Local Controllability of Quasilinear Integrodifferential Evolution Systems in Banach Spaces

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Sufficient conditions for local controllability of quasilinear integrodifferential systems in Banach spaces are established. The results are obtained by using the analytic semigroup theory and the Schauder fixed-point theorem. An example is provided to illustrate the theory.

Key Words: controllability, quasilinear integrodifferential system, analytic semigroup, fixed point theorem.

1. INTRODUCTION

It is well known that the systems described by partial differential equations can be expressed as abstract differential equations [21]. These equations occur in various fields of study, and each system can be represented...
by different forms of differential or integrodifferential equations in Banach spaces. For example, quasilinear integrodifferential equations have occurred during the study of the nonlinear behavior of elastic strings [17]. A derivation of this equation for finite strings is given in [20] and the global existence, boundedness, and regularity of solutions are established in [10]. Abstract quasilinear evolution equations have been studied by many authors and have been applied to partial differential equations [9, 11, 24]. Recently, Bahuguna [1], Oka [18], and Oka and Tanaka [19] discussed the existence of solutions of quasilinear integrodifferential equations in Banach spaces. An equation of the form

$$u_t(t, x) + \Psi(u(t, x)) = \int_0^t b(t-s)\Psi(u(s, x))ds + f(t, x),$$

$$t \in [0, T], \quad x \in R$$

$$u(0, x) = \phi(x), \quad x \in R,$$

occurs in a nonlinear conservation law with memory. So, it is interesting to investigate the controllability problem for these types of equations in Banach spaces.

2. PRELIMINARIES

Consider the quasilinear integrodifferential evolution system of the form
\[
\begin{aligned}
\dot{x}(t) + A(t, x(t))x(t) &= Bu(t) + f(t, x(t), \int_0^t g(t, s, x(s)) \, ds), \quad t \in J, \\
x(0) &= x_0,
\end{aligned}
\]  
where the state $x(\cdot)$ takes values in a Banach space $X$ and the control $u(\cdot)$ in $L^2(J, U)$, a Banach space of all admissible controls with $U$ as a Banach space and $J = [0, T]$. Here $-A$ is the infinitesimal generator of an analytic semigroup and $B$ is a bounded linear operator from $U$ into $X$. The nonlinear operators $f : J \times X \times X \to X$ and $g : J \times J \times X \to X$ are uniformly bounded and continuous in all of their arguments.

Let $r > 0$ and take $B_r = \{ y \in X : \|y\| < r \}$, and assume the following conditions:

(i) The operator $A_0 = A(0, x_0)$ is a closed operator with domain $D$ dense in $X$ and
\[
\| (\lambda I - A_0)^{-1} \| \leq C [\| \lambda \| + 1]^{-1}
\]
for all $\lambda$ with Re $\lambda \leq 0$ and $C > 0$.

(ii) The operator $A_0^{-1}$ is a completely continuous operator in $X$.

(iii) For some $\alpha \in [0, 1)$ and for any $y \in B_r$, the operator $A(t, A_0^{-\alpha}y)$ is well defined on $D$ for all $t \in J$. Furthermore, for any $t, \tau \in J$ and for $y, z \in B_r$,
\[
\| [A(t, A_0^{-\alpha}y) - A(\tau, A_0^{-\alpha}z)] A_0^{-1} \| \leq C_1 [\| t - \tau \|^{\epsilon} + \| y - z \|^{\rho}],
\]
where $0 < \epsilon \leq 1, 0 < \rho \leq 1$.

(iv) For every $t, \tau \in J$ and $y, z, u, v \in X$,
\[
\| f(t, A_0^{-\alpha}y, u) - f(\tau, A_0^{-\alpha}z, v) \| \leq C_2 [\| t - \tau \|^{\epsilon} + \| y - z \|^{\rho} + \| u - v \|].
\]

(v) For every $t, \tau \in J$ and $y, z \in B_r$,
\[
\| g(t, s, A_0^{-\alpha}y) - g(\tau, s, A_0^{-\alpha}z) \| \leq C_3 [\| t - \tau \|^{\epsilon} + \| y - z \|^{\rho}].
\]

(vi) $x_0 \in D(A_0^\beta)$ for any $\beta > \alpha$ and
\[
\| A_0^\beta x_0 \| < r.
Then the system (1) has a continuously differentiable local solution of the form [9, 24]

\[
x(t) = U_x(t, 0)x_0 + \int_0^t U_x(t, s)Bu(s)\, ds \\
+ \int_0^t U_x(t, s)f\left(s, x(s), \int_0^s g(s, \tau, x(\tau))\, d\tau\right)\, ds,
\]

where \(U_x(t, s)\) is a fundamental solution corresponding to \(A_x(t) = A(t, A_{\alpha}^0 x(t))\) and all estimates for fundamental solutions derived in [9, 24] hold uniformly in the set \(Q = \{x \in Y : \|x(t) - x(\tau)\| \leq K|t - \tau|^\eta\\} \) for \(t, \tau \in [0, T], x(0) = A_{\alpha}^0 x_0\},

where \(K\) is a positive constant, \(\eta\) is any number satisfying \(0 < \eta < \beta - \alpha\), and \(Y\) is a Banach space \(C(J, X)\) with usual sup norm. From (vi), we observe that if \(x \in Q\) then \(x \in B_r\). Hence the operator \(A_x(t)\) is well defined and satisfies the conditions

\[
\|A_x(t) - A_x(\tau)\| \leq C_4\left(|t - \tau|^{\gamma} + \|x(t) - x(\tau)\|^{\rho}\right)
\]

\[
\leq C_4|t - \tau|^{\mu},
\]

where \(\mu = \min\{\epsilon, \rho\eta\}\).

Further, if \(x(0) = A_{\alpha}^0 x_0\),

\[
A_x(0) = A(0, A_{\alpha}^0 x(0)) = A(0, A_{\alpha}^0 A_{\alpha}^0 x_0) = A(0, x_0) = A_0,
\]

and it follows that for every \(t \in [0, T]\) and \(\lambda\) with \(\text{Re}\lambda \leq 0\)

\[
\|\lambda I - A_x(t)\|^{-1} \leq C_5[|\lambda| + 1]^{-1},
\]

\[
\|A_x(t) - A_x(\tau)A_x^{-1}(s)\| \leq C_7|t - \tau|^{\mu},
\]

for any \(t, \tau, s \in [0, T]\).

We further assume the following conditions:

(vii) There exists a constant \(M_0 > 0\) such that

\[
\|U_x(t, s)\| \leq M_0, \quad 0 \leq s \leq t \leq T, \quad x \in B_r.
\]

(viii) The linear operator \(W\) from \(L^2(J, U)\) into \(X\), defined by

\[
Wu = \int_0^T U_x(T, s)Bu(s)\, ds,
\]

for each fixed \(x \in Y\), induces an invertible operator \(\tilde{W}\) defined on \(L^2(J, U)/\ker W\), and there exist positive constants \(M_1, M_2 > 0\) such that \(\|B\| \leq M_1\) and \(\|\tilde{W}^{-1}\| \leq M_2\). The construction of the bounded invertible operator \(\tilde{W}^{-1}\) in the general Banach space is outlined in [23].
(ix) There exists a constant $M_3 > 0$ such that

$$\left\| \int_0^t g(t, s, A_0^{-\alpha} x(s)) \, ds \right\| \leq M_3.$$ 

Let us take

$$f_x(t) = f \left( t, A_0^{-\alpha} x(t), \int_0^t g(t, s, A_0^{-\alpha} x(s)) \, ds \right).$$

Then, it follows that the function $f_x(t)$ is Holder continuous such that

$$\left\| f_x(t) - f_x(\tau) \right\| \leq C_8 |t - \tau|^\mu.$$ 

Since $f_x(0) = f(0, A_0^{-\alpha} x(0), 0)$ is independent of $x$, we have from the above inequality that

$$\left\| f_x(t) \right\| \leq M_4, \quad M_4 > 0. \quad (3)$$

Up to now and in the subsequent discussion the $C_i$'s are positive constants.

**Definition 2.1.** The system (1) is said to be locally controllable on the interval $J$ if for every $x_0, x_T \in Z$ a subset of $X$, there exists a control $u \in L^2(J, U)$ such that a continuously differentiable local solution $x(t)$ of (1) satisfies $x(T) = x_T$.

In order to study the controllability problem, we must introduce a set $S$ of functions $x(t)$ and a transformation $z_x = \Phi_x$ defined by $z_x = A_0^{-\alpha} z$, where $z$ is the unique solution of

$$\frac{dz}{dt} + A_x(t) z = B u(t) + f_x(t)$$

$$z(0) = x_0.$$ 

We then show that $\Phi$ has a fixed point; that is, there is a function $y \in S$ such that $\Phi y = y$, and so $x = A_0^{-\alpha} y$ is the required solution of our problem (1).

### 3. MAIN RESULT

**Theorem 3.1.** If the hypotheses (i)–(ix) are satisfied, then the system (1) is locally controllable on $J$.

**Proof.** For each $x \in Y$, we define a control

$$u(t) = \tilde{W}^{-1} \left[ x_T - U_x(T, 0)x_0 - \int_0^T U_x(T, s)f_x(s) \, ds \right] (t) \quad (4)$$
and take the set \( S \) in \( Y \) by
\[
S = \left\{ x \in Y : \|x(t) - x(s)\| \leq K|t - s|^{\eta}, \right. \\
\text{for any } t, s \in [0, T], x(0) = A_0^0 x_0 \} ,
\]
where \( K \) is a positive constant and \( \eta \) is any number satisfying \( 0 < \eta < \beta - \alpha \). Clearly \( S \) is a closed convex and bounded subset of \( Y \). Using the control we shall show that the operator \( \Phi: S \to Y \) defined by
\[
\Phi x(t) = A_0^0 U_x(t, 0) x_0 + A_0^0 \int_0^t U_x(t, s) B \tilde{W}^{-1} \\
\times \left[ x_T - U_x(T, 0) x_0 - \int_0^T U_x(T, \tau) f_x(\tau) d\tau \right](s) ds \\
+ A_0^0 \int_0^t U_x(t, s) f_x(s) ds
\]
has a fixed point. This fixed point is the solution of Eq. (2). First we show that \( \Phi \) maps \( S \) into itself. Obviously, \( \Phi x(0) = A_0^0 x_0 \).

For any \( 0 \leq \alpha < \beta \leq 1 \) and \( 0 \leq t_1 \leq t_2 \leq T \) we have
\[
\|\Phi x(t_1) - \Phi x(t_2)\| \leq A_0^0 \| U_x(t_1, 0) - U_x(t_2, 0) \| A_0^\beta \| A_0^\beta x_0 \| \\
+ A_0^0 \int_0^{t_1} \left[ U_x(t_1, s) - U_x(t_2, s) \right] B \tilde{W}^{-1} \\
\times \left[ x_T - U_x(T, 0) x_0 - \int_0^T U_x(T, \tau) f_x(\tau) d\tau \right](s) ds \\
+ A_0^0 \int_{t_1}^{t_2} U_x(t_2, s) B \tilde{W}^{-1} \left[ x_T - U_x(T, 0) x_0 \\
- \int_0^T U_x(T, \tau) f_x(\tau) d\tau \right] ds \\
+ \left\| \int_0^{t_1} U_x(t_1, s) f_x(s) ds - \int_0^{t_2} U_x(t_2, s) f_x(s) ds \right\|.
\]

From our assumptions, we have
\[
\| A_0^0 \left[ U_x(t_1, 0) - U_x(t_2, 0) \right] A_0^\beta \| \leq C_9 |t_1 - t_2|^{\beta - \alpha} ,
\]
\[
\| f_x(t_1) - f_x(t_2) \| \leq C_8 |t_1 - t_2|^\mu ,
\]
and
\[
\left\| A_0^\alpha \left[ \int_0^{t_1} U_x(t_1, s) f_x(s) ds - \int_0^{t_2} U_x(t_2, s) f_x(s) ds \right] \right\| \leq C_{10} |t_1 - t_2|^{1 - \alpha} .
\]
Thus,
\[
\|\Phi x(t_1) - \Phi x(t_2)\| \\
\leq rC_\delta|t_1 - t_2|^{\beta - \alpha} + \int_0^{t_2} \|A_0^\alpha[U_x(t_1, s) - U_x(t_2, s)]\| \|B\| \|\tilde{W}^{-1}\| \\
\times \left[\|x_T\| + \|U_x(T, 0)\| \|x_0\| + \int_0^T \|U_x(T, \tau)\| \|f_x(\tau)\| d\tau\right] ds \\
+ \int_0^{t_2} \|A_0^\alpha U_x(t_2, s)\| \|B\| \|\tilde{W}^{-1}\| \\
\times \left[\|x_T\| + \|U_x(T, 0)\| \|x_0\| + \int_0^T \|U_x(T, \tau)\| \|f_x(\tau)\| d\tau\right] ds \\
+ \left\|\int_0^{t_1} U_x(t_1, s)f_x(s) ds - \int_0^{t_2} U_x(t_2, s)f_x(s) ds\right\| \\
\leq K|t_1 - t_2|^\eta \quad \text{for some } K > 0, \eta < \beta - \alpha.
\]

Hence \(\Phi\) maps \(S\) into itself.

Next we show that this operator is continuous on the space \(Y\). Let \(x_1, x_2 \in S\) and set \(z_1 = A_0^{-\alpha}\Phi x_1, z_2 = A_0^{-\alpha}\Phi x_2\). Thus,
\[
\frac{dz_i}{dt} + A_{x_i}(t)z_i = Bu_i(t) + f_{x_i}(t) \\
z_i(0) = x_0,
\]
where
\[
u_i(t) = \tilde{W}^{-1}\left[x_T - U_x_x(T, 0)x_0 - \int_0^T U_x_x(T, s)f_x(s) ds\right](t), \quad i = 1, 2.
\]

Therefore,
\[
\frac{d}{dt}(z_1 - z_2) + A_{x_1}(t)(z_1 - z_2) = \left[A_{x_2}(t) - A_{x_1}(t)\right]z_2 + B[u_1(t) - u_2(t)] \\
+ f_{x_1}(t) - f_{x_2}(t).
\]

It is easy to see that the functions \(A_{x_1}(t)z_2(t)\) and \(A_0A_{x_2}^{-1}(t)\) are uniformly Holder continuous, and so
\[
A_0z_2(t) = \left[A_0A_{x_2}^{-1}(t)\right]A_{x_1}(t)z_2(t)
\]
is uniformly Holder continuous. Similarly, the functions
\[
B[u_1(t) - u_2(t)] \text{ and } f_{x_1}(t) - f_{x_2}(t)
\]
are also uniformly Holder continuous in $[\tau, T]$, $\tau > 0$. Hence we have

$$[z_1(t) - z_2(t)] = U_{x_1}(t, \tau)[z_1(\tau) - z_2(\tau)] + \int_{\tau}^{t} U_{x_1}(t, s)[[A_{x_2}(s) - A_{x_1}(s)]z_2(s)$$

$$+ B[u_1(t) - u_2(t)] + [f_{x_1}(s) - f_{x_2}(s)]ds.$$

Since

$$A_0 \int_{0}^{t} U_{x_2}(t, s)f_{x_2}(s)ds$$

and

$$A_0 \int_{0}^{t} U_{x_1}(t, s)Bu_2(s)ds$$

are bounded functions, it follows that

$$\|A_0z_2(t)\| \leq C_1 t^{\beta - 1}.$$  

Hence we can take $\tau \to 0$ in the above equation and we get

$$[\dot{z}_1(t) - \dot{z}_2(t)] = \int_{0}^{t} U_{x_1}(t, s)\lambda[A_{x_2}(s) - A_{x_1}(s)]z_2(s)$$

$$+ B[u_1(t) - u_2(t)] + [f_{x_1}(s) - f_{x_2}(s)]\lambda ds.$$  

Since $z_1 = A_0^{-\alpha}\Phi x_1$ and $z_2 = A_0^{-\alpha}\Phi x_2$, from (iii), (iv), and (v) it follows that

$$\|\Phi x_1(t) - \Phi x_2(t)\| \leq \int_{0}^{t} \|A_0^{-\alpha}U_{x_1}(t, s)\| \|[[A_{x_2}(s) - A_{x_1}(s)]z_2(s)]$$

$$+ \|B\| \|u_1(t) - u_2(t)\| + \|f_{x_1}(s) - f_{x_2}(s)\|ds$$

$$\leq \int_{0}^{t} C_{12}[t - s]^{-\alpha}\{C_{13}\|x_1(s) - x_2(s)\|^p s^{\beta - 1}$$

$$+ C_{14}\|x_1(s) - x_2(s)\|^p\}ds.$$  

Hence

$$\|\Phi x_1 - \Phi x_2\|_Y \leq K^*\|x_1 - x_2\|_Y$$  

for some $K^* > 0$.

This shows that $\Phi : S \to Y$ is continuous. We shall show that this operator is completely continuous. We now claim that the set $\Phi S$ is contained in a compact subset of $Y$. Indeed, the functions $x(t)$ of $S$ are uniformly bounded and equicontinuous. By the Arzela–Ascoli Theorem it is sufficient to show that for each $t$ the set $\{\Phi x(t) : x \in S\}$ is contained in a compact subset of $X$. For each $t \in [0, T]$, we can write $\Phi x(t) = A_0^{-\gamma} A_0^\gamma \Phi x(t)$ ($0 < \gamma < \beta - \alpha$). Since $\{A_0^\gamma \Phi x(t) : x \in S\}$ is a bounded subset of $X$, and since $A_0^{-\gamma}$ is completely continuous, it follows that the set $\{\Phi x(t) : x \in S\}$ is contained
in a compact subset of $X$. Therefore, by the Schauder fixed point theorem, $\Phi$ has a fixed point $z \in S$ such that $\Phi z(t) = z(t)$, which satisfies

$$z(t) = A_0^0 U_z(t, 0) x_0 + A_0^0 \int_0^t U_z(t, s) BW^{-1} \times \left[ x_T - U_z(T, 0) x_0 - \int_0^T U_z(T, \tau) f_z(\tau) d\tau \right] ds$$

$$+ A_0^0 \int_0^t U_z(t, s) f_z(s) d\tau.$$

Then $x(t) = A_0^0 z(t)$ satisfies $x(T) = x_T$. Thus the system (1) is locally controllable on $J$.

4. EXAMPLE

Consider the nonlinear parabolic integrodifferential equation

$$\frac{\partial z}{\partial t} + \sum_{|a|=2m} a_a (x, t, z, Dz, \ldots, D^{2m-1} z) D^a z$$

$$= u(x, t) + f \left( x, t, z, Dz, \ldots, D^{2m-1} z, \right.$$

$$\times \int_0^t k(x, t, s, z, Dz, \ldots, D^{2m-1} z) ds \left. \right)$$

$$\frac{\partial^j z}{\partial \nu^j} = 0 \quad \text{on } S_T = \{(x, t) : x \in \partial \Omega, 0 < t \leq T, 0 \leq j \leq m - 1 \}$$

$$z(x, 0) = 0 \quad \text{on } \Omega_0 = \{(x, 0) : x \in \partial \Omega \}$$

in a cylinder $Q_T = \Omega \times (0, T)$ with coefficients in $Q_\Omega$, where $\Omega$ is a bounded domain in $R^n$, $\partial \Omega$ is the boundary of $\Omega$, $\nu$ is the outward normal, and $u(x, t)$ is the control parameter. Let $\Gamma$ be a bounded set in $R^m$. Now we have to prove that there exists a control $u \in L^2(\gamma, \Gamma)$ which steers the system (7) from any specified initial state to the final state in a subspace. Here the parabolicity means that for any vector $y \neq 0$ and for arbitrary values of $z, Dz, \ldots, D^{2m-1} z$,

$$(-1)^m \text{ Re} \left\{ \sum_{|a|=2m} a_a (x, t, z, Dz, \ldots, D^{2m-1} z) y^a \right\} \geq C |y|^{2m}, \quad C > 0.$$

If $z_0(x) \in C^{2m-1}(\Omega)$, then

$$A_0 z = \sum_{|a|=2m} a_a (x, t, z_0(x), Dz_0(x), \ldots, D^{2m-1} z_0(x)) D^a z$$
is a strongly elliptic operator with continuous coefficients. So the condition (i) holds. Let us take $X$ to be $L^p(\Omega)$, $1 < p < \infty$. Then $A_{0}^{-1}$ maps bounded subsets of $L^p(\Omega)$ into bounded subsets of $W^{2m,p}(\Omega)$, so it is a completely continuous operator in $L^p(\Omega)$. Further, if $(2m - 1)/2m < \alpha < 1$, then [9]

$$|D^\beta A_{0} - \alpha z|^\Omega_{0,p} \leq C|z|^\Omega_{0,p}, \quad 0 \leq |\beta| \leq 2m - 1,$$

where $C$ depends only on a bound on the coefficients $A_{0}$, on a module of strong ellipticity, and on a modulus of continuity of the leading coefficients. Here the norm is defined as

$$|z|^\Omega_{j,p} = \left\{ \sum_{|\alpha| \leq j} \int_{\Omega} |D^\alpha z(x)|^p dx \right\}^{1/p}$$

for any nonnegative integer $j$ and a real number $p$, $1 \leq p < \infty$. It follows that if $f, k$, and $a_\alpha$ are continuously differentiable in all variables, then (iii) and (iv) hold with $\sigma = \rho = 1$. Hence there exists a fundamental operator solution $U_{z}(t, s)$ and a solution for the system (7) locally, that is, in a subspace. Further, assume that the operator $W$ from $L^2(J, \Gamma)$ into $X$ given by

$$Wu = \int_{0}^{T} U_{z}(T, s)u(x, s) \, ds$$

satisfies the condition(viii). The nonlinear function $f$ satisfies the conditions (v) and (vii). Further, all of the conditions of the above theorem are satisfied. Hence the system (7) is controllable on $J$.

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