) and \( H \) are hemimorphic if \( G \) is isomorphic to \( H \) or to \( H^* \). Given \( k > 0 \), the digraphs \( G = (V, A) \) and \( H = (V, B) \) are \( k \)-hemimorphic if for every \( X \subseteq V \), with \( |X| \leq k \), \( G[X] \) and \( H[X] \) are hemimorphic. A class \( C \) of digraphs is \( k \)-recognizable if every digraph \( k \)-hemimorphic to a digraph of \( C \) belongs to \( C \).

1. Introduction

A directed graph or simply digraph \( G \) consists of a finite and nonempty set \( V \) of vertices together with a prescribed collection \( A \) of ordered pairs of distinct vertices, called the set of the arcs of \( G \). Such a digraph is denoted by \( (V, A) \). For example, given a set \( V \), \( (V, \{\}) \) is the empty digraph on \( V \) whereas \( (V, (V \times V) - \{(x, x); x \in V\}) \) is the complete digraph on \( V \). Given a digraph \( G = (V, A) \), with each nonempty subset \( X \) of \( V \) associate the subdigraph \((X, A \cap (X \times X))\) of \( G \) induced by \( X \) denoted by \( G[X] \).

In another respect, given digraphs \( G = (V, A) \) and \( G' = (V', A') \), a bijection \( f \) from \( V \) onto \( V' \) is an isomorphism from \( G \) onto \( G' \) provided that for any \( x, y \in V \), \( (x, y) \in A \) if and only if \( (f(x), f(y)) \in A' \). Two digraphs are then isomorphic if there exists an isomorphism from one onto the other. Finally, a digraph \( H \) embeds into a digraph \( G \) if \( H \) is isomorphic to a subdigraph of \( G \).

With each digraph \( G = (V, A) \) associate its dual \( G^* = (V, A^*) \) and its complement \( \overline{G} = (V, \overline{A}) \) defined as follows. Given \( x \neq y \in V \), \( (x, y) \in A^* \) if \( (y, x) \in A \), and \( (x, y) \in \overline{A} \) if \( (x, y) \notin A \). The digraph \( \overline{G} = (V, \overline{A}) \) is then defined by \( \overline{A} = A - A^* \). Given digraphs \( G \) and \( H \) and \( H \) are hemimorphic if \( G \) is isomorphic to \( H \) or to \( H^* \). Given an integer \( k > 0 \), consider digraphs \( G = (V, A) \) and \( H = (V, B) \). The digraphs \( G \) and \( H \) are \( k \)-hemimorphic if for every subset \( X \) of \( V \), with \( |X| \leq k \), the subdigraphs \( G[X] \) and \( H[X] \) are hemimorphic.

A digraph \( G \) is \( k \)-forced (up to duality) if \( G \) and \( G^* \) are the only digraphs \( k \)-hemimorphic to \( G \).

We need some notations. Let \( G = (V, A) \) be a digraph. For \( x \neq y \in V \), \( x \rightarrow_G y \) or \( y \leftarrow_G x \) means \( (x, y) \in A \) and \( (y, x) \notin A \), \( x \rightarrow_G y \) means \( (x, y) \in A \) and \( (y, x) \in A \) and \( x \cdot \circ_G y \) means \( (x, y) \in A \) and \( (y, x) \notin A \). For \( x \in V \) and \( V \subseteq V \), \( x \rightarrow_G V \) signifies that for every

* Corresponding author.
E-mail addresses: aboussairi@hotmail.com (A. Boussaïri), ille@iml.univ-mrs.fr (P. Ille).

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Theorem 1 (\cite{5,9}). Let \( Q = (V, A) \) be an indecomposable poset. For every poset \( Q' = (V, A') \), if \( C(Q') = C(Q) \), then \( Q' = Q \) or \( Q' = Q^* \).

Given a poset \( Q \), any digraph \( G \), 3-hemimorphic to \( Q \), is a poset such that \( C(G) = C(Q) \). Therefore, every indecomposable poset is 3-forced. To obtain an analogue of Theorem 1 for the tournaments, the comparability digraph is replaced by the \( C_3 \)-structure. Given a tournament \( T = (V, A) \), the family of the subsets \( X \) of \( V \), such that \( T[X] \) is isomorphic to \( C_3 \), is called the \( C_3 \)-structure of \( T \) and denoted by \( C_3(T) \).

Theorem 2 (\cite{11}). Let \( T = (V, A) \) be an indecomposable tournament. For every tournament \( T' = (V, A') \), if \( C_3(T') = C_3(T) \), then \( T' = T \) or \( T' = T^* \).

In other words, every indecomposable tournament is 3-forced. To generalize the two theorems above, we have to disallow the embedding of the following digraphs and its dual. The digraph \( (\{0, 1, 2\}, \{(0, 2), (2, 0), (0, 1)\}) \) is denoted by \( F \). The digraphs \( F \) and \( F^* \) are called flags. A digraph \( G \) is then said to be without flags when \( F \) and \( F^* \) do not embed into \( G \).

Theorem 3 (\cite{11}). An indecomposable digraph without flags is 3-forced.

The flags are generalized in the following way. Given an integer \( n \geq 4 \), consider a permutation \( \sigma \) of \( \{0, \ldots, n - 2\} \). The digraph \( F_n(\sigma) \) is defined on \( \{0, \ldots, n - 1\} \) in the following manner:

\begin{enumerate}
\item \( F_n(\sigma)[\{0, \ldots, n - 2\}] \) is the total order \( \sigma(0) < \cdots < \sigma(n - 2) \);
\item given \( m \in \{0, \ldots, n - 2\} \), either \( m \) is even and \( (m, n - 1) \) and \( (n - 1, m) \) are arcs of \( F_n(\sigma) \) or \( m \) is odd and \( (m, n - 1) \) and \( (n - 1, m) \) are not.
\end{enumerate}

Given \( n \geq 4 \), \( F_n(\text{id}[-(n-2)]) \) is simply denoted by \( F_n \) (see Fig. 1). For \( k \geq 2 \), the digraphs \( F_{2k} \) and \( F_{2k}^* \) (resp. \( F_{2k+1} \) and \( F_{2k+1}^* \)) are called generalized flags. By definition, \( F_n(\text{id}[-1]) \equiv F \). We may verify that for a permutation \( \sigma \) of \( \{0, \ldots, n - 2\} \), where \( n \geq 3 \), \( F_n(\sigma) \) is decomposable if and only if there is \( i \in \{0, \ldots, n - 3\} \) such that \( \sigma(i) \) and \( \sigma(i + 1) \) share the same parity. Therefore, the generalized flags are indecomposable. Furthermore, given an indecomposable digraph \( G \), if \( i \) is an interval of \( G \), then the digraph obtained from \( G \) by reversing all the arcs included in \( i \), is 3-hemimorphic to \( G \). Sometimes, intervals are created in this way so that the obtained digraph equals neither \( G \) nor \( G^* \). For instance, given \( n \geq 4 \), consider the generalised flag \( F_4 \) and an integer \( i > 0 \) such that \( 2i \leq n - 2 \). Clearly, \( \{1, \ldots, 2i\} \) is an interval of \( F_n \). From \( F_n \), we obtain by reversing the arcs contained in \( \{1, \ldots, 2i\} \) the digraph \( F_n(\sigma_i) \), where \( \sigma_i \) is the permutation of \( \{0, \ldots, n - 2\} \) which interchanges \( j \) and \( 2i - j + 1 \) for \( 1 \leq j \leq 2i \). The pair \( \{0, 2i\} \) forms an interval of \( F_n(\sigma_i) \). Consequently, the generalized flags are not 3-forced since \( F_n(\sigma_i) \) differ regarding the indecomposability. Incidentally, the problem of the recognition of the class of indecomposable digraphs occurs. Precisely, given \( k > 0 \), a class \( C \) of digraphs is \( k \)-recognizable if every digraph \( k \)-hemimorphic to a digraph of \( C \) belongs to \( C \) as well. As showing by \( F_n \) and \( F_n(\sigma) \), the class of indecomposable digraphs is not \( k \)-recognizable. We re-examine these counter-examples with the following observation: \( \{0, \ldots, 2i\} \) is an interval of \( F_n \) and for every \( x \in \{0, \ldots, n - 1\} - \{0, 2i\} \), we have \( (x, 0) \not\equiv F_n(\sigma_i)(x, 2i) \) if and only if \( 0 < x < 2i \). Generally, consider an indecomposable digraph \( G = (V, A) \). Given vertices \( \alpha \) and \( \beta \) of \( G \) such that \( \alpha \not\equiv c \beta \), the pair \( \{\alpha, \beta\} \) is weakly separated if \( \{\alpha, \beta\} \subseteq c \) and if \( \alpha \not\equiv c \beta \). The main result consists of the following characterization.

Theorem 4. Let \( G \) be an indecomposable digraph 3-hemimorphic to \( G \) if and only if \( G \) admits a weakly separated pair.

As an immediate consequence, we obtain:

Theorem 5. The class of indecomposable digraphs is 4-recognizable.
2. The Gallai decomposition theorem

We begin with a well-known property of the intervals. Given a digraph $G = (V, A)$, if $X$ and $Y$ are disjoint intervals of $G$, then $(x, y) \in c^I(x', y')$ for any $x, x' \in X$ and $y, y' \in Y$. This property leads to consider interval partitions of $G$, that is, partitions of $V$, all the elements of which are intervals of $G$. The elements of such a partition $P$ become the vertices of the quotient $G/P = (P, A/P)$ of $G$ by $P$ defined as follows: given $X \neq Y \in P$, $(X, Y) \in A/P$ if $(x, y) \in A$ for $x \in X$ and $y \in Y$. To state the Gallai decomposition theorem below, we need the following strengthening of the notion of interval. Given a digraph $G = (V, A)$, a subset $X$ of $V$ is a strong interval [5,9] of $G$ provided that $X$ is an interval of $G$ and for each interval $Y$ of $G$, we have: if $X \cap Y \neq \emptyset$, then $X \subseteq Y$ or $Y \subseteq X$. The family of the maximal strong intervals under inclusion which are distinct from $V$ is denoted by $P(G)$.

Theorem 6 ([5,9]). Given a digraph $G = (V, A)$, with $|V| \geq 2$, the family $P(G)$ constitutes an interval partition of $G$. Moreover, the corresponding quotient $G/P(G)$ is a complete digraph or an empty digraph or a total order or an indecomposable digraph.

The next result follows from Theorem 3.

Corollary 7 ([1]). Given digraphs $G$ and $H$ without flags, if $G$ and $H$ are 3-hemimorphic, then $P(G) = P(H)$.

3. Proof of Theorems 4 and 5

Lemma 8. Consider 3-hemimorphic digraphs $G = (V, A)$ and $H = (V, B)$. Given an interval $I$ of $G$ such that $|I| \geq 2$, if $\overrightarrow{G}[I]/P(\overrightarrow{G}[I])$ is not a total order, then $I$ is an interval of $H$.

Proof. Given $x \in V - I$, since $I$ is an interval of $G$, we have: $x \leftarrow c I$ or $x \cdots c I$ or $x \rightarrow c I$. In the first two instances, it follows from the 2-hemimorphy that $x \leftarrow H I$ or $x \cdots H I$. In the last two ones, since $\overrightarrow{G}[I]/P(\overrightarrow{G}[I])$ is not a total order, $P(\overrightarrow{G}[I \cup \{x\}]) = I \cup \{x\}$ As $\overrightarrow{G}$ and $\overrightarrow{H}$ are 3-hemimorphic digraphs without flags, it follows from Corollary 7 that $P(\overrightarrow{H}[I \cup \{x\}]) = I \cup \{x\}$. Consequently, either $x \rightarrow H I$ or $x \leftarrow H I$. \qed

Corollary 9. Consider 3-hemimorphic digraphs $G$ and $H$. If $G$ is indecomposable, then for every interval $I$ of $H$, $H[I]$ is a total order.

Proof. Consider an interval $I$ of $H$. By the previous lemma, $\overrightarrow{H}[I]/P(\overrightarrow{H}[I])$ is a total order. We denote the elements of $P(\overrightarrow{H}[I])$ by $X_1, \ldots, X_q$ in such a way that $\overrightarrow{H}[I]/P(\overrightarrow{H}[I])$ is the total order $X_1 < \cdots < X_q$. For a contradiction, suppose that there is $i \in \{1, \ldots, q\}$ such that $|X_i| \geq 2$. Since $I$ is an interval of $H$, $X_i$ is also. It follows from the preceding lemma that $\overrightarrow{H}[X_i]/P(\overrightarrow{H}[X_i])$ is a total order as well. By interchanging $H$ and $H^*$, we can assume that $i < q$. By denoting by $Y$ the largest element of $\overrightarrow{H}[X_i]/P(\overrightarrow{H}[X_i])$, we obtain that $Y \cup X_{i+1}$ would be an interval of $\overrightarrow{H}[I]$, which contradicts the fact that $X_i$ is a strong interval of $\overrightarrow{H}[I]$. Consequently, for each $i \in \{1, \ldots, q\}$, $|X_i| = 1$, that is, $H[I]$ is a total order. \qed

Theorem 10. Consider 3-hemimorphic digraphs $G = (V, A)$ and $H = (V, B)$. If $G$ is indecomposable and if $H$ is decomposable, then there exist $\alpha \neq \beta \in V$ such that $[\alpha, \beta]$ is an interval of $H$ which is weakly separated in $G$.

Proof. Given a non-trivial interval $I$ of $H$, by the preceding corollary, $H[I]$ is a total order. Denote by $\alpha$ and $\beta$ the first two elements of this total order, with $\alpha \rightarrow c \beta$. Clearly, $[\alpha, \beta]$ is an interval of $H$. Consider the smallest interval $\overrightarrow{I}$ of $\overrightarrow{G}$ containing $\alpha$ and $\beta$. We use Theorem 6. Firstly, suppose that $\overrightarrow{G}[\overrightarrow{I}]/P(\overrightarrow{G}[\overrightarrow{I}])$ is empty. Since $[\alpha, \beta]$ is directed, there is an element of $P(\overrightarrow{G}[\overrightarrow{I}])$ containing $\alpha$ and $\beta$, which contradicts the minimality of $\overrightarrow{I}$. Secondly, assume that $\overrightarrow{G}[\overrightarrow{I}]/P(\overrightarrow{G}[\overrightarrow{I}])$ is indecomposable. As $\overrightarrow{G}[\overrightarrow{I}]$ and $\overrightarrow{H}[\overrightarrow{I}]$ are 3-hemimorphic digraphs without flags, it follows from Corollary 7 that $P(\overrightarrow{G}[\overrightarrow{I}]) = P(\overrightarrow{H}[\overrightarrow{I}])$. Since $[\alpha, \beta]$ is an interval of $H$, $[\alpha, \beta]$ is an interval of $\overrightarrow{H}[\overrightarrow{I}]$. We obtain the same contradiction.
because \( \overrightarrow{H(J)} / P(\overrightarrow{G(J)}) \) is indecomposable by Theorem 3. Therefore, \( \overrightarrow{G(J)} / P(\overrightarrow{G(J)}) \) is a total order. We denote the elements of \( P(\overrightarrow{G(J)}) \) by \( X_1, \ldots, X_q \) in such a way that the corresponding quotient is \( X_1 < \cdots < X_q \). By the minimality of \( J \), \( \alpha \in X_1 \) and \( \beta \in X_q \). As previously noticed, \( P(\overrightarrow{H(J)}) = \{X_1, \ldots, X_q\} \) and hence \( \overrightarrow{H(J)} / P(\overrightarrow{H(J)}) \) is a total order as well. Since \( \{\alpha, \beta\} \) is an interval of \( H \), \( \{\alpha, \beta\} \) is an interval of \( \overrightarrow{H(J)} \). As \( X_1 \) and \( X_q \) are strong intervals of \( \overrightarrow{H(J)} \), \( \{\alpha, \beta\} = X_1 \cup X_q \) or, equivalently, \( X_1 = \{\alpha\} \) and \( X_q = \{\beta\} \). To conclude, we verify that \( \{\alpha, \beta\} \) is weakly separated in \( G \). It suffices to show that for every \( x \in V - \{\alpha, \beta\} \), \( (x, \alpha) \not\equiv_C (x, \beta) \) if and only if \( x \in \overrightarrow{J} - \{\alpha, \beta\} \). Clearly, if \( x \in \overrightarrow{J} - \{\alpha, \beta\} \), then \( \alpha \rightarrow C x \rightarrow C \beta \) and hence \( (x, \alpha) \not\equiv_C (x, \beta) \). Conversely, consider an element \( u \) of \( V - \overrightarrow{J} \). If \( (u, \alpha) \) is directed, then \( (u, \alpha) \equiv_C (u, \beta) \) because \( \overrightarrow{J} \) is an interval of \( \overrightarrow{G} \). Otherwise, \( (u, \alpha) \equiv_C (u, \beta) \) because \( \{\alpha, \beta\} \) is an interval of \( H \). \( \square \)

The proof of the main result follows.

**Proof of Theorem 4.** Consider an indecomposable digraph \( G = (V, A) \). If there is a decomposable digraph 3-hemimorphic to \( G \), then, by Theorem 10, \( G \) possesses a weakly separated pair. Conversely, consider \( \alpha \neq \beta \in V \) such that \( \{\alpha, \beta\} \) is a weakly separated pair of \( G \). Since \( \{\alpha, \beta\} \cup S_C((\alpha, \beta)) \) is an interval of \( \overrightarrow{G} \), \( \{\beta\} \cup S_C((\alpha, \beta)) \) is also. Consequently, by reversing all the arcs contained in \( \{\beta\} \cup S_C((\alpha, \beta)) \), we obtain a digraph \( H \) which is 3-hemimorphic to \( G \). The pair \( \{\alpha, \beta\} \) is then an interval of \( H \) and thus \( H \) is decomposable. \( \square \)

The next result follows from Theorem 10.

**Corollary 11.** Consider 3-hemimorphic digraphs \( G = (V, A) \) and \( H = (V, B) \) such that \( G \) is indecomposable and \( H \) is decomposable. There exists a subset \( X \) of \( V \), with \( |X| = 4 \), such that \( G[X] \) is indecomposable and \( H[X] \) is decomposable. More precisely, \( G[X] \) is isomorphic to \( F_4 \) (resp. \( F_2 \)) and \( H[X] \) is isomorphic to \( F_4(\sigma) \) (resp. \( F_2(\sigma) \)), where \( \sigma \) is the permutation of \( \{0, 1, 2\} \) which interchanges either 0 and 1 or 2 and 1 and 2.

**Proof.** By Theorem 10, there are \( \alpha \neq \beta \in V \) such that \( \{\alpha, \beta\} \) is an interval of \( H \) which is weakly separated in \( G \). If \( \{\alpha, \beta\} \cup S_C((\alpha, \beta)) = V \), then \( \{\beta\} \cup S_C((\alpha, \beta)) \) would be an interval of \( G \). Consequently, \( \{\alpha, \beta\} \cup S_C((\alpha, \beta)) \neq V \) and hence \( \{\alpha, \beta\} \cup S_C((\alpha, \beta)) \) is an interval of \( \overrightarrow{G} \) and not of \( G \). Therefore, there exist \( s \in S_C((\alpha, \beta)) \) and \( u \notin \{\alpha, \beta\} \cup S_C((\alpha, \beta)) \), such that \( \{\alpha, \beta, s\} \) is an interval of \( \overrightarrow{G} \) \( \{\alpha, \beta, s\} \) and not of \( G[\{\alpha, \beta, s\}] \). It follows that \( \{\alpha, u\}, \{s, u\} \) and \( \{\beta, u\} \) are not directed. For example, assume that \( \alpha \leftarrow C u \). Since \( u \notin S_C((\alpha, \beta)) \), \( \beta \leftarrow C u \) and, necessarily, \( s \cdots \alpha \). Furthermore, \( G[\{\alpha, \beta, s\}] \) is the total order \( \alpha < s < \beta \) or \( \beta < s < \alpha \) because \( s \in S_C((\alpha, \beta)) \). In both cases, \( G[\{\alpha, \beta, s\}] \) is isomorphic to \( F_4 \). As \( G \) and \( H \) are 3-hemimorphic, we have \( \alpha \leftarrow_H u, s \cdots_H u, \beta \leftarrow_H u \) and \( H[\{\alpha, \beta, s\}] \) is a total order. To end, it is sufficient to recall that \( \{\alpha, \beta\} \) is an interval of \( H[\{\alpha, \beta, s\}] \). \( \square \)

Theorem 5 is directly deduced. Finally, Corollary 11 leads to the following.

**Remark 12.** To obtain Theorem 5, it is not necessary to assume that the considered digraphs \( G = (V, A) \) and \( H = (V, B) \) to be 4-hemimorphic. It suffices to require that \( G \) and \( H \) are 2-hemimorphic and that for every subset \( X \) of \( V \), with \( |X| = 3 \) or 4, the subdigraphs \( G[X] \) and \( H[X] \) are both indecomposable or not.

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**References**