MATHEMATICS

INTEGRALS INVOLVING APPELL'S FUNCTION F_4

BY

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1. In this note we have evaluated a number of infinite integrals involving Appell's function F_4 with the help of a theorem recently given by SHARMA [2].

Throughout this note we use the following notations:

(1) $$k_\mu\{f(x); p\} = \int_0^\infty (px)^{\frac{1}{2}} K_\mu(px)\, f(x)\, dx$$

(2) $$H_\nu\{f(x); y\} = \int_0^\infty (xy)^{\frac{1}{2}} J_\nu(xy)\, f(x)\, dx.$$

2. The following results will be required.

If $f(x)$ and $H_\nu\{f(x); y\}$ both belong to $L(0, \infty)$, and if $R(\lambda \pm \mu + \nu \pm \sigma) > 0$, $R(\frac{1}{2} + \nu) > 0$, $R(p+a) > 0$, then [(2), p. 112]

(3) $$\left\{\begin{aligned} & k_\mu\{x^{\lambda-2} K_\sigma(ax)\, f(x); p\} = \sum_{\sigma, -\sigma} \frac{2^{\lambda-3} a^\sigma\, \Gamma(-\sigma)}{p^{\lambda+\nu+\sigma-\frac{1}{2}}\, \Gamma(1+\nu)} \times \\ & \quad \times \Gamma\{\tfrac{1}{2}(\lambda+\sigma+\nu \pm \mu)\} \int_0^\infty y^{\nu+\frac{1}{2}} H_\nu\{f(x); y\} \times \\ & \quad \times F_4\left[\tfrac{1}{2}(\lambda+\sigma+\nu-\mu), \tfrac{1}{2}(\lambda+\sigma+\nu+\mu); 1+\nu, 1+\sigma; -\frac{y^2}{p^2}, \frac{a^2}{p^2}\right] dy. \end{aligned}\right.$$

If $R(\lambda \pm \nu \pm \varrho - 2\alpha_t) > 0$, $t = 1, \ldots, m$; $R(p+c) > 0$, $2(m+n) \geqslant l+q$, then [(3), p. 73]

(4) $$\left\{\begin{aligned} & \int_0^\infty x^{\lambda-1} K_\nu(px)\, K_\varrho(cx)\, G_{l,q}^{m,n}\left[\frac{4}{x^2}\,\middle|\, \begin{matrix}\alpha_1, \ldots, \alpha_l \\ \beta_1, \ldots, \beta_q\end{matrix}\right] dx \\ & = \sum_{\nu, -\nu} \sum_{r=0}^\infty \frac{2^{\lambda-3}\, \pi\, p^{\nu+2r}}{\lfloor r\, \sin(-\nu\pi)\, \Gamma(\nu+1+r)\, c^{\lambda+\nu+2r}} \times \\ & \quad \times G_{l,q+2}^{m+2,n}\left[c^2 \,\middle|\, \begin{matrix} \alpha_1, \ldots, \alpha_l \\ \frac{\lambda+\nu+\varrho}{2}+r, \frac{\lambda+\nu-\varrho}{2}+r, \beta_1, \ldots, \beta_q \end{matrix}\right]. \end{aligned}\right.$$

Also [(3), p. 74]

$$(5)\quad \begin{cases} \displaystyle\sum_{\nu,-\nu}\sum_{r=0}^{\infty} \frac{\pi^{\frac{1}{2}}\operatorname{cosec}(-2\nu\pi)}{\lfloor r\, \Gamma(2\nu+1+r)} G_{p,q+2}^{m+2,n}\left[z\,\middle|\,\begin{matrix}\alpha_1,\ldots,\alpha_p\\ a+\nu+r,\, b+\nu+r,\, \beta_1,\ldots,\beta_q\end{matrix}\right] \\ \displaystyle = 2^{1-a-b}\, G_{p+2,q+4}^{m+4,n}\left[4z\,\middle|\,\begin{matrix}\alpha_1,\ldots,\alpha_p,\frac{a+b}{2},\frac{a+b+1}{2}\\ a\pm\nu,\, b\pm\nu,\, \beta_1,\ldots,\beta_q\end{matrix}\right]. \end{cases}$$

3. The main result to be proved is

$$(6)\quad \begin{cases} \displaystyle\int_0^\infty y^{\varrho+\nu} G_{l,q}^{m,n}\left[p^2y^2\,\middle|\,\begin{matrix}\alpha_1,\ldots,\alpha_l\\ \beta_1,\ldots,\beta_q\end{matrix}\right] F_4\left[\alpha,\beta;1+\nu,1+\sigma;-\frac{y^2}{b^2},\frac{a^2}{b^2}\right]dy \\ \displaystyle = \frac{b^{\nu+\varrho+1}\,\Gamma(1+\nu)}{2\Gamma(\alpha)\,\Gamma(\beta)}\sum_{r=0}^{\infty}\frac{(a/b)^{2r}}{\lfloor r\,(\sigma+1)_r}\times \\ \displaystyle\times\, G_{l+2,q+2}^{m+2,n+1}\left[b^2p^2\,\middle|\,\begin{matrix}\frac{1-\varrho-\nu}{2},\alpha_1,\ldots,\alpha_l,\frac{1-\varrho+\nu}{2}\\ \alpha-\frac{\nu+\varrho+1}{2}+r,\beta_1-\frac{\nu+\varrho+1}{2}+r,\beta_1,\ldots,\beta_q\end{matrix}\right] \end{cases}$$

for $R(b)>0$, $0\leqslant n\leqslant l$, $0\leqslant m\leqslant q$, $2(m+n)>l+q$, $|\arg p|<(m+n-\frac{1}{2}l-\frac{1}{2}q)\pi/2$, $-2\min b_j-1<\varrho+\nu<(2\alpha+1-2\max a_i)$ or $(2\beta+1-2\max a_i)$, $i=1,\ldots,n$, $j=1,\ldots,m$.

Proof. If we start with

$$f(x)=2^\varrho x^{-\varrho-\frac{1}{2}}\, G_{l+2,q}^{m,n+1}\left[\frac{4d^2}{x^2}\,\middle|\,\begin{matrix}\frac{1-\varrho-\nu}{2},\alpha_1,\ldots,\alpha_l,\frac{1-\varrho+\nu}{2}\\ \beta_1,\ldots,\beta_q\end{matrix}\right],$$

then [(1), p. 91, eq. 20]

$$H_\nu\{f(x);y\}=y^{\varrho-\frac{1}{2}}\, G_{l,q}^{m,n}\left[d^2y^2\,\middle|\,\begin{matrix}\alpha_1,\ldots,\alpha_l\\ \beta_1,\ldots,\beta_q\end{matrix}\right].$$

Therefore the result (3) gives us

$$\sum_{\sigma,-\sigma}\frac{2^{\lambda-\varrho-3}\,a^\sigma\,\Gamma(-\sigma)\,\Gamma\{\frac{1}{2}(\lambda+\sigma+\nu\pm\mu)\}}{p^{\lambda+\nu+\sigma}\,\Gamma(1+\nu)}\times\int_0^\infty y^{\varrho+\nu}\, G_{l,q}^{m,n}\left[d^2y^2\,\middle|\,\begin{matrix}\alpha_1,\ldots,\alpha_l\\ \beta_1,\ldots,\beta_q\end{matrix}\right]\times$$

$$\times\, F_4\left[\tfrac{1}{2}(\lambda+\sigma+\nu-\mu),\tfrac{1}{2}(\lambda+\sigma+\nu+\mu);1+\nu,1+\sigma;-\frac{y^2}{p^2},\frac{a^2}{p^2}\right]dy=$$

$$=\int_0^\infty x^{\lambda-\varrho-2}\,K_\sigma(ax)\,K_\mu(px)\,G_{l+2,q}^{m,n+1}\left[\frac{4d^2}{x^2}\,\middle|\,\begin{matrix}\frac{1-\varrho-\nu}{2},\alpha_1,\ldots,\alpha_l,\frac{1-\varrho+\nu}{2}\\ \beta_1,\ldots,\beta_q\end{matrix}\right]dx.$$

Evaluating the integral on the right by (4), replacing d by p, p by b, and on substituting α for $\frac{1}{2}(\lambda+\sigma+\nu-\mu)$, β for $\frac{1}{2}(\lambda+\sigma+\nu+\mu)$ we arrive at the result (6).

4. Particular cases: Many interesting particular cases of the result (6) are given below. These particular cases thus gives us the various integral transforms of the Appell's function F_4.

(i) Taking $m=1$, $q=2$, $n=l=0$, $\beta_1=\frac{\mu+\delta}{2}$, $\beta_2=\frac{\delta-\mu}{2}$, we obtain

$$(7)\quad \left\{\begin{aligned} &\int_0^\infty y^{\varrho+\nu+\delta} J_\mu(py)\, F_4\left[\alpha,\beta;\,1+\nu,\,1+\sigma;\,-\frac{y^2}{b^2},\frac{a^2}{b^2}\right]dy = \\ &= \frac{p^{-\delta}\, b^{\nu+\varrho+1}\,\Gamma(1+\nu)}{2^{1-\delta}\,\Gamma(\alpha)\,\Gamma(\beta)} \sum_{r=0}^{\infty} \frac{(a/b)^{2r}}{\lfloor r\,(\sigma+1)_r} \times \\ &\times G_{2,4}^{3,1}\left[\frac{b^2p^2}{4}\middle|\begin{matrix}\frac{1-\varrho-\nu}{2},\ \frac{1-\varrho+\nu}{2}\\ \alpha-\frac{\nu+\varrho+1}{2}+r,\beta-\frac{\nu+\varrho+1}{2}+r,\frac{\delta+\mu}{2},\frac{\delta-\mu}{2}\end{matrix}\right],\end{aligned}\right.$$

for $p>0$, $R(b)>0$, $R(\nu+1)>0$, $R(\varrho+\nu+\mu+\delta+1)>0$, $R(\varrho+\nu+\delta-2\alpha-\frac{1}{2})<$ <0, $R(\varrho+\nu+\delta-2\beta-\frac{1}{2})<0$.

Taking $\varrho=1$, $\delta=0$, $\mu=\nu$ in (7), we get

$$(8)\quad \left\{\begin{aligned} &\int_0^\infty y^{\nu+1} J_\nu(py)\, F_4\left[\alpha,\beta;\,1+\nu,\,1+\sigma;\,-\frac{y^2}{b^2},\frac{a^2}{b^2}\right]dy = \\ &= \frac{b^{\alpha+\beta}\, p^{\alpha+\beta-\nu-\sigma-2}\,\Gamma(\nu+1)\,\Gamma(\sigma+1)}{2^{\alpha+\beta-\nu-\sigma-2}\, a^\sigma\,\Gamma(\alpha)\,\Gamma(\beta)} K_{\alpha-\beta}(bp)\, I_\sigma(ap),\end{aligned}\right.$$

for $p>0$, $R(b)>0$, $R(\nu+1)>0$, $R(\frac{1}{2}+\nu-2\alpha)<0$, $R(\frac{1}{2}+\nu-2\beta)<0$.

(8) is a well-known result due to W. N. Bailey [(2), p. 110, eq. 15].

(ii) Taking $m=q=2$, $n=l=0$, $\beta_1=\frac{\delta+\mu}{2}$, $\beta_2=\frac{\delta-\mu}{2}$, we get

$$(9)\quad \left\{\begin{aligned} &\int_0^\infty y^{\varrho+\nu+\delta} K_\mu(py)\, F_4\left[\alpha,\beta;\,1+\nu,\,1+\sigma;\,-\frac{y^2}{b^2},\frac{a^2}{b^2}\right]dy = \\ &= \frac{b^{\nu+\varrho+1}\,\Gamma(1+\nu)}{2^{3-\delta}\,p^\delta\,\Gamma(\alpha)\,\Gamma(\beta)} \sum_{r=0}^{\infty} \frac{(a/b)^{2r}}{\lfloor r\,(\sigma+1)_r} \times \\ &\times G_{2,4}^{4,1}\left[\frac{b^2p^2}{4}\middle|\begin{matrix}\frac{1-\varrho-\nu}{2},\ \frac{1-\varrho+\nu}{2}\\ \alpha-\frac{\nu+\varrho+1}{2}+r,\beta-\frac{\nu+\varrho+1}{2}+r,\frac{\delta+\mu}{2},\frac{\delta-\mu}{2}\end{matrix}\right],\end{aligned}\right.$$

for $R(p)>0$, $R(b)>0$, $R(\delta+\varrho+\nu\pm\mu+1)>0$.

When $\mu=\pm\frac{1}{2}$, (9) yields the following result:

$$
(10)\quad \left\{
\begin{aligned}
&\int_0^\infty y^{\delta+\varrho+\nu-\frac{1}{2}}\, e^{-py}\, F_4\left[\alpha, \beta;\ 1+\nu,\ 1+\sigma;\ -\frac{y^2}{b^2}, \frac{a^2}{b^2}\right] dy = \\
&\qquad = \frac{2^{\delta-5/2}\, b^{\nu+\varrho+1}\, \Gamma(1+\nu)}{\pi^{\frac{1}{2}}\, p^{\delta-\frac{1}{2}}\, \Gamma(\alpha)\, \Gamma(\beta)} \sum_{r=0}^{\infty} \frac{(a/b)^{2r}}{\lfloor r\, (\sigma+1)_r} \times \\
&\quad \times G^{4,1}_{2,4}\left[\frac{b^2p^2}{4}\left|\begin{matrix} \frac{1-\varrho-\nu}{2},\ \frac{1-\varrho+\nu}{2} \\ \alpha-\frac{\nu+\varrho+1}{2}+r,\ \beta-\frac{\nu+\varrho+1}{2}+r,\ \frac{\delta}{2}+\frac{1}{4},\ \frac{\delta}{2}-\frac{1}{4}\end{matrix}\right.\right],
\end{aligned}\right.
$$

for $R(p)>0,\ R(b)>0,\ R(\delta+\varrho+\nu+\frac{1}{2})>0$.

(iii) Taking $m=2,\ n=0,\ l=1,\ q=3,\ \alpha_1=\frac{\delta-\mu-1}{2},\ \beta_1=\frac{\delta-\mu}{2},\ \beta_2=\frac{\delta+\mu}{2},$ $\beta_3=\frac{\delta-\mu-1}{2}$, then

$$
(11)\quad \left\{
\begin{aligned}
&\int_0^\infty y^{\varrho+\nu+\delta}\, Y_\nu(py)\, F_4\left[\alpha, \beta;\ 1+\nu,\ 1+\sigma;\ -\frac{y^2}{b^2}, \frac{a^2}{b^2}\right] dy = \\
&\qquad = \frac{2^{\delta-1}\, b^{\nu+\varrho+1}\, \Gamma(1+\nu)}{p^{\delta}\, \Gamma(\alpha)\, \Gamma(\beta)} \sum_{r=0}^{\infty} \frac{(a/b)^{2r}}{\lfloor r\, (\sigma+1)_r} \times \\
&\quad \times G^{4,1}_{3,5}\left[\frac{b^2p^2}{4}\left|\begin{matrix} \frac{1-\varrho-\nu}{2},\ \frac{\delta-\mu-1}{2},\ \frac{1-\varrho+\nu}{2} \\ \alpha-\frac{\nu+\varrho+1}{2}+r,\ \beta-\frac{\nu+\varrho+1}{2}+r,\ \frac{\delta-\mu}{2},\ \frac{\delta+\mu}{2},\ \frac{\delta-\mu-1}{2}\end{matrix}\right.\right],
\end{aligned}\right.
$$

for $p>0,\ R(b)>0,\ R(\varrho+\nu+\delta\pm\mu+1)>0,\ R(\varrho+\nu+\delta-2\alpha-\frac{1}{2})<0,$ $R(\varrho+\nu+\delta-2\beta-\frac{1}{2})<0$.

(iv) Taking $m=n=l=1,\ q=3,\ \alpha_1=\frac{1+\delta+\mu}{2},\ \beta_1=\frac{1+\delta+\mu}{2},\ \beta_2=\frac{\delta-\mu}{2},$ $\beta_3=\frac{\delta+\mu}{2}$, we get

$$
(12)\quad \left\{
\begin{aligned}
&\int_0^\infty y^{\varrho+\nu+\delta}\, H_\mu(py)\, F_4\left[\alpha, \beta;\ 1+\nu,\ 1+\sigma;\ -\frac{y^2}{b^2}, \frac{a^2}{b^2}\right] dy = \\
&\qquad = \frac{2^{\delta-1}\, b^{\nu+\varrho+1}\, \Gamma(1+\nu)}{p^{\delta}\, \Gamma(\alpha)\, \Gamma(\beta)} \sum_{r=0}^{\infty} \frac{(a/b)^{2r}}{\lfloor r\, (\sigma+1)_r} \times \\
&\quad \times G^{3,2}_{3,5}\left[\frac{b^2p^2}{4}\left|\begin{matrix} \frac{1-\varrho-\nu}{2},\ \frac{1+\delta+\mu}{2},\ \frac{1-\varrho+\nu}{2} \\ \alpha-\frac{\nu+\varrho+1}{2}+r,\ \beta-\frac{\nu+\varrho+1}{2}+r,\ \frac{1+\delta+\mu}{2},\ \frac{\delta-\mu}{2},\ \frac{\delta+\mu}{2}\end{matrix}\right.\right],
\end{aligned}\right.
$$

for $p>0,\ R(b)>0,\ R(\varrho+\nu+\delta+\mu+2)>0,\ R(\varrho+\nu+\delta-2\alpha-\frac{1}{2})<0,$ $R(\varrho+\nu+\delta-2\beta-\frac{1}{2})<0,\ R(\varrho+\nu+\delta+\mu-2\alpha)<0,\ R(\varrho+\nu+\delta+\mu-2\beta)<0.$

(v) Taking $m=q=2, n=0, l=1, \alpha_1=1-k+l, \beta_1=l+m+\frac{1}{2}, \beta_2=l-m+\frac{1}{2}$, then

$$(13)\quad \begin{cases} \int_0^\infty y^{l+\frac{\varrho+\nu-1}{2}} e^{-(py)/2} W_{k,m}(py)\, F_4\left[\alpha,\beta;1+\nu,1+\sigma;-\frac{y}{b^2},\frac{a^2}{b^2}\right]dy= \\ \quad =\frac{b^{\nu+\varrho+1}\,\Gamma(1+\nu)}{p^l\,\Gamma(\alpha)\,\Gamma(\beta)}\sum_{r=0}^\infty \frac{(a/b)^{2r}}{\underline{|r}\,(\sigma+1)_r}\times \\ \times G_{3,4}^{4,1}\left[b^2p\,\middle|\, \begin{matrix} \frac{1-\varrho-\nu}{2},\ 1-k+l,\ \frac{1-\varrho+\nu}{2} \\ \alpha-\frac{\nu+\varrho+1}{2}+r,\ \beta-\frac{\nu+\varrho+1}{2}+r,\ l+m+\frac{1}{2},\ l-m+\frac{1}{2}\end{matrix}\right], \end{cases}$$

for $R(p)>0,\ R(b)>0,\ R(\nu+1)>0,\ R(\varrho+\nu+2l\pm 2m+2)>0$.

(vi) Finally taking $m=1, n=l=q=2, \beta_1=0$ we get

$$(14)\quad \begin{cases} \int_0^\infty y^{\varrho+\nu}\, {}_2F_1[1-\alpha_1,1-\alpha_2;1-\beta_2;-p^2y^2]\, F_4\left[\alpha,\beta;1+\nu,1+\sigma;-\frac{y^2}{b^2},\frac{a^2}{b^2}\right]dy= \\ \quad =\frac{b^{\nu+\varrho+1}\,\Gamma(1+\nu)\,\Gamma(1-\beta_2)}{2\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(1-\alpha_1)\,\Gamma(1-\alpha_2)}\sum_{r=0}^\infty \frac{(a/b)^{2r}}{\underline{|r}\,(\sigma+1)_r}\times \\ \times G_{4,4}^{3,3}\left[b^2p^2\,\middle|\, \begin{matrix} \frac{1-\varrho-\nu}{2},\ \alpha_1,\ \alpha_2,\ \frac{1-\varrho+\nu}{2} \\ \alpha-\frac{\nu+\varrho+1}{2}+r,\ \beta-\frac{\nu+\varrho+1}{2}+r,\ 0,\ \beta_2\end{matrix}\right], \end{cases}$$

for $R(b)>0$, $R(\varrho+\nu+1)>0$, $R(\varrho+\nu+2\alpha_1-2\alpha-1)<0$, $R(\varrho+\nu+2\alpha_1-2\beta-1)<0$, $R(\varrho+\nu+2\alpha_2-2\alpha-1)<0$, $R(\varrho+\nu+2\alpha_2-2\beta-1)<0$.

The above result is more general than the result recently given by Sharma [(2), p. 111].

Comparing (14) with [(2), p. 111], we get

$$(15)\quad \begin{cases} \sum_{r=0}^\infty \frac{(-1)^r(a/b)^{2r}}{\underline{|r}\,(\sigma+1)_r} G_{2,2}^{2,2}\left[b^2p^2\,\middle|\, \begin{matrix} \alpha_1,\ \alpha_2 \\ \alpha-\nu-1+r,\ \beta-\nu-1+r\end{matrix}\right]= \\ \quad =\sum_{\alpha_1,\alpha_2} (bp)^{2\alpha_1-2}\,\Gamma(\alpha_1-\alpha_2)\,\Gamma(\alpha-\alpha_1-\nu)\,\Gamma(\beta-\alpha_1-\nu)\times \\ \quad \times F_4\left[\alpha-\alpha_1-\nu,\beta-\alpha_1-\nu;1+\sigma,\alpha_2-\alpha_1+1;-\frac{a^2}{b^2},\frac{1}{p^2b^2}\right]. \end{cases}$$

Replacing ν by $-\nu$, putting $b=ia$, $\sigma=2\nu$, multiplying both sides by cosec $(-2\nu\pi)$ and summing for $\nu, -\nu$, we have

$$(16)\quad \left\{ \begin{aligned} &\sum_{\nu,-\nu}\sum_{r=0}^{\infty} \frac{\operatorname{cosec}(-2\nu\pi)}{\lfloor r\, \Gamma(2\nu+1+r)} G^{2,2}_{2,2}\left[b^2p^2 \,\middle|\, \begin{matrix} \alpha_1, \alpha_2 \\ \alpha-1+\nu+r,\ \beta-1+\nu+r \end{matrix}\right] = \\ &= \sum_{\nu,-\nu}\sum_{\alpha_1,\alpha_2} \frac{\operatorname{cosec}(-2\nu\pi)\,\Gamma(\alpha_1-\alpha_2)\,\Gamma(\alpha-\alpha_1+\nu)\,\Gamma(\beta-\alpha_1+\nu)}{(bp)^{2-2\alpha_1}\,\Gamma(2\nu+1)} \times \\ &\times F_4\left[\alpha-\alpha_1+\nu,\ \beta-\alpha_1+\nu;\ 1+2\nu,\ \alpha_2-\alpha_1+1;\ 1, \frac{1}{p^2b^2}\right] = \\ &= \pi^{-\frac{1}{2}}\, 2^{3-\alpha-\beta}\, G^{4,2}_{4,4}\left[4\,b^2p^2 \,\middle|\, \begin{matrix} \alpha_1, \alpha_2, \frac{\alpha+\beta-2}{2}, \frac{\alpha+\beta-1}{2} \\ \alpha-1\pm\nu,\ \beta-1\pm\nu \end{matrix}\right] \end{aligned} \right.$$

by (5).

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