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L_{μ}^{p}-Boundedness of the Pseudo-differential Operator Associated with the Bessel Operator

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An L_{μ}^{p}-boundedness result for the pseudo-differential operator associated with the Bessel operator is obtained. © 2001 Academic Press

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1. INTRODUCTION

Zemanian [14] extended the Hankel transformation

$$(h_{\mu}\phi)(x) = \int_{0}^{\infty} (xy)^{1/2} J_{\mu}(xy)\phi(y)\,dy, \qquad \mu \geq -1/2 \tag{1.1}$$

to distributions belonging to H'_{μ}, the dual of test function space H_{μ} consisting of all complex-valued infinitely differentiable functions ϕ defined on $I = (0, \infty)$ satisfying

$$\gamma_{n,k}^{\mu}(\phi) = \sup_{x \in I} \left| x^{n} \left(x^{-1} \frac{d}{dx} \right)^{k} x^{-\mu-1/2}\phi(x) \right| < \infty, \qquad \forall n, k \in \mathbf{N}_{0}. \tag{1.2}$$

It was shown by Zemanian that the Hankel transformation h_{μ} is an automorphism on the space H_{μ} and the generalized Hankel transformation h'_{μ} is an automorphism on H'_{μ}.

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The pseudo-differential operator associated with the Bessel operator studied in [9–11] is defined by

$$(h_{\mu,a}\phi)(x) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi) a(x,\xi)(h_\mu \phi)\, d\xi, \qquad \phi \in H_\mu \quad (1.3)$$

where the symbol $a(x, \xi)$ is defined as follows.

The function $a(x, \xi)$: $C^\infty(I \times I) \to \mathbb{C}$ belongs to class H^m if and only if for each $q \in \mathbf{N}_0$ there exists $D = D_{\alpha, m, q, \gamma}$ such that

$$(1+x)^q \left| \left(x^{-1} \frac{d}{dx} \right)^\gamma \left(\xi^{-1} \frac{d}{d\xi} \right)^\alpha a(x,\xi) \right| \le D(1+\xi)^{m-2\alpha}, \quad (1.4)$$

where m is a fixed real number.

For $1 \le p < \infty$, we define $L_\mu^p(I)$ as the Banach space of all measurable functions f on I which satisfy

$$\|f\|_p = \left(\int_0^\infty x^{\mu+1/2} |f(x)|^p\, dx \right)^{1/p} < \infty. \quad (1.5)$$

Now, let $f \in L_\mu^1(I)$, $g \in L_\mu^p(I)$, and define its associated function by

$$(\tau_x f)(y) = f(x,y) = \int_0^\infty f(z) D_\mu(x,y,z)\, dz, \qquad 0 < x,\ y < \infty, \quad (1.6)$$

where

$$D_\mu(x,y,z) = \int_0^\infty t^{-\mu-1/2} (xt)^{1/2} J_\mu(xt)(yt)^{1/2} J_\mu(yt)(zt)^{1/2} J_\mu(zt)\, dt. \quad (1.7)$$

Then the Hankel convolution defined by

$$(f\#g)(x) = \int_0^\infty f(x,y) g(y)\, dy \quad (1.8)$$

satisfies the following norm inequality [9]

$$\|f\#g\|_p \le \|f\|_1 \|g\|_p. \quad (1.9)$$

The Hankel convolution has been extended to distributions by Marrero and Betancor [6] and Betancor and Rodriguez-Mesa [1].

The L^p-boundedness of the classical pseudo-differential operator has been investigated by Fefferman [2], Kato [4], Nagase [7], Wong [12], and others. Wong's analysis depends heavily on a result of Hörmander [3]. In this paper an analogue of Hörmander's result is established first then the proof of the main theorem on the L^p_μ-boundedness of the p.d.o. $h_{\mu,a}$ is constructed. Using this L^p_μ-boundedness property we prove that $h_{\mu,a}$ is a bounded linear operator from

$$W_\mu^{m,p} \to W_\mu^{0,p} \qquad \text{and} \qquad W_\mu^{s,p} \to W_\mu^{s-m,p}.$$

These spaces are defined in [11].

We shall use notation and terminology of [14]. The differential operators N_μ, M_μ, and S_μ are defined by

$$N_\mu = N_{\mu,x} = x^{\mu+1/2}\left(\frac{d}{dx}\right)x^{-\mu-1/2} \tag{1.10}$$

$$M_\mu = M_{\mu,x} = x^{-\mu-1/2}\left(\frac{d}{dx}\right)x^{\mu+1/2} \tag{1.11}$$

$$S_\mu = S_{\mu,x} = M_\mu N_\mu = \frac{d^2}{dx^2} + \frac{1-4\mu^2}{4x^2}. \tag{1.12}$$

From [14, p. 139] we have the following relations for any $\phi \in H_\mu$,

$$h_{\mu+1}(-x\phi) = N_\mu h_\mu \phi \tag{1.13}$$

$$h_\mu(N_\mu \phi) = -y h_\mu \phi \tag{1.14}$$

$$h_\mu(S_\mu \phi) = (-y)^2 h_\mu \phi \tag{1.15}$$

$$\left(x^{-1}\frac{d}{dx}\right)^k \left(x^{-\mu-1/2}\psi\phi\right) = \sum_{\nu=0}^{k}\binom{k}{\nu}\left(x^{-1}\frac{d}{dx}\right)^\nu \psi \left(x^{-1}\frac{d}{dx}\right)^{k-\nu} \times\left(x^{-\mu-1/2}\phi(x)\right). \tag{1.16}$$

The next result is due to Koh and Zemanian [5]

$$S^r_{\mu,x}\phi(x) = \sum_{j=0}^{r} b_j x^{2j+\mu+1/2}\left(x^{-1}\frac{d}{dx}\right)^{r+j}\left(x^{-\mu-1/2}\phi(x)\right), \tag{1.17}$$

where b_j are certain constants.

2. AN INTEGRAL REPRESENTATION

In this section we obtain an integral representation for the p.d.o. $h_{\mu,a}$ which will be useful in the proof of the L^p_μ-boundedness result.

LEMMA 2.1. *Assume that the symbol $a(x,\xi)$ is defined by* (1.4) *with* $m=0$. *Let*

$$K(x,z)=\int_0^\infty (z\xi)^{1/2}J_\mu(z\xi)\xi^{\mu+1/2}a(x,\xi)\,d\xi, \tag{2.1}$$

as a distribution in $H'_\mu(I)$. *Then,*

(i) *for each* $x\in I$, $K(x,\cdot)$ *is a function defined on* I;

(ii) *for each sufficiently large positive integer* k, *there exists a positive constant* $C_{\mu,k,q}$ *such that* $|K(x,z)|\le C_{\mu,k,q}(1+x)^{-q}(1+z^2)^{-k}$.

Proof. Let k be a positive integer greater than 1. Then using the formula (1.15) we get

$$\begin{aligned}K(x,z)&=\int_0^\infty (z\xi)^{1/2}J_\mu(z\xi)\xi^{\mu+1/2}a(x,\xi)\,d\xi\\&=\int_0^\infty (z\xi)^{1/2}J_\mu(z\xi)(1+z^2)^{-k}(1-S_{\mu,\xi})^k(\xi^{\mu+1/2}a(x,\xi))\,d\xi.\end{aligned}$$

Now using formula (1.17) we can write

$$\begin{aligned}K(x,z)&=\int_0^\infty (z\xi)^{1/2}J_\mu(z\xi)(1+z^2)^{-k}\\&\quad\times\left(\sum_{r=0}^{k}\binom{k}{r}(-1)^r S^r_{\mu,\xi}(\xi^{\mu+1/2}a(x,\xi))\right)d\xi\\&=\int_0^\infty (z\xi)^{1/2}J_\mu(z\xi)(1+z^2)^{-k}\\&\quad\times\sum_{r=0}^{k}\sum_{j=0}^{r}\binom{k}{r}(-1)^r\xi^{2j+\mu+1/2}b_j\left(\xi^{-1}\frac{d}{d\xi}\right)^{r+j}a(x,\xi)\,d\xi.\end{aligned}$$

Since $a(x,\xi)\in H^0$, using (1.4), it follows that $K(x,z)$ is a continuous function, and

$$\begin{aligned}|K(x,z)|&\le A_\mu(1+z^2)^{-k}\left(\sum_{r=0}^{k}\sum_{j=0}^{r}|b_j|\binom{k}{r}\right)\int_0^\infty(1+\xi)^{2j+\mu+1/2}(1+x)^{-q}\\&\quad\times D_{r+j}(1+\xi)^{-2(r+j)}\,d\xi\\&\le A_\mu(1+z^2)^{-k}(1+x)^{-q}\left(\sum_{r=0}^{k}\sum_{j=0}^{r}|b_j|\binom{k}{r}\right)\int_0^\infty(1+\xi)^{2j+\mu+1/2}\\&\quad\times D_{r+j}(1+\xi)^{-2(r+j)}\,d\xi\end{aligned}$$

choosing $r>\mu/2+\frac{3}{4}$, where D_{r+j} is a constant.

Thus there exists a positive constant $C_{\mu,k,q}$ such that

$$|K(x,z)| \le C_{\mu,k,q}(1+z^2)^{-k}(1+x)^{-q}.$$

∎

THEOREM 2.2. *For each fixed $x \in I$ and $\phi \in H_\mu(I)$, $\mu \ge -1/2$, the p.d.o. can be expressed in the form*

$$(h_{\mu,a}\phi)(x) = \int_0^\infty \left(\int_0^\infty K(x,z) D_\mu(x,y,z)\, dz \right) \phi(y)\, dy, \qquad (2.2)$$

where $K(x,z) = h_\mu(\xi^{\mu+1/2} a(x,\xi))$ in the distributional sense.

Proof. We recall from [6, p. 354] that τ_x, $0 < x < x_0$, is a continuous linear mapping from H_μ into itself. Hence $h_\mu(\tau_x\phi)$ is well defined for $\phi \in H_\mu$. Moreover, in view of definition (1.8),

$$\begin{aligned} h_\mu(\tau_x\phi)(\xi) &= \int_0^\infty (z\xi)^{1/2} J_\mu(z\xi)(\tau_x\phi)(z)\, dz \\ &= \int_0^\infty (z\xi)^{1/2} J_\mu(z\xi) \left(\int_0^\infty \phi(y) D_\mu(x,y,z)\, dy \right) dz \\ &= \int_0^\infty \phi(y)\, dy \int_0^\infty (z\xi)^{1/2} J_\mu(z\xi) D_\mu(x,y,z)\, dz \\ &= \int_0^\infty \xi^{-\mu-1/2}(x\xi)^{1/2} J_\mu(x\xi)(y\xi)^{1/2} J_\mu(y\xi)\phi(y)\, dy \\ &= (x\xi)^{1/2} J_\mu(x\xi)\, \xi^{-\mu-1/2}(h_\mu\phi)(\xi); \end{aligned}$$

so that

$$(h_\mu\phi)(\xi) = \left((x\xi)^{1/2} J_\mu(x\xi)\right)^{-1} \xi^{\mu+1/2} h_\mu(\tau_x\phi)(\xi) \in H_\mu.$$

Now, define the p.d.o. $h_{\mu,a}$ in the sense of distribution as

$$\begin{aligned} (h_{\mu,a}\phi)(x) &= \langle (x\xi)^{1/2} J_\mu(x\xi) a(x,\xi), (h_\mu\phi)(\xi) \rangle \\ &= \langle (x\xi)^{1/2} J_\mu(x\xi) a(x,\xi), \\ &\qquad \left((x\xi)^{1/2} J_\mu(x\xi)\right)^{-1} \xi^{\mu+1/2} h_\mu(\tau_x\phi)(\xi) \rangle \\ &= \langle \xi^{\mu+1/2} a(x,\xi), h_\mu(\tau_x\phi)(\xi) \rangle \end{aligned}$$

$$= \langle h_\mu(\xi^{\mu+1/2}a(x,\xi)), (\tau_x\phi)(z)\rangle$$
$$= \langle K(x,z), (\tau_x\phi)(z)\rangle$$
$$= \int_0^\infty K(x,z)(\tau_x\phi)(z)\,dz$$
$$= \int_0^\infty K(x,z)\left(\int_0^\infty D_\mu(x,y,z)\phi(y)\,dy\right)dz$$
$$= \int_0^\infty \left(\int_0^\infty K(x,z)D_\mu(x,y,z)\,dz\right)\phi(y)\,dy.$$

▮

3. THE L_μ^p-BOUNDEDNESS THEOREM

THEOREM 3.1. *Let $\theta \in C^k(I)$, $k \geq 1$, be such that there is a positive constant B for which*

$$\left|\left(\xi^{-1}\frac{d}{d\xi}\right)^\alpha \theta(\xi)\right| \leq B(1+\xi)^{-2\alpha}, \qquad \alpha \leq k/2. \tag{3.1}$$

If

$$f(x) = \int_0^\infty (x\xi)^{1/2}J_\mu(x\xi)\xi^{\mu+1/2}\theta(\xi)\,d\xi \tag{3.2}$$

then $f \in L_\mu^p(I)$, $1 \leq p < \infty$, $\mu \geq -1/2$.

Proof. Proceeding as in the proof of Lemma 2.1 we obtain

$$|f(x)| \leq (1+x^2)^{-k}\sum_{r=0}^{k}\sum_{j=0}^{r}\binom{k}{r}|b_j|\int_0^\infty |(x\xi)^{1/2}J_\mu(x\xi)|\xi^{2j+\mu+1/2}$$
$$\times\left|\left(\xi^{-1}\frac{d}{d\xi}\right)^{r+j}\theta(\xi)\right|d\xi$$
$$\leq (1+x^2)^{-k}\sum_{r=0}^{k}\sum_{j=0}^{r}\binom{k}{r}A_{\mu,j}\left(\int_0^\infty (1+\xi)^{\mu-2r+1/2}\,d\xi\right)$$
$$\leq B_{\mu,k}(1+x^2)^{-k},$$

choosing $r > \mu/2 + \frac{3}{4}$. Therefore, for $k > 1$ we have

$$\|f(x)\|_{L_\mu^p(I)} \leq B_{\mu,k} \|(1+x^2)^{-k}\|_{L_\mu^p(I)} < \infty$$

as k may be chosen large enough. ∎

THEOREM 3.2. *Let θ be the same as in Theorem 3.1; then for $1 \leq p < \infty$, there exists a positive constant $C = C(p, \mu)$ such that*

$$\|(h_{\mu,\theta}\phi)(x)\|_{L_\mu^p(I)} \leq C\|\phi\|_{L_\mu^p(I)}, \qquad \phi \in H_\mu.$$

Proof. By definition we have

$$(h_{\mu,\theta}\phi)(x) = h_\mu^{-1}\big[\theta(\xi)(h_\mu\phi)\big](x), \qquad \phi \in H_\mu.$$

Now, assume that there exists f such that

$$h_\mu^{-1}\big[\theta(\xi)(h_\mu\phi)(\xi)\big](x) = (f\#\phi)(x).$$

Then invoking inequality (1.9) and Theorem 3.1 we get

$$\begin{aligned}\|(h_{\mu,\theta}\phi(x)\|_{L_\mu^p(I)} &\leq \left\|h_\mu^{-1}(\xi^{\mu+1/2}\theta(\xi))\right\|_{L_\mu^1(I)}\|\phi\|_{L_\mu^p(I)}\\ &\leq C\|\phi\|_{L_\mu^p(I)}, \qquad \phi \in H_\mu.\end{aligned}$$

∎

The following theorem contains the basic results on $L_\mu^p(I)$-boundedness.

THEOREM 3.3. *Let $a(x,\xi)$ be a symbol in H^0. Then for $1 < p < \infty$ and $\mu \geq -1/2$, $h_{\mu,a}$: $L_\mu^p(I) \to L_\mu^p(I)$ is a bounded linear operator.*

Proof. We write $I = (0,\infty)$ as a union of intervals with disjoint interiors, i.e.,

$$I = \bigcup_{m \in N_0} Q_m,$$

where $Q_m = [m, m+1]$. Let η be a smooth function on $\mathbf{R}$ such that

$$\begin{aligned}\eta(x) &= 0 \qquad \text{for } x \in (-\infty,-1) \cup (2,\infty)\\ &= 1 \qquad \text{for } x \in [0,1]\end{aligned}$$

and

$$\left|\left(x^{-1}\frac{d}{dx}\right)^k x^{-\mu-1/2}\eta(x)\right| < \infty \qquad \forall k = 0,1,2,\dots.$$

Define

$$a_m(x, \xi) = \eta(x - m) a(x, \xi), \qquad (x, \xi) \in I \times I.$$

Then by [9] for $\phi \in H_\mu(I)$, we have

$$(h_{\mu, a_m}\phi)(x) = \eta(x - m)(h_{\mu, a}\phi)(x) \in H_\mu(I).$$

and obviously,

$$\int_{Q_m} |(h_{\mu, a}\phi)(x)|^p x^{\mu + 1/2}\, dx \leq \int_0^\infty |(h_{\mu, a_m}\phi)(x)|^p x^{\mu + 1/2}\, dx. \quad (3.3)$$

Since $a_m(x, \xi)$ has compact support with respect to x, it follows from the inversion formula for Hankel transformation and Fubini's theorem that

$$\begin{aligned}(h_{\mu, a_m}\phi)(x) &= \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi) a_m(x, \xi)(h_\mu\phi)(\xi)\, d\xi \\ &= \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi) \\ &\quad \times \left(\int_0^\infty (x\lambda)^{1/2} J_\mu(x\lambda)(h_\mu a_m)(\lambda, \xi)\, d\lambda \right)(h_\mu\phi)(\xi)\, d\xi \\ &= \int_0^\infty (x\lambda)^{1/2} J_\mu(x\lambda) \\ &\quad \times \left(\int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)(h_\mu a_m)(\lambda, \xi)(h_\mu\phi)(\xi)\, d\xi \right) d\lambda,\end{aligned} \quad (3.4)$$

where

$$(h_\mu a_m)(\lambda, \xi) = \int_0^\infty (x\lambda)^{1/2} J_\mu(x\lambda) a_m(x, \xi)\, dx. \quad (3.5)$$

The following estimate will be needed in the proof of the theorem.

LEMMA 3.4. *For all $\alpha \in \mathbf{N}_0$ and $N \in \mathbf{N}_0$, there is a positive constant $C_{\alpha, N}$, depending upon α and N only such that*

$$\left| \left(\xi^{-1} \frac{d}{d\xi} \right)^\alpha (h_\mu a_m)(\lambda, \xi) \right| \leq C_{\alpha, N}(1 + \xi)^{-2\alpha}(1 + \lambda^{2N})^{-1}, \quad (3.6)$$

where $\lambda, \xi, \in I$.

Proof. Let $\beta \in \mathbf{N}_0$ be arbitrary. Then proceeding as in the proof of Lemma 2.1 we have

$$(-\lambda^2)^{\beta}\left(\xi^{-1}\frac{d}{d\xi}\right)^{\alpha}(h_\mu a_m)(\lambda,\xi)$$

$$= (-\lambda^2)^{\beta}\left(\xi^{-1}\frac{d}{d\xi}\right)^{\alpha}\int_0^\infty (x\lambda)^{1/2}J_\mu(x\lambda)a_m(x,\xi)\,dx$$

$$= (-\lambda^2)^{\beta}\left(\xi^{-1}\frac{d}{d\xi}\right)^{\alpha}\int_0^\infty (x\lambda)^{1/2}J_\mu(x\lambda)a(x,\xi)\eta(x-m)\,dx$$

$$= \left(\xi^{-1}\frac{d}{d\xi}\right)^{\alpha}\int_0^\infty (x\lambda)^{1/2}J_\mu(x\lambda)S^{\beta}_{\mu,x}[a(x,\xi)\eta(x-m)]\,dx.$$

Now using the formula (1.17) and estimate (1.4) with $m = 0$ and the fact that $|x^{1/2}J_\mu(x)|$ is bounded by a constant A_μ for all x, we have

$$\left|(-\lambda^2)^{\beta}\left(\xi^{-1}\frac{d}{d\xi}\right)^{\alpha}(h_\mu a_m)(\lambda,\xi)\right|$$

$$\le \sum_{j=0}^{\beta}\sum_{r=0}^{j}\binom{j}{r}\int_0^\infty\left|\left(x^{-1}\frac{d}{dx}\right)^{\alpha}\eta(x-m)\right||b_j|A_\mu(1+\xi)^{-2\alpha}$$

$$\times x^{2j+\mu+1/2}(1+x)^{-q}\,dx$$

$$\le \sum_{j=0}^{\beta}\sum_{r=0}^{j}\binom{j}{r}A_{\mu,j,r,m}(1+\xi)^{-2\alpha}\int_0^\infty(1+x)^{-q+2j+2\mu+1}\,dx$$

$$\le A_{\beta,\mu,m}(1+\xi)^{-2\alpha},$$

since the integral is finite as q may be chosen large enough. Setting $\beta = 0$ and $\beta = N$ in turn we get the desired inequality:

$$\left|\left(\xi^{-1}\frac{d}{d\xi}\right)^{\alpha}(h_\mu a_m)(\lambda,\xi)\right| \le C_{\alpha,N}(1+\xi)^{-2\alpha}(1+\lambda^{2N})^{-1}.$$

∎

Now, this lemma and Theorem 3.2 imply that the operator $\phi \to \tilde{h}_{\mu,\lambda}\phi$, defined on H_μ by

$$\left(\tilde{h}_{\mu,\lambda}\phi\right)(x) = \int_0^\infty (x\xi)^{1/2}J_\mu(x\xi)(h_\mu a_m)(\lambda,\xi)(h_\mu\phi)(\xi)\,d\xi \quad (3.7)$$

can be extended as a bounded linear operator on $L^p_\mu(I)$. Moreover, proceeding as in the proof of Theorem 3.1 and using Lemma 3.4, for any positive integers N and k, we have

$$|h_\mu^{-1}\big(\xi^{\mu+1/2}(h_\mu a_m)(\lambda,\xi)\big)| \le B_{\mu,k,N}(1+x^2)^{-k}(1+\lambda^{2N})^{-1}.$$

Hence, using (1.9) we get

$$\begin{aligned}
&\big\|\big(\tilde{h}_{\mu,\lambda}\phi\big)(x)\big\|_{L^p_\mu(I)}\\
&\quad=\left\{\int_0^\infty x^{\mu+1/2}|h_\mu^{-1}\big(\xi^{\mu+1/2}(h_\mu a_m)(\lambda,\xi)\#\phi\big)(x)|^p\,dx\right\}^{1/p}\\
&\quad\le\big\|h_\mu^{-1}\big(\xi^{\mu+1/2}(h_\mu a_m)(\lambda,\xi)\big)\big\|_{L^1_\mu}\|\phi\|_{L^p_\mu}\\
&\quad\le B_{\mu,k,N}(1+\lambda^{2N})^{-1}\big\|(1+x^2)^{-k}\big\|_{L^1_\mu}\|\phi\|_{L^p_\mu}. \qquad (3.8)
\end{aligned}$$

Therefore, using (3.4) and (3.8) we get

$$\begin{aligned}
&\big\|(h_{\mu,a_m},\phi)(x)\big\|_{L^p_\mu(I)}\\
&\quad=\left\{\int_0^\infty x^{\mu+1/2}\left|\int_0^\infty (x\lambda)^{1/2}J_\mu(x\lambda)\big(\tilde{h}_{\mu,\lambda}\phi\big)(x)\,d\lambda\right|^p dx\right\}^{1/p}\\
&\quad\le A_\mu\int_0^\infty\left(\int_0^\infty x^{\mu+1/2}|\big(\tilde{h}_{\mu,\lambda}\phi\big)(x)|^p\,dx\right)^{1/p}d\lambda\\
&\quad\le A_\mu B'_{\mu,N}\left(\int_0^\infty(1+\lambda^{2N})^{-1}\,d\lambda\right)\|\phi\|_{L^p_\mu(I)}\\
&\quad\le C_{\mu,N}2^{N-1}\left(\int_0^\infty(1+\lambda^2)^{-N}\,d\lambda\right)\|\phi\|_{L^p_\mu}\\
&\quad\le\beta_{\mu,N}\|\phi\|_{L^p_\mu(I)},\qquad \phi\in H_\mu(I). \qquad (3.9)
\end{aligned}$$

Now, let $Q_m^* = [m, m+2]$ and $Q_m^{**} = [m, m+3]$ for $m \in \mathbf{N}_0$. Let $\psi \in C_0^\infty(I)$ be such that $0 \le \psi(x) \le 1$ for all $x \in I$, $\operatorname{supp}\psi \subseteq Q_m^*$, and $\psi(x) = 1$ in a neighbourhood $x \in Q_m^*$. Write $\phi = \phi_1 + \phi_2$, where $\phi_1 = \psi\phi$ and $\phi_2 = (1-\psi)\phi$.

Then

$$h_{\mu,a}\phi = h_{\mu,a}\phi_1 + h_{\mu,a}\phi_2.$$

Let us set

$$I_m = \int_{Q_m}\big|\big(h_{\mu,a}\phi\big)(x)\big|^p x^{\mu+1/2}\,dx$$

and

$$J_m = \int_{Q_m} |(h_{\mu,a}\phi_2)(x)|^p x^{\mu+1/2}\,dx.$$

Then

$$I_m = \int_{Q_m} |(h_{\mu,a}\phi_1)(x) + (h_{\mu,a}\phi_2)(x)|^p x^{\mu+1/2}\,dx. \tag{3.10}$$

By Theorem 2.2 and Lemma 2.1 there is a positive constant $C_{\mu,N}$ such that for all $x \in Q_m$,

$$\begin{aligned}
&|(h_{\mu,a}\phi_2)(x)| \\
&\quad \le \left| \int_{I-Q_m^*} \left(\int_0^\infty k(x,y) D_\mu(x,y,z)\,dy \right) \phi_2(z)\,dz \right| \\
&\quad \le \left| \int_{I-Q_m^*} \left(\int_0^\infty |k(x,y)| D_\mu(x,y,z)\,dy \right) |\phi_2(z)|\,dz \right| \\
&\quad \le \left(\int_{I-Q_m^*} \left(\int_0^\infty C_{\mu,N}(1+x)^{-q}(1+y^2)^{-k} D_\mu(x,y,z)\,dy \right) |\phi_2(z)|\,dz \right) \\
&\quad \le C_{\mu,N}(1+m)^{-q} \left(\int_{I-Q_m} \left(\int_0^\infty (1+y^2)^{-k} D_\mu(x,y,z)\,dy \right) |\phi_2(z)|\,dz \right).
\end{aligned}$$

Since, $x \in Q_m$ therefore,

$$(1+x)^{-q} \le (1+m)^{-q}.$$

Then

$$\begin{aligned}
&|(h_{\mu,a}\phi_2)(x)| \\
&\quad \le 2C_{\mu,N}(1+m)^{-q} \left(\int_0^\infty \left(\int_0^\infty (1+y^2)^{-k} D_\mu(x,y,z)\,dy \right) |\phi(z)|\,dz \right) \\
&\quad \le C'_{\mu,N}(1+m)^{-q} (f\#|\phi|)(x),
\end{aligned}$$

where $f = (1 + y^2)^{-k}$, $k > 1$. Therefore,

$$\int_{Q_m} |(h_{\mu,a}\phi_2)(x)|^p x^{\mu+1/2}\, dx$$

$$\le \left(C'_{\mu,N}\right)^p (1+m)^{-qp} \int_{Q_m} |(f\#|\phi|)(x)|^p x^{\mu+1/2}\, dx$$

$$\le (1+m)^{-qp} \left(C'_{\mu,N}\right) \left(\|f\|_{L^1_{\mu(1)}}\right)^p \left(\|\phi\|_{L^p_{\mu(I)}}\right)^p. \tag{3.11}$$

Then from (3.9), using (3.10) we have

$$\int_{Q_m} |(h_{\mu,a}\phi)(x)|^p x^{\mu+1/2}\, dx \le (1+m)^{-qp} D_{\mu,N,P} \left(\|\phi\|_{L^p_{\mu(I)}}\right)^p.$$

Summing over m, we have

$$\int_0^\infty |(h_{\mu,a}\phi)(x)|^p x^{\mu+1/2}\, dx \le 2^p (1 + D_{\mu,N,p}) \times \left(\sum_{m=0}^{\infty} (1+m)^{-qp} \right) \left(\|\phi\|_{L^p_{\mu(I)}}\right)^p. \tag{3.12}$$

Since q may be chosen large, we get the desired inequality for all $\phi \in H_\mu(I)$. Moreover, $H_\mu(I)$ is dense in $L^p_\mu(I)$ [11, p. 108] and the result (3.11) can be extended to all $\phi \in L^p_\mu(I)$. ∎

4. AN APPLICATION

In this section an application of Theorem 3.3 is given.

THEOREM 4.1. *Let $a(x, \xi)$ be a symbol in H^m, $s \in \mathbf{R}$, $\mu \ge -1/2$, and $1 < p < \infty$. Then the p.d.o. $h_{\mu,a}$ is a bounded linear operator $W_\mu^{m,p} \to W_\mu^{0,p}$ and also from*

$$W_\mu^{s,p} \to W_\mu^{s-m,p}.$$

Proof. We consider at first the following linear operators:

$$H_\mu^{-s}\colon W_\mu^{s,p} \to W_\mu^{0,p}$$

$$h_{\mu,a} H_\mu^m\colon W_\mu^{0,p} \to W_\mu^{0,p}$$

$$H_\mu^{s-m}\colon W_\mu^{0,p} \to W_\mu^{s-m,p}.$$

The first and third operators are bounded [11, p. 101] and the second operator is bounded by Theorem 3.3. Therefore $H_\mu^{s-m} h_{\mu,a} H_\mu^{m-s}$ is a bounded linear operator from $W_\mu^{s,p}$ into $W_\mu^{s-m,p}$. Also by [11, p. 101] the operators $H_\mu^{m-s}: W_\mu^{s,p} \to W_\mu^{m,p}$ and $H_\mu^{s-m}: W_\mu^{0,p} \to W_\mu^{s-m,p}$ are isometric and onto. Hence $h_{\mu,a}: W_\mu^{m,p} \to W_\mu^{0,p}$ must be a bounded linear operator.

To prove the second part we note that $H_\mu^{m-s} h_{\mu,a}$ is a pseudo-differential operator with symbol in H^s. Hence we can find a positive constant B such that

$$\|(h_{\mu,a}\phi)(x)\|_{s-m,p} = \|H_\mu^{m-s}(h_{\mu,a}\phi)(x)\|_p \leq B\|\phi\|_{s,p} \qquad \forall \phi \in W_\mu^{s,p}.$$

∎

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