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# Finitely forcible graphons

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ABSTRACT

We investigate families of graphs and graphons (graph limits) that are determined by a finite number of prescribed subgraph densities. Our main focus is the case when the family contains only one element, i.e., a unique structure is forced by finitely many subgraph densities. Generalizing results of Turán, Erdős–Simonovits and Chung–Graham–Wilson, we construct numerous finitely forcible graphons. Most of these fall into two categories: one type has an algebraic structure and the other type has an iterated (fractal-like) structure. We also give some necessary conditions for forcibility, which imply that finitely forcible graphons are "rare", and exhibit simple and explicit non-forcible graphons.



**Contents**



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## 1. Introduction

We define the *density* $t(F,G)$ of a simple graph $F$ in a simple graph $G$ as the probability that a random map $V(F) \to V(G)$ is a graph homomorphism, i.e., it preserves edges.

A classical theorem by Turán implies that if a simple graph $G$ has edge density larger than $1 - \frac{1}{k}$, then it contains a complete $(k+1)$-graph; furthermore, if the edge density is $1 - \frac{1}{k}$ and the density of complete $(k+1)$-graphs is 0, then $G$ is a complete $k$-partite graph with equal color classes. Here two subgraph densities force a unique structure on a graph $G$. Stability theorems (Erdős and Simonovits [21,10]) imply that if the densities are "close" to the above values, then the structure of the graph is "close" to the complete $k$-partite graph.

Another interesting theorem of this type is by Chung, Graham and Wilson [5] asserting that if the edge density of $G$ is "close" to 1/2 and the 4-cycle density is "close" to 1/16, then $G$ is quasi-random, which means (among many other nice properties) that then the density of an arbitrary fixed graph $F$ is "close" to $2^{-|E(F)|}$.

The second theorem is different from the first one in two important ways. First, this pair of subgraph densities can never be attained by finite graphs (they can be approximated with arbitrary precision). Second, the structure forced by the two subgraph densities is not as well defined as in the first example. Motivated by their results, Chung, Graham and Wilson introduced the notion of a forcing family, which is any set of graphs that can be used to force quasi-randomness in a similar way. They ask which graph families are forcing families.

Our paper goes in a slightly different direction. Instead of asking which graph families can be used to force quasi-randomness, we ask which structures can be forced by prescribing the densities of finitely many subgraphs. For this reason we will define forcing families more generally.

Most of the time we consider finite simple graphs, i.e., graphs without loops and multiple edges; where we allow multiple edges, we emphasize this by talking about multigraphs (we never need loops).

**Definition 1.1.** Let $F_1, F_2, \ldots, F_k$ be simple graphs and $a_1, a_2, \ldots, a_k$ be real numbers in $[0,1]$. We say that the set $\{(F_i, a_i)\colon\ i = 1, \ldots, k\}$ is a *forcing family* if there is a sequence of simple graphs $\{G_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} t(F_i, G_n) = a_i$ for $1 \leqslant i \leqslant k$, and for every such graph sequence $\lim_{n\to\infty} t(F, G_n)$ exists for every simple graph $F$.

We note that if $G'$ is obtained from $G$ by replacing every node with the same number of twin nodes, then $t(F, G') = t(F, G)$ for every simple graph $F$. Hence in the definition above, we could restrict our attention to graph sequences with $\lim_{n\to\infty} |V(G_n)| = \infty$.

This definition immediately implies that there is a simple graph parameter (a function $r : \mathcal{F} \mapsto [0,1]$ on the set $\mathcal{F}$ of finite simple graphs) such that $\lim_{n\to\infty} t(F, G_n) = r(F)$ whenever $\{G_n\}_{n=1}^{\infty}$ satisfies $\lim_{n\to\infty} t(F_i, G_n) = a_i$ for $1 \leqslant i \leqslant k$. The graph parameter $r$ encodes the unique structure which is forced.

Let $C_n$, $K_n$ and $P_n$ denote the cycle, complete graph and path with $n$ nodes, respectively. The result of Chung, Graham and Wilson mentioned above says in this language that $\{(K_2, 1/2), (C_4, 1/16)\}$ is a forcing family. The graph parameter describing the limit is $r(F) = 2^{-|E(F)|}$. Simple graph sequences satisfying the conditions in the definition are called *quasi-random*.

The forced structure in Definition 1.1 is best described as the limit of a graph sequence, using the newly developed theory of convergent graph sequences [2,3].

Let $\mathcal{W}$ denote the set of bounded symmetric measurable functions of the form $W : [0,1]^2 \to \mathbb{R}$, and let $\mathcal{W}_0 \subset \mathcal{W}$ consist of those functions with range in $[0,1]$. The elements of $\mathcal{W}$ are called *graphons*.

A graphon $W$ is a *stepfunction* if there is a partition $\{S_1, \ldots, S_n\}$ of $[0,1]$ into measurable sets such that $W$ is constant on each product set $S_i \times S_j$. Every graph $G$ can be represented by a function $W_G \in \mathcal{W}_0$: Let $V(G) = \{1, \ldots, n\}$. Split the interval $[0,1]$ into $n$ equal intervals $J_1, \ldots, J_n$, and for $x \in J_i$, $y \in J_j$ define $W_G(x, y) = \mathbf{1}_{ij\in E(G)}$.

The notion of subgraph densities can be extended to graphons: for a graph $F = (V, E)$ and graphon $W \in \mathcal{W}$, we define

$$t(F, W) = \int\limits_{[0,1]^V} \prod_{ij\in E} W(x_i, x_j) \prod_{i\in V} dx_i. \tag{1}$$

(This definition is meaningful for multigraphs $F$ too.) These quantities were introduced in [15] and it was proved that if in a graph sequence $(G_1, G_2, \ldots)$ the densities of every fixed graph form a convergent sequence, then there is a graphon $W \in \mathcal{W}_0$ such that $t(F, G_n) \to t(F, W)$ for every graph $F$.

If two graphons have the same simple subgraph densities, then they are called *weakly isomorphic* (see Section 2.1 for more on this relation). It follows then that also the densities of multigraphs in them are also equal.

In this paper we reformulate the problem of forcing in terms of measurable functions. An immediate advantage of this can already be seen from the simpler definition of finite forcing:

**Definition 1.2.** Let $\mathcal{A} \subseteq \mathcal{W}$. Let $F_1, F_2, \ldots, F_k$ be simple graphs and $a_1, a_2, \ldots, a_k \in [0,1]$. We say that the set $\{(F_i, a_i){:}\ i = 1 \ldots k\}$ is a *forcing family* in $\mathcal{A}$ if there is a unique (up to weak isomorphism) graphon $W \in \mathcal{A}$ with $t(F_i, W) = a_i$ for every $1 \leqslant i \leqslant k$. In this case say that $W$ is *finitely forcible* (in $\mathcal{A}$), and the family $\{F_i{:}\ i = 1 \ldots k\}$ is a *forcing family for W* (in $\mathcal{A}$).

Due to the identity

$$t(F_1F_2, W) = t(F_1, W)t(F_2, W) \tag{2}$$

(where $F_1F_2$ denotes the disjoint union of $F_1$ and $F_2$), every finitely forcible graphon can be forced by a family of connected graphs.

The two main choices for $\mathcal{A}$ will be $\mathcal{A} = \mathcal{W}$ and $\mathcal{A} = \mathcal{W}_0$. Definition 1.1 is equivalent with Definition 1.2 when $\mathcal{A} = \mathcal{W}_0$.

If a graphon $W \in \mathcal{W}_0$ is finitely forcible in $\mathcal{W}$, then it is also finitely forcible in $\mathcal{W}_0$, but the reverse is open. A forcing family for $W$ in $\mathcal{W}_0$ is not necessarily forcing $W$ in $\mathcal{W}$. For example, one can show that the constraints $t(P_3, W) = 1/4$ and $t(C_4, W) = 1/16$ force the function $W \equiv 1/2$ among functions in $\mathcal{W}_0$, but allows also the function $W \equiv -1/2$ in $\mathcal{W}$. (Nevertheless, the graphon $1/2$ is finitely forcible in $\mathcal{W}$, for example, by adding the constraint $t(K_2, W) = 1/2$.)

Besides the advantage of a simpler definition, the new language enables us to specify the structure which is forced and to use analytic methods together with algebraic ones. In this language the Chung–Graham–Wilson theorem says that if $t(K_2, W) = 1/2$ and $t(C_4, W) = 1/16$ for $W \in \mathcal{W}_0$, then $W$ is the constant $1/2$ function. A generalization of this was proved by Lovász and Sós in [14]: every stepfunction is finitely forcible in $\mathcal{W}_0$.

After the Lovász–Sós result it remained open for a few years whether only stepfunctions are finitely forcible. This is true in a one-variable analogue of the forcing problem: a bounded function $f : [0,1] \to \mathbb{R}$ is forced by finitely many moments if and only if it is a stepfunction. In this paper we show that in the 2-variable case more complicated structures can be forced. One family of these structures is the indicator function of a level set of a monotone symmetric 2-variable polynomial. Our other main example has an iterated (fractal like) structure.

So being a stepfunction is not a characterization of finitely forcible graphons. The examples mentioned above are, however, stepfunctions in a weaker sense, i.e., their range is finite. Even this is not necessary: In Section 3 we develop a class of operations on graphons that preserve finite forcibility in $\mathcal{W}$, and applying these to the first type of examples, we construct finitely forcible graphons whose range is a continuum. In the finite language this implies the surprising fact that we can create an extremal problem involving the densities of finitely many subgraphs such that in the unique asymptotically optimal solution a continuous spectrum of probabilities for quasi-randomness appears.

One implication of these results could be that they provide prototypes of possible extremal graphs other than the usual ones modeled by stepfunctions.

Of course, instead of forcing a given graphon by a finite number of density constraints, we can also try to force various properties. Let $\mathcal{B} \subseteq \mathcal{A} \subseteq \mathcal{W}$ be closed under weak isomorphism. We say that $\mathcal{B}$ is *finitely forcible in* $\mathcal{A}$ if there exists a family $\{(F_i, a_i)\colon\ i = 1\dots k\}$ (where the $F_i$ are simple graphs and $a_i \in [0,1]$) such that a graphon $W \in \mathcal{A}$ satisfies the constraints $t(F_i, W) = a_i$ $(i = 1\dots k)$ if and only if it is in $\mathcal{B}$. While the study of this generalization is not the goal of this paper, we do need certain facts about forcing some simple properties, which are discussed in Section 4.

Since finitely forcible graphons can be described by finitely many real numbers, we believe that they are very special. However, a full characterization seems to be very difficult. In fact, it is not easy to find necessary conditions for finite forcibility. We present a few in this paper and formulate many open problems in this direction.

## 2. Preliminaries

### 2.1. Graphons

Recall that a graphon is a symmetric bounded measurable function $W : [0,1]^2 \to \mathbb{R}$. Sometimes it is more convenient to use two-variable functions on other probability spaces than $[0,1]$; this does not add real generality, however, since a 2-variable function on any probability space can be replaced by an "equivalent" graphon on $[0,1]$; see [1] for more details.

Also recall that two graphons are weakly isomorphic if have the same simple subgraph densities. We denote by $[W]$ the set of all graphons weakly isomorphic to $W$. In [1] various characterizations of weakly isomorphic graphons were given, of which we need the following. For a graphon $W$ and map $\varphi : [0,1] \to [0,1]$, define $W^\varphi(x,y) = W(\varphi(x), \varphi(y))$.

**Theorem 2.1.** *Two graphons $U$ and $W$ are weakly isomorphic if and only if there are measure preserving maps $\varphi, \psi : [0,1] \to [0,1]$ such that $U^\varphi = V^\psi$.*

Every graphon defines a kernel operator $T_W : L_1[0,1] \to L_\infty[0,1]$, by

$$(T_W f)(x) = \int_0^1 W(x,y) f(y)\, dy.$$

We can also consider $T_W$ as an operator $L_\infty[0,1] \to L_1[0,1]$ or $L_2[0,1] \to L_2[0,1]$. In the latter case it is a Hilbert–Schmidt operator, and hence it has a discrete spectrum $\{\lambda_1, \lambda_2, \dots\}$ such that the eigenvalues tend to 0 (in particular, every nonzero eigenvalue has finite multiplicity). Furthermore, it has a spectral decomposition

$$W(x,y) \sim \sum_k \lambda_k f_k(x) f_k(y),$$

where $f_k$ is the eigenfunction belonging to the eigenvalue $\lambda_k \neq 0$ with $\|f\|_2 = 1$. It can be shown that the eigenfunctions $f_k$ are bounded. The $\sim$ sign indicates that the series on the right may not be convergent pointwise, only in $L_2$; but one has

$$\sum_{k=1}^{\infty} \lambda_k^2 = \int_{[0,1]^2} W(x,y)^2 \, dx \, dy \leqslant \|W\|_\infty^2.$$

The spectral decomposition can be used, among others, to compute the inner product of $W$ with any function $U \in L_2[0,1]^2$:

$$\int_{[0,1]^2} W(x,y)U(x,y) \, dx \, dy = \sum_k \lambda_k \int_{[0,1]^2} f_k(x) f_k(y) U(x,y) \, dx \, dy.$$

The *cut-norm* introduced in [11] is defined for $W \in \mathcal{W}$ by

$$\|W\|_\square = \sup_{S,T \subset [0,1]} \left| \int_{S \times T} W(x,y) \, dx \, dy \right|,$$

where the supremum goes over measurable subsets of $[0,1]$. This is within a factor of 4 to the operator norm $\|T_W\|_{L_\infty[0,1] \to L_1[0,1]}$.

Let $W_1, W_2, \ldots \in \mathcal{W}$ be graphons (a finite or countable sequence), and let $a_1, a_2, \ldots$ be positive real numbers such that $\sum_i a_i = 1$. We use $\sum_i a_i W_i$ to denote the pointwise linear combination. We also define the weighted direct sum $W = \bigoplus_i (a_i) W_i$ as the graphon on $[0,1]$ as follows: we break the $[0,1]$ interval into intervals $J_1, J_2, \ldots$ of length $a_1, a_2, \ldots$, take homothetical maps $\phi_i : J_i \to [0,1]$ and define $W(x,y) = W_i(\phi_i(x), \phi_i(y))$ if $(x,z) \in J_i \times J_i$ for some $i$, and $W(x,y) = 0$ otherwise. See [12] for more on weighted direct sums and decomposing graphons into connected components.

Somewhat confusingly, we can introduce three "product" operations on graphons, and we will need all three of them. Let $U, W \in \mathcal{W}$. We denote by $UW$ their product as functions, i.e.,

$$UW(x,y) = U(x,y)W(x,y).$$

We denote by $U \circ W$ the product of $U$ and $W$ as kernel operators, i.e.,

$$(U \circ W)(x,y) = \int_0^1 U(x,z)W(z,y) \, dz.$$

Finally, we denote by $U \otimes W$ their tensor product; this is defined as a function $[0,1]^2 \times [0,1]^2 \to [0,1]$ by

$$(U \otimes W)(x_1, x_2, y_1, y_2) = U(x_1, y_1)W(x_2, y_2).$$

However, we can consider any measure preserving map $\phi : [0,1] \to [0,1]^2$, and define the graphon

$$(U \otimes W)^\phi(x,y) = (U \otimes W)\big(\phi(x), \phi(y)\big).$$

These graphons are weakly isomorphic for all $\phi$, and so we can call any of them the tensor product of $U$ and $W$. We note that the tensor product has the nice property $t(F, U \otimes W) = t(F,U)t(F,W)$ for every graph $F$.

We denote the $n$-th power of a graphon according to these three multiplications by $W^n$ (pointwise power), $W^{\circ n}$ (operator power), and $W^{\otimes n}$ (tensor power).

*2.2. Labeled, colored, and quantum graphs*

Suppose that the edges of a graph $F$ are partitioned into two sets $E_+$ and $E_-$. The triple $\widehat{F} = (V, E_+, E_-)$ will be called a *signed graph*. For $W \in \mathcal{W}$, we define

$$t(\widehat{F}, W) = \int\limits_{[0,1]^V} \prod_{ij\in E_+} W(x_i, x_j) \prod_{ij\in E_-} \bigl(1 - W(x_i, x_j)\bigr) \prod_{i\in V} dx_i. \tag{3}$$

If all edges are signed "+", then $t(\widehat{F}, W) = t(F, W)$. If all edges are signed "−", then $t(\widehat{F}, W) = t(F, 1 - W)$. In general, $t(\widehat{F}, W)$ can be expressed as

$$t(\widehat{F}, W) = \sum_{Y\subseteq E_-} (-1)^{|Y|} t\bigl((V, E_+ \cup Y), W\bigr). \tag{4}$$

For a simple graph $F$, let $\widehat{F}$ denote the complete graph on $V(F)$ in which the edges of $F$ are signed "+" and the other edges are signed "−". Let $G$ be a simple graph and let $W_G$ denote the associated graphon. Then $t(\widehat{F}, W_G)$ is the probability that a random map $V(F) \to V(G)$ preserves both adjacency and nonadjacency. If $F$ is fixed and $|V(G)| \to \infty$, then this is asymptotically $t_{\mathrm{ind}}(F, G)$, the probability that a random injection $V(F) \to V(G)$ is an embedding as an induced subgraph.

Let $F = (V, E)$ be a *k-labeled multigraph*, i.e., a multigraph with $k$ specified nodes labeled $1, \dots, k$ and any number of unlabeled nodes. Let $V_0 = V \setminus [k]$ be the set of unlabeled nodes. For $W \in \mathcal{W}$, we define a function $t_k(F, W) : [0, 1]^k \to \mathbb{R}$ by

$$t_k(F, W)(x_1, \dots, x_k) = \int\limits_{x\in[0,1]^{V_0}} \prod_{ij\in E} W(x_i, x_j) \prod_{i\in V_0} dx_i.$$

Note that $t_0(F, W) = t(F, W)$. We can extend this notation to signed graphs $\widehat{F}$ to get $t_k(\widehat{F}, W)$ in the obvious way. If all nodes of a signed graph $\widehat{F} = (V, E_+, E_-)$ are labeled, then

$$t_k(\widehat{F}, W)(x_1, \dots, x_k) = \prod_{ij\in E_+} W(x_i, x_j) \prod_{ij\in E_-} \bigl(1 - W(x_i, x_j)\bigr).$$

If a $(k-1)$-labeled graph $F'$ arises from a $k$-labeled graph $F$ by unlabeling node $k$ (say), then

$$t_{k-1}\bigl(F', W\bigr)(x_1, \dots, x_{k-1}) = \int\limits_{[0,1]} t_k(F, W)(x_1, \dots, x_k)\, dx_k. \tag{5}$$

A simple but very useful inequality from [15] relates the cut norm to subgraph densities: for every simple graph $F$ and $U, W \in \mathcal{W}$,

$$\bigl|t(F, U) - t(F, W)\bigr| \leqslant 4\bigl|E(F)\bigr| \max\bigl\{\|U\|_\infty, \|W\|_\infty\bigr\}^{|E(F)|-1} \|U - W\|_\square. \tag{6}$$

(One can extend this to simple signed graphs $F$ at the cost of including $\|1 - U\|_\infty$ and $\|1 - W\|_\infty$ in the max on the right-hand side.) We will also use a related (in fact, simpler) inequality: For all $U, W \in \mathcal{W}$ and every simple graph $F$ with $k$ nodes we have trivially

$$\bigl\|t_k(F, U) - t_k(F, W)\bigr\|_\infty \leqslant \bigl|E(F)\bigr| \max\bigl\{\|U\|_\infty, \|W\|_\infty\bigr\}^{|E(F)|-1} \|U - W\|_\infty,$$

and hence for all $1 \leqslant j \leqslant k$,

$$\bigl\|t_j(F, U) - t_j(F, W)\bigr\|_\infty \leqslant \bigl|E(F)\bigr| \max\bigl\{\|U\|_\infty, \|W\|_\infty\bigr\}^{|E(F)|-1} \|U - W\|_\infty. \tag{7}$$

Let $F_1$ and $F_2$ be two $k$-labeled multigraphs. Their *product* is the $k$-labeled multigraph $F_1F_2$ defined as follows: we take their disjoint union, and then identify nodes with the same label (retaining the labels). Clearly this multiplication is associative and commutative. Note that for $k \geqslant 2$ the graph

$F_1F_2$ can have multiple edges even if the graphs $F_1$ and $F_2$ are simple. For two 0-labeled graphs, $F_1F_2$ is their disjoint union.

For two $k$-labeled graphs $F_1$ and $F_2$ the following generalization of (2) holds:

$$t_k(F_1F_2, W) = t_k(F_1, W)t_k(F_2, W) \tag{8}$$

where the multiplication on the right-hand side is just the pointwise product of two real functions with the same domain.

A *k-labeled quantum graph* is a formal finite linear combination with real coefficients of $k$-labeled multigraphs. Multigraphs that occur in a $k$-labeled quantum graph with nonzero coefficient will be called its *constituents*. A 0-labeled quantum graph will be called simply a *quantum graph*. For a $k$-labeled [quantum] graph $f$, let $\widetilde{f}$ denote the 0-labeled [quantum] graph obtained from $f$ by removing all labels. A $k$-labeled quantum graph is *simple*, if it is a linear combination of simple graphs. A $k$-labeled graph is *connected*, if every connected component contains a labeled node. A $k$-labeled quantum graph is *connected*, if every constituent is connected.

We can extend the definition of the product to the product of two $k$-labeled quantum graphs by distributivity. This way $k$-labeled quantum graphs form a commutative algebra $\mathcal{G}_k$; the graph with $k$ nodes, all labeled, and no edge is the unit element in this algebra.

We can define $t_k(f, W): [0,1]^k \to \mathbb{R}$ for every $k$-labeled quantum graph $f$ so that it is linear in $f$. Then (8) will remain valid.

We can identify a signed graph $\widehat{F} = (V, E_+, E_-)$ with the quantum graph

$$\sum_{Y \subseteq E_-} (-1)^{|Y|}(V, E_+ \cup Y),$$

then the two possible definitions of $t(\widehat{F}, W)$ give the same result by (4).

Recall that a graphon $W$ is *finitely forcible* (in $\mathcal{A} \subseteq \mathcal{W}$), if $W \in \mathcal{A}$ and there are a finite number of simple graphs $F_1, \dots, F_k$ so that whenever a graphon $U \in \mathcal{A}$ satisfies

$$t(F_i, U) = t(F_i, W) \quad (i = 1, \dots, k) \tag{9}$$

then $W$ and $U$ are weakly isomorphic. We could be more general and allow quantum graphs in the forcing family. This would not lead to more finitely forcible graphons, but we could use forcing constraints of the special form $t(f_i, W) = 0$, where $f_i$ is a quantum graph.

### 2.3. Moments of one-variable functions

We will need a certain one-variable version of finite forcing. Let $\mathbf{w} = (w_1, \dots, w_r)$, where $w_1, \dots, w_r: [0,1] \to \mathbb{R}$ are bounded measurable functions. For $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{Z}_+^r$, we define the joint moment of $\mathbf{u}$ by

$$M(\mathbf{w}, \mathbf{a}) = \int_{[0,1]} w_1(x)^{a_1} \cdots w_r(x)^{a_r}\, dx.$$

The following theorem follows from classical results of Karlin [13] (see also [8]). For any map $\phi: [0,1] \to [0,1]$ and any $w: [0,1] \to \mathbb{R}$, we set $w^\phi(x) = w(\phi(x))$.

**Theorem 2.2.** *Let $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{Z}_+^r$, and suppose that $M(\mathbf{w}, \mathbf{a}_j)$ exists for all $j = 1, \dots, m$. Then there is a vector $\mathbf{u} = (u_1, \dots, u_r)$ of stepfunctions such that*

$$M(\mathbf{u}, \mathbf{a}_j) = M(\mathbf{w}, \mathbf{a}_j) \quad (j = 1, \dots, m).$$

A certain converse of this theorem is also true:

**Proposition 2.3.** *Let $u_1, \dots, u_r : [0,1] \to \mathbb{R}$ be stepfunctions. Then there is a finite set of vectors $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{Z}_+^r$ such that the values $M(\mathbf{u}, \mathbf{a}_j)$ $(j = 1, \dots, m)$ uniquely determine the functions $u_i$ up to a measure preserving transformation of* $[0,1]$.

**Proof.** First we prove this for $r = 1$:

**Claim 2.4.** *Every stepfunction $u : [0,1] \to \mathbb{R}$ with $k$ steps is uniquely determined, up to a measure preserving transformation in the variable, by its first $2k$ moments.*

Let $u$ have $k$ steps, of sizes $\alpha_1, \dots, \alpha_k$, on which the value of $u$ is $\beta_1, \dots, \beta_k$, respectively. Then

$$M(u, a) = \sum_{i=1}^{k} \alpha_i \beta_i^a.$$

Let $f : [0,1] \to \mathbb{R}$ be a function such that $M(u, a) = M(f, a)$ for $a = 1, \dots, 2k$. Consider the polynomial

$$p(x) = \prod_{i=1}^{k} (x - \beta_i)^2 = \sum_{j=0}^{2k} c_j x^j.$$

Then

$$\int_0^1 p\big(f(x)\big)\,dx = \sum_{j=0}^{2k} c_j M(f, j) = \sum_{j=0}^{2k} c_j M(u, j) = \int_0^1 p\big(u(x)\big)\,dx = 0.$$

Since $p$ is a square, this implies that $p(f(x)) = 0$ almost everywhere. Hence $f(x) \in \{\beta_1, \dots, \beta_k\}$ almost everywhere, i.e., up to a set of measure 0, $f$ is a stepfunction attaining the same values as $u$.

Let $\alpha_i' = \lambda(f^{-1}(\beta_i))$ (where $\lambda$ denotes the Lebesgue measure). Then

$$\sum_{i=1}^{k} \alpha_i \beta_i^a = M(u, a) = M(f, a) = \sum_{i=1}^{k} \alpha_i' \beta_i^a \quad (1 \leqslant a \leqslant k),$$

or

$$\sum_{i=1}^{k} \big(\alpha_i - \alpha_i'\big) \beta_i^a = 0 \quad (1 \leqslant a \leqslant k).$$

Considering this as a system of linear equations on the differences $\alpha_i - \alpha_i'$, the determinant of the system is nonzero, which implies that $\alpha_i - \alpha_i' = 0$ for all $i$. So $f$ differs from $u$ in a measure preserving transformation only.

Now we turn to the general case. Let $u_i$ have $k_i$ steps, and set $k = k_1 k_2 \dots k_r$. We claim that if $v_1, \dots, v_r$ is a system of functions for which

$$M(\mathbf{v}, \mathbf{a}) = M(\mathbf{u}, \mathbf{a})$$

for all $a \in \mathbb{Z}_+^r$ with $\sum_i a_i \leqslant 2k$, then there is a measure preserving transformation of $[0,1]$ which transforms $v_i$ into $u_i$ for all $i$.

First, by specifying the first $2k_i$ moments of each $u_i$ separately, we get that $v_i$ is a stepfunction obtained from $u_i$ by a measure preserving transformation of the variable. We want to argue that we can use the same measure preserving transformation for every $i$.

Let $\varepsilon > 0$ be a sufficiently small real number (smaller than the minimum difference between two distinct values of any $u_i$, divided by the maximum of $\|u_i\|_\infty$), and consider the function

$$u = u_1 + \varepsilon u_2 + \dots + \varepsilon^{r-1} u_r.$$

Then $u$ is a stepfunction with at most $k$ steps, and the $j$-th moment of $u$ can be expressed as a polynomial of the joint moments $M(\mathbf{u},\mathbf{a})$ of the $u_i$ with $\sum_i a_i = j$. Hence the first $2k$ moments of the function

$$v = v_1 + \varepsilon v_2 + \dots + \varepsilon^{r-1} v_r$$

match the first $2k$ moments of $u$, and so there is a measure preserving transformation $\phi$ with $v^\phi = u$. By the choice of $\varepsilon$, the steps of $v_1$ must be mapped by $\phi$ onto the steps of $u_1$, and so $v_1^\phi = u_1$. But then $(v - v_1)^\phi = u - u_1$, and we get similarly that $v_2^\phi = u_2$, and so on. □

### 2.4. *Typical points of graphons*

We need some technical results about 2-variable functions. In this section $W$ denotes a measurable function $[0,1]^2 \to [0,1]$ (not necessarily symmetric). Let $R(W)$ denote the set of 1-variable functions $\{W(x,.)\colon\ x \in [0,1]\}$. Clearly $R(W)$ inherits a topology from $L_1[0,1]$, and it also inherits a probability measure $\pi$ from $[0,1]$.

**Definition 2.5.** Let $T(W)$ be the set of functions $f \in L_1[0,1]$ such that every neighborhood of $f$ intersects $R(W)$ in a set with positive measure. A point $x \in [0,1]$ will be called *typical* if $W(x,.) \in T(W)$ and *atypical* otherwise.

**Lemma 2.6.** *Let $W : [0,1]^2 \to [0,1]$ be a measurable function. Then almost every point of $[0,1]$ is typical.*

**Proof.** If $g \notin T(W)$, then there is an open neighborhood $U_g$ of $g$ in $L_1[0,1]$ such that $\pi(U_g) = 0$. Let $U = \bigcup_{g \notin T(W)} U_g$. Since $L_1[0,1]$ is separable, it contains a countable dense set $D$, and then every set $U_g$ is the union of all balls with rational radius centered at a point in $D$ contained in $U_g$. Hence $U$ equals the union of a countable number of such balls contained in a $U_g$, $g \notin T(W)$, and hence, it is also the union of a countable subfamily $\{U_{g_i}\colon\ i \in \mathbb{N}\}$ ($g_i \notin T(W)$). Hence $\pi(U) = 0$. If $x$ is atypical, then $W(x,.) \in U$, and $|\{x\colon\ W(x,.) \in U\}| = \pi(U) = 0$. □

**Remark 2.7.** The proof above can be modified so that instead of $[0,1]$, it works for graphons defined on any probability space $\Omega$. In this general case $L_1[\Omega]$ is not necessarily separable, but we can replace it by the linear space generated by functions $W(x,.)$, which is separable.

We need the following property of typical points.

**Lemma 2.8.** *Let $W$ be a graphon, and let $f$ be a $k$-labeled quantum graph such that the labeled nodes are independent in each multigraph constituting $f$. Assume that $t_k(f,W) = 0$ almost everywhere. Then $t_k(f,W)(x_1,\dots,x_k) = 0$ for every $k$-tuple of typical points.*

**Proof.** We may assume that $\|W\|_\infty \leqslant 1$. Suppose that $t_k(f,W)(x_1,\dots,x_k) = \varepsilon > 0$ with $x_i$ typical. Let $f = \sum_i \alpha_i F_i$, $c_f = \sum_i |\alpha_i| \cdot |E(F_i)|$ and $\delta = \varepsilon/(2c_f)$. By the definition of typical points, there are sets $Z_i \subseteq [0,1]$ with positive measure such that $\|W(x_i,.) - W(z,.)\|_1 \leqslant \delta$ for all $z \in Z_i$. We claim that for every choice of points $z_i \in Z_i$, we have

$$\bigl|t_k(f,W)(x_1,\dots,x_k) - t_k(f,W)(z_1,\dots,z_k)\bigr| \leqslant \frac{\varepsilon}{2}. \tag{10}$$

Clearly it suffices to verify this for the case when $f = F$ is a multigraph. Let $u_1v_1,\dots,u_qv_q$ be the edges of $F$ incident with the labeled nodes; say, $v_r$ is labeled but $u_r$ is not (here we use the assumption about $f$). Let $u_{q+1}v_{q+1},\dots,u_mv_m$ be the other edges of $F$, and $U = V \setminus \{1,\dots,k\}$. Using variables $y_u$ for the unlabeled nodes, we have

$$\begin{aligned}&\left|t_k(f,W)(x_1,\dots,x_k)-t_k(f,W)(z_1,\dots,z_k)\right|\\&\quad=\left|\int\limits_{[0,1]^U}\sum_{j=1}^{q}A_j(x,y)\big(W(y_{u_j},x_{v_j})-W(y_{u_j},z_{v_j})\big)B_j(z,y)\,dy\right|,\end{aligned}$$

where

$$A_j(x,y)=\prod_{i<j}W(y_{u_i},x_{v_i})$$

and

$$B_j(z,y)=\prod_{j<i\leqslant q}W(y_{u_i},z_{v_i})\prod_{i>q}W(y_{u_i},y_{v_i}).$$

Hence

$$\begin{aligned}&\left|t_k(f,W)(x_1,\dots,x_k)-t_k(f,W)(z_1,\dots,z_k)\right|\\&\quad\leqslant\sum_{j=1}^{q}\int\limits_{[0,1]^U}\left|A_j(x,y)\right|\cdot\left|W(y_{u_j},x_{v_j})-W(y_{u_j},z_{v_j})\right|\cdot\left|B_j(z,y)\right|dy\\&\quad\leqslant\sum_{j=1}^{q}\int\limits_{[0,1]^U}\left|W(y_{u_j},x_{v_j})-W(y_{u_j},z_{v_j})\right|dy\\&\quad=\sum_{j=1}^{q}\left\|W(x_{v_j},.)-W(z_{v_j},.)\right\|_1\leqslant q\delta\leqslant\frac{\varepsilon}{2}.\end{aligned}$$

This proves (10), which in turn implies that $t_k(f,W)(z_1,\dots,z_k)\neq 0$. Since this holds for all $z_i\in Z_i$, we get that $t_k(f,W)=0$ cannot hold almost everywhere, a contradiction. □

**Remark 2.9.** We can use Lemma 2.6 to define a "normalization" of graphons: by modifying a graphon on a set of measure 0, we can obtain one in which every point is typical. Lemma 2.8 implies then that if $t_k(f,W)=0$ almost everywhere, then it is identically 0.

## 3. Operations on graphs and graphons

We discuss various operations on graphs and graphons in connection with forcing.

### 3.1. Labeling, unlabeling and contraction

We start with a trivial consequence of (5).

**Lemma 3.1.** *Assume that $t_k(g,W)=0$ holds almost everywhere for some $k$-labeled quantum graph $g$, and let $g_1$ be obtained by unlabeling $r+1,\dots,k$ in $g$. Then $t_r(g_1,W)=0$ almost everywhere. In particular, $t(\widetilde{g},W)=0$.*

The condition that $t_k(g,W)=0$ for some $k$-labeled quantum graph $g$ (where $k>0$) seems to carry much more information than a condition that $t(f,W)=0$ for an unlabeled quantum graph $f$. However, there is a way to translate labeled constraints to unlabeled constraints. First we state a simple version.

**Lemma 3.2.** *Let $f$ be a $k$-labeled quantum graph and $d$, a positive even integer. Let $f^d$ be the $d$-th power of $f$ in the algebra $\mathcal{G}_k$. Then for any $W \in \mathcal{W}$, $t(\widetilde{f^d}, W) = 0$ if and only if $t_k(f, W) = 0$ almost everywhere.*

(By Lemma 3.1, the same conclusion holds if only some of the nodes are unlabeled.)

**Proof.** We have

$$t\big(\widetilde{f^d}, W\big) = \int\limits_{[0,1]^k} t_k\big(f^d, W\big)(x)\,dx = \int\limits_{[0,1]^k} \big(t_k(f, W)(x)\big)^d\,dx,$$

so this is 0 if and only if $t_k(f, W)(x) = 0$ for almost all $x \in [0,1]^k$. □

Let $F$ be a $k$-labeled multigraph, and let $\mathcal{P} = \{S_1, \dots, S_m\}$ be a partition of $[k]$. We say that $\mathcal{P}$ is *legitimate* for $F$, if each set $S_i$ is independent in $F$. If this is the case, then we define the $m$-labeled multigraph $F/\mathcal{P}$ by identifying the nodes in each $S_i$, and labeling the obtained node with $i$. For a $k$-labeled quantum graph $f$, we say that the partition $\mathcal{P}$ of $[k]$ is *legitimate for $f$* if it is legitimate for every constituent. Then we can define $f/\mathcal{P}$ by linear extension.

**Lemma 3.3.** *Let $f$ be a $k$-labeled quantum graph and $\mathcal{P}$, a legitimate partition for $f$ with $r$ classes. Let $W \in \mathcal{W}$, and suppose that $t_k(f, W) = 0$ almost everywhere. Then $t_r(f/\mathcal{P}, W) = 0$ almost everywhere.*

**Proof.** If $k = 2$ and $\mathcal{P}$ identifies the two labels, then the lemma follows from Lemma 2.8, since $t_1(f/\mathcal{P}, W)(x) = t_2(f, W)(x, x)$ is 0 whenever $x$ is a typical point, i.e., almost everywhere. (We could also invoke Theorem 1.6 in [17] here.)

Now for an arbitrary $k \geqslant 2$, it suffices to prove the case when $\mathcal{P}$ identifies a single pair of labeled nodes, say 1 and 2 (so $r = k-1$). If $t_k(f, W) = 0$ almost everywhere, then by (8) $t_k(f^2, W) = 0$ almost everywhere. Let $h$ be obtained from $f^2$ by unlabeling $3, \dots, k$, then by Lemma 3.1 it follows that $t_2(h, W) = 0$ almost everywhere. By the above, we have $t_1(h', W) = 0$ almost everywhere, where $h'$ is the 1-labeled quantum graph obtained from $h$ by identifying labels 1 and 2. Again by Lemma 3.1, we have $t(\widetilde{h'}, W) = 0$. But $\widetilde{h'}$ can be obtained from $(f/\mathcal{P})^2$ by unlabeling all nodes, and hence by Lemma 3.2 it follows that $t_{k-1}(f/\mathcal{P}, W) = 0$ almost everywhere. □

One drawback of Lemma 3.2 is that $f^r$ may have multiple edges, even if $f$ does not. The construction in the next lemma gets around this.

**Lemma 3.4.** *For every simple $k$-labeled quantum graph $f$ there is a simple unlabeled quantum graph $g$ such that for any $W \in \mathcal{W}$, $t(g, W) = 0$ if and only if $t_k(f, W) = 0$ almost everywhere.*

**Proof.** For every $k$-labeled quantum graph $g$, consider the product (in the algebra $\mathcal{G}_k$) of all constituents, and define $\mathrm{Lab}(g)$ as the subgraph of this induced by the labeled nodes.

We prove the lemma by induction on the chromatic number $\chi(\mathrm{Lab}(f))$. If $\chi(\mathrm{Lab}(f)) = 1$, then the labeled nodes are independent in every constituent, and hence we can take $g = \widetilde{f^2}$ and use Lemma 3.2.

Suppose that $\chi(\mathrm{Lab}(f)) = r > 1$, and let $[k] = S_1 \cup \dots \cup S_r$ be an $r$-coloring of $\mathrm{Lab}(f)$, and let $S_r = \{k-q+1, \dots, k\}$. We glue together two copies of $f$ along $S_r$. Formally, let $f_1$ be obtained from $f$ by increasing the labels in $S_r$ by $k-q$ (the labels not in $S_r$ are not changed), and by adding isolated nodes labeled $k-q+1, \dots, 2k-2q$ to every constituent of $f$. Let $f_2$ be obtained from $f$ by increasing all labels by $k-q$, and by adding isolated nodes labeled $1, \dots, k-q$ to every constituent of $f$. So $f_1$ and $f_2$ are $(2k-q)$-labeled quantum graphs. Form the product $f_1 f_2$ and remove the labels $2k-2q+1, \dots, 2k-q$, to get a $(2k-2q)$-labeled quantum graph $h$.

**Claim 3.5.** *For every $W \in \mathcal{W}$, $t_k(f, W) = 0$ almost everywhere if and only if $t_{2k-2q}(h, W) = 0$ almost everywhere.*

The "if" part is obvious, since $t_k(f,W)=0$ almost everywhere implies $t_{2k-q}(f_1,W)=t_{2k-q}(f_2,W)=0$ almost everywhere, which implies $t_{2k-q}(f_1f_2,W)=0$ almost everywhere, which in turn implies $t_{2k-2q}(h,W)=0$ almost everywhere.

To prove the "only if" part, note that two labeled nodes whose labels correspond to the same label in $f$ are never adjacent, so we can identify these labels in $h$ to get $f^2$ (with the labels in $S_r$ removed). So $t_{2k-2q}(h,W)=0$ almost everywhere implies by Lemmas 3.3 and 3.1 that $t(\widetilde{f^2},W)=0$, and hence by Lemma 3.2, we get that $t_k(f,W)=0$ almost everywhere. This proves the claim.

Thus it suffices to replace the constraint $t_{2k-2q}(h,W)=0$ by an unlabeled constraint. This can be done by induction, since $\chi(\mathrm{Lab}(h))\leqslant r-1$. $\square$

**Corollary 3.6.** *Suppose that for $W\in\mathcal{W}$ there is a family $\{f_1,\dots,f_m\}$, where $f_i$ is a simple $k_i$-labeled quantum graph such that $t_{k_i}(f_i,W)=0$ almost everywhere, and the constraints $t_{k_i}(f_i,U)=0$, $U\in\mathcal{W}$ imply that $U$ is weakly isomorphic to $W$. Then $W$ is finitely forcible in $\mathcal{W}$. Similar assertion holds for forcing in $\mathcal{W}_0$.*

### 3.2. *The adjoint of an operator*

Let $\mathcal{F}$ denote the set of simple graphs (up to isomorphism), and let $\mathcal{Q}$ be the linear space of simple quantum graphs.

**Definition 3.7.** Let $\mathbf{F}:\mathcal{W}\to\mathcal{W}$ be an operator (not necessarily linear) preserving weak isomorphism, and let $\mathbf{F}^*:\mathcal{Q}\to\mathcal{Q}$ be a linear map. We say that the map $\mathbf{F}^*$ is an *adjoint* of $\mathbf{F}$ if

$$t\big(g,\mathbf{F}(W)\big)=t\big(\mathbf{F}^*(g),W\big)$$

for every $g\in\mathcal{Q}$ and $W\in\mathcal{W}$. (Note that it is enough to define $\mathbf{F}^*$ on simple graphs and extend it linearly to quantum graphs.) We denote the set of functionals which have an adjoint by $\mathcal{D}$.

It is clear from this definition that the elements of $\mathcal{D}$ form a semigroup with respect to composition.

**Example 3.8.** Fix a real number $\alpha$ and let $\mathbf{F}$ denote the functional defined by $\mathbf{F}(W)=\alpha W$. It is easy to see that $\mathbf{F}$ has an adjoint defined for simple graphs $G$ by $\mathbf{F}^*(G)=\alpha^{|E(G)|}G$.

**Example 3.9.** Let $\beta$ be a real number and $\mathbf{F}(W)=W+\beta$. Then $\mathbf{F}$ has an adjoint defined by

$$\mathbf{F}^*(G)=\sum_{Z\subseteq E(G)}\beta^{|E(G)\setminus Z|}\big(V(G),Z\big).$$

**Example 3.10.** Let $U\in\mathcal{W}$ be a fixed function and define $\mathbf{F}(W)$ as the tensor product $U\otimes W$. Then $\mathbf{F}^*(G)=t(G,U)G$ defines an adjoint of $\mathbf{F}$.

**Example 3.11.** Let $\mathbf{F}(W)=W^{\otimes k}$ be the $k$-th tensor power of $W$ (for a fixed $k\geqslant 1$). Then an adjoint $\mathbf{F}^*$ can be defined by letting $\mathbf{F}^*(G)$ be the disjoint union of $k$ copies of $G$.

**Example 3.12.** Let $p(z)=\sum_{k=1}^n a_kz^k$ be a real valued polynomial. We define $\mathbf{F}(W)$ as $p(W)$ where $W$ is substituted into $p$ as an integral kernel operator. For any graph $G=(V,E)$, we define

$$\mathbf{F}^*(G)=\sum_{\mathbf{k}\in[n]^E}a_{\mathbf{k}}G^{(\mathbf{k})},$$

where for $\mathbf{k}\in[n]^E$ we define $a_{\mathbf{k}}=\prod_{e\in E}a_{k_e}$, and $G^{(\mathbf{k})}$ is the graph obtained from $G$ by subdividing each edge $e$ by $k_e-1$ nodes.

We show that $\mathbf{F}^*$ is an adjoint of $\mathbf{F}$. We use that for $\mathbf{k} \in [n]^E$,

$$t\big(G^{(\mathbf{k})}, W\big) = \int\limits_{[0,1]^{V(G)}} \prod_{ij \in E(G)} W^{\circ k_{ij}}(x_i, x_j)\, dx.$$

Hence

$$t\big(G, p(W)\big) = \int\limits_{[0,1]^{V(G)}} \prod_{ij \in E(G)} \left(\sum_{k=1}^{n} a_k W^{\circ k}(x_i, x_j)\right) dx$$
$$= \int\limits_{[0,1]^{V(G)}} \sum_{\mathbf{k} \in [n]^E} a_{\mathbf{k}} \prod_{ij \in E(G)} W^{\circ k_{ij}}(x_i, x_j)\, dx = \sum_{\mathbf{k} \in [n]^E} t\big(G^{(\mathbf{k})}, W\big) = t\big(\mathbf{F}^*(G), W\big).$$

**Example 3.13.** Let $H$ be a simple 2-labeled graph which has an automorphism interchanging the labeled nodes. Then $\mathbf{F}_H(W)(x, y) = t_2(H, W)(x, y)$ is a symmetric 2-variable function in $x, y$. Let $\mathbf{F}^*_H(G)$ be the graph obtained from $G$ by replacing each edge by a copy of $H$ where the labeled nodes of $H$ are identified with the endpoints of the edge. Then $\mathbf{F}^*_H$ is an adjoint of $\mathbf{F}_H$.

As a special case, if $H = K_3^{\bullet\bullet}$ denotes the triangle with two labeled nodes, then $\mathbf{F}_H(W) = (W \circ W)W$, and $\mathbf{F}^*_H(G)$ can be constructed by doubling each edge of $G$ and subdividing one copy of each edge.

**Lemma 3.14.** *Let $W \in \mathcal{W}$ be finitely forcible in $\mathcal{W}$, let $\mathbf{F} \in \mathcal{D}$, and assume that $\mathbf{F}^{-1}([W])$ is finite (up to weak isomorphism). Then every element in $\mathbf{F}^{-1}([W])$ is finitely forcible in $\mathcal{W}$.*

**Proof.** If $W$ can be forced by the constraints $t(F_i, W) = a_i$ $(i = 1, \dots, k)$, then the set $\mathbf{F}^{-1}([W])$ can be forced by the constraints $t(\mathbf{F}^*(F_i), U) = a_i$. Let $\mathbf{F}^{-1}([W])$ have $m$ elements up to weak isomorphism, then the equivalence class of each element can be distinguished from the others by at most $m - 1$ further graph density constraints. $\square$

Applying this lemma with Examples 3.8 and 3.9, we get:

**Corollary 3.15.** *If $W \in \mathcal{W}$ is finitely forcible (in $\mathcal{W}$), then so is $\alpha W + \beta$ for $\alpha, \beta \in \mathbb{R}$.*

**Corollary 3.16.** *For every finitely forcible graphon $W \in \mathcal{W}$ there are numbers $\alpha \neq 0$ and $\beta$ such that $\alpha W + \beta$ is in $\mathcal{W}_0$ and is finitely forcible in $\mathcal{W}_0$.*

For our next corollary, we need a simple lemma. (Here we substitute a graphon into a polynomial as a kernel operator.)

**Lemma 3.17.** *Let $p$ be a polynomial which is a bijection on $\mathbb{R}$ with $p(0) = 0$. Then:*

(a) *If $p(W_1) = p(W_2)$ almost everywhere, then $W_1 = W_2$ almost everywhere.*
(b) *If $p(W_1)$ and $p(W_2)$ are weakly isomorphic, then so are $W_1$ and $W_2$.*

**Proof.** (a) Let $U \in \mathcal{W}$, and consider any function $W \in \mathcal{W}$ with $p(W) = U$ almost everywhere. Let

$$W(x, y) \sim \sum_{i=1}^{\infty} \mu_i f_i(x) f_i(y)$$

be the spectral decomposition of $W$, where $\{f_i\}_{i=1}^{\infty}$ is an orthonormal system of functions in $L_2[0,1]$ and $\sum_i \mu_i^2 < \infty$. Then

$$U(x, y) \sim \sum_{i=1}^{\infty} p(\mu_i) f_i(x) f_i(y)$$

is a spectral decomposition of $U$, since $p$ is Lipshitz on any bounded interval and hence $\sum_i p(\mu_i)^2 < \infty$. Since the spectral decomposition of $U$ is unique (up to an orthogonal basis transformation in the eigensubspaces) and $p$ is injective, and we see that the $\mu_i$ and $f_i$ are determined by $U$ (again, up to an orthogonal basis transformation in the eigensubspaces), and so $W$ is determined by $U$.

(b) Assume that $p(W_1)$ and $p(W_2)$ are weakly isomorphic. By Theorem 2.1, this implies that there are measure preserving maps $\varphi, \psi : [0, 1] \to [0, 1]$ such that $p(W_1)^\varphi = p(W_2)^\psi$ almost everywhere. It is easy to check that $p(W_1)^\varphi = p(W_1^\varphi)$, so we get that $p(W_1^\varphi) = p(W_2^\psi)$ almost everywhere. By (a), this means that $W_1^\varphi = W_2^\psi$ almost everywhere, and so $W_1$ and $W_2$ are weakly isomorphic. □

**Corollary 3.18.** *Let $p$ be a polynomial which is a bijection on $\mathbb{R}$ with $p(0) = 0$. If $p(W)$ is finitely forcible for some $W \in \mathcal{W}$, then so is $W$.*

**Proof.** Let $\mathbf{F}(W) = p(W)$. By Lemma 3.17(b), $\mathbf{F}^{-1}([p(W)])$ is finite up to weak isomorphism (in fact, has at most one element). Hence by Lemma 3.14, $W$ is finitely forcible. □

## 4. Finitely forcible properties

As mentioned in the introduction, instead of forcing specific graphons by a finite number of subgraph densities, we can more generally ask which properties of graphons can be forced this way. Clearly, every such property is invariant under weak isomorphism, and also closed under convergence. (More generally, it is closed in the cut-norm [15,3], but we don't need this in this paper.)

Some important properties are finitely forcible, but some others are not. It is sometimes the case, however, that in the presence of some other condition, such properties become finitely forcible. The property that $W$ is 0/1 valued is an example (to be discussed below).

### *4.1. Regularity*

We call a graphon *d-regular*, or *regular of degree d* ($0 \leqslant d \leqslant 1$), if

$$\int_0^1 W(x, y)\, dy = d$$

for almost all $0 \leqslant x \leqslant 1$. This condition can also be written as $t_1(K_2^\bullet, W)(x) = d$, where $K_2^\bullet$ denotes the single edge with one endnode labeled. These graphons can be forced by two subgraph density constraints:

$$t(K_2, W) = d, \qquad t(P_3, W) = d^2,$$

since equality holds in the Cauchy–Schwarz estimate

$$t(P_3, W) = \int_0^1 t\big(K_2^\bullet, W\big)^2 dx \geqslant \left( \int_0^1 t\big(K_2^\bullet, W\big)\, dx \right)^2 = t(K_2, W)^2.$$

Regular graphons (without specifying the degree $d$) can be forced by the constraint $t(P_3, W) = t(K_2, W)^2$.

*4.2. Zero-one valued functions*

Trivially, $W \in \mathcal{W}$ is 0/1 valued almost everywhere if and only if $t_2(\widehat{C}_2, W) = 0$, where $\widehat{C}_2$ is the 2-labeled signed multigraph on 2 nodes with 2 parallel edges, one signed "+" and one signed "−". By Lemma 3.2, this is equivalent to the single numerical equation $t(\widehat{B}_4, W) = 0$, where $\widehat{B}_4$ is the unlabeled signed multigraph on 2 nodes with 4 parallel edges, 2 signed "+" and 2 signed "−".

So we can "force" the property of being 0/1 valued using multigraphs, but we cannot express it in terms of simple graphs. This follows from the observation that if $G(n, 1/2)$ is the Erdős–Rényi random graph with $n$ nodes and edge density 1/2, and $W_n = W_{G(n,1/2)}$, then with probability 1, $W_n$ tends to the identically 1/2 function $U_{1/2}$ in the $\|.\|_\square$ norm, which implies by (6) that $t(F, W_n) \to t(F, U_{1/2})$ for every simple graph $F$. So every constraint of the form $t(F, W_n) = 0$, where $F$ is a simple graph, is inherited by $U_{1/2}$, which is not 0/1 valued.

It makes sense to formulate sufficient conditions for being 0/1 valued. Here is a useful one.

**Lemma 4.1.** *Let $\widehat{F}$ be a signed bipartite graph on n nodes, all labeled. Let $\widehat{F}'$ be obtained by unlabeling all the nodes. Suppose that for some $W \in \mathcal{W}$ we have*

$$t_n(\widehat{F}, W) = 0 \tag{11}$$

*almost everywhere. Then $W(x, y) \in \{0, 1\}$ almost everywhere. If $W \in \mathcal{W}_0$, then it suffices to assume that*

$$t\big(\widehat{F}', W\big) = 0. \tag{12}$$

**Proof.** By Lemma 3.3, (11) implies that for the 2-labeled signed multigraph $J$ obtained by identifying each color class of $F$, we have $t_2(J, W) = 0$. This clearly implies that $W$ is 0/1 valued.

If $W \in \mathcal{W}_0$, then (12) implies that in the integral

$$t\big(\widehat{F}', W\big) = \int\limits_{[0,1]^{V(F)}} t_n(\widehat{F}, W)(x)\, dx$$

the integrand is 0 almost everywhere. This means that (11) holds. □

*4.3. Monotonicity*

Let $\mathcal{M}_0$ denote the set of functions $[0, 1]^2 \to \{0, 1\}$ that are monotone decreasing in both variables, and let $\mathcal{M}$ be the set of graphons which are weakly isomorphic to some function in $\mathcal{M}_0$. (These graphons have been studied by Diaconis, Holmes and Janson [7] as limits of threshold graphs.) In this section we show that the set $\mathcal{M}$ is finitely forcible in $\mathcal{W}$.

Let $\widehat{C}_4$ denote a signed 4-labeled 4-cycle, with two opposite edges signed "+", the other two signed "−". Let $\widehat{C}_4'$ be obtained from $\widehat{C}_4$ by unlabeling all its nodes.

**Lemma 4.2.** *Let $W \in \mathcal{W}$, then $W \in \mathcal{M}$ if and only if*

$$t_4(\widehat{C}_4, W) = 0 \tag{13}$$

*almost everywhere. If $W \in \mathcal{W}_0$, then it is enough to assume that*

$$t\big(\widehat{C}_4', W\big) = 0 \tag{14}$$

*almost everywhere.*

**Proof.** It is easy to see that (13) and (14) hold for every $W \in \mathcal{M}$.

Next we prove if that $W \in \mathcal{W}_0$ satisfies (14) then $W \in \mathcal{M}$. By Lemma 4.1, $W$ is 0/1 valued almost everywhere, and we may assume that it is 0/1 valued. Let $N(x)$ denote the support of the function $W(x, .)$ $(x \in [0, 1])$.

**Claim 4.3.** *For almost all $x, y \in [0,1]$, either $\lambda(N(x) \setminus N(y)) = 0$ or $\lambda(N(y) \setminus N(x)) = 0$.*

Indeed, for every $u \in N(x) \setminus N(y)$ and $v \in N(y) \setminus N(x)$, the 4-tuple $(x, u, y, v)$ satisfies $W(x,u)(1 - W(u,y))W(y,v)(1 - W(v,x)) > 0$, and so $t(\widehat{C}'_4, W) = 0$ implies that the measure of pairs $(x, y)$ for which there is a positive measure of such pairs $(u, v)$ must be 0.

By the Monotone Reordering Theorem, there is a monotone decreasing function $f : [0,1] \to [0,1]$ and a measure preserving map $\varphi : [0,1] \to [0,1]$ such that $\lambda(N(x)) = f(\varphi(x))$ almost everywhere.

We can change $W$ on a set of measure 0 so that $\lambda(N(x)) = f(\varphi(x))$ for all $x$. Since $f$ is monotone decreasing, it follows that if $\varphi(x) \leqslant \varphi(y)$, then $\lambda(N(x)) \geqslant \lambda(N(y))$, and so Claim 4.3 implies that $\lambda(N(y) \setminus N(x)) = 0$.

Let

$$A = \bigl\{(x, y, u) \in [0,1]^3 \colon\ W(x,u) = 0,\ W(y,u) = 1,\ \varphi(x) \leqslant \varphi(y)\bigr\}$$

and $A(u) = \{(x, y)\colon\ (x, y, u) \in A\}$. Claim 4.3 implies that for almost all pairs $(x, y)$, $\lambda\{u\colon\ (x, y, u) \in A\} = 0$, which implies that $\lambda_3(A) = 0$. This also implies that for almost all $u \in [0,1]$, $\lambda_2(A(u)) = 0$.

Let $K(u) = \varphi^{-1}[0, f(\varphi(u))]$. For every point $u$ with $\lambda_2(A(u)) = 0$ we must have $\lambda(N(u) \bigtriangleup K(u)) = 0$; indeed, we have $\lambda(N(u)) = f(\varphi(u)) = \lambda(K(u))$, and so if $\lambda(N(u) \bigtriangleup K(u)) > 0$, then $\lambda(N(u) \setminus K(u)) > 0$ and $\lambda(K(u) \setminus N(u)) > 0$. But every $x \in K(u) \setminus N(u)$ and $y \in N(u) \setminus K(u)$ satisfies $(x, y) \in A(u)$, a contradiction.

So we know that for almost all $u$, $\lambda(N(u) \bigtriangleup K(u)) = 0$. Hence for almost all pairs $(x, u)$, $W(x,u) = 1$ if and only if $x \in K(u)$, i.e., $\varphi(x) \leqslant f(\varphi(u))$.

Consider the function $U(x, y) = \mathbf{1}_{y \leqslant f(x), x \leqslant f(y)}$. This is clearly symmetric and monotone decreasing in both variables. We claim that $W = U^{\varphi}$ almost everywhere. We have seen that for almost all pairs $x, y$, $W(x,y) = 1$ if and only if $\varphi(x) \leqslant f(\varphi(y))$. By the symmetry of $W$, this is also equivalent (for almost all pairs $x, y$) to $\varphi(y) \leqslant f(\varphi(x))$. So (for almost all pairs $x, y$) $W(x,y) = 1$ if and only if $U(\varphi(x), \varphi(y)) = 1$.

This shows that $W$ is weakly isomorphic to $U \in \mathcal{M}_0$.

Finally, assume that $W \in \mathcal{W}$ satisfies (13). By Lemma 4.1, $W$ is 0/1 valued almost everywhere, so $W \in \mathcal{W}_0$. Since it trivially satisfies (14), it follows that $W \in \mathcal{M}$. □

## 5. Finitely forcible graphons I: polynomials

### 5.1. Positive supports of polynomials

**Theorem 5.1.** *Let $p$ be a real symmetric polynomial in two variables, which is monotone decreasing on $[0,1]^2$. Then the function $W(x,y) = \mathbf{1}_{p(x,y) \geqslant 0}$ is finitely forcible in $\mathcal{W}$.*

**Proof.** We in fact prove that the equations

$$t_4(\widehat{C}_4, U) = 0 \tag{15}$$

and

$$t(K_{a,b}, U) = t(K_{a,b}, W) \quad \bigl(1 \leqslant a, b \leqslant 2\deg(p) + 2\bigr) \tag{16}$$

form a forcing family for $W$ in $\mathcal{W}$. The condition on the monotonicity of $p$ implies that $U = W$ satisfies (15). It is trivial that the other equations are satisfied by $U = W$.

Let $U \in \mathcal{W}$ be any graphon satisfying (15)–(16). By Lemma 4.2 we may assume that $U$ is 0/1 valued and monotone decreasing. Let $S_U = \{(x, y)\colon\ U(x,y) = 1\}$.

We have

$$t(K_{a,b}, U) = \int\limits_{[0,1]^a} \int\limits_{[0,1]^b} \prod_{i=1}^{a} \prod_{j=1}^{b} U(x_i, y_j)\, dy\, dx.$$

Split this integral according to which $x_i$ and which $y_j$ is the largest. Restricting the integral to, say, the domain where $x_1$ and $y_1$ are the largest, we have that whenever $U(x_1, y_1) = 1$ then also $U(x_i, y_j) = 1$ for all $i$ and $j$, and hence

$$\int\limits_{x_1\in[0,1]} \int\limits_{x_2,\dots,x_a\leqslant x_1} \int\limits_{y_1\in[0,1]} \int\limits_{y_2,\dots,y_b\leqslant y_1} \prod_{i=1}^{a}\prod_{j=1}^{b} U(x_i, y_j)\,dy\,dx$$

$$= \int\limits_{x_1\in[0,1]} \int\limits_{y_1\in[0,1]} U(x_1, y_1)x_1^{a-1}y_1^{b-1}\,dy_1\,dx_1 = \int\limits_{(x,y)\in S_U} x^{a-1}y^{b-1}\,dy\,dx.$$

Hence

$$t(K_{a,b}, U) = ab \int\limits_{(x,y)\in S_U} x^{a-1}y^{b-1}\,dy\,dx.$$

By Stokes' Theorem, we can rewrite this as

$$t(K_{a,b}, U) = b \int\limits_{\partial S_U} x^a y^{b-1} n_1(x, y)\,ds,$$

where $ds$ is the arc length of $\partial S$ and $n = (n_1, n_2)$ is the outward normal of $\partial S$. (Since $\partial S$ is the graph of a monotone function, this normal exists almost everywhere.) Interchanging the roles of $x$ and $y$, and adding, we get

$$\int\limits_{\partial S_U} x^a y^b \big(n_1(x, y) + n_2(x, y)\big)\,ds = \frac{1}{a+1}t(K_{a+1,b}, U) + \frac{1}{b+1}t(K_{a,b+1}, U). \tag{17}$$

Now consider the following integral:

$$I(U) = \int\limits_{\partial S_U} xyp(x, y)^2\big(n_1(x, y) + n_2(x, y)\big)\,ds.$$

By (17), this can be expressed as a linear combination of the values $t(K_{a,b}, U)$, where $a, b \leqslant 2\deg(p) + 1$ and the coefficients depend only on $a, b$ and $p$. Hence it follows that $I(U) = I(W) = 0$.

On the other hand, the integrand in $I(U)$ is clearly 0 on the axes, and it is nonnegative on the rest of the boundary. Hence it must be identically 0, which means that $\partial(S_U)$ must be contained in the union of the axes and the curve $p = 0$. But this clearly implies that $U = W$ except perhaps on the boundary. □

The following special case is perhaps the simplest. Define the *half-graphon* by $W_h(x, y) = \mathbf{1}_{x+y\leqslant 1}$.

**Corollary 5.2.** *The half-graphon is finitely forcible in $\mathcal{W}$.*

In fact, by a variation of the argument above, one can prove that the following equations force the half-graphon:

$$t(\widehat{C}_4, W) = 0, \tag{18}$$

and

$$t(P_3, W) - t(K_2, W) + 1/6 = 0. \tag{19}$$

Clearly, the left-hand side of (18) is always nonnegative. It is easy to show that in (19), the left had side is nonnegative, provided equality holds in (18).

Half-graphons are natural limits of *half-graphs*, defined by $V(G) = [n]$ and $E(G) = \{ij\colon\ i + j \leqslant n\}$. This implies the following graph-theoretic extremal result.

**Corollary 5.3.** *Among all simple graphs with no induced matching with 2 edges, the difference $t(P_3,.) - t(K_2,.)$ is asymptotically minimal for half-graphs.*

### 5.2. Continuous range

Applying the results of Section 3 to the half-graphon $U(x,y) = \mathbf{1}_{x+y>1}$, we get an interesting finitely forcible graphon.

**Proposition 5.4.** *There exists a graphon $W \in \mathcal{W}$ satisfying $W + \frac{1}{2}(W \circ W \circ W) = U$, it is finitely forcible in $\mathcal{W}$, and its range consists of two nontrivial intervals.*

**Proof.** Let $\lambda$ be a nonzero eigenvalue of $U$ and let $f$ be the corresponding eigenfunction with unit norm. Then

$$\lambda f(x) = \int_{1-x}^{1} f(y)\,dy.$$

From this integral equation one can calculate that the eigenvalues and eigenfunctions are

$$\lambda_k = \frac{2}{(4k+1)\pi}, \qquad f_k(x) = \sqrt{2}\sin\left(\frac{4k+1}{2}\pi x\right), \quad k \in \mathbb{Z}.$$

In particular, the eigenfunctions are analytic and uniformly bounded.

Let $g$ be the inverse function of $x \mapsto x + \frac{1}{2}x^3$ on the real line. We have

$$x - g(x) = \frac{xg(x)^2}{2 + g(x)^2},$$

which implies that $|g(x)| < |x|$ and $|x - g(x)| < \frac{1}{2}|x|g(x)^2 < \frac{1}{2}|x|^3$. Since the series $\sum_k |\lambda_k|^3$ is convergent, and the $f_k$ are bounded, this implies that

$$P(x,y) = \sum_{k\in\mathbb{Z}} \big(\lambda_k - g(\lambda_k)\big) f_k(x) f_k(y)$$

is uniformly absolute convergent. Thus $P(x,y)$ is well defined and analytic in $x$ and $y$. It is clear that $P$ is not constant.

Note that we have

$$\big|P(x,y)\big| \leqslant \sum_{k\in\mathbb{Z}} 2\big|\lambda_k - g(\lambda_k)\big| < \sum_{k\in\mathbb{Z}} |\lambda_k|^3 = \left(\frac{2}{\pi}\right)^3 \frac{7}{8}\zeta(3) < \frac{1}{2}. \tag{20}$$

Now we can define $W = U - P$. Clearly this function is bounded and symmetric. Furthermore, the spectral decomposition of $W$ is

$$W(x,y) = \sum_{k\in\mathbb{Z}} \lambda_k f_k(x) f_k(y) - \sum_{k\in\mathbb{Z}} \big(\lambda_k - g(\lambda_k)\big) f_k(x) f_k(y) = \sum_{k\in\mathbb{Z}} g(\lambda_k) f_k(x) f_k(y),$$

from which it follows that $W + \frac{1}{2}(W \circ W \circ W) = U$. Corollary 3.18 implies that $W$ is finitely forcible.

Clearly $W$ is continuous over $U^{-1}(0)$ and over $U^{-1}(1)$. Inequality (20) implies that the range of $W$ over $U^{-1}(0)$ is an interval contained in $(-1/2, 1/2)$, while its range over $U^{-1}(1)$ is an interval contained in $(1/2, 3/2)$. So we get two disjoint intervals. □

We can invoke Corollary 3.16 to transform $W$ into an element from $\mathcal{W}_0$. This implies the following.

**Corollary 5.5.** *There is finitely forcible function in $\mathcal{W}_0$ whose range is of continuum cardinality.*

## 6. Finitely forcible graphons II: complement reducible graphons

### 6.1. The finite case

A simple graph is called *complement reducible*, or for short, a *CR-graph*, if it can be constructed starting from a single node, by repeated application of disjoint union and complementation. (These graphs are often called *cographs*.) One of the many known characterizations of these graphs is the following [6]; see [4] for more on these graphs.

**Proposition 6.1.** *A simple graph is complement reducible if and only if it does not contain a path $P_4$ on 4 nodes as an induced subgraph.*

Let $\widehat{P}_4$ denote the graph $K_4$ in which the edges of a path of length 3 are signed "+", and the remaining edges are signed "−". Then the condition in the proposition can be rephrased as

$$t(\widehat{P}_4, G) = 0.$$

Every CR-graph $G$ can be described by a rooted tree $T$ in a natural way [6]: each node of $T$ represents a CR-graph; the leaves represent single nodes, the root represents $G$, and the children of each internal node represent the connected components of the graph represented by the node, complemented. Another way of describing this connection is that $G$ is defined on the leaves of $T$, and two nodes of $G$ are connected by an edge if and only if their last common ancestor is at an odd distance from the root.

### 6.2. CR graphons and trees

We define a *CR-graphon* as any graphon $W$ with $t(\widehat{P}_4, W) = 0$. Also we extend the notion of CR-graphs to infinite graphs by the requirement that no four nodes induce a path of length four.

One can construct CR-graphons from trees, but (unlike in the finite case) not all CR-graphons arise this way (we shall see later that all regular CR-graphons do). Let $T$ be any (possibly infinite) rooted tree, in which every non-leaf node has at least two and at most countably many children, except possibly the root, which may have only one child. Let $\Omega = \Omega_T$ be the set of maximal paths starting at the root $r$ (they are either infinite or end at a leaf). For each node $v$, let $C_v$ be the set of its children, and let $\Omega_v$ denote the set of paths in $\Omega$ passing through $v$. The sets $\Omega_v$ generate a $\sigma$-algebra $\mathcal{A}_T$.

We define a (simple) graph on node set $\Omega$ by connecting two nodes if the last common node of the corresponding paths is at odd depth (where the depth of the root $r$ is 0). We also define the adjacency function $U_T : \Omega \times \Omega \to \{0, 1\}$ by letting $U_t(x, y) = 1$ if and only if $x$ and $y$ are adjacent. It is clear that $U_T$ is measurable with respect to $\mathcal{A} \times \mathcal{A}$.

If we choose any probability measure $\pi$ on $(\Omega_T, \mathcal{A}_T)$, this completes the construction of a graphon $(\Omega_T, \mathcal{A}_T, \pi, U_T)$, which is clearly a CR-graphon. Note that such a measure can be specified through the values

$$f(v) = \pi(\Omega_v).$$

It is clear that these values satisfy

$$f(r) = 1, \qquad f(u) \geqslant 0, \quad \text{and} \quad f(u) = \sum_{v \in C_u} f(v). \tag{21}$$

Conversely, every function satisfying (21) defines a probability measure on $(\Omega_T, \mathcal{A}_T)$.

### 6.3. Regular CR-graphons

Of special interest for us will be regular CR-graphons. Our first goal is to prove:

**Theorem 6.2.** *Every regular CR-graphon $W$ can be represented (up to weak isomorphism) as $(\Omega_T, \mathcal{A}_T, \pi, U_T)$ where $T$ is a locally finite tree and $\pi$ is a probability measure on $\mathcal{A}_T$.*

The proof of this theorem will need several lemmas. Recall from [15] that for every graphon $W : [0,1]^2 \mapsto [0,1]$ there is a random graph model $\mathbb{G}(W,n)$ on node set $\{1,2,\dots,n\}$, created as follows: We pick independent uniform random points $x_1, x_2, \dots, x_n \in [0,1]$ and connect two distinct nodes $i$ and $j$ with probability $W(x_i, x_j)$. Let $d_i(\mathbb{G}(W,n))$ denote the degree of $i$ in the resulting graph.

**Lemma 6.3.** *If a graphon $W$ is $d$-regular, then with probability at least $1 - 2ne^{-(n-1)\varepsilon^2/2}$ we have*

$$\left|\frac{d_i(\mathbb{G}(W,n))}{n-1} - d\right| < \varepsilon$$

*simultaneously for all $1 \leqslant i \leqslant n$.*

**Proof.** It is easy to see that for every $1 \leqslant i \leqslant n$ the value $d_i(\mathbb{G}(W,n))$ is the sum of $n-1$ independent random variables all taking 1 with probability $d$ and 0 with probability $1-d$. This implies by the Chernoff–Hoeffding Inequality that

$$P\left(\left|\frac{d_i(\mathbb{G}(W,n))}{n-1} - d\right| \geqslant \varepsilon\right) \leqslant 2e^{-(n-1)\varepsilon^2/2}.$$

This means that the probability that there exists at least one number $1 \leqslant i \leqslant n$ with $|d_i(\mathbb{G}(W,n))/(n-1) - d| \geqslant \varepsilon$ is at most $2ne^{-(n-1)\varepsilon^2/2}$. $\square$

**Definition 6.4.** We say that $\{G_i\}_{i=1}^{\infty}$ is a *degree-uniformly* convergent sequence of simple graphs with limiting degree $0 \leqslant d \leqslant 1$ if it is convergent, $\lim_{i\to\infty} |V(G_i)| = \infty$ and

$$\lim_{i\to\infty} \frac{d_{\max}(G_i)}{|V(G_i)|} = \lim_{i\to\infty} \frac{d_{\min}(G_i)}{|V(G_i)|} = d.$$

**Lemma 6.5.** *If a CR-graphon $W$ is $d$-regular then there is a sequence of CR-graphs $\{G_n\}_{n=1}^{\infty}$ that degree-uniformly converges to $W$.*

**Proof.** The Borel–Cantelli lemma together with Lemma 6.3 implies that with probability one

$$\lim_{i\to\infty} \frac{d_{\max}(\mathbb{G}(W,n))}{n-1} = \lim_{i\to\infty} \frac{d_{\min}(\mathbb{G}(W,n))}{n-1} = d.$$

It is clear that with probability 1 $\mathbb{G}(W,n)$ does not contain an induced $P_4$, and hence it is a CR-graph for every $n$. We also know [15] that with probability 1 it converges to $W$ as $n \to \infty$. These facts imply that the sequence $\{\mathbb{G}(W,n)\}_{n=1}^{\infty}$ satisfies the conditions with probability 1. $\square$

**Lemma 6.6.** *Let $G$ be a finite disconnected CR-graph on $n \geqslant 2$ nodes such that $|d(v)/n - d| \leqslant \varepsilon$ for every $v \in V(G)$ with some $d \in [0,1]$. Then every connected component of $G$ has size less than $(\frac{2}{3} + \frac{4}{3}\varepsilon)n$.*

**Proof.** Let $G'$ be a connected component of $G$ of maximal size. Let $n = |V(G)|$ and $a = |V(G')|$. We may assume that $a > 1$ (else, the assertion is trivial). Let $v \in V(G) \setminus V(G')$, then $d(v) < n - a$. Since $G'$ is a connected CR-graph, there is a node $w \in V(G')$ with degree at least $a/2$. Then by our assumption

$$2\varepsilon n \geqslant d(w) - d(v) > \frac{a}{2} - (n-a) = \frac{3}{2}a - n.$$

This implies that $a < (\frac{2}{3} + \frac{4}{3}\varepsilon)n$. $\square$

**Lemma 6.7.** *Let $W$ be a $d$-regular CR-graphon. Then either $W$ or $1 - W$ can be decomposed as a weighted direct sum of at least two regular CR-graphons.*

**Proof.** By Lemma 6.5, there is a sequence of CR-graphs $\{G_n\}_{n=1}^{\infty}$ that degree-uniformly converges to $W$. For each $n$, either $G_n$ or $\overline{G}_n$ is disconnected, since $G_n$ is a CR-graph. We may assume, by restricting ourselves to a subsequence, that either $G_n$ is disconnected for all $n$, or $\overline{G}_n$ is disconnected for all $n$. By complementing if necessary, we may assume that $G_n$ is disconnected for all $n$.

Let $H_{n,1}, \dots, H_{n,k_n}$ be the connected components of $G_n$. Since the convergence is degree-uniform, it follows that for any $0 < d' < d$, all degrees of $G_n$ are larger than $d'|V(G_n)|$ if $n$ is large enough, and then trivially $|V(H_{n,i})| \geqslant d'|V(G_n)|$. This implies that $k_n$ remains bounded, and so by going to a subsequence again, we may assume that $k_n = k$ is independent of $n$. By the same token, we may assume that $|V(H_{n,i})|/|V(G_n)|$ has a limit $a_i$ as $n \to \infty$. Clearly $a_i \geqslant d$ and $\sum_i a_i = 1$. We may also assume that for each $1 \leqslant i \leqslant k$, the sequence of graphs $(H_{n,i})_{n=1}^{\infty}$ is convergent. Let $W_i$ denote its limit graphon. It is straightforward to check that $W_i$ is a regular CR-graphon. Furthermore, the weighted direct sum $\bigoplus_i (a_i) W_i$ is the limit of the graphs $G_n$. By the uniqueness of the limit, $W$ is weakly isomorphic to $\bigoplus_i (a_i) W_i$. $\square$

Now we are able to prove Theorem 6.2.

**Proof.** Assume that $W = \bigoplus_{i=1}^{k} (a_i) W_i$, $k \geqslant 2$. We may assume that the $W_i$ cannot be written as weighted direct sums in a nontrivial way. We build a tree by starting with a root corresponding to $W$, having $k$ children corresponding to $1 - W_1, \dots, 1 - W_k$. If any of these functions is almost everywhere 0, then this node will be a leaf. Else, we continue building the tree from this node as root.

If $W$ cannot be written as a weighted direct sum of at least two regular CR-graphons, then by Lemma 6.7, $1 - W$ can be, and we start the tree with a root with a single child, corresponding to $1 - W$.

This way we obtain a tree $T$, where each node $v$ is labeled by a regular CR-graphon $W_v$. For each node of the tree constructed this way, we define $f(v)$ as the product of the weights of the graphons along the path from the root to $v$. It is straightforward to check that the $W$ is weakly isomorphic to the graphon $U_T$ with the probability distribution defined by $f$. $\square$

Let $W$ be a regular CR-graphon represented by the tree $T$ with a measure $\mu$ on $\Omega_T$. For a node $v \in V(T)$ let $f(v) := \mu(\Omega_v)$. We assume that $f(v) > 0$, since parts of the tree with 0 weight can be deleted. We observe that for every $v \in V(T)$, the subtree $T_v$ of $T$ rooted at $v$, with the same local distributions (i.e., with node weights $f(u)/f(v)$), defines another regular CR-graphon. This implies that the value

$$c(v) = \int\limits_{\Omega_v} U_{T_v}(x, y)\, d\mu(y)$$

is the same for all $x \in \Omega_v$. The degree of the graphon on $T_v$ is $d(v) = c(v)/f(v)$. (Note however that, depending on the parity of the depth of $v$, either $U_{T_v}$ or $1 - U_{T_v}$ is an induced sub-graphon of $U_T$.) For $u \in V(T)$ and $v \in C_u$ we have

$$c(u) + c(v) = f(v), \tag{22}$$

and for every leaf $u$ (if any)

$$c(u) = 0. \tag{23}$$

The following simple lemma gives some conditions that $f$ and $c$ satisfy.

**Lemma 6.8.** *Let $W_T$ be a regular CR-graphon.*

(a) *If $u \in V(T)$ has $r$ children, then $c(u) \leqslant \frac{1}{r} f(u)$.*
(b) *If $u \in V(T)$, $v \in C_u$ and $v$ has $r$ children, then $f(u) \geqslant (2 - \frac{1}{r}) f(v)$.*

**Proof.** (a) Let $v_1, v_2, \ldots, v_r$ be the children of $u$. By (22), $f(v_i) = c(u) + c(v_i) \geqslant c(u)$, and summing this over $i$, we get $f(u) \geqslant rc(u)$.

(b) Let $v'$ be a sibling of $v$. Using (22) and (a),

$$f(v') = c(u) + c(v') \geqslant c(u) = f(v) - c(v) \geqslant \left(1 - \frac{1}{r}\right) f(v),$$

and so

$$f(u) \geqslant f(v) + f(v') \geqslant \left(2 - \frac{1}{r}\right) f(v). \qquad \square$$

**Lemma 6.9.** *Let $T$ be a locally finite tree such that no node except possibly the root has exactly one child. Let $c, f : V(T) \to \mathbb{R}_+$ be two functions satisfying* (21), (22) *and* (23)*. Then the probability measure defined by $f$ gives a regular CR-graphon on $T$.*

**Proof.** By (21), the function $f$ defines a probability measure $\pi$ on $(\Omega, \mathcal{A})$. Let $x = (v_0, v_1, v_2, \ldots)$ be a maximal path starting at the root $r = v_0$. Lemma 6.8(b) implies that if this path is infinite, then $f(v_n) \to 0$, and part (a) of the same lemma implies that $c(v_n) \to 0$.

The path $x$ is connected to all paths $y$ that branch off from $x$ at $v_1$, $v_3$, $v_5$, etc. The $\pi$-measure of these paths is $(f(v_1) - f(v_2)) + (f(v_3) - f(v_4)) + \cdots$, which by (22) can be written as

$$\big(c(v_0) + c(v_1)\big) - \big(c(v_1) + c(v_2)\big) + \big(c(v_2) + c(v_3)\big) - \cdots = c(v_0)$$

(if the path ends at a leaf, then we use (23)). This is indeed independent of the path. $\square$

Now we are able to prove the second main result in this section:

**Theorem 6.10.** *For every locally finite rooted tree $T$ there is a unique regular CR-graphon on $T$.*

**Proof.** *Existence.* First we prove this for a finite tree, by induction on the depth. For a single node, the function $U \equiv 0$ is a regular CR-graphon.

Suppose that the tree has more than one node, and let $u_1, \ldots, u_k$ be the children of the root. By induction, we find regular CR-graphons on $T_{u_1}, \ldots, T_{u_k}$, with degrees $d_1, \ldots, d_k$. Note that since $u_i$ is either a leaf or has at least two children, we must have $d_i < 1$. Let

$$d = \frac{1}{\sum_i \frac{1}{1-d_i}}, \qquad a_i = \frac{d}{1-d_i},$$

then scaling the measure of $T_{u_i}$ by $a_i$, complementing each $T_{u_i}$ and taking their disjoint union, we get a $d$-regular CR-graphon on $T$.

Now suppose that $T$ is infinite, and let $T_k$ denote the tree obtained by deleting all nodes farther than $k$ from the root. By the above, there is a regular CR-graphon on $T_k$, which yields two functions $f^k$ and $c^k$ on $V(T_k)$ satisfying (21), (22) and (23). We can select a subsequence of the indices $k$ such that $f^k(v)$ tends to some $f(v)$ and $c^k(v)$ tends to some $c(v)$ as $k$ ranges through this subsequence. Clearly, the functions $f$ and $c$ also satisfy (21), (22) and (23), and so by Lemma 6.9, they yield a regular CR-graphon on $T$.

*Uniqueness.* Suppose that there are two weightings $f, f'$ of the nodes of $T$ such that they both define a regular CR-graphon.

Let $P = (v_0, v_1, v_2, \ldots)$ be an infinite path starting from the root, then we get by (22)

$$f(v_1) - f(v_2) + \cdots + (-1)^{k+1} f(v_k) = c(v_0) + (-1)^{k+1} c(v_k).$$

The sequence $f(v_k)$ is monotone decreasing and it tends to 0. Hence $c(v_k) \to 0$, and

$$c(v_0) = \sum_{k=1}^{\infty} (-1)^{k+1} f(v_k).$$

In the other weighting,

$$c'(v_0) = \sum_{k=1}^{\infty} (-1)^{k+1} f'(v_k).$$

Let

$$z_i = \frac{f'(v_i) f(v_{i-1})}{f'(v_{i-1}) f(v_i)},$$

then $f'(v_k) = z_1 \dots z_k f(v_k)$. Thus

$$c'(v_0) = \sum_{k=1}^{\infty} (-1)^{k+1} z_1 \cdots z_k f(v_k) = z_1\bigl(f(v_1) - z_2\bigl(f(v_2) - \cdots\bigr)\bigr). \tag{24}$$

Choose the path $v_0, v_1, \dots$ as follows. Given $v_i$, choose $v_{i+1} \in C_{v_i}$ so that $z_{i+1}$ is as large as possible if $i$ is odd, and as small as possible if $i$ is even. Clearly $z_{i+1} \geqslant 1$ if $i$ is odd, and $z_{i+1} \leqslant 1$ if $i$ is even. This is clearly possible. Raising $z_1$ to 1, then lowering $z_2$ to 1, then raising $z_3$ to 1, etc., increases the expression in (24), and hence

$$c'(v_0) \leqslant c(v_0). \tag{25}$$

Since the reverse inequality follows similarly, we get that $c'(v_0) = c(v_0)$. From the fact that equality holds in (25), we get that all $z_1 = 1$, which in turn implies that $f'(v_1) = f(v_1)$ for any child $v_1$ of $v_0$.

Applying the same argument to the graphons $W_{T_{v_1}}$, $v \in C_{v_0}$, then to their children, etc., we get that $f(v) = f'(v)$ for all $v$. □

### 6.4. *Forcible regular CR-graphons with irrational edge densities*

We start with an easy observation:

**Lemma 6.11.** *If a $d$-regular CR-graphon is a stepfunction, then $d$ is rational.*

**Proof.** By induction on the depth of the tree. □

In contract to this, we prove that for every $\alpha \in [0,1]$ there is a regular CR-graphon of degree $\alpha$ which is finitely forcible. As Lemma 6.11 shows, for irrational $\alpha$ such a graphon in not a stepfunction.

Let $P_3^2$ be the disjoint union of two copies of $P_3$. Let $\mathcal{Z}$ denote the set of regular CR-graphons that don't contain any induced copy of $P_3^2$ and its complement.

**Lemma 6.12.** *The set $\mathcal{Z}$ consists of those regular CR-graphons whose representing tree has the property that every node has at most one child which is not a leaf.*

**Proof.** Let $W$ be an element in $\mathcal{Z}$. First of all note that a graphon without an induced copy of $P_3$ is the disjoint union of complete graphons. It follows that a graphon without an induced copy of $P_3^2$ has at most one connected component which is not complete. Applying this for the sub-graphons (and their complements) corresponding to the nodes of the tree of $W$ we get that $W$ satisfies the condition. The other direction is trivial. □

The previous lemma shows that the tree of an element in $\mathcal{Z}$ is one (possibly infinite) path with additional leaves hanging from its nodes. Thus the structure is determined by the integer sequence $n_1, n_2, \ldots$ where $n_k$ is the number of leaves at the $k$-th level.

We start with a simple example. Let $\alpha = (3-\sqrt{5})/2$. Note that $\alpha = (1-\alpha)^2$. There exists a unique graphon $W$ which is the disjoint union of a clique of size $\alpha$ and a version of the complement of $W$ scaled to the size $1-\alpha$. The graphon $W$ has an iterated structure. The choice of $\alpha$ guarantees that $W$ is $\alpha$-regular. We show that $W$ is finitely forcible.

**Lemma 6.13.** *The graphon $W$ is the only element of $\mathcal{Z}$ with degree $\alpha$ and thus it is finitely forcible.*

**Proof.** Let $W'$ be another graphon with the above property. Since $1/3 < \alpha < 1/2$, we have that $W' = (\alpha)1 \oplus (1-\alpha)W''$, where $W''$ has density $\alpha/(1-\alpha) = 1-\alpha$. This shows that we can inductively continue the process with the complement $W''$ and finally obtain the desired iterated structure. □

The previous argument can be easily generalized to other irrational values of $\alpha$.

**Proposition 6.14.** *For every irrational number $0 < \alpha < 1$ there is exactly one graphon in $\mathcal{Z}$ with edge density $\alpha$ and so this graphon is finitely forcible.*

**Proof.** Let $\alpha$ be a number between 0 and 1/2 (the case $\alpha > 1/2$ is similar). Let $W$ be a graphon from $\mathcal{Z}$ with edge density $\alpha$. Since $\alpha < 1/2$, the graphon $W$ is weighted direct sum of $n_1$ cliques, all with weight $\alpha$ and a connected element $W'$ of $\mathcal{Z}$ with weight $0 < s < 1$ and with edge density between 1/2 and 1. To guarantee edge density $\alpha$ in this component, $s$ has to be between $\alpha$ and $2\alpha$. Consequently $n_1$ is the unique natural number such that $\alpha \leqslant 1 - n_1\alpha < 2\alpha$. The complement $1 - W'$ is an element from $\mathcal{Z}$ with edge density smaller than 1/2 and we can continue the process. □

The reader can see that the numbers $n_1, n_2, \ldots$ are uniquely determined by $\alpha$ so it is a natural question to ask what these numbers are. An elementary calculation shows that these numbers are basically the numbers occurring in the (unique) continued fraction expansion of $\alpha$. The only exception is the first number, which is shifted by one:

$$\alpha = \cfrac{1}{n_1 + 1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{\ddots}}}}.$$

This shows that one can force graphons that encode an arbitrary sequence of natural numbers in a very structural way.

### *6.5. Forcing the binary tree*

In this section we prove that the regular CR-graphon $U_{T_2}$ is finitely forcible, where $T_2$ is the complete binary tree $B$ of infinite depth, with a root of degree 1 added to comply with our previous definitions. We note that $U_{T_2}$ has the following alternative description: Consider the space $V(C_4)^{\mathcal{N}}$ with the uniform probability measure. We connect two nodes $x$ and $y$ of $V(C_4)^{\mathcal{N}}$ if for the first coordinate where they differ, say $i \in \mathcal{N}$, $x_i$ and $y_i$ are connected in $C_4$. The graphon $U_{T_2}$ can be called the *infinite lexicographic power* of $C_4$.

Let us define the following signed labeled graphs: $B$ is $K_3$ with one node labeled 1, the incident edges signed "+", and the opposite edge signed "−"; and $C$ and $D$ are obtained from $K_2^\bullet$ and $B$, respectively, by adding a new node labeled 2, and connecting it to the unlabeled nodes by edges signed "−". Also consider the signed graphs $\overline{B}$, $\overline{C}$ and $\overline{D}$ obtained from $B$, $C$ and $D$ by switching the "+" and "−" signs on the edges.

**Proposition 6.15.** *Let $W$ be a graphon satisfying*

$$t(\widehat{P}_4, W) = 0, \qquad t_1\big(K_2^\bullet, W\big)(x) = \frac{2}{3},$$
$$t_1(B, W)(x) = \frac{8}{45}, \qquad t_1(\overline{B}, W)(x) = \frac{2}{45}$$

*and*

$$2t_2(C, W)^2(x, y) = 5t_2(D, W)(x, y), \qquad 2t_2(\overline{C}, W)^2(x, y) = 5t_2(\overline{D}, W)(x, y)$$

*almost everywhere. Then $W$ is weakly isomorphic to $U_{T_2}$.*

**Proof.** It is straightforward to check that the graphon $U_{T_2}$ satisfies these identities.

The first two identities mean that $W$ is a regular CR-graphon with degree 2/3. By Theorem 6.2 we know that $W$ can be represented by a locally finite tree $T$ and so we can assume that $W = U_T$. The edge density 2/3 guarantees that the root $r$ of $T$ has one child $q$, but $q$ must have at least 2 children, i.e., $U_T$ is a connected graphon and $1 - U_T$ has at least 2 components. Let $v$ be any child of $q$, and let $\Omega' = \Omega \setminus \Omega_v$. We also know that $v$ has at least two children $v_1, v_2$. Let $x \in \Omega_{v_1}$ and $y \in \Omega_{v_2}$. By the definition of $U_T$, $U_T(x, z) = 0$ for $z \in \Omega_{v_2}$, $U_T(y, z) = 0$ for $z \in \Omega_{v_1}$ and $U_T(x, z) = U_T(y, z) = 1$ for $z \in \Omega'$. Hence

$$\begin{aligned} t_2(C, U_T)(x, y) &= \int_\Omega U_T(x, z)\big(1 - U_T(z, y)\big)\, dz = \int_{\Omega_{v_1}} U_T(x, z)\, dz \\ &= \int_\Omega U_T(x, z)\, dz - \mu\big(\Omega'\big) = t_1\big(K_2^\bullet, U_T\big)(x) - \mu\big(\Omega'\big) = \frac{2}{3} - \mu\big(\Omega'\big). \end{aligned} \tag{26}$$

Similarly,

$$\begin{aligned} t_2(D, U_T)(x, y) &= \int_{\Omega\times\Omega} U_T(x, z)\big(1 - U_T(z, y)\big)U_T(x, u)\big(1 - U_T(u, y)\big)\big(1 - U_T(z, u)\big)\, du\, dz \\ &= \int_{\Omega_{v_1}\times\Omega_{v_1}} U_T(x, z)U_T(x, u)\big(1 - U_T(z, u)\big)\, du\, dz \\ &= t_1(B, U_T)(x) - \int_{\Omega'\times\Omega'} 1 - U_T(z, u)\, du\, dz. \end{aligned}$$

Here

$$\int_{\Omega'\times\Omega'} 1 - U_T(z, u)\, du\, dz = \int_{\Omega\times\Omega'} 1 - U_T(z, u)\, du\, dz = \int_{\Omega'} 1 - t_1\big(K_2^\bullet, U_T\big)(u)\, du = \frac{1}{3}\mu\big(\Omega'\big),$$

and so

$$t_2(D, U_T)(x, y) = \frac{8}{45} - \frac{1}{3}\mu\big(\Omega'\big). \tag{27}$$

Using our conditions, we get from (26) and (27)

$$2\left(\frac{2}{3} - \mu\big(\Omega'\big)\right)^2 = 2t_2(C, U_T)(x, y)^2 = 5t_2(D, U_T)(x, y) = \frac{8}{9} - \frac{5}{3}\mu\big(\Omega'\big).$$

This simplifies to the equation $2\mu(\Omega')^2 = \mu(\Omega')$. Since $\mu(\Omega') \neq 0$ (as $q$ has at least two children), we get $\mu(\Omega') = 1/2$ and so $\mu(\Omega_v) = 1/2$. This is true for every child of $q$, and hence there are exactly two children, both with weight 1/2.

To finish, it is easy to check that the complement of the graphon $U_{T_v}$ satisfies the same identities as listed in the statement for either child $v$ of $q$. Iterating the argument, we get that $T$ is a complete binary tree. □

Using Lemma 3.4, we get:

**Corollary 6.16.** *The graphon $U_{T_2}$ is finitely forcible.*

## 7. Necessary conditions for finite forcing

### 7.1. Infinite rank

We define the *rank* of a graphon $W$ as its rank as a kernel operator. In other words, the rank of $W$ is the least nonnegative integer $r$ such that there are measurable functions $w_i : [0,1] \to \mathbb{R}$ and reals $\lambda_i$ $(i = 1, \dots, r)$ such that

$$W(x, y) = \sum_{k=1}^{r} \lambda_k w_k(x) w_k(y) \tag{28}$$

almost everywhere. If no such integer $r$ exists, then we say that $W$ has infinite rank.

**Theorem 7.1.** *If $W$ has finite rank, then for every finite list $F_1, \dots, F_m$ of simple graphs there is a stepfunction $U$ such that*

$$t(F_i, U) = t(F_i, W) \quad (i = 1, \dots, m).$$

**Proof.** We know that $W$ has a decomposition (28), where we may assume that the $\lambda_k$ are the eigenvalues and the $w_k$ are the corresponding eigenfunctions of $W$ as a kernel operator. In this case, all the $w_k$ are bounded and all the moments $M(\{w_1, \dots, w_r\}, k)$ are finite.

Fix a simple graph $F = (V, E)$. For a map $\varphi : E \to [r]$, $t \in [r]$, and $i \in V$, let $d_t(\varphi, i)$ denote the number of edges $e \in E$ incident with $i$ for which $\varphi(e) = t$, and set $\lambda_\varphi = \prod_{ij \in E} \lambda_{\varphi(ij)}$. Then

$$\begin{aligned} t(F, W) &= \int\limits_{[0,1]^V} \prod_{ij \in E} W(x_i, x_j)\, dx = \int\limits_{[0,1]^V} \prod_{ij \in E} \left( \sum_{k=1}^{r} \lambda_k w_k(x_i) w_k(x_j) \right) dx \\ &= \int\limits_{[0,1]^V} \sum_{\varphi \in [r]^E} \lambda_\varphi \prod_{ij \in E} w_{\varphi(ij)}(x_i) w_{\varphi(ij)}(x_j)\, dx \\ &= \int\limits_{[0,1]^V} \sum_{\varphi \in [r]^E} \lambda_\varphi \prod_{i \in V} \prod_{t \in [r]} w_t(x_i)^{d_t(\varphi, i)}\, dx = \sum_{\varphi \in [r]^E} \lambda_\varphi \prod_{i \in V} \int_0^1 \prod_{t \in [r]} w_t(y)^{d_t(\varphi, i)}\, dy \\ &= \sum_{\varphi \in [r]^E} \lambda_\varphi \prod_{i \in V} M\big(w, d(\varphi, i)\big). \end{aligned}$$

So if $(u_1, \dots, u_r)$ is another set of functions that satisfy

$$M\big(u, d(\varphi, i)\big) = M\big(w, d(\varphi, i)\big) \tag{29}$$

for every $1 \leqslant j \leqslant m$, $i \in V(F_j)$ and $\varphi : V(F_j) \to [r]$, then the function

$$U = \sum_{t=1}^{r} \lambda_t u_t(x) u_t(y)$$

satisfies $t(F_j, U) = t(F_j, W)$ for all $j = 1, \dots, m$. By Theorem 2.2 there is a system of functions $u$ satisfying (29) which are stepfunctions, and then $U$ is also a stepfunction. $\square$

**Corollary 7.2.** *Every finitely forcible graphon is either a stepfunction or has infinite rank.*

In view of Theorem 5.1, the following corollary of this theorem may be surprising:

**Corollary 7.3.** *Assume that $W \in \mathcal{W}_0$ can be expressed as a non-constant polynomial in $x$ and $y$. Then $W$ is not finitely forcible.*

### 7.2. Weak homogeneity

For every graph $F = (V, E)$, and node $i \in V$, let $F^i$ denote the 1-labeled quantum graph obtained by labeling $i$ by 1, and for every edge $ij \in E$, let $F^{ij}$ denote the 2-labeled quantum graph obtained from $F$ by deleting the edge $ij$, and labeling $i$ by 1 and $j$ by 2. Let $F^\dagger = \sum_{i \in V} F^i$ and $F^\ddagger = \frac{1}{2} \sum_{i,j:\ ij \in E} F^{ij}$ (each edge contributes two terms, since its endpoints can be labeled in two ways). We extend the operators $F \to F^\dagger$ and $F \to F^\ddagger$ linearly to all quantum graphs.

**Example 7.4.** Clearly $C_n^\ddagger = n P_n^{\bullet\bullet}$, where $P_n^{\bullet\bullet}$ denotes the path on $n$ nodes with its endpoints labeled. So

$$t_2\big(C_n^\ddagger, W\big) = n W^{\circ(n-1)}.$$

These operations were introduced by Razborov [19,20] in the proof of conjectures about the minimum number of triangles in simple graphs with given edge density. For us, their significance is in the following formulas.

We consider $\mathcal{W}$ as a Banach space with the $L_\infty$ norm. Let $U_t$, $0 \leqslant t \leqslant 1$ be a family of functions in $\mathcal{W}$. We say that $U_t$ is *differentiable* if for every $t \in [0, 1]$ there exists a function $\dot{U}_t \in \mathcal{W}$ such that

$$\left\| \frac{1}{s-t}(U_s - U_t) - \dot{U}_t \right\|_\infty \to 0 \quad \big(s \in [0,1],\ s \to t\big).$$

**Lemma 7.5.** *Let $U_t$, $0 \leqslant t \leqslant 1$ be a uniformly bounded differentiable family of functions in $\mathcal{W}$ and $F = (V, E)$, a simple graph. Then the function $t(F, U_t)$ is differentiable as a function of $t$, and*

$$\frac{d}{dt} t(F, U_t) = \big\langle \dot{U}_t, t_2\big(F^\ddagger, U_t\big)\big\rangle.$$

**Proof.** Suppose that $\|U_s\|_\infty \leqslant C$ for some real number $C$. Write

$$t(F, U_{t+h}) - t(F, U_t) = \big(t(F, U_t + h\dot{U}_t) - t(F, U_t)\big) + \big(t(F, U_{t+h}) - t(F, U_t + h\dot{U}_t)\big). \tag{30}$$

Here the first term is a polynomial in $h$:

$$t(F, U_t + h\dot{U}_t) - t(F, U_t) = h\big\langle \dot{U}_t, t_2\big(F^\ddagger, U_t\big)\big\rangle + O\big(h^2\big),$$

while the second term can be estimated by (6) and the definition of differentiation:

$$\big|t(F, U_{t+h}) - t(F, U_t + h\dot{U}_t)\big| \leqslant \big|E(F)\big| C^{|E(F)|-1} \|U_{t+h} - U_t - h\dot{U}_t\|_\infty = o(h).$$

So indeed

$$\frac{1}{h}\big(t(F, U_{t+h}) - t(F, U_t)\big) \to \big\langle \dot{U}_t, t_2\big(F^\ddagger, U_t\big)\big\rangle \quad (h \to 0). \qquad \square$$

We use this formula to derive a necessary condition for finite forcibility.

**Lemma 7.6.** *Suppose that $W \in \mathcal{W}$ is forced (in $\mathcal{W}$) by the simple graphs $F_1, \dots, F_m$. Also suppose that there exists an open ball $\mathcal{U} \subseteq \mathcal{W}$ about $W$ and a Lipschitz map $\Phi : \mathcal{U} \to \mathcal{W}$ such that for all $U \in \mathcal{U}$ the function $\Phi(U)$ satisfies $\langle \Phi(U), t_2(F_i^\ddagger, U)\rangle = 0$ ($i = 1, \dots, m$). Then $\langle \Phi(W), t_2(F^\ddagger, W)\rangle = 0$ for every simple graph $F$.*

**Proof.** By classical results on differential equations in Banach spaces (see e.g. [22]), there exists a $b > 0$ and a differentiable family $\{U_s\colon\ s \in [-b, b]\}$ of functions in $\mathcal{U}$ satisfying the differential equation

$$\dot{U}_s = \Phi(U_s), \qquad U_0 = W.$$

Lemma 7.5 shows that for every simple graph $F$,

$$\frac{d}{ds} t(F, U_s) = \big\langle \dot{U}_s, t_2\big(F^\ddagger, U_s\big)\big\rangle = \big\langle \Phi(U_s), t_2\big(F^\ddagger, U_s\big)\big\rangle.$$

In particular, we have

$$\frac{d}{ds} t(F_i, U_s) = \big\langle \Phi(U_s), t_2\big(F_i^\ddagger, U_s\big)\big\rangle = 0$$

for $i = 1, \dots, m$, and hence $t(F_i, U_s) = t(F_i, U_0) = t(F_i, W)$ for all $\in [0, c]$. Since the graphs $F_i$ force $W$, it follows that the $U_s$ are weakly isomorphic to $W$, and so $t(F, U_s) = t(F, W)$ for every $F$. But then $\langle \Phi(W), t_2(F^\ddagger, W)\rangle = \frac{d}{ds} t(F, U_s)|_{s=0} = 0$ as claimed. □

Let $\mathcal{L}(W)$ be the linear space generated by 2-variable functions $t_2(F^\ddagger, W)$ (modulo zero sets). Inequality (7) implies that $t_2(F^\ddagger, W) \in \mathcal{W}$ for all $F$. Due to the identity

$$t_2\big((F_1F_2)^\ddagger, W\big) = t(F_1, W) t_2\big(F_2^\ddagger, W\big) + t(F_2, W) t_2\big(F_1^\ddagger, W\big), \tag{31}$$

the space $\mathcal{L}(W)$ is generated by functions $t_2(F^\ddagger, W)$ where $F$ is connected.

**Proposition 7.7.** *Let $W$ be a finitely forcible graphon. Then $\mathcal{L}(W)$ has finite dimension if and only if $W$ is a stepfunction.*

**Proof.** If $W$ is a stepfunction, then every function $t_2(F^\ddagger, W)$ is a stepfunction with the same steps, and so $\mathcal{L}(W)$ is finite dimensional. Conversely, if $\mathcal{L}(W)$ is finite dimensional, then by Example 7.4, the functions $W^{\circ k} \in \mathcal{L}(W)$ are linearly dependent, and so $W$ satisfies a polynomial equation as an operator. This means that it has a finite number of different nonzero eigenvalues. Since every nonzero eigenvalue has finite multiplicity, $W$ has finite rank. By Corollary 7.2, $W$ is a stepfunction. □

**Remark 7.8.** Suppose that $\mathcal{L}(W)$ has finite dimension and the functions $t_2(F_1^\ddagger, W), \dots, t_2(F_k^\ddagger, W)$ generate it. Informally, this means that every infinitesimal change in $W$ that preserves $t(F_1, W), \dots, t(F_k, W)$, also preserves $t(F, W)$ for every $F$; we could say that $W$ is *infinitesimally finitely forcible*. Proposition 7.7 says that graphons that are both finitely forcible and infinitesimally finitely forcible are exactly the stepfunctions.

Our examples of finitely forcible non-stepfunctions (e.g., half-graphs) show that there are graphons which are finitely forcible but not infinitesimally finitely forcible. We don't know if the converse is true.

**Lemma 7.9.** *Suppose that $W \in \mathcal{W}$ is forced (in $\mathcal{W}$) by the simple graphs $F_1, \dots, F_m$. Then either $t_2(F_1^\ddagger, W), \dots, t_2(F_m^\ddagger, W)$ are linearly dependent, or they generate $\mathcal{L}(W)$.*

**Proof.** Suppose not, then there is a simple graph $F_{m+1}$ such that $t_2(F_1^\ddagger, W), \dots, t_2(F_m^\ddagger, W)$ and $t_2(F_{m+1}^\ddagger, W)$ are linearly independent. For $U \in \mathcal{W}$, set $h_k(U) = t_2(F_k^\ddagger, U)$. Let $\Phi(U)$ denote the component of $h_{m+1}(U)$ orthogonal to the subspace spanned by $h_1(U), \dots, h_m(U)$.

We need the following technical claim.

**Claim 7.10.** *There is an open ball $\mathcal{U}$ in $\mathcal{W}$ centered at the function $W$ such that $\Phi : \mathcal{W} \to \mathcal{W}$ is Lipschitz on $\mathcal{U}$.*

Inequality (7) implies that there is a neighborhood $\mathcal{U}$ of $W$ in $\mathcal{W}$ such that the functions $h_1(U), \dots, h_{m+1}(U)$ are linearly independent for $U \in \mathcal{U}$. We may assume that $U$ is an open ball such that $\|U\|_\infty \leqslant 2\|W\|_\infty$ for all $U \in \mathcal{U}$.

Let $(g_1(U), \dots, g_{m+1}(U))$ be the Gram–Schmidt orthogonalization of $(h_1(U), \dots, h_{m+1}(U))$, then $\Phi(U) = g_{m+1}(U)$. For a function $H \in \mathcal{W}$, let $\Psi_H : \mathcal{W} \to \mathcal{W}$ denote the orthogonal projection onto the subspace orthogonal to $H$. Consider the functions

$$g_{k,r}(U) = \Psi_{g_r(U)} \dots \Psi_{g_1(U)} h_k(U).$$

Then we have

$$g_{k,r+1}(U) = \Psi_{g_{r+1}(U)} g_{k,r}(U),$$

and

$$g_{k,0}(U) = h_k(U), \qquad g_{k+1,k}(U) = g_{k+1}(U).$$

We prove by induction on $k$ and $r$ $(r < k)$ that there is a constant $c_{k,r} > 0$ and an open ball $\mathcal{U}_{k,r}$ about $W$ such that if $\{U_s\colon 0 \leqslant s \leqslant 1\} \subseteq \mathcal{U}_{k,r}$ is a differentiable family, then

$$\left\| \frac{d}{ds} g_{k,r}(U_s) \right\|_\infty \leqslant c_{k,r} \left\| \frac{d}{ds} U_s \right\|_\infty.$$

This will imply that $\Phi = g_{m+1}$ is Lipschitz.

First, for the functions $g_{k,0}(U) = h_k(U) = t_2(F_k^\ddagger, U)$ this follows from inequality (7).

Let $k \geqslant 1$, $r < k$, and suppose that we know the existence of $c_{r+1,0}$ and of $c_{k+1,r}$. We prove that $c_{k+1,r+1}$ exists.

Set $G = g_{k+1,r}(U_s)$ and $H = g_{r+1}(U_s)$, then (denoting differentiation by dot)

$$\begin{aligned} \frac{d}{ds} \Psi_H G &= \frac{d}{ds} \left( G - \frac{\langle G, H \rangle}{\langle H, H \rangle} H \right) \\ &= \dot{G} - \frac{\langle \dot{G}, H \rangle}{\langle H, H \rangle} H - \frac{\langle G, \dot{H} \rangle}{\langle H, H \rangle} H + 2 \frac{\langle G, H \rangle \cdot \langle H, \dot{H} \rangle}{\langle H, H \rangle^2} + \frac{\langle G, H \rangle}{\langle H, H \rangle} \dot{H}. \end{aligned} \tag{32}$$

There is a constant $a > 0$ such that $\|G\|_\infty, \|H\|_\infty \leqslant a$ for all $s$. Furthermore, $\|\dot{G}\|_\infty \leqslant c_{k+1,r} \|\dot{U}_s\|_\infty$ and $\|\dot{H}\|_\infty \leqslant c_{r+1,0} \|\dot{U}_s\|_\infty$ by induction. Since

$$\left| \frac{d}{ds} \langle H, H \rangle \right| = 2 \left| \langle H, \dot{H} \rangle \right| \leqslant 2 \|H\|_\infty \|\dot{H}\|_\infty < b \|\dot{U}_s\|_\infty$$

for some constant $b > 0$, it follows that $\langle g_{r+1}(U), g_{r+1}(U) \rangle$ is a Lipschitz function of $U$ (as a real-valued function), and hence if $\mathcal{U}_{k+1,r+1}$ is small enough, then $\langle H, H \rangle \geqslant c > 0$ for all $U \in \mathcal{U}_{k+1,r+1}$. Hence (32) implies that $c_{k+1,r+1}$ exists, which proves the claim.

By Lemma 7.6, we have $\langle \Phi(W), \Phi(W) \rangle = \langle \Phi(W), t(F_{m+1}^\ddagger, W) \rangle = 0$, which is a contradiction. $\square$

In particular, it follows that the functions $t_2(F^\ddagger, W)$ are linearly dependent, which implies that finitely forcible graphons are in a sense "homogeneous".

**Corollary 7.11.** *Let $W \in \mathcal{W}$ be finitely forcible. Then there is a nonzero simple 2-labeled connected quantum graph $f$ such that $t_2(f, W) = 0$ almost everywhere.*

**Proof.** As remarked before, every finitely forcible graphon can be forced by connected graphs $F_1, \dots, F_m$. Then either the functions $t_2(F_1^\ddagger, W), \dots, t_2(F_m^\ddagger, W)$ are linearly dependent, or else they span $\mathcal{L}(W)$, and so for any simple connected graph $F_{m+1}$ they span $t_2(F^{\ddagger}_{m+1}, W)$. In either case we get a finite set $F_1, \dots, F_k$ of (distinct) connected graphs such that the functions $t_2(F^{\ddagger}_1, W), \dots, t_2(F^{\ddagger}_k, W)$ are linearly dependent. Suppose that $\sum_i a_i t_2(F^{\ddagger}_i, W) = 0$, then $t_2(f, W) = 0$ for $f = \sum_i F^{\ddagger}_i$.

It is clear that $f$ is a simple connected 2-labeled quantum graph. It is also clear that $f \neq 0$: From each constituent of $F^{\ddagger}_i$ we can reconstruct $F_i$ by connecting the labeled nodes by an edge and deleting the labels. Hence constituents coming from different $F^{\ddagger}_i$ are different and they cannot cancel each other. $\square$

The last corollary can be used to show that "most" graphons are not finitely forcible.

**Theorem 7.12.** *The set of finitely forcible graphons is of first category in $L_2([0,1]^2)$.*

**Proof.** We claim that for a fixed set $\{F_1, \dots, F_k\}$ of connected simple 2-labeled graphs, the set of graphons $W$ for which there is a nonzero quantum graph $f = \sum_{i=1}^k a_i F_i$ composed of these $F_i$ satisfying an equation $t_2(f, W) = 0$ is nowhere dense. Let us fix a $W$, we want to show that an arbitrary neighborhood of $W$ contains a graphon $W'$ such that $t_2(F_1, W'), \dots, t_2(F_k, W')$ are linearly independent. This will be enough, since $t_2$ is continuous and so there is an open set $\mathcal{U}$ in the neighborhood such that $t_2(F_1, U), \dots, t_2(F_k, U)$ are linearly independent for all $U \in \mathcal{U}$.

Lemma 5 of [9] implies that there are graphons $U_1, \dots, U_k$ such that the matrix $(t(F_i, U_j))_{i,j=1}^k$ is nonsingular. We may assume that $\|W\|_\infty, \|U_1\|_\infty, \dots, \|U_k\|_\infty \leqslant 1$. For $0 < \varepsilon < 1/k$, define $W^\varepsilon = (1 - k\varepsilon)W \oplus (\varepsilon)U_1 \oplus \dots \oplus (\varepsilon)U_k$ (so the components of $W^\varepsilon$ are $W, U_1, \dots, U_k$, scaled by $1 - k\varepsilon, \varepsilon, \dots, \varepsilon$).

First we show that $W^\varepsilon \to W$ in $L_2[0,1]^2$ if $\varepsilon \to 0$. Let $W_\varepsilon = (1 - k\varepsilon)W \oplus (k\varepsilon)0$. Then

$$\left\|W^\varepsilon - W_\varepsilon\right\|_2^2 = \varepsilon^2\left(\|U_1\|_2^2 + \dots + \|U_k\|_2^2\right) \longrightarrow 0 \quad (\varepsilon \to 0),$$

so it suffices to show that $\|W - W_\varepsilon\|_2 \to 0$. This is easy if $W$ is a stepfunction with interval steps, and it follows for general $W$ as these can be approximated by such stepfunctions in $L_2$.

Now suppose that $t_2(F_1, W^\varepsilon), \dots, t_2(F_k, W^\varepsilon)$ are linearly dependent, so that there are real numbers $a_i$ such that

$$\sum_{i=1}^k a_i t_2\left(F_i, W^\varepsilon\right)(x, y) = 0$$

for all $x, y \in [0,1]$. If we integrate only over the points in the interval $(1 - k\varepsilon + (j-1)\varepsilon, 1 - k\varepsilon + j\varepsilon)$, we get that

$$\sum_{i=1}^k a_i \varepsilon^{|V(F_i)|} t(F_i, U_j) = 0 \quad (j = 1, \dots, k)$$

(here we use that every connected component of each $F_i$ contains a labeled node). But this contradicts the nonsingularity of the matrix $(t(F_i, U_j))_{i,j=1}^k$. $\square$

If $W$ is only finitely forced in $\mathcal{W}_0$, then we get a weaker condition:

**Lemma 7.13.** *Suppose that $W \in \mathcal{W}_0$ is forced in $\mathcal{W}_0$ by the simple graphs $F_1, \dots, F_m$. Also suppose that there exists a neighborhood $\mathcal{U} \subseteq L_\infty[0,1]$ of the all-1 function and a Lipschitz map $\Phi : \mathcal{U} \to L_\infty[0,1]$ such that for all $U \in \mathcal{U}$, $\langle \Phi(U), t_1(F^{\dagger}_i, U)\rangle = 0$ $(i = 1, \dots, m)$. Then $\langle \Phi(W), t_1(F^{\dagger}, W)\rangle = 0$ for every simple graph $F$.*

**Proof.** Since we are not using this lemma, we only sketch the proof: instead of changing the values of the function $W$, we change the variable: we consider $U_s(x, y) = W(\phi_s(x), \phi_s(y))$, where $\phi_s : [0,1] \to$

$[0, 1]$ is a smooth monotone Lipschitz function with $\phi_s(0) = 0$ and $\phi_s(1) = 1$. Then Lemma 7.5 is replaced by the formula

$$\frac{1}{ds}t(F, U_s) = \left\langle \frac{1}{ds}\phi_s, t_1\left(F^\dagger, U_s\right)\right\rangle.$$

The proof is concluded by a similar argument solving a differential equation as the proof of Lemma 7.6. □

**Corollary 7.14.** *Let $W \in \mathcal{W}_0$ be finitely forcible in $\mathcal{W}_0$. Then there is a simple connected nonzero 1-labeled quantum graph $f \neq 0$ such that $t_1(f, W) = 0$ almost everywhere.*

## 8. Open problems and further directions

It does not seem easy to characterize finitely forcible functions. Let us offer a few conjectures. The next question might be easy but the examples and theorems in the present paper don't answer it.

**Question 1.** Is there a non-constant continuous (or smooth) function on $[0, 1]^2$ which is finitely forcible? (As we have seen, the simplest candidates, namely polynomial functions, don't work.)

We believe that in Theorem 5.1, the assumption that $p$ is monotone can be omitted:

**Conjecture 2.** *For every symmetric 2-variable polynomial $p$, the function $\mathbf{1}_{p(x,y)\geqslant 0}$ is finitely forcible in $\mathcal{W}$. (Using ad hoc tricks, the proof given in Section 5.1 can be extended to some non-monotone polynomials, for example, to $(1/2 - x - y)(3/2 - x - y)$.)*

We can try to generalize the results of Section 5.1 to more variables. Here is an interesting special case:

**Question 3.** Is the following graphon finitely forcible: the underlying probability space is the uniform distribution on the surface of the unit sphere $S^2$, and $W(x, y) = 1$ if $x$ and $y$ are closer than 90°, and $W(x, y) = 0$ otherwise?

It is not clear whether the two notions of forcibility we have considered are really different.

**Question 4.** If a function $W \in \mathcal{W}_0$ is finitely forcible in $\mathcal{W}_0$, is it also finitely forcible in $\mathcal{W}$?

We don't know too much about algebraic operations which generate new forcible functions. For example, it is unreasonable to expect that the sum of two forcible functions is forcible, since the sum depends on the concrete representation of the graphons (not just on their weak isomorphism types). However the next question is natural.

**Question 5.** Is the tensor product $U \otimes W$ of two finitely forcible graphons $U$ and $W$ forcible?

Corollary 6.16 suggests the following problem:

**Question 6.** For which finite graphs $G$ is the infinite lexicographic power of $G$ finitely forcible?

Our motivation for the study of finitely forcible graphons was to understand the structure of extremal graphs. This would be fully justified by the following conjecture:

**Conjecture 7.** *If a finite set of constraints of the form $t(F_i, W) = a_i$ $(i = 1, \ldots, k)$ is satisfied by some graphon, then it is satisfied by a finitely forcible graphon. This conjecture would imply the (imprecise) fact that* every extremal graph problem has a finitely forcible solution.

Let us state the problem mentioned in Remark 7.8:

**Question 8.** Is every infinitesimally finitely forcible graphon also finitely forcible (and hence, a step-function)?

The topology of the set $T(W)$ introduced in Section 2.4 gives rise to some interesting problems. It is easy to see that $R(W) \cap T(W)$ is dense in $T(W)$ (in the topology of $L_1[0,1]$), and if two graphons $W$ and $U$ are weakly isomorphic then $T(U)$ is homeomorphic to $T(W)$.

Surprisingly, in each of the finitely forcible examples of this paper $T(W)$ is a finite dimensional compact topological space. For positive supports of monotone polynomials, $T(W)$ is homeomorphic with the interval $[0,1]$. The topology of the regular CR-graphon corresponding to the binary tree is the Cantor set $\{0,1\}^\infty$. The examples constructed in Section 6.4 correspond to the one-point compactification of the natural numbers.

This topological space was introduced and studied in [18], where it was proved that if $t(F,W) = 0$ for some signed bipartite graph $F$, then $T(W)$ is finite dimensional and compact. The following conjectures would lead to the same conclusion from a different assumption.

**Conjecture 9.** *If $W$ is finitely forcible in $\mathcal{W}_0$ then $T(W)$ is a compact space.* (*We can't even prove that $T(W)$ is locally compact.*)

**Conjecture 10.** *If $W$ is finitely forcible then $T(W)$ is finite dimensional.* (*We intentionally do not specify which notion of dimension is meant here—a result concerning any variant would be interesting.*) *Note that Corollary* 7.2 *implies that the linear hull of $T(W)$ is infinite dimensional unless $T(W)$ is a finite set.*

*In our examples $T(W)$ is either* 0*-dimensional of* 1*-dimensional. This is probably due to the fact that we have only found very simple examples.*

**Question 11.** Is there a finitely forcible graphon $W$ such that $T(W)$ is homeomorphic with $[0,1]^2$? (A positive answer would follow from a positive answer to Question 5.)

One can also consider a more direct notion of dimension. We define the *dimension of the graphon* $W$ as the infimum of all $c > 0$ such that for every $\varepsilon > 0$ there is a stepfunction $W_\varepsilon$ with $O((1/\varepsilon)^c)$ steps such that $\|W - W_\varepsilon\|_\square \leqslant \varepsilon$. It was shown in [16] that the dimensions $W$ and $T(W \circ W)$ are related.

The dimension of $W$ can be described in terms of the number of classes in weak Szemerédi partitions (introduced by Freeze and Kannan [11]). So a positive answer to Conjectures 7 and 10 would imply that extremal graph problems have solution with efficient (polynomial-size) weak Szemerédi partitions. This could explain (in a weak sense) why Szemerédi partitions are so important in extremal graph theory.

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