JOURNAL OF ALGEBRA **129**, 439–445 (1990)

A 2-Categorical Pasting Theorem

A. J. POWER*

Department of Pure Mathematics, University of Sydney, NSW 2006, Australia

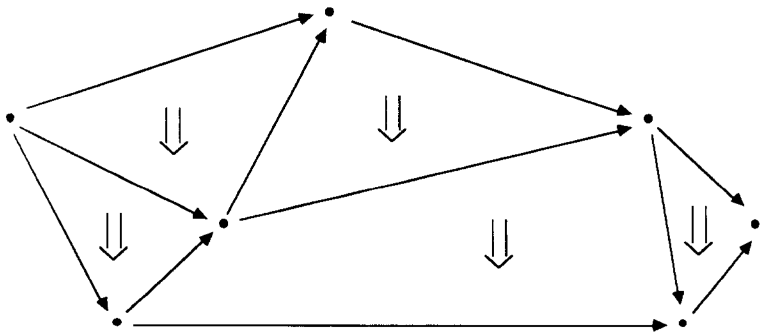
Communicated by Saunders MacLane

Received April 27, 1988

Pasting is the fundamental operation within a 2-category. However, until recently, it had never been proved that the operation of pasting is well defined. In fact, a 2-categorical pasting theorem had not even been properly formulated. Herein, we state and prove such a result. This brings together two strands of the current work of the Sydney Category Theory school: taking the general theory of n-categories, and applying its ideas to the particular case of 2-categories, with heavy use of the techniques of Graph Theory. This provides a more solid foundation for the development of 2-categories. © 1990 Academic Press, Inc.

1. INTRODUCTION

The fundamental operation within a 2-category is that of pasting, as introduced by Bénabou in [1]. The idea is as follows: we draw a picture such as

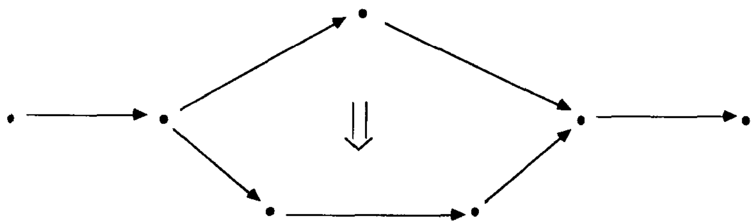


Given a 2-category $\mathscr{A}$, each vertex is labelled by a 0-cell in $\mathscr{A}$, each arc is labelled by a 1-cell in $\mathscr{A}$, and each face is labelled by a 2-cell, such that

* The author is funded by the Australian National Research Fellowship Scheme.

0021-8693/90 $3.00

sources and targets are preserved. Then, the above "pasting diagram" is meant to indicate a vertical composite of horizontal composites of 2-cells of the form



There is usually a choice in the order in which the composites are taken. Kelly and Street asserted, in their careful introduction to the theory of 2-categories [5], that the result is independent of the choice of the order. That assertion is correct, but the authors soon realised that they had neither a proof nor a precise statement of the fact. Their assertion, although fundamental to the foundation of the theory of 2-categories, remained not only unproved, but without a precise formulation, for many years.

Since the publication of [5], there have been several attempts to produce a pasting theorem, including unpublished partial results by Greg Bird, Bob Walters, and probably others. In 1987, inspired by Street's work on n-categories, Michael Johnson included an n-categorical pasting theorem in his thesis [4]. However, the restriction of that work to 2-categories does not yield a theorem and proof with the flavour of that adumbrated in [5].

Meanwhile, Max Kelly, other colleagues, and I have been writing a series of articles on "two-dimensional universal algebra," for which the notion of 2-category is fundamental. So, the serious deficiency in the foundation of the theory of 2-categories, plus the work of Johnson and Street, has inspired this paper, which may be seen both as part of the series with Kelly (see [2]) and as part of the n-categorical work of Street and others (see [6]).

The techniques of Graph Theory are fundamental here. In particular, the notion of "plane graph" or "graph in the plane" is central. To the best of my knowledge, this is a first substantial connection between Graph Theory and Category Theory. This work is also related to current work by Street on parity complexes [7], but it is not yet clear precisely what is the nature of the relationship.

In Section 2 herein, we produce the necessary graph-theoretic results; in Section 3 we state and prove the theorem. For the 2-categorical notation used, the reader is referred to [5]; the graph-theoretic notation is that of [3].

2. Graph-Theoretic Preliminaries

2.1. For the purposes of this paper, a *graph* is a (non-empty) connected, finite, directed graph. A *path* is an alternating sequence of vertices and arcs $v_0 a_1 v_1 \cdots v_n$ such that a_i has endpoints v_{i-1} and v_i and such that all of the vertices v_i are distinct. A *directed path* is a path for which each a_i has head v_i.

2.2. A *plane graph* is a graph together with a drawing of the graph in the plane so that its arcs touch only at their ends; more precisely, it is a graph together with a topological embedding of its geometric realization in the plane. We assume throughout the paper that the plane is oriented with the usual orientation.

2.3. A plane graph G partitions the rest of the plane into a finite number of connected regions; these regions are called the *faces* of G. Each plane graph has exactly one unbounded face, called the *exterior face*; the other faces are called *interior faces.* A face F is said to be *incident* with the vertices and arcs in its boundary. We regard the boundary of a face F as an alternating sequence of vertices and arcs $v_0 a_1 v_1 \cdots v_n$ such that $v_0 = v_n$, the endpoints of a_i are v_{i-1} and v_i, and if one moves from v_0 via a_1 to v_1, et cetera, one always has F on one's right hand side; an arc in the boundary appears in the sequence either once or twice, dependent upon whether F is on both sides of it or only on one side of it: some pictures appear in [3]; one such picture is given by two concentric circles joined by part of a radius.

2.4. Given a path γ from u to v, the corresponding path from v to u is denoted by γ^*.

2.5. A *plane graph G with source s and sink t* is a plane graph G with distinct vertices s and t in the exterior face of G, such that for each vertex v, there are directed paths from s to v and from v to t.

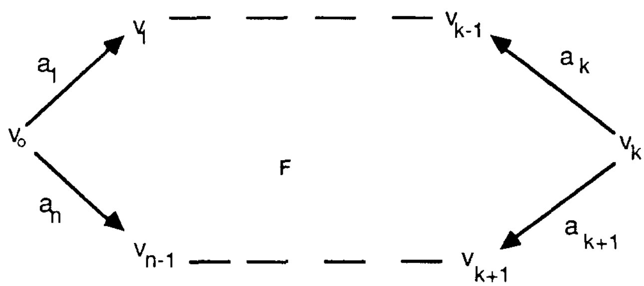
2.6. Proposition. *Let G be a plane graph with source s and sink t. Then the following are equivalent*:

(i) *for each face F, there exist distinct vertices $s(F)$ and $t(F)$ and directed paths $\sigma(F)$ and $\tau(F)$ from $s(F)$ to $t(F)$ such that the boundary of F is $\sigma(F)(\tau(F))^*$.*

(ii) *G contains no directed cycle.*

Proof. ($\Rightarrow$) Suppose otherwise. Let γ be a directed cycle in G such that γ encloses a minimum number of faces. Suppose γ encloses the region R. Without loss of generality, assume γ has the clockwise orientation. By the hypothesis, R must consist of more than one face. Let a_0, from u to v, be an arc in γ, and let F be the face inside R that is incident with a_0. Since G has source s, there is a directed path α from s to $s(F)$. Since s is in the exterior face of G and $s(F)$ is in $\bar{R}$, it follows from planarity that α touches γ. Let w be the last vertex on α that also lies on γ. Then, we have a path that starts at v, follows γ to w, then follows α to $s(F)$: call this path β. Since G has sink t, there is a path δ from $t(F)$ to t. It follows, by planarity again, that δ touches β. So, if we start at $t(F)$, follow δ until it first touches β, then follow β to $s(F)$, then follow $\tau(F)$ to $t(F)$, we have a directed closed curve that does not leave $\bar{R}$ but does not contain the arc a_0. Hence, we can construct a directed cycle that does not leave $\bar{R}$ but does not enclose F, a contradiction to the minimality of γ.

($\Leftarrow$) Let F be a face of G, with boundary $v_0 a_1 v_1 \cdots v_n$. Since s and t are distinct and G is connected, it follows that $n \neq 0$. Since G has no directed cycle, we may assume that a_1 and a_n both have tail v_0. Suppose that there exists $1 < k < n-1$ such that a_k and a_{k+1} both have tail v_k:



Now, suppose $v_0 = v_k$. There must be directed paths from v_1 and v_{n-1} to t, say γ and γ', respectively. Since F is a face and G is planar and $v_0 = v_k$, it follows that either γ or γ' must touch v_0, yielding a directed cycle at v_0, a contradiction. Hence, $v_0 \neq v_k$. So without loss of generality, $v_k \neq s$.

There exist directed paths β from v_{k-1} to t, β' from v_{k+1} to t, α from s to v_0, and α' from s to v_k. Note that there can be no arc into s, because s is a source and G has no directed cycle; so s cannot lie on β or β'. Hence, s does not lie on the closed curve $a_k\beta + a_{k+1}\beta'$; so by planarity, either α or α' must cut $a_k\beta + a_{k+1}\beta'$. Moreover, if α' touched β or β', we would have a directed cycle at v_k given by parts of α' and of either $a_k\beta$ or $a_{k+1}\beta'$, a contradiction. Hence, α' cannot touch β or β'. So, without loss of generality, assume α touches β. Thus, we have a directed path δ from v_{k-1} to v_0. There can be no arc out of t, so t cannot lie on δ. Moreover, if v_1 was on δ, we would have a directed cycle at v_0 given by a_1 and part of δ, a contradiction. So v_1, and similarly v_{n-1}, cannot lie on δ.

Let γ be a directed path from v_1 to t, and let γ' be a directed path from

v_{n-1} to t. By planarity again, either γ or γ' must touch δ. If γ touches δ, then starting at v_1, following γ until it first touches δ, then following δ to v_0, then following a_1 to v_1, yields a directed cycle at v_1, a contradiction. Similarly, if γ' touches δ, we would have a directed cycle at v_{n-1}, also a contradiction. Hence, no such k exists.

It follows that there exists $0 < r < n$ such that a_i has head v_i if and only if $1 \leqslant i \leqslant r$. Since G has no directed cycle, it follows that $v_0 a_1 \cdots v_r$ and $v_n a_n v_{n-1} \cdots v_r$ are directed paths that comprise the boundary of F.

2.7. COROLLARY. *Let G be a plane graph with source s and sink t such that for every interior face F, there exist distinct vertices $s(F)$ and $t(F)$ and directed paths $\sigma(F)$ and $\tau(F)$ from $s(F)$ to $t(F)$ such that the boundary of F is $\sigma(F)(\tau(F))^*$. Then*

(1) *the exterior face of G also satisfies the above condition, and*

(2) *for every interior face F, the directed paths $\sigma(F)$ and $\tau(F)$ touch only at $s(F)$ and $t(F)$.*

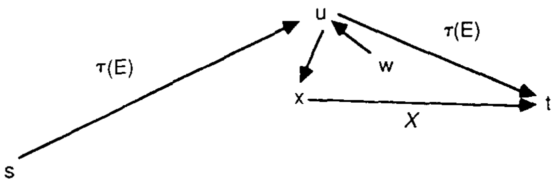
Proof. (1) In the proof of (i) $\Rightarrow$ (ii) in Proposition 2.6, condition (i) was only invoked for interior faces.

(2) The vertices s and t are both on the exterior face of G, and there are directed paths from s to $s(F)$ and from $t(F)$ to t. Since G is planar and has no directed cycle, the result follows.

2.8. *Notation.* Given G as above, for each interior face F of G, we call $\sigma(F)$ the *domain* of F, and we call $\tau(F)$ the *codomain* of F. Since the plane is oriented, we may assume that s lies to the left of the page and that t lies to the right. If E denotes the exterior face of G, then $\tau(E)$ is the top path from s to t, and dually, $\sigma(E)$ is the bottom path from s to t. Observe that we do not define a domain or codomain of the exterior face: what could best be described as the domain of G is $\tau(E)$, not $\sigma(E)$. Let a_0, from u to v, be an arc in $\tau(E)$. Then, order the arcs incident with u by calling a_0 the zeroth arc at u, and proceeding clockwise.

2.9. LEMMA. *Let a_0, from u to v, lie on $\tau(E)$. Then, in the order of arcs incident with u, for each $n > 0$, we do not have a_n an arc with head u and a_{n+k} an arc with tail u.*

Proof. Suppose otherwise. Let a_n be from w to u; let a_{n+k} be from u to x. There is a directed path χ from x to t. Since $\tau(E)$ is the top path from s to t, and G has no directed cycle, it follows that w lies inside the region bounded by a_{n+k}, χ, and $\tau(E)$, but s lies outside it.



So, by planarity, any directed path from s to w yields a directed cycle at u, a contradiction.

2.10. PROPOSITION. *If G has at least two faces, there exists an interior face F such that $\sigma(F)$ lies entirely on $\tau(E)$.*

Proof. Start at s and proceed along $\tau(E)$.

Either s is of the form $s(F)$ for some interior F, or there is a unique arc at s. In the latter case, $G-s$ satisfies all hypotheses of G, so replace G by $G-s$ and continue. In the former case, suppose $\sigma(F)$ and $\tau(E)$ agree until the vertex v. By Lemma 2.9, either $v=t(F)$, in which case we are done, or $v=s(F')$ for some F' with boundary commencing on $\tau(E)$. Since $v \neq s$, F' must be an interior face. So, continue. Ultimately, we must have a face as desired, because there is no arc out of t.

3. THE PASTING THEOREM

3.1. A *pasting scheme* is a plane graph G with source and sink such that for every interior face F, there exist distinct vertices $s(F)$ and $t(F)$ and directed paths $\sigma(F)$ and $\tau(F)$ from $s(F)$ to $t(F)$ such that the boundary of F is $\sigma(F)(\tau(F))^*$.

Equivalently, by Proposition 2.6 and Corollary 2.7, a pasting scheme is a plane graph G with source and sink such that G has no directed cycle.

3.2. A *labelling* of a pasting scheme G in a 2-category $\mathscr{A}$ is a computad morphism from G to $\mathscr{A}$, i.e., for each vertex, arc, and face in G other than the exterior face E, the assignment of a k-cell in $\mathscr{A}$, for $k=0, 1, 2$ respectively, preserving domains and codomains.

3.3. THEOREM. *Every labelling of a pasting scheme has a unique composite.*

Proof. Induction on the number of faces of G.

If G has one or two faces, the result is trivial.

If G has $n+1$ faces, the existence of a composite follows from Proposition 2.10 and the inductive hypothesis.

For unicity, suppose that composites are obtained in two ways; i.e., there are at least two ways of passing from $\tau(E)$ to $\sigma(E)$ by passing through one interior face at a time. Either both procedures commence with the same face, in which case we are done by induction; or the procedures start with different faces, F and F' say. In the latter case, $\sigma(F)$ and $\sigma(F')$ must be disjoint except at $s(F)$ or $t(F)$. Using that fact, it is routine to apply the axioms for a 2-category and the inductive hypothesis to conclude unicity.

ACKNOWLEDGMENTS

I thank the Sydney Category Theory school in general and Ross Street in particular for advice and encouragement. I also thank the referee for helpful suggestions and advice.

REFERENCES

1. J. BÉNABOU, Introduction to bicategories, *in* "Lecture Notes in Mathematics, Vol. 47," pp. 1–77, Springer-Verlag, New York/Berlin, 1967.
2. R. BLACKWELL, G. M. KELLY, AND A. J. POWER, Two-dimensional monad theory, *J. Pure Appl. Algebra* **59** (1989), 1–41.
3. J. A. BONDY AND U. S. R. MURTY, "Graph Theory with Applications," Macmillan Co., New York, 1976.
4. M. JOHNSON, "Pasting Diagrams in *n*-Categories with Applications to Coherence Theorems and Categories of Paths," Ph.D. thesis, University of Sydney, 1987.
5. G. M. KELLY AND R. H. STREET, Review of the elements of 2-categories, *in* "Lecture Notes in Mathematics, Vol. 420," pp. 75–103, Springer-Verlag, New York/Berlin, 1974.
6. R. H. STREET, The algebra of oriented simplexes, *J. Pure Appl. Algebra* **49** (1987), 283–336.
7. R. H. STREET, "Parity Complexes," Macquarie Mathematics Report No. 88-0015, 1988.