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Schur and Weyl Functors

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0. GENERAL INTRODUCTION

The Schur and Weyl functors are the functorial generalisation of the Schur respectively Weyl modules in the representation theory of general linear groups, this also explains their names. Both types of functors are defined over any commutative ring with 1, and they are parametrised by Young diagrams. They were defined and studied in [20, 1], further work can be found in [2, 5].

The Schur and Weyl functors are endofunctors on the category of finitely generated projective modules, and they are universally defined, i.e., they commute with change of base ring. Special cases of Schur functors are the symmetric and exterior powers, in fact these are the extreme cases in some sense. Weyl functors are the duals of Schur functors in a natural sense, the divided power is the dual of the symmetric power in this sense and the exterior power is self-dual. Both Schur and Weyl functors also arise naturally in the study of exterior and symmetric powers.

To illustrate the relevance of Schur and Weyl functors in multilinear algebra and in general representation theory of groups, we mention the following: For each Young diagram there is a special universally defined natural transformation from the Weyl to the Schur functor which can be characterised up to sign. In case the base ring contains the rationals it is in fact an equivalence. The images of these transformations constitute over an infinite field a complete irredundant system of irreducible polynomial endofunctors on the category of finite dimensional vector spaces. And when applied to the natural representation of the endomorphism monoid, or the automorphism group, of a finite dimensional vector space, they give a complete set of irreducible polynomial representations of this monoid respectively group. (For the definition of polynomial functors and representations see [18, 13].) Moreover when the base field is finite similar results hold for the natural transformations for a certain subset of the set of all Young

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diagrams. And in this case the polynomial condition is not a restrictive condition. The above results will be proved in a planned follow up of this paper; see however [17].

Since the Schur and Weyl functors are universally defined general results about them rely on combinatorial observations. They are elementary in fact and go back to the 19th century. The combinatorics we need to deal with Schur functors is *precisely* given by letter place algebras, as developed in [12, 7, 8]. For Weyl functors we shall develop another, but in many respects similar, kind of letter place algebra based on exterior powers, as opposed to symmetric powers which are used for ordinary letter place algebras. For recent generalisations of these algebras see [A_1, A_2, A_3].

This paper is divided into two chapters. In the first chapter we construct explicit universal functorial generalisations of the classical branching and Clebsch–Gordan rule for Schur modules, generalising results in [1, 2, 5]. In our setup the Clebsch–Gordan rule is a special case of the branching rule. The corresponding rules for Specht modules for the symmetric groups (Murnaghan–Nakayama resp. Littlewood–Richardson rule) can be seen as summands, thus generalising [8, 14].

In the second chapter we prove similar results but now for Weyl functors, generalising [1, 4]. Each chapter has its own introduction.

We observe that the filtrations involving Schur and Weyl functors constructed in the first two papers are in agreement with more general existence theorems for algebraic groups, as in [9, 10]. Our filtrations make perfect sense over non-commutative rings with 1 provided one restricts oneself to *two*-sided summands of finite direct sums of copies of the base ring (as opposed to finitely generated projective modules).

The results in this paper form part of the author's thesis [15].

Some general notation:

A: a commutative ring with 1.

A-mod: the category of finitely generated *projective* A-modules.

M, N: elements of A-mod.

$\mathbb{N}_0$: $\mathbb{N} \cup \{0\}$.

n: an element of $\mathbb{N}_0$.

Λ^n, S^n, D^n: the nth exterior, symmetric respectively divided power seen as endofunctors on A-mod.

S_n: the symmetric group on $\{1, 2, ..., n\}$.

$M(n, A)$: the monoid of $n \times n$ matrices over A.

I. Filtrations Involving Schur Functors

As mentioned in the general introduction, we shall, in this first chapter, use letter place algebras as the combinatorial devise to derive results for Schur functors. Basic are the results that a Schur functor is universally free on free modules and that the symmetric power applied on a tensor product admits a universal filtration with as subquotients tensor products of Schur functors. This latter results generalises Cauchy's determinental formula and shows that Schur functors arise naturally in the study of symmetric powers. Typical consequences are a filtration for tensor products of symmetric powers and for "good" base rings also, see [16], a formula for a Schur functor applied on a tensor product of symmetric powers (so-called (outer) plethysms).

We shall give a universal filtration for a Schur functor applied to a direct sum with a subquotients tensor products of Schur functors again. Immediate consequences are the branching and Clebsch–Gordan rule for Schur modules for general linear groups.

I.1. *The Combinatorics*

We start by summarising the results for letter place algebras we shall need and, to make this possible, a checklist for the notation.

[7, pp. 163–165]: $\langle\ \rangle_A$, $\underline{k}$, $\alpha \models k$, S_α, $\lambda \vdash k$, $\alpha \leqslant \beta$, $\alpha \trianglelefteq \beta$, T_λ, $\overset{*}{T}_\lambda$, $\sim_c$, $\leqslant_c$, $\leqslant_r$, $T^\lambda\ (\leqslant m)$, $ST^\lambda(\leqslant m)$, $T^\lambda(\alpha)$, $ST^\lambda(\alpha)$, (S, T), $BT^\lambda(\alpha, \beta)$, $SBT^\lambda(\alpha, \beta)$, $BT(\alpha, \beta)$, $SBT(\alpha, \beta)$, $A_m{}^n$, $(S \mid T)$, $D'_L(a', a)$, $D_P{}^r(b', b)$. In case $\alpha \models k$, or $\lambda \vdash k$, we shall allow k to be zero. A proper partition is called a *Young diagram*. When $\lambda \vdash 0$ we put $\lambda = (0)$ and $\lambda_1 = 0$. The associate of a Young diagram λ shall be called the *conjugate* of λ and is denoted by $\tilde{\lambda}$. When $\lambda \vdash 0$, $\tilde{\lambda} \vdash 0$. The lexicographic order for partitions shall be extended: if $\alpha \models k$ and $\beta \models l$ then $\alpha \leqslant \beta$ means $k < l$ or ($k = l$ and $\alpha \leqslant \beta$). For any partition α, T_α and $\overset{*}{T}_\alpha$ shall have the obvious meaning. We view $A_m{}^n$ as a left $A[\mathrm{End}_A(A^m) \times \mathrm{End}_A(A^n)]$-algebra in the obvious way, extending the action of $Gl(m, A) \times Gl(n, A)$ in [7].

[7, p. 169]: By $C_L(S)$ and $C_P(T)$ we shall mean the decoulered Capelli operators $\delta \circ C_P(T)$.

[7, pp. 172, 173]: $(\boxed{S} \mid T)$, $(S \mid \boxed{T}\,)$, $(\boxed{S} \mid \boxed{T}\,)$. By $D_L(S, T_\lambda)$, $D_P(T, T_\lambda)$, etc., we shall mean the decoulered operators $\delta \circ\ D_L(S, T_\lambda)$ respectively $\delta \circ D_P(T, \overset{*}{T}_\lambda)$.

[7, p. 186]: For $S \in T(1^n)$, $P(S)$ and $Q(S)$ denote $\underline{H(S)}$, respectively $\underline{V(S)}^A$.

[8, p. 163]: T^{tr}.

[8, p. 167]: $\nu\backslash\lambda$, $T^{\nu\backslash\lambda}$ $(\leqslant m)$, $T^{\nu\backslash\lambda}(\alpha)$, $ST^{\nu\backslash\lambda}$ $(\leqslant m)$, $ST^{\nu\backslash\lambda}(\alpha)$, τ_k. We shall delete the brackets in $(\nu)\supset(\lambda)$, $(\nu)\backslash(\lambda)$, $T|_{(\lambda)}$, and $T|_{(\nu)\backslash(\lambda)}$. Furthermore, $BT^{\nu\backslash\lambda}(\alpha,\beta)$, $\underset{*}{S}BT^{\nu\backslash\lambda}$ $(\leqslant m,\beta)$, etc., shall have the obvious meaning, similarly for $T_{\nu\backslash\lambda}$, $\overset{*}{T}_{\nu\backslash\lambda}$, $C_P(\overset{*}{T}_{\nu\backslash\lambda})$, $D_L(S, T_{\nu\backslash\lambda})$, etc. In the following cases we allow λ to be any partition: $\nu\supset\lambda$, $\nu\backslash\lambda$, $T^{\nu\backslash\lambda}(...)$, $BT^{\nu\backslash\lambda}(...,...)$, $T_{\nu\backslash\lambda}$ and $\overset{*}{T}_{\nu\backslash\lambda}$. The orders $\leqslant_r$ and $\leqslant_c$ shall be extended to skew shapes $\nu\backslash\lambda$ in the obvious way. Since $\nu\backslash\lambda$ can be seen as a partition as well, namely $(\nu_1-\lambda_1, \nu_2-\lambda_2, ..., \nu_{\tilde{\lambda}_1}-\lambda_{\tilde{\lambda}_1}, \nu_{\tilde{\lambda}_1+1}, ..., \nu_{\tilde{\nu}_1})$, bideterminants for bitableaux of skew shape make perfect sense, as well as symmetrised bideterminants.

We have summarised the results we use about bideterminants, and operators on them, in the following theorem. (All bideterminants are computed in a letter place algebra where they make sense.

THEOREM 1.1. *Let $r, s\in\mathbb{N}_0$, let ν and λ be Young diagrams with $\nu\supsetneqq\lambda$, let $\gamma\models n$, $\delta\models n$ and let $(U, T)\in BT^{\nu\backslash\lambda}(\gamma,\delta)$.*

(a) (Straightening, [19] or [12]). Suppose $\lambda=(0)$ and $\tilde{\nu}_1\geqslant 2$ and let $i\in\{1,2,...,\tilde{\nu}_1-1\}$ and $j\in\{1,2,...,\nu_{i+1}\}$. Write the ih and $(i+1)$th row of U like this, where $d=\nu_{i+1}-j$:

$$\begin{matrix} a_1 & a_2 & \cdots & a_{j-1} & c_{j+1} & c_{j+2} & \cdots & c_{\nu_i+1} \\ c_1 & c_2 & \cdots & c_{j-1} & c_j & b_1 b_2 & \cdots & b_d. \end{matrix}$$

Suppose one has

$$\begin{matrix} a_1 < a_2 < \cdots < a_{j-1} < c_{j+1} < c_{j+2} < \cdots < c_{\nu_i+1} \\ \text{/\!\!\wedge} \quad \text{/\!\!\wedge} \qquad\qquad \text{/\!\!\wedge} \quad\; \vee \qquad\qquad\qquad \\ c_1 < c_2 < \cdots < c_{j-1} < \; c_j \; < \; b_1 \; < \cdots < b_d \end{matrix}$$

then

$$\sum_{\sigma\in R} \operatorname{sgn}(\sigma)\cdot(U^\sigma\,|\,T)\in\langle(V\,|\,W)\,|\,(V,W)\in BT^\rho(\gamma,\delta)\text{ for some }\rho\vdash n$$
$$\text{with }\rho>\nu\rangle_{\mathbb{Z}\cdot 1_A}.$$

Here $R=\{\sigma\in S_{\nu_i+1}\,|\,\sigma(1)<\sigma(2)<\cdots<\sigma(j)$ and $\sigma(j+1)<\sigma(j+2)<\cdots<\sigma(\nu_i+1)\}$ and U^σ is the tableau obtained from U by replacing, in the ith and $(i+1)$th row, c_l by $c_{\sigma(l)}$ for all l. Observe that $U^\sigma<_r U$ if $\sigma\neq 1$.

In case $T=T_\nu$ the span is zero.

Similar results hold for the place side.

(b) ([19] or [12]) If $\lambda=(0)$ then

$$(U\mid T)\in\langle (V\mid T)\mid V\in ST^{\nu}(\gamma),\ V\leqslant_r U\rangle_{\mathbb{Z}\cdot 1_A}$$
$$+\ \langle (V\mid W)\mid (V,W)\in SBT^{\rho}(\gamma,\delta)\text{ for some }\rho\vdash n\text{ with }\rho>\nu\rangle_{\mathbb{Z}\cdot 1_A}.$$

In case $T=T_\nu$ the second span is zero.
Similar results hold for the place side.

(c) (Compare [12].) Suppose U is standard. If $\lambda=(0)$ or the entries of U are mutually distinct then $C_L(U)(U\mid T)=(T_{\nu\setminus\lambda}\mid T)$. If $\lambda=(0)$ and $U'\in ST^{\nu}(\gamma)$ with $U'>_c U$ then $C_L(U)(U'\mid T)=0$.
Similar results hold for the place side.

(d) ([19] or [12].) $\{(S\mid T)\mid (S,T)\in\bigcup_{\nu\vdash n} SBT^{\nu}\ (\leqslant r,\leqslant s)\}$ is an A-independent system.

(e) (Compare [1], or see below.) Put $\alpha=(\lambda_1,\lambda_2,\ldots,\lambda_{\tilde{\lambda}_1},0^{\tilde{\nu}_1-\tilde{\lambda}_1})$, $\beta=\nu\setminus\lambda$ as partitions. Suppose there is an $i\in\{1,2,\ldots,\tilde{\nu}_1-1\}$ such that $\beta_i\neq 0$, $\beta_{i+1}\neq 0$, and $\nu_{i+1}>\alpha_i$, (i.e., the ith and $(i+1)$th and $(i+1)$th row meet). Let $j\in\{1,2,\ldots,\beta_{i+1}\}$ with $j>\alpha_i-\alpha_{i+1}$ and write the ith and $(i+1)$th row of U like

$$\begin{array}{ccccccccccc} & & a_1 & a_2 & \cdots & a_e & c_{j+1} & c_{j+2} & \cdots & & c_f \\ c_1c_2 & \cdots & c_{j-e} & c_{j-e+1} & \cdots & c_{j-1} & c_j & b_1 & b_2 & \cdots & b_d \end{array},$$

where $d=\beta_{i+1}-j$, $e=j-1-(\alpha_i-\alpha_{i+1})$ and $f=\beta_i+j-e$.
Suppose one has

$$\begin{array}{ccccccccccccccc} & & a_1 & < & a_2 & < \cdots < & a_e & < c_{j+1} < c_{j+2} < & \cdots & < c_f \\ & & \geqslant & & \geqslant & & \geqslant & \vee & & \\ c_1<c_2<\cdots<c_{j-e} & < & c_{j-e+1} & < \cdots < & c_{j-1} & < & c_j & < b_1 & <b_2<\cdots & <b_d \end{array}$$

then $\sum_{\sigma\in R}\operatorname{sgn}(\sigma)(U^{\sigma}\mid T_{\nu\setminus\lambda})=0$. (This generalises (a) in case $\lambda=(0)$.)

Here $R=\{\sigma\in S_f\mid \sigma(1)<\sigma(2)<\cdots<\sigma(j)$ and $\sigma(j+1)<\sigma(j+2)<\cdots<\sigma(f)\}$ and U^{σ} is the tableau obtained from U by replacing, in the ith and $(i+1)$th row, c_l by $c_{\sigma(l)}$ for all l. Observe that $U^{\sigma}<_r U$ for $\sigma\neq 1$.

Always, $(U\mid T_{\nu\setminus\lambda})\in\langle (V\mid T_{\nu\setminus\lambda})\mid V\in ST^{\nu\setminus\lambda}(\gamma),\ V\leqslant_r U\rangle_{\mathbb{Z}\cdot 1_A}$.

And $\{(S\mid T_{\nu\setminus\lambda})\mid S\in ST^{\nu\setminus\lambda}\ (\leqslant r)\}$ is an A-independent system.

Similar results hold for the place side.

(f) (Compare [6], [7], and [1], see the second chapter for the relevance of the latter.) Put $\alpha=(\tilde{\lambda}_1,\tilde{\lambda}_2,\ldots,\tilde{\lambda}_{\lambda_1},0^{\nu_1-\lambda_1})$ and $\beta=\tilde{\nu}\setminus\tilde{\lambda}$, as partitions. Suppose there is an $i\in\{1,2,\ldots,\nu_1-1\}$ such that $\beta_i\neq 0$, $\beta_{i+1}\neq 0$ and $\tilde{\nu}_{i1}>\alpha_i$ (i.e., the ith and $(i+1)$th column meet). Let $j\in\{1,2,\ldots,\beta_{i+1}\}$ be such that $j>\alpha_i-\alpha_{i+1}$ and let $k\in\{0,1,2,\ldots,\beta_{i+1}-j\}$ and write the ith and $(i+1)$th column of U like

$$
\begin{array}{cc}
 & c_1 \\
 & c_2 \\
 & \vdots \\
a_1 & c_{j-e} \\
a_2 & \\
\vdots & \vdots \\
a_e & c_{j-1} \\
c_{j+k+1} & c_j \\
c_{j+k+2} & \vdots \\
\vdots & c_{j+k} \\
\vdots & b_1 \\
\vdots & b_2 \\
\vdots & \vdots \\
 & b_d \\
c_f &
\end{array}
\quad , \qquad \text{where} \quad
\begin{array}{l}
d=\beta_{i+1}-(j+k) \\
e=j-1-(\alpha_i-\alpha_{i+1}) \\
f=\beta_i-e+j+k
\end{array}
$$

Suppose one has

$$
\begin{array}{ccc}
 & & c_1 \\
 & & \rotatebox{90}{$\geqslant$} \\
 & & c_2 \\
 & & \rotatebox{90}{$\geqslant$} \\
 & & \vdots \\
 & & \rotatebox{90}{$\geqslant$} \\
a_1 & < & c_{j-e} \\
\rotatebox{90}{$\geqslant$} & & \rotatebox{90}{$\geqslant$} \\
a_2 & < & c_{j-e+1} \\
\rotatebox{90}{$\geqslant$} & & \rotatebox{90}{$\geqslant$} \\
\vdots & \vdots & \vdots \\
\rotatebox{90}{$\geqslant$} & & \rotatebox{90}{$\geqslant$} \\
a_e & < & c_{j-1} \\
 & & \rotatebox{90}{$\geqslant$} \\
c_{j+k+1} & \geqslant & c_j \\
\rotatebox{90}{$\geqslant$} & & \| \\
c_{j+k+2} & & c_{j+1} \\
\rotatebox{90}{$\geqslant$} & & \| \\
\cdot & & \vdots \\
\cdot & & \| \\
\cdot & & c_{j+k} \\
\cdot & & \wedge \\
\cdot & & b_1 \\
\cdot & & \rotatebox{90}{$\geqslant$} \\
\cdot & & b_2 \\
\cdot & & \rotatebox{90}{$\geqslant$} \\
\cdot & & \vdots \\
\cdot & & \rotatebox{90}{$\geqslant$} \\
\rotatebox{90}{$\geqslant$} & & b_d \\
c_f & &
\end{array}
$$

Then $\sum_{\sigma \in Q} r_\sigma \cdot (\boxed{{}^{\sigma}U} \mid T) = 0$.

Here Q is a transversal for S_f/S_ε, where $m=b_d$ and $\varepsilon\in(\mathbb{N}_0)^m$ is such that $\varepsilon_l=\#\{t\in\underline{f}\mid c_t=l\}$ all $l\in\underline{m}$. And ${}^\sigma U$ is the tableau obtained from U by replacing, in the ith and $(i+1)$th column, c_t by $c_{\sigma(t)}$ for all $t\in\underline{f}$. Observe that $a_e<c_{j+k+1}$ and ${}^\sigma U<_c U$ for $\sigma\notin S_\varepsilon$. In order to define $r_\sigma\in\mathbb{Z}\cdot 1_A$ let A, ${}^\sigma A$, B, ${}^\sigma B\in(\mathbb{N}_0)^m$ be such that $A_l=\#\{t\mid a_t=l\}$, ${}^\sigma A_l=\#\{t\in\underline{f}\setminus\underline{j+k}\mid c_{\sigma(t)}=l\}$, $B_l=\#\{t\in\underline{d}\mid b_t=l\}$, and ${}^\sigma B_l=\#\{t\in\underline{j+k}\mid c_{\sigma(t)}=l\}$, for $l\in\underline{m}$. Then

$$r_\sigma=\prod_{l\in\{a_1,a_2,\ldots,a_e\}}\left(\binom{A_l+{}^\sigma A_l}{A_l}\cdot\prod_{l\in\{b_1,b_2,\ldots,b_d\}}\binom{B_l+{}^\sigma B_l}{B_l}\right).$$

Moreover $(\boxed{U}\mid T)\in\langle(\boxed{S}\mid T\mid S\in ST_{\nu\setminus\lambda}(\gamma),\ S\leqslant_c U\rangle_{\mathbb{Z}\cdot 1_A}$, and

$$(\boxed{U}\mid T)\in\langle(\boxed{U}\mid W)\mid W\in ST^{\nu\setminus\lambda}(\delta),\ W\leqslant_r T\rangle_{\mathbb{Z}\cdot 1_A}.$$

Also, $\{\boxed{S}\mid \overset{*}{T}_{\nu\setminus\lambda})\mid S\in ST^{\nu\setminus\lambda}\ (\leqslant r)\}$ is an A-independent system.

Similar results for the place side.

(g) (compare [7].) $D_L(U,\overset{*}{T}_{\nu\setminus\lambda})(\overset{*}{T}_{\nu\setminus\lambda}\mid T)=(U\mid T)$,

$$D_L(U,T_{\nu\setminus\lambda})(T_{\nu\setminus\lambda}\mid T)=(\boxed{U}\mid T).$$

Similar results hold for the place side.

Remark. The reader is not supposed to understand the proofs of the above results except the following. The formula in (e) implies the result for $(U\mid T_{\nu\setminus\lambda})$ and the formula in (f) implies the first result about $(U\mid \overset{*}{T}_{\nu\setminus\lambda})$. To see this observe that when in a bitableau there is a tableau with two equal elements in one of its rows then its bideterminant is zero. Also rearranging in a tableau, in a bitableau, the elements of one of its rows in increasing order makes the tableau smaller, if something has happened, and changes the bideterminant by the signature of the permutation used. Hence, for arbitrary U, $(U\mid T_{\nu\setminus\lambda})=\pm(W\mid T_{\nu\setminus\lambda})$, where $W\leqslant_r U$ and either W is standard or the formula in (e) applies to W. In this last case the formula implies $(W\mid T_{\nu\setminus\lambda})\in\langle(V\mid T_{\nu\setminus\lambda})\mid V\leqslant_r W$ and either V is standard or the formula is (e) applies to $V\rangle_{\mathbb{Z}\cdot 1_A}$. Hence, by an induction argument

$$(U\mid T_{\nu\setminus\lambda})\in\langle(V\mid T_{\nu\setminus\lambda})\mid V\in ST^{\nu\setminus\lambda}(\gamma),\ V\leqslant_r U\rangle_{\mathbb{Z}\cdot 1_A}.$$

Concerning (f) one can argue in a similar way with the difference that now the column lexicographic order is important and rearranging elements in columns of the symmetrised tableau does not change the symmetrised bideterminant.

We shall also assume that the reader is able to prove (g).

We shall now make the connection between [1] and letter place algebras. Let $\nu \vdash n$, $a \in \mathbb{N}_0$, $a \leqslant n$, $\lambda \vdash a$. Put $\alpha = \tilde{\nu} \backslash \tilde{\lambda}$ and $\beta = \nu \backslash \lambda$, as partitions. Define natural transformations $\bigotimes_{i=1}^{\nu_1} \Lambda^{\alpha_i} \to (-)^{\otimes (n-a)} \to \bigotimes_{j=1}^{\nu_1} S^{\beta_j}$ as follows. Define the first one by ($m_{i,j} \in M$ all i, j):

$$\bigotimes_{i=1}^{\nu_1} (m_{i,1} \otimes m_{i,2} \otimes \cdots \otimes m_{i,\alpha_i}) \mapsto \sum_{\sigma \in S_\alpha} \left(\operatorname{sgn}(\sigma) \bigotimes_i m_{i,\sigma_i(1)} \otimes \cdots \otimes m_{i,\sigma_i(\alpha_i)} \right).$$

And $\otimes_i x_i$, for $x_i \in M^{\otimes \alpha_i}$ all $i \leqslant \nu_1$, means the image of $x_1 \otimes x_2 \otimes \cdots \otimes x_{\nu_1}$ in $M^{\otimes (n-\alpha)}$. Define the second one by ($m_i \in M$ all i):

$$m_1 \otimes m_2 \otimes \cdots \otimes m_{n-a} \mapsto \bigotimes_{j=1}^{\tilde{\nu}_1} (m_{1,j} \cdot m_{2,j} \cdot \cdots \cdot m_{\beta_j, j}), \qquad \text{where for all } (i, j),$$

$m_{i,j} = m_{l(i,j)}$ with $l(i, j) = \sum_{k=1}^{n(i,j)-1} \alpha_k + j - (\tilde{\nu}_{n(i,j)} - \alpha_{n(i,j)})$, where $n(i,j) = \nu_j - \beta_j + i$. To understand this second map imagine the elements $m_1, m_2, \ldots, m_{n-a}$ are the entries of a tableau of shape $\tilde{\nu} \backslash \tilde{\lambda}$,

$$\begin{matrix} m_1 m_2 & \cdots & m_{\alpha_1} \\ m_{\alpha_1+1} & \cdots & m_{\alpha_1+\alpha_2} \\ \vdots & & \vdots \\ m_{c+1} & \cdots & m_{n-a} \end{matrix}, \qquad \text{where} \quad c = \sum_{i=1}^{\nu_1 - 1} \alpha_i.$$

Then the second map constructs for each column the product (in the symmetric algebra) of its entries and then constructs the tensor products of the results.

Let us denote, as in [1], the composition of the above transformations by $d_{\tilde{\nu} \backslash \tilde{\lambda}}$ then by Theorem 1.1 (c) there is a commutative diagram, for each $r \in \mathbb{N}_0$,

$$\begin{array}{ccc} \Lambda^{\alpha_1}(A^r) \otimes \Lambda^{\alpha_2}(A^r) \otimes \cdots \otimes \Lambda^{\nu\alpha_{\nu_1}}(A^r) & \xrightarrow{d_{\tilde{\nu}\backslash\tilde{\lambda}}(A^r)} & S^{\beta_1}(A^r) \otimes \cdots \otimes S^{\beta_{\tilde{\nu}_1}}(A^r) \\ f \downarrow \cong & & g \downarrow \cong \\ \langle (S \mid \overset{*}{T}_{\tilde{\nu}\backslash\tilde{\lambda}}) \mid S \in ST^{\tilde{\nu}\backslash\tilde{\lambda}}(\leqslant r) \rangle_A & \xrightarrow{C_P(\overset{*}{T}_{\tilde{\nu}\backslash\tilde{\lambda}})} & \langle (S \mid T) \mid (S, T) \in BT(\leqslant r, \beta) \rangle_A, \end{array}$$

where, of course, f and g are the natural A-linear maps defined by

$$\bigotimes_{i=1}^{\nu_1} (e_{n_{i,1}} \wedge e_{n_{i,2}} \wedge \cdots \wedge e_{n_{i,\alpha_i}}) \mapsto \left(\begin{matrix} n_{1,1} n_{1,2} & \cdots & n_{1,\alpha_1} \\ n_{2,1} n_{2,2} & \cdots & n_{2,\alpha_2} \\ \vdots & & \vdots \\ n_{\nu_1,1} n_{\nu_1,2} & \cdots & n_{\nu_1,\alpha_{\nu_1}} \end{matrix} \middle| \overset{*}{T}_{\tilde{\nu}\backslash\tilde{\lambda}} \right),$$

respectively

$$\bigotimes_{j=1}^{\nu_1} (e_{m_{1,j}} \cdot e_{m_{2,j}} \cdots e_{m_{\beta_j,j}}) \mapsto \prod_j \begin{pmatrix} m_{1,j} & j \\ m_{2,j} & j \\ \vdots & \vdots \\ m_{\beta_j,j} & j \end{pmatrix}.$$

Here $n_{i,j}$, $m_{i,j} \in \{1, 2, ..., r\}$ all i, j and $(e_1, e_2, ..., e_r)$ is the natural basis of A^r. By Theorem 1.1(e), $\operatorname{Im} C_P(\overset{*}{T}_{\tilde{\nu}\setminus\tilde{\lambda}}) = \langle (S \mid T_{\tilde{\nu}\setminus\tilde{\lambda}}) \mid S \in ST^{\tilde{\nu}\setminus\tilde{\lambda}} (\leqslant r) \rangle_A$, which is (universally) free by Theorem 1.1(e). Slightly generalising [1]:

Definition. Let ν and λ be Young diagrams with $\nu \supset \lambda$, then $S_A^{\nu\setminus\lambda} := \operatorname{Im}(d_{\tilde{\nu}\setminus\tilde{\lambda}} : A\text{-mod} \to A\text{-mod})$, and $S_A^{\nu\setminus\lambda}$ shall be called the (*skew*) *Schur functor* for the (skew) diagram $\nu\setminus\lambda$ over A.

Observe that the image of $S_A^{\nu\setminus\lambda}$ is indeed in A-mod because $S_A^{\nu\setminus\lambda}$ is a functor and $S_A^{\nu\setminus\lambda}(M)$ is free of finite rank when M is. Also $B \otimes S_A^{\nu\setminus\lambda} \cong S_B^{\nu\setminus\lambda}(B \otimes -)$ for a unitary commutative A-algebra B. We shall drop the suffix "A" in $S_A^{\nu\setminus\lambda}$ when there is no danger of confusion. Clearly $S^\lambda(A^m) = 0$ when $\tilde{\lambda}_1 > m$.

Examples. $S^{(n)} = S^n$ and $S^{(1^n)} \cong \Lambda^n$ (the "extreme" cases).

Remark. $S^{\nu\setminus\lambda}(A^r)$ is the Schur module $L_{\tilde{\nu}\setminus\tilde{\lambda}}(A^r)$ in [1], it corresponds to $schur_A(\tilde{\nu}\setminus\tilde{\lambda})$ in [3], and, if $\lambda = (0)$, to $\mathscr{W}_r^\nu(A)$ in [7]. Also $S^\nu = \bigwedge^{\tilde{\nu}}$ in the notation of [20]. We use $S^{\nu\setminus\lambda}(A^r)$ rather than $S^{\tilde{\nu}\setminus\tilde{\lambda}}(A^r)$ to stay in line with the notation for Schur modules (or: dual Weyl modules, or: induced modules) in the representation theory of $Gl(r, A)$, where A^r is the natural representation of $Gl(r, A)$.

Consider the diagram involving $d_{\tilde{\nu}\setminus\tilde{\lambda}}(A^r)$ for $r = n - a$.

Clearly, $\langle (S \mid \overset{*}{T}_{\tilde{\nu}\setminus\tilde{\lambda}}) \mid S \in T^{\tilde{\nu}\setminus\tilde{\lambda}}(1^{n-a}) \rangle_A \cong A[S_{n-a}] \otimes_{A[S_\alpha]} {}_{\text{sgn}}A$ as $A[S_{n-a}]$-module, where ${}_{\text{sgn}}A$ is one-dimensional representation of S_α with as character the signature. Also $\langle (S \mid T) \mid (S, T) \in BT(1^{n-a}, \beta) \rangle_A \cong A[S_{n-a}] \otimes_{A[S_\beta]} {}_{\text{triv}}A$ as $A[S_{n-a}]$-module where ${}_{\text{triv}}A$ is the trivial representation of S_β.

Convention (Notation as Above). We shall identify $A[S_{n-a}] \otimes_{A[S_\alpha]} {}_{\text{sgn}}A$ with the subspace of $\bigotimes_i \Lambda^{\alpha_i}(A^{n-a})$ corresponding to $\langle (S \mid \overset{*}{T}_{\tilde{\nu}\setminus\tilde{\lambda}}) \mid S \in T^{\tilde{\nu}\setminus\tilde{\lambda}}(1^{n-a}) \rangle_A$, via f. Similarly we identify $A[A_{n-a}] \otimes_{A[S_\beta]} {}_{\text{triv}}A$ with the subspace of $\bigotimes_j S^{\beta_j}(A^{n-a})$ corresponding to $\langle (S \mid T) \mid (S, T) \in BT(1^{n-a}, \beta) \rangle_A$.

So there is a commutative diagram:

$$\begin{array}{ccc} A[S_{n-a}] \otimes_{A[S_\alpha]} ({}_{\text{sgn}}A) & \xrightarrow{e_{\tilde{\nu}\setminus\tilde{\lambda}}} & A[S_{n-a}] \otimes_{A[S_\beta]} ({}_{\text{triv}}A) \\ \downarrow \cong & & \downarrow \cong \\ \langle (S \mid \overset{*}{T}_{\tilde{\nu}\setminus\tilde{\lambda}}) \mid S \in ST^{\tilde{\nu}\setminus\tilde{\lambda}}(1^{n-a}) \rangle_A & \xrightarrow{C_P(\overset{*}{T}_{\tilde{\nu}\setminus\tilde{\lambda}})} & \langle (S \mid T) \mid (S, T) \in BT(1^{n-a}) \rangle_A. \end{array}$$

By Theorem 1.1(e), $\operatorname{Im} C_P(\overset{*}{T}_{\tilde{\nu}\setminus\tilde{\lambda}}) = \langle (S \mid T_{\tilde{\nu}\setminus\tilde{\lambda}}) \mid S \in ST^{\tilde{\nu}\setminus\tilde{\lambda}}(1^{n-a}) \rangle_A$ which is (universally) free by Theorem 1.1(e).

DEFINITION. Let ν and λ be Young diagrams with $\nu \supset \lambda$, then $\mathscr{S}_{\nu\setminus\lambda}(A) := \operatorname{Im}(e_{\tilde{\nu}\setminus\tilde{\lambda}})$, and $\mathscr{S}_{\nu\setminus\lambda}(A)$ is called the (*skew*) *Specht module* of S_{n-a} for the (skew) diagram $\nu\setminus\lambda$ over A. It is denoted by $S^{\nu\setminus\lambda}$ in [14] and, when $\lambda = (0)$, by $\mathscr{S}_\nu(A)$ in [7].

PROPOSITION 1.2 (Compare [1] and [20]). *Let ν and λ by Young diagrams with $\nu \supsetneqq \lambda$. Put $\alpha = (\lambda_1, \lambda_2, ..., \lambda_{\tilde{\lambda}_1}, 0^{\tilde{\nu}_1 - \tilde{\lambda}_1})$ and $\beta = \nu\setminus\lambda$, as partitions. Then* $\ker d_{\nu\setminus\lambda}(M)$ *is generated by all the elements*:

$$\sum_{\sigma \in R} \operatorname{sgn}(\sigma)(x_1 \otimes x_2 \otimes \cdots \otimes x_{i-1}$$
$$\otimes (a_1 \wedge a_2 \wedge \cdots \wedge a_e \wedge c_{\sigma(j+1)} \wedge c_{\sigma(j+2)} \wedge \cdots \wedge c_{\sigma(f)})$$
$$\otimes (c_{\sigma(1)} \wedge c_{\sigma(2)} \wedge \cdots \wedge c_{\sigma(j)} \wedge b_1 \wedge b_2 \wedge \cdots \wedge b_d)$$
$$\otimes x_{i+2} \otimes \cdots \otimes x_{\tilde{\nu}_1}),$$

where i, j, σ, d, e, and f are as in Theorem 1.1(e), *$x_k \in \Lambda^{\beta_k}(M)$ for all k, and $a_1, a_2, ..., a_e, c_1, c_2, ..., c_f, b_1, b_2, ..., b_d \in M$.*

Proof. Consider the diagram involving ${}_*d_{\nu\setminus\lambda}(A^r)$ and $C_p(\overset{*}{T}_{\nu\setminus\lambda})$. By Theorem 1.1 (parts (c) and (e)), $\ker C_P(\overset{*}{T}_{\nu\setminus\lambda})$ contains the elements $\sum_\sigma \operatorname{sgn}(\sigma)(U^\sigma \mid \overset{*}{T}_{\nu\setminus\lambda})$. But by Theorem 1.1(e), $\operatorname{Im}(C_P(\overset{*}{T}_{\nu\setminus\lambda}))$ has as basis $\{(S \mid T_{\nu\setminus\lambda}) \mid S \in ST^{\nu\setminus\lambda} (\leqslant r)\}$. Hence by the remark following Theorem 1.1, $\ker C_P(\overset{*}{T}_{\nu\setminus\lambda})$ is generated by the elements $\sum \operatorname{sgn}(\sigma)(U^\sigma \mid \overset{*}{T}_{\nu\setminus\lambda})$. By the commutativity of the diagram the elements in Proposition 1.2 generate $\ker(d_{\nu\setminus\lambda}(M))$, in case $M = A^r$. The general case now follows immediately. ∎

I.2. *The Filtrations*

The Schur functors turn up in a natural way studying symmetric powers. For this, first observe that there is an isomorphism of (graded) $A[\operatorname{End}_A(A^m) \times \operatorname{End}_A(A^n)]$-algebras $A_m{}^n \xrightarrow{\sim} S(A^m \otimes_A A^n)$ given by $(i \mid j) \mapsto e_i \otimes f_j$ ($i \in \underline{m}$, $j \in \underline{n}$), where $(e_1, e_2, ..., e_m)$ respectively $(f_1, f_2, ..., f_n)$ are the natural bases of A^m and A^n. Here $S(A^m \otimes A^n)$ is, of course, the symmetric algebra of $A^m \otimes A^n$.

PROPOSITION 2.3 (Slight Refinement of [1], See Also [12]). *The functor $S^n(-\otimes-)$: A-mod $\times$ A-mod $\to$ A-mod admits an explicit filtration by subfunctors*:

$$S^n(-\otimes-) = L^{\mu^1} \supset L^{\mu^2} \supset \cdots \supset L^{\mu^{l+1}} = 0$$

with $l \in \mathbb{N}$, $\mu^1 < \mu^2 < \cdots < \mu^l$ *are all Young diagrams for* n *and for all* $i \in \underline{l}$ *there is an equivalence*

$$S^{\tilde{\mu}^i} \otimes S^{\tilde{\mu}^i} \xrightarrow{\sim} L^{\mu^i}/L^{\mu^{i+1}}.$$

Moreover for each $\nu \vdash n$, $\bigotimes_{i=1}^{\tilde{\nu}_1} S^{\nu_i} : A\text{-mod} \to A\text{-mod}$ *admits an explicit filtration by subfunctors*:

$$\bigotimes_i S^{\nu_i} = K^{\lambda_1} \supset K^{\lambda_2} \supset \cdots \supset K^{\lambda^{m+1}} = 0$$

with $m \in \mathbb{N}$, $\lambda^1 < \lambda^2 < \cdots < \lambda^m$ *are all elements of* $\{\lambda \vdash n \mid \tilde{\lambda} \trianglerighteq \nu\}$ *and for all* $i \in \underline{m}$ *there is an equivalence,*

$$S^{\tilde{\lambda}^i} \otimes F_i \xrightarrow{\sim} K^{\lambda^i}/K^{\lambda^{i+1}},$$

where F_i *is the constant functor with value the free* A*-module of rank* $\# ST^{\lambda^i}(\nu)$. *In case* $\nu = (1^n)$, S_n *acts by natural transformations on* $\bigotimes_i S^{\nu_i} = (-)^{\otimes n}$ *(by permuting the tensor factors), and the* K^{λ^i} *can be chosen to be* S_n*-invariant such that the* S_n*-action on* $K^{\lambda^i}/K^{\lambda^{i+1}}$ *corresponds to an action on* F_i *turning its value into the* $\mathcal{S}_{\tilde{\lambda}^i}(A)$.

Proof. Let $\mu \vdash n$ and let $\varphi_\mu = \varphi_\mu(M, N) : \bigotimes_{i=1}^{\tilde{\mu}_1} \Lambda^{\mu_i}(M) \otimes \bigotimes_{i=1}^{\tilde{\mu}_1} \Lambda^{\mu_i}(N) \to S^n(M \otimes N)$ be the natural A-homomorphism defined by

$$\bigotimes_i (m_{i,1} \wedge m_{i,2} \wedge \cdots \wedge m_{i,\mu_i}) \otimes \bigotimes_i (n_{i,1} \wedge n_{i,2} \wedge \cdots \wedge n_{i,\mu_i})$$

$$\mapsto \prod_i \det((m_{i,r} \otimes n_{i,s})_{r,s \leqslant \mu_i})$$

for $m_{i,r} \in M$ and $n_{i,s} \in N$ all i, r, and s.

When M and N are A-free modules of rank r respectively s, then φ_μ corresponds to the A-linear map (see above the definition of $S^{\nu \setminus \lambda}$):

$$\langle (S \mid \mathring{T}_\mu) \mid S \in T^\mu (\leqslant r) \rangle_A \otimes \langle (\mathring{T}_\mu \mid T) \mid T \in T^\mu (\leqslant s) \rangle_A$$

$$\to \left\langle (S \mid T) \mid (S, T) \in \bigcup_{u \vdash n} BT^\mu (\leqslant r, \leqslant s) \right\rangle_A$$

$$\text{defined by} \quad (S \mid \mathring{T}_\mu) \otimes (\mathring{T}_\mu \mid T) \mapsto (S \mid T).$$

Moreover $\varphi_\mu(x \otimes y) \in \sum_{\tau > \mu} \operatorname{Im} \varphi_\tau$ for every generator x of $\ker d_\mu(M)$ described in Proposition 1.2 and every $y \in \bigotimes_i \Lambda^{\mu_i}(N)$, by specialising Theorem 1.1(a).

Similarly $\varphi_\mu(x \otimes y) \in \sum_{\tau > \mu} \operatorname{Im} \varphi_\tau$ for every generator y of $\ker d_\mu(N)$, as described in Proposition 1.2, and every $x \in \bigotimes_i \Lambda^{\mu_i}(M)$.

Hence, the quotient homomorphism $\bar{\varphi}_\mu : \bigotimes_i \Lambda^{\mu_i}(M) \otimes \bigotimes_i \Lambda^{\mu_i}(N) \to S^n(M \otimes N)/\sum_{\tau > \mu} \operatorname{Im} \varphi_\tau$ factorises through a natural map $c_\mu : S^{\tilde{\mu}}(M) \otimes S^{\tilde{\mu}}(N) \to S^n(M \otimes N)/\sum_{\tau > \mu} \operatorname{Im} \varphi_\tau$. When M and N are free A-modules of rank r respectively s, the c_μ coresponds by Theorem 1.1(b) to the A-linear map,

$$\langle (S \mid T_\mu) \mid S \in ST^\mu (\leqslant r) \rangle_A \otimes \langle (T_\mu \mid T) \mid T \in ST^\mu (\leqslant s) \rangle_A$$

$$\to \left\langle (S \mid T) \mid (S, T) \in \bigcup_{\tau \leqslant \mu} SBT^\tau (\leqslant r, \leqslant s) \right\rangle_A$$

$$\text{defined by} \quad (S \mid T_\mu) \otimes (T_\mu \mid T) \mapsto (S \mid T).$$

This however is by Theorem 1.1(d) an isomorphism onto its image. It follows that c_μ is an isomorphism onto its image in general. Observe that $\varphi_{(1^n)}$ is surjective hence $\sum_\mu \operatorname{Im} \varphi_\mu = S^n(M \otimes N)$. Observe that the definition of φ_μ actually defines a natural transformation $\Phi_\mu : \bigotimes_i \Lambda^{\mu_i}(-) \otimes \bigotimes_i \Lambda^{\mu_i}(-) \to S^n(- \otimes -)$. Hence, c_μ generalises to a natural transformation $C_\mu : S^{\tilde{\mu}} \otimes S^{\tilde{\mu}} \to S^n(- \otimes -)/\sum_{\tau > \mu} \operatorname{Im} \Phi_\tau$ which is an equivalence onto its image. Hence, put $L^\mu = \sum_{\tau > \mu} \operatorname{Im} \Phi_\tau$, then the L^μ make up the desired filtration.

Concerning $\bigotimes_i S^{\nu_i}$ observe that there is an equivalence

$$\bigoplus_{(\alpha_1, \alpha_2, \ldots, \alpha_n) \models n} \bigotimes_{i=1}^{n} S^{\alpha_i} \xrightarrow{\sim} S^n(- \otimes A^n)$$

defined for M by $\sum_\alpha \bigotimes_i \prod_{j=1}^{\alpha_i} m_{i,j} \mapsto \sum_\alpha \prod_{i,j} (m_{i,j} \otimes e_i)$, where $m_{i,j} \in M$ all i, j and $(e_1, e_2, \ldots, e_n)$ is the natural basis of A^n. For $M = A^r$ this map corresponds to the decomposition according to place content,

$$\bigoplus_\alpha \langle (S \mid T) \mid (S, T) \in BT) \in BT (\leqslant r, \alpha) \rangle_A$$

$$= \left\langle (S \mid T) \mid (S, T) \in \bigcup_{\mu \vdash n} BT^\mu (\leqslant r, \leqslant n) \right\rangle_A .$$

From the construction of the filtration for $S^n(- \otimes -)$ it now follows that the desired filtration for $\bigotimes_i S^{\nu_i}$ is obtained as the "summand" of the one for $S^n(- \otimes A^n)$, the summand corresponding to $\alpha = \nu$. The dimension of the values of the constant functors follows from Theorem 1.1 (parts (b) and (d)).

The filtration for $\nu = (1^n)$ given above has the desired properties. ∎

Remark. As a direct summand of the filtration for $\bigotimes_i S^{\nu_i}(A^n)$ there is a filtration for $A[S_n] \otimes_{A[S_\nu]} \mathrm{triv} A$, corresponding to latter content (1^n), which is in fact Young's rule in [7]. The special case $\nu = (1^n)$ yields a bimodule filtration for the group ring $A[S_n]$.

The way skew Schur functors arise is:

THEOREM 1.4. *Let* $\nu \vdash n$, $L \in A$-mod.

(a) (*Slight refinement of* [1].) *There are explicit subfunctors* F_k *for* $k \in \{0, 1, 2, ..., n\}$ *of* $S^\nu(-\oplus-): A\text{-mod} \times A\text{-mod} \to A\text{-mod}$ *such that* $\bigoplus_{k=0}^n F_k = S^\nu(-\oplus-)$. *And such that for each* k, F_k *admits an explicit filtration by subfunctors,*

$$F_k = G^{\lambda^1} \supset G^{\lambda^2} \supset \cdots G^{\lambda^{l(k)+1}} = 0$$

with $l(k) \in \mathbb{N}$, $\lambda^1 < \lambda^2 < \cdots < \lambda^{l(k)}$ *are all elements of* $\{\lambda \vdash k \mid \tilde{\nu} \supset \lambda\}$ *and for all* $i \in \underline{l(k)}$ *there is an equivalence*:

$$S^{\tilde{\lambda}^i} \otimes S^{\nu \setminus \tilde{\lambda}^i} \simeq G^{\lambda^i}/G^{\lambda^{i+1}}.$$

(b) *Suppose there is an exact sequence* $0 \to N \to M \to L \to 0$ *in* A-mod *then* $S^\nu(M)$ *admits a natural explicit filtration,*

$$S^\nu(M) = P^{\mu^1} \supset P^{\mu^2} \supset \cdots \supset P^{\mu^{l+1}} = 0$$

with $l \in \mathbb{N}$, $\mu^1 < \mu^2 < \cdots < \mu^l$ *are all elements of* $I = \bigcup_{k=0}^n \{\mu \vdash k \mid \tilde{\nu} \supset \mu\}$ *and for all* $i \in \underline{l}$ *there is a natural isomorphism,*

$$S^{\tilde{\mu}^i}(N) \otimes S^{\nu \setminus \tilde{\mu}^i}(L) \simeq P^{\mu^i}/P^{\mu^{i+1}}.$$

Here "natural" means natural with respect to commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N' & \longrightarrow & M' & \longrightarrow & L' & \longrightarrow & 0 \end{array}$$

in A-mod *with exact rows.*

Proof. (a) For $k \in \{0, 1, ..., n\}$ put $P(k) = \{\alpha \in \mathbb{N}_0)^{\nu_1} \mid \sum_{i=1}^{\nu_1} \alpha_i = k$, $\alpha_i \leqslant \tilde{\nu}_i$ all $i\}$. For each $\alpha \in P(k)$ we shall define a natural homomorphism,

$$db_\alpha = db_\alpha(N, L): \bigotimes_{i=1}^{\nu_1} \Lambda^{\alpha_i}(N) \otimes \bigotimes_{i=1}^{\nu_1} \Lambda^{\beta_i}(L) \to S^\nu(N \oplus L),$$

where $\beta_i = \tilde{\nu}_i - \alpha_i$ all i. To do this let b_α: $\bigotimes_{i=1}^{\nu_1} \Lambda^{\alpha_i}(N) \otimes \bigotimes_{i=1}^{\nu_1} \Lambda^{\beta_i}(L) \to \bigotimes_{i=1}^{\nu_1} \Lambda^{\tilde{\nu}_i}(N \oplus L)$ be the natural A-homomorphism defined by

$$\bigotimes_i (n_{i,1} \wedge n_{i,2} \wedge \cdots \wedge n_{i,\alpha_i}) \otimes \bigotimes_i (l_{i,1} \wedge \cdots \wedge l_{i,\beta_i})$$

$$\mapsto \bigotimes_i ((n_{i,1}, 0) \wedge (n_{i,2}0) \wedge \cdots \wedge (n_{i,\alpha_i}) \wedge (0, l_{i,1}) \wedge \cdots \wedge (0, l_{i,\beta_i}))$$

for $n_{i,j} \in N$ and $l_{i,j} \in L$ all i, j.

Now put $db_\alpha = d_{\tilde\nu}(N \oplus L) \circ b_\alpha \colon \bigotimes_i A^{\alpha_i}(N) \otimes \bigotimes_i A^{\beta_i}(L) \to S^\nu(N \oplus L)$, and put $K_k = \sum_{\alpha \in P(k)} \operatorname{Im} db_\alpha$. The K_k is clearly a natural submodule of $S^\nu(N \oplus L)$ and $\sum_{k=0}^n K_k = S^\nu(N \oplus L)$. By the following claim this sum is a direct sum. Let $k \in \{0, 1, ..., n\}$ and let $\lambda \vdash k$ be such that $\tilde\nu \supset \lambda$, then by b_λ and db_λ we shall mean b_α respectively db_α with $\alpha = (\lambda_1, \lambda_2, ..., \lambda_{\tilde\lambda_1}, 0^{\nu_1 - \tilde\lambda_1})$.

Claim 1. For all $\alpha \in P(k)$, $\operatorname{Im} db_\alpha \subset \sum_{\tau \vdash k, \tau \geqslant \alpha} \operatorname{Im} db_\tau$, especially $K_k = \sum_{\lambda \vdash k} \operatorname{Im} db_\lambda$. Moreover $\sum_{k=0}^n K_k = \bigoplus_{k=0}^n K_k (= S^\nu(N \oplus L))$.

Proof of Claim 1. Clearly one may assume that N and L are free A-modules, of rank r respectively s say. But in that case db_α corresponds to the A-linear map (see above the definition of $S^{\nu\backslash\lambda}$),

$$\begin{gathered}\langle (S \mid \overset{*}{T}_\alpha) \mid S \in T^\alpha (\leqslant r) \rangle_A \otimes \langle (V \mid \overset{*}{T}_{\tilde\nu\backslash\alpha}) \mid V \in T^{\tilde\nu\backslash\alpha} (\leqslant s) \rangle_A \\ \to (U \mid T_{\tilde\nu}) \mid U \in T^{\tilde\nu} (\leqslant (r+s)) \rangle_A \\ \text{defined by} \quad (S \mid \overset{*}{T}_\alpha) \otimes (V \mid \overset{*}{T}_{\tilde\nu\backslash\alpha}) \mapsto (S\tau_r(V) \mid T_{\tilde\nu}).\end{gathered}$$

So the image of db_α corresponds to

$$\begin{gathered}\langle (U \mid T_{\tilde\nu}) \mid U \in T^{\tilde\nu} (\leqslant (r+s)), \text{the shape of the entries of } U \\ \text{which are at most } r \text{ is } \alpha \rangle_A.\end{gathered}$$

By Theorem 1.1 (parts (a), (b), and (d)) the assertions follow.

Claim 2. Let $\lambda \vdash k$ be such that $\tilde\nu \supset \lambda$, and put $\beta = \tilde\nu \backslash \lambda$. Then $db_\lambda(x) \in \sum_{\tau \vdash k, \tau \geqslant \alpha} \operatorname{Im} db_\tau$ for all $x \in ((\bigotimes_{l=1}^{\tilde\lambda_1} A^{\lambda_l}(N) \otimes \operatorname{Ker} d_{\tilde\nu\backslash\lambda}(L)) \cup (\operatorname{Ker} d_\lambda(N) \otimes \bigotimes_{i=1}^{\nu_1} A^{\beta_i}(L)))$.

Proof of Claim 2. We shall do the case $x \in \bigotimes_l A^{\lambda_l}(N) \otimes \operatorname{Ker} d_{\tilde\nu\backslash\lambda}(L)$, the other case can be handled similarly. So let $y_l \in A^{\lambda_l}(N)$ all l, and let z be one of the generators of $\operatorname{Ker} d_{\tilde\nu\backslash\lambda}(L)$ described in Proposition 1.2, say z corresponds to (i, j), here i refers to the ith row and j to the jth column, see Theorem 1.1(e). Then $b_\lambda(\bigotimes_l y_l \otimes z)$ looks, at first glance, like a generator of $\ker d_{\tilde\nu}(N \oplus L)$ for the same pair (i, j) as described in Proposition 1.2. If it is such a generator then $db_\lambda(\bigotimes_l y_l \otimes z) = 0$ so we are clearly done then. However, this only happes when $i > \tilde\lambda_1$. When $i \leqslant \tilde\lambda_1$ then $d_\lambda(\bigotimes_l y_l \otimes z)$ only involves permutations which permute elements of L whereas the corresponding generator of $\operatorname{Ker} d_{\tilde\nu}(N \oplus L)$ also involves permutations which interchange elements of N with elements of L. But fortunately these extra terms correspond to terms in $\sum_{\alpha \in P(k), \alpha > \lambda} \operatorname{Im} b_\alpha$. Hence their $d_{\tilde\nu}(N \oplus L)$-images lie in $\sum_{\tau \vdash k, \tau > \lambda} \operatorname{Im} db_\tau$ by Claim 1. But clearly $d_{\tilde\nu}(N \oplus L)$ $(b_\lambda(\bigotimes_l y_l \otimes z) + \text{extra terms}) = 0$, so $db_\lambda(x)$ is in $\sum_{\tau \vdash k, \tau > \lambda} \operatorname{Im} db_\tau$ as desired.

By Claim 1, $\sum_{\tau \vdash k, \tau > \lambda} \operatorname{Im} db_\tau$ is a natural submodule of K_k for every $\lambda \vdash k$, for all k. By Claim 2 the quotient map $\bigotimes_l \Lambda^{\lambda_l}(N) \otimes \bigotimes_i \Lambda^{\beta_i}(L) \to K_k / \sum_{\tau \vdash k, \tau > \lambda} \operatorname{Im} db_\tau$ induced by db_λ factors through a natural A-homomorphism:

$$c_\lambda : S^{\tilde\lambda}(N) \otimes S^{\nu \backslash \tilde\lambda}(L) \to K_k \Big/ \sum_{\tau \vdash k, \tau > \lambda} \operatorname{Im} db_\tau .$$

Claim 3. For all $\lambda \vdash k$ with $\tilde\nu \supset \lambda$, c_λ is an isomorphism onto its image.

Proof of Claim 3. Clearly one may assume that N and L are free A-modules, of rank r respectively s say. From the proof of Claim 1 it follows that, by Theorem 1.1(b), c_λ corresponds to the map

$$\langle (S \mid T_\lambda) \mid S \in ST^\lambda (\leqslant r) \rangle_A \otimes \langle V \mid T_{\tilde\nu \backslash \lambda}) \mid V \in ST^{\tilde\nu \backslash \lambda} (\leqslant s) \rangle_A$$

$\to \langle (U \mid T_{\tilde\nu}) \mid U \in ST^{\tilde\nu}(\leqslant (r+s))$, the shape of the entries of U which are atmost r is $\lambda \rangle_A$

defined by $(S \mid T_\lambda) \otimes (V \mid T_{\tilde\nu \backslash \lambda}) \mapsto (S_{\tau_r(V)} \mid T_{\tilde\nu})$.

By Theorem 1.1, parts (d) and (e), this latter map is an isomorphism onto its image, as desired.

Observe that the definition of db_α defines in fact a natural transformation $DB_\alpha : \bigotimes_i \Lambda^{\alpha_i}(-) \otimes \bigotimes_i \Lambda^{\beta_i}(-) \to S^\nu(- \otimes -)$. Hence, K_k generalises to a subfunctor F_k of $S^\nu(- \oplus -)$. And c_λ generalises to a natural transformation,

$$C_\lambda : S^{\tilde\lambda} \otimes S^{\nu \backslash \tilde\lambda} \to F_k \Big/ \sum_{\tau \vdash k, \tau > \lambda} \operatorname{Im} db_\tau ,$$

which is an equivalence onto its image. Now put $G^\lambda = \sum_{\tau \vdash k, \tau \geqslant \lambda} \operatorname{Im} db_\tau$ for all $\lambda \vdash k$, $k \in \{0, 1, ..., n\}$, then the G^λ's make up the desired filtration.

(b) Let $0 \to N \xrightarrow{\gamma} M \xrightarrow{\delta} L \to 0$ be the exact sequence. The map δ splits by an A-homomorphism. Now using γ and this splitting one can define, in a similar way as in the proof of (a), the maps b_α and db_α. However, b_α and db_α are not natural maps in general, and K_k and $\sum_{\tau > \lambda} \operatorname{Im} db_\tau$ are not necessarily natural submodules. But the first two claims remain valid. Moreover when

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & N' & \longrightarrow & M' & \longrightarrow & L' & \longrightarrow & 0 \end{array}$$

is a commutative diagram of A-homomorphisms, with exact rows, then for each $\mu \in I$ and each $x \in \bigotimes_{j=1}^{\tilde{\mu}_1} A^{\mu_j}(N) \otimes \bigotimes_{i=1}^{\nu_1} A^{\beta_i}(L)$,

$$S^{\nu}(g)(db_{\mu}(x)) = db_{\mu}\left[\left(\bigotimes_j A^{\mu_j}(f) \otimes \bigotimes_i A^{\beta_i}(h)\right)(x)\right] + y \qquad \text{with}$$

$$y \in \sum_{m=k+1}^{n} \sum_{\alpha \in P(m)} \operatorname{Im} db_{\alpha} \quad \text{and} \quad \beta = \tilde{\nu} \backslash \mu.$$

Hence, $\sum_{\tau \in I, \tau > \mu} \operatorname{Im} db_{\tau}$ is a natural submodule of $S^{\nu}(M)$ for all $\mu \in I$ and by Claims 1 and 2, db_{μ} induces a natural A-homomorphism,

$$S^{\tilde{\mu}}(N) \otimes S^{\nu \backslash \tilde{\mu}}(L) \to S^{\nu}(M) \Big/ \sum_{\tau \in I, \tau > \mu} \operatorname{Im} db_{\tau}.$$

Now replace c_{λ} and λ in Claim 3 by this map respectively μ then it remains valid. Hence, put $P^{\mu} = \sum_{\tau \in I, \tau \geqslant \mu} \operatorname{Im} db_{\tau}$ then the P^{μ} make up the desired filtration. ∎

Remarks. (1) Theorem 1.4 generalises the well-known results for exterior and symmetric powers (the cases $\nu = (1^n)$ respectively (n)).

(2) Theorem 1.4 can be generalised to skew Young diagrams. For this, let $m \in \{0, 1, ..., n\}$, $\lambda \vdash m$, $\nu \supset \lambda$ and replace S^{ν} by $S^{\nu \backslash \lambda}$, $\{0, 1, ..., n\}$ by $\{m, m+1, ..., n\}$, $\bigoplus_{k=0}^{n}$ by $\bigoplus_{k=m}^{n}$, $\{\lambda \vdash k \mid \tilde{\nu} \supset \lambda\}$ by $\{\mu \vdash k \mid \tilde{\nu} \supset \mu \supset \tilde{\lambda}\}$, $S^{\tilde{\lambda}^i}$ by $S^{\tilde{\lambda}^i \backslash \lambda}$ and I by $\bigcup_{k=m}^{n} \{\mu \vdash k \mid \tilde{\nu} \supset \mu \supset \tilde{\lambda}\}$. A proof can be given in a similar way, compare [1].

Modules for skew diagrams were defined first in characteristic zero by prescribing their composition factors. Hence, a convincing argument for the adjective "skew" is a universal filtration for skew Schur functors with (ordinary) Schur functors as subquotients.

THEOREM 1.5. *Let $a \in \mathbb{N}$, $\nu \vdash n$, $\lambda \vdash a$, suppose $\nu \supsetneqq \lambda$, and put $\beta = \nu \backslash \lambda$ (as a partition). Then $S^{\nu \backslash \lambda}$ admits an explicit filtration by subfunctors,*

$$S^{\nu \backslash \lambda} = N^{T^1} \supset N^{T^2} \supset \cdots \supset N^{T^{l+1}} = 0$$

with $l \in \mathbb{N}$, $T^1 <_c T^2 <_c \cdots <_c T^l$ are all elements of $L = \{T \in ST^{\mu}(\beta) \mid \mu \vdash (n-a),\ \varphi^{\mu}_{\nu \backslash \lambda}(T) \in ST^{\tilde{\nu} \backslash \tilde{\lambda}}(\tilde{\mu})\}$ and for all $i \in \underline{l}$ this is an equivalence,

$$S^{\tilde{\mu}^i} \simeq N^{T^i}/N^{T^{i+1}}, \qquad \text{where } \mu^i \text{ is the shape of } T^i.$$

Here $\varphi^{\mu}_{\nu \backslash \lambda}: ST^{\mu}(\beta) \to T^{\tilde{\nu} \backslash \tilde{\lambda}}(\tilde{\mu})$ is the map defined by the following: for every $T \in ST^{\mu}(\beta)$ and every $j \leqslant \tilde{\nu}_1$ the jth-column of $\varphi^{\mu}_{\nu \backslash \lambda}(T)$ contains exactly the numbers of the columns of T containing j arranged in (weakly) increasing order from top to bottom. See Remark 1 below the proof of this theorem concerning this map.

Proof. Let $T \in L$ and let μ be its shape. Let $[\tilde{\nu} \backslash \tilde{\lambda}]$ denote the tableau of shape $\tilde{\nu} \backslash \tilde{\lambda}$ such that $[\tilde{\nu} \backslash \tilde{\lambda}]_{i,j} = (i, j)$ all i, j, so the entries reflect their slots. Let $[\mu]$ of shape μ have a similar definition. Let $E(\tilde{\nu} \backslash \tilde{\lambda})$ and $E(\mu)$ denote the set of entries of $[\tilde{\nu} \backslash \tilde{\lambda}]$ respectively $[\mu]$.

First we construct a bijection f_T: $E(\tilde{\nu} \backslash \tilde{\lambda}) \simeq E(\mu)$ such that for all i, j, k: $\#(f_T(i\text{th row of } [\tilde{\nu} \backslash \tilde{\lambda}]) \cap (j\text{th column of } [\mu])) \leqslant 1$, and $f_T(k\text{th-column of } [\tilde{\nu} \backslash \tilde{\lambda}]) =$ the set of slots in T containing k.

Put $Y = \varphi^{\mu}_{\nu \backslash \lambda}(T)$ then the elements in the first row of Y, so $Y_{1, \tilde{\lambda}_1 + 1}, ..., Y_{1, \tilde{\nu}_1}$, are numbers of columns of T containing the entries $\tilde{\lambda}_1 + 1$, $\tilde{\lambda}_2 + 1, ..., \tilde{\nu}_1$, respectively. These latter entries are, because Y is standard, bottom elements in their columns, say their slots are $x_{\tilde{\lambda}_1 + 1}, ..., x_{\tilde{\lambda}_2 + 1}, ..., x_{\tilde{\nu}_1}$, respectively.

Set $f_T((1, j)) = x_j$ for all $j \in \{\tilde{\lambda}_1 + 1, ..., \tilde{\nu}_1\}$. We proceed with the entries of T which are left now. The entries in the second row of Y determine a new sequence of bottom elements and their slots shall be $f_T((2, \tilde{\lambda}_2 + 1)), ..., f_T((2, \tilde{\nu}_2))$, as we did with the first row of Y. Continuing this way we find the announced bijection f_T. It is the inverse of the one in [8, p. 169].

For $i \leqslant \nu_1$, put $Q_i = f_T(i\text{th row of } [\tilde{\nu} \backslash \tilde{\lambda}])$, so $E(\mu) = \coprod_i Q_i$. Order the elements of Q_i by $: x \leqslant y \Leftrightarrow f_T^{-1}(x) \leqslant f_T^{-1}(y)$ in lexicographic order.

For $j \in \tilde{\mu}_1$ and $i \in \tilde{\nu}_1$ put $Q_{i,j} = Q_i \cap (j\text{th-row of } [\tilde{\mu}])$ so $Q_i = \coprod_j Q_{i,j}$ all i. Let for a set Q, $S(Q)$ denote the symmetric group on Q. Put for $i \leqslant \nu_1$, $P_i = \{\sigma \in S(Q_i) \mid \text{for all } j \text{ and all } x, y \in Q_{i,j}: x \leqslant y \Leftrightarrow \sigma(x) \leqslant \sigma(y)\}$, and put $P = \prod_{i=1}^{\nu_1} P_i$. Let for all $\rho \in P$, T^{ρ} be the tableau of shape μ such that for all $i \leqslant \nu_1$ and all $x \in Q_i$, $(T^{\rho})_x = T_{\rho_i(w)}$.

Now let b_T: $\bigotimes_{j=1}^{\tilde{\mu}_j} A^{\mu_j}(M) \to S^{n-a}(M \otimes A^n)$ be the natural A-homomorphism defined by

$$\alpha \mapsto \sum_{\rho \in P} \text{sgn}(\rho) \cdot \varphi_{\mu}\left(\alpha \otimes \bigotimes_j \left(e_{(T^{\rho})_{j,1}} \wedge e_{(T^{\rho})_{j,2}} \wedge \cdots \wedge \left(e_{(T^{\rho})_{j,\mu_j}}\right)\right)\right),$$

where $(e_1, e_2, ..., e_n)$ is the natural basis of A^n and φ_{μ} is the natural A-homomorphism in the proof of Proposition 1.3. Then the image of b_T is contained in the image of the natural injective A-homomorphism: $\bigotimes_{i=1}^{\tilde{\nu}_1} S^{\beta_i}(M) \to S^{n-a}(M \otimes A^n)$ defined by $\bigotimes_i (m_{i,1} \cdot m_{i,2} \cdot \cdots \cdot m_{i,\beta_i}) \mapsto \prod_i ((m_{i,1} \otimes e_i) \cdot (m_{i,2} \otimes e_i) \cdot \cdots \cdot (m_{i,\beta_i} \otimes e_i))$, for $m_{i,j} \in M$ all i and j, simply because the content of T is β.

In order to understand "b_T": $\bigotimes_j A^{\mu_j}(M) \to \bigotimes_j S^{\beta_i}(M)$ we shall describe it in the case $M = A^r$. In this case the map corresponds to (see the proof of Proposition 1.3)

$$\langle (U \mid \overset{*}{T}_{\mu}) \mid T^{\mu} (\leqslant r) \rangle_A \to \langle (U \mid V) \mid (U, V) \in BT(\leqslant r, \beta) \rangle_A$$

defined by $(U \mid \overset{*}{T}_{\mu}) \mapsto \sum_{\rho \in P} \text{sgn}(\rho) \cdot (U \mid T^{\rho})$.

Claim 1. (a) If $T \in L$ then $\sum_{\rho \in P} \operatorname{sgn}(\rho) \cdot (U \mid T^{\rho})$ is contained in

$$\langle (S \mid T_{\tilde{\nu} \setminus \tilde{\lambda}}) \mid S \in ST^{\tilde{\nu} \setminus \tilde{\lambda}} (\leqslant r) \rangle_{\mathbb{Z} \cdot 1_A} \qquad \text{for all} \quad U \in T^{\mu} (\leqslant r).$$

(b) If $T \in L$ and $U \in ST^{\tilde{\nu} \setminus \tilde{\lambda}}$ $(\leqslant r)$, then $\sum_{\rho \in P} \operatorname{sgn}(\rho) \cdot (U \mid T^{\rho}) = (U \mid T) + \sum b_{v,w}(V \mid W)$ for certain $b_{v,w} \in \mathbb{Z} \cdot 1_A$, where the sum is over $I(U, T) := \{(V, W) \in SBT\ (\leqslant r, \beta) \mid W >_c T \text{ or } (W = T \text{ and } V >_c U)\}$.

(c) (Converse to (a) and (b)). Let $0 \neq x \in \langle (S \mid T_{\tilde{\nu} \setminus \tilde{\lambda}}) \mid S \in ST^{\tilde{\nu} \setminus \tilde{\lambda}} (\leqslant r) \rangle_A$ and write $x = a_{U', T'}(U' \mid T') + \sum a_{v,w}(V \mid W)$ for $a_{U',T'} \in A \setminus \{0\}$, $a_{v,w} \in A$ all (V, W), where $(U', T') \in SBT$ $(\leqslant r, \beta)$ and the summation is over $I(U', T')$, see (b). Then $T' \in L$. (One can write x like this by Theorem 1.1(b) and the expression is unique by Theorem 1.1(d).)

"*Proof*" *of Claim* 1. Parts (a) and (b) can be proved with similar techniques as [8, (11.21) respectively (11.22)(1)]. Part (c) can be proved in a similar way as [8, (11.13)] because of Theorem 1.1(f).

By specialising Claim 1(a) we find that $\operatorname{Im} b_T$ is in fact contained in $S^{\nu \setminus \lambda}(M)$.

Claim 2. (a) $\sum_{T \in L} \operatorname{Im} b_T = S^{\nu \setminus \lambda}(M)$.

(b) $b_T(\operatorname{Ker} d_{\mu}(M)) \subset \sum_{V >_c T} \operatorname{Im} b_V$.

Moreover let $\bar{b}_T \colon \bigotimes_j \Lambda^{\mu_j}() \to S^{\nu \setminus \lambda}(M) / \sum_{V >_c T} \operatorname{Im} b_V$ be the map induced by b_T, then $\bar{b}_T$ factorises through a natural map

$c_T \colon S^{\tilde{\mu}}(M) \to S^{\nu \setminus \lambda}(M) \Big/ \sum_{V >_c T} \operatorname{Im} b_V$ which is an isomorphism onto its image.

Proof of Claim 2. Clearly one may assume that M is a free A-module, of rank r say, so we can use the description of b_T just above Claim 1.

Let us say that for the $x \in N := \langle (S \mid T_{\tilde{\nu} \setminus \tilde{\lambda}}) \mid S \in ST^{\tilde{\nu} \setminus \tilde{\lambda}}\ (\leqslant r) \rangle_A$ in Claim 1(c), (U', T') is the leading bitableau of x with coefficient $b_{U',T'}$. Then by applying Claim 1 (parts (b) and (c)) repeatedly one finds that $\sum_{V \geqslant_c T} \operatorname{Im} b_V$ is generated by all elements of N with leading bitableau (U, W) with coefficient 1 for some $(U, W) \in SBT$ $(\leqslant r, \beta)$ with $W \in L$ and $W \geqslant_c T$.

Hence, (a) follows from Claim 1(c). Moreover, select for each $U \in ST^{\mu}$ $(\leqslant r)$ an element of N with leading tableau $(U \mid T)$ with coefficient 1 (this is possible by Claim 1(b)), then the classes of these elements form a basis for $\operatorname{Im} \bar{b}_T$ by Theorem 1.1 (parts (b) and (d)), which we will use below to prove (b).

But one can also use parts (b) and (c) of Claim 1 to show that $b_T(\sum_{\sigma \in R} \operatorname{sgn}(\sigma)(U^{\sigma} \mid \mathring{T}_{\mu})) = \sum_{\sigma \in R} \operatorname{sgn}(\sigma) \sum_{\rho \in P} (U^{\sigma} \mid T^{\rho}) \in \sum_{V >_c T} \operatorname{Im} d_V$, where we use the notation from Theorem 1.1(a), (with ν replaced by μ

there). By the proof of Proposition 1.2 (with ν replaced by μ and $\lambda=(0)$ there) one now sees that $b_T(\operatorname{Ker} d_\mu(A^r)) \subset \sum_{V>_c T} \operatorname{Im} d_V$.

Hence, the factorisation of $\bar{b}_T$, moreover $c_T(U|T_\mu)$ is by Claim 1(b) an element of N with leading tableau (U, T) with coefficient 1, for all $U \in ST^\mu$ $(\leqslant r)$. Hence, the latter elements form a basis for $\operatorname{Im} \bar{b}_T$, and since the $(U|T_\mu)$ generate $\langle (U|T_\mu) | U \in ST^\mu\ (\leqslant r)\rangle_A$ by Theorem 1.1(b), we have proved (b).

Observe that the definition of b_T defines in fact a natural transformation, $B_T\colon \bigotimes_j A^{\mu_j} \to S^{n-a}(-\otimes A^n)$. Hence c_T generalises to a natural transformation, $C_T\colon S^{\tilde{\mu}} \to S^{\nu\backslash\lambda}/\sum_{V>_c T} \operatorname{Im} B_V$ which is an equivalence onto its image.

Now put $N^T = \sum_{V \geqslant_c T} \operatorname{Im} b_V$ then the N^T make up the desired filtration. ∎

Remarks. (1) One can define $\varphi^\mu_{\nu\backslash\lambda}$ on tableaux of shape μ of which the elements in each column weakly increase from top to bottom. This extension is a bijection onto its image, the inverse has a similar definition and is used, and denoted by φ_μ, in [8, p. 170].

(2) Theorem 1.5 together with Theorem 1.4(a) describe $S^\nu(-\oplus-)$ for a Young diagram ν, by means of a filtration with a subquotients tensor products of Schur functors. Especially, one obtains a filtration of the Schur module $S^\nu(A^k)$ for $Gl(k, A)$ over A ($k \in \mathbb{N}_0$), when it is seen as a representation of a subgroup of type $Gl(l, A) \times Gl(k-l, A)$, (embedded as diagonal block matrices), in terms of tensor products of Schur modules. So we obtain a generalisation of the classical branching rule for $Gl(k, A)$ to arbitrary commutative rings A with 1. A a summand for $k=n$ (n is such that $\nu \vdash n$) one finds a generalised Murnaghan–Nakayama rule by projecting on "letter content" (1^n). The Murnaghan–Nakayama rule describes the restriction of Specht modules for S_n to (Young) subroups of type $S_k \times S_{n-k}$ in terms of Specht modules. So we have generalised [14] to any commutative ring with 1. In a similar way one derives a generalised Murnaghan–Nakayama rule for generalised Specht modules, as mentioned in [16].

COROLLARY 1.5 (Compare [5]). *Let* $m \in \mathbb{N}$, $\lambda \vdash n$ *and* $\mu \vdash m$. *Then* $S^\lambda \otimes S^\mu$: A-mod $\to$ A-mod *admits an explicit filtration by subfunctors,*

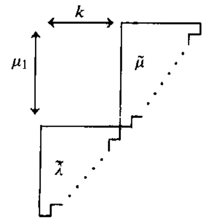
$$S^\lambda \otimes S^\mu = L^{T^1} \supset L^{T^2} \supset \cdots \supset L^{T^{l+1}} = 0$$

with $l \in \mathbb{N}$, $T^1 <_c T^2 <_c \cdots <_c T^l$ *are all elements of* $\{T \in ST^\nu(w\backslash\rho) \mid \nu \vdash (n+m),\ \varphi^\nu_{w\backslash\rho}(T) \in ST^{\tilde{w}\backslash\tilde{\varphi}}(\tilde{\nu})\}$ *and for all* $i \in \underline{l}$ *there is an equivalence*

$$S^{\tilde{\nu}^i} \simeq M^{T^i}/M^{T^{i+1}}, \qquad \text{where } \nu^i \text{ is the shape of } T^i.$$

Here $\rho = (\mu_1, \mu_1, \ldots, \mu_1) \vdash (k \cdot \mu_1)$ *and* $\omega = (\mu_1 + \lambda_1, \mu_1 + \lambda_2, \ldots, \mu_1 + \lambda_k, \mu_1, \mu_2, \ldots, \mu_{\tilde{\mu}_1}) \vdash (n+m+k \cdot \mu_1)$, *where* $k = \tilde{\lambda}_1$.

Proof. The skew shape $\tilde{\omega}\backslash\tilde{\rho}$ looks like



so the rows of "$\tilde{\lambda}$" do not meet those of "$\tilde{\mu}$." Hence, $S^{\lambda}\otimes S^{\mu}\cong S^{\omega\backslash\rho}$, so apply Theorem 1.5. ∎

Remark. (1) For the connection between the tableaux parametrising the subquotients, [5] and lattice permutations see [8].

(2) Corollary 1.5 generalises, and its proof as well, to skew Schur functors.

Corollary 1.5 implies a filtration of the tensor product of Schur modules $S^{\lambda}(A^k)\otimes S^{\mu}(A^k)$ for $Gl(k, A)$ over A with Schur modules as subquotients. So one obtains a generalisation of the classical Clebsch–Gordan rule for $Gl(k, A)$ to arbitrary commutative rings A with 1. Also by the proof of Corollary 1.5, $S^{\lambda}(A^{n+m})\otimes S^{\mu}(A^{n+m})\cong S^{\omega\backslash\rho}(A^{n+m})$, using the notation of Corollary 1.5. By projecting on "letter content" (1^{n+m}) of $S^{\omega\backslash\rho}(A^{n+m})$ one finds the representation: $\mathrm{Ind}_{S_n\times S_m}^{S_{n+m}}(\mathscr{S}_{\lambda}(A)\otimes\mathscr{S}_{\mu}(A))$. So the filtration in Corollary 1.5 yields a filtration for the induced representation of S_{n+m} with Specht modules as subquotients, by projecting on content (1^{n+m}). Thus recovering the (generalised) Littlewood–Richardson rule for Specht modules in [8]. The induction $S_n\times S_m\to S_{n+m}$ is adjoint to the restriction $S_{n+m}\to S_n\times S_m$, this is nicely reflected by the proof Theorem 1.5 and the one for the Littlewood–Richardson rule in [8]: the proofs use bijections inverse to each other.

II. Filtrations Involving Weyl Functors

We will now introduce another kind of letter place algebras, based on the exterior power as opposed to the symmetric power which is basic for the letter place algebras in the first chapter. We shall derive combinatorial results for these new letter place algebras analogous to those for ordinary letter place algebras. A non-characteristic-free start was already made in [11]. With these results we construct filtrations analogous to those in the first paper, but now involving Weyl functors instead of Schur functors.

The Weyl functors are the contravariant duals of Schur functors, this notion of duality shall be defined below. Over algebras over the rationals the corresponding Weyl and Schur functor are equivalent. But over algebras over a finite field, for example, they differ significantly in general.

Since Weyl functors are contravariant dual to Schur functors, the existence of filtrations involving Weyl functors follows in many cases from the corresponding one involving Schur functors. As before, it is the explicitness which is the main point about the filtrations.

We finish with a sketch of another natural kind of letter place algebras, based on the divided power, and show briefly its relevance for Weyl functors.

II.1 *The Combinatorics*

DEFINITIONS. Let $n, m \in \mathbb{N}_0$, then ${}^n_m A$ shall denote the quotient of the free non-commutative algebra with generators $]i|j[$, for $i \in \underline{m}$ and $j \in \underline{n}$, by the ideal generated by the squares. The class of $]i|j[$ shall be denoted by $[i|j]$, ($i \in m$, $j \in \underline{n}$), and the "i" on the left shall be called a *letter* and the "j" on the right a *place*. And ${}^n_m A$ shall be called the *exterior letter place algebra* for $(\underline{m}, \underline{n})$. The aldjective "exterior" is explained by the A-isomorphism ${}^n_m A \cong \Lambda(A^m \otimes_A A^n)$ defined by $[i|j] \mapsto e_i \otimes f_j$, for $i \in \underline{m}$, $j \in \underline{n}$, where $(e_1, e_2, \ldots, e_m)$ and $(f_1, f_2, \ldots, f_n)$ are the natural bases for A^m respectively A^n. Via this isomorphism ${}^n_m A$ becomes an $A[\mathrm{End}_A(A^m) \times \mathrm{End}_A(A^n)]$-algebra. One can define polarisation operators and decoulering on exterior letter place algebras as in [7]. We shall denote them by the same symbols because of the following lemma ("generic case"):

LEMMA 2.1. *Let k, m, $n \in \mathbb{N}_0$. Then if $k \leqslant n$ there is an A-isomorphism,*

$$\left\langle \prod_{i=1}^{k} (a_i|i) \mid a_i \in \underline{m} \text{ all } i \right\rangle_A \cong \langle [a_1|1] \cdot [a_2|2] \cdot \cdots \cdot [a_k|k] \mid a_i \in \underline{m} \text{ all } i \rangle_A$$

defined by

$$\prod_i (a_i|i) \mapsto [a_1|1] \cdot [a_2] \cdot \cdots \cdot [a_k|k], \qquad \text{for all } \quad a_i \in \underline{m} \text{ all } i.$$

This isomorphism commutes with all (decoulered) letter polarisation operators. A similar holds, when $k \leqslant m$, for (decoulered) place polarisation operators with respect to the A-isomorphism defined in a similar way:

$$\left\langle \prod_{i=1}^{k} (i|b_i) \mid b_i \in \underline{n} \text{ all } i \right\rangle_A \cong \langle [1|b_1] \cdot [2|b_2] \cdot \cdots \cdot [k|b_k] \mid b_i \in \underline{n} \text{ all } i \rangle_A.$$

Hence, we can also define (decoulered) Capelli operator for standard tableaux and (decoulered) operators like $D_L(S, T_{\nu\setminus\lambda})$ and $D_P(T, \dot{T}_{\nu\setminus\lambda})$ etc., in a similar way. However, instead of bideterminants we shall use, what we shall call, bipermanents:

DEFINITIONS. Let k, m, $n \in \mathbb{N}_0$ and $(U, V) \in BT^{(k)}$ $(\leqslant m, \leqslant n)$, (a one-rowed bitableau). Then we define:

$$[U \mid T] := [U_{\sigma(1)} \mid V_1] \cdot [U_{\sigma(2)} \mid V_2] \cdot \cdots \cdot [U_{\sigma(k)} \mid V_k],$$

where R_U is a transversal for S_k/H_U ith $H_U = \{\sigma \in S_k \mid U_{\sigma(i)} = U_i$ for all $i \in \underline{k}\}$ (the row stabilisor of U).

Observe that permutations of entries in U do not alter $[U \mid V]$ whereas a permutation of the entries of V alters $[U \mid V]$ by its signature. In fact when V contains two equal entries then $[U \mid V] = 0$. Also, when the entries of U are mutually distinct then $[U \mid V] = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \cdot [U_1 \mid V_{\sigma(1)}] \cdot [U_2 \mid V_{\sigma(2)}] \cdot \cdots \cdot [U_k \mid V_{\sigma(k)}]$, so it can be seen as a determinant as well in this case. It should be clear now that results for letters do not necessarily imply similar results for places in exterior letter place algebras, as opposed to ordinary letter place algebras.

Let, more generally, $(\alpha_1, \alpha_2, ..., \alpha_l) \models k$ and $(U, V) \in BT^{\alpha}$ $(\leqslant m, \leqslant n)$. Then we define $[U \mid V] := [U_{1,*} \mid V_{1,*}] \cdot [U_{2,*} \mid V_{2,*}] \cdot \cdots \cdot [U_{l,*} \mid V_{l,*}]$, where $U_{i,*}$ and $V_{i,*}$ denote the ith row of U respectively V. For skew shapes we view, as for bideterminants, the skew shape as a partition.

Now $[U \mid V]$ shall be called the *bipermanent* of (U, V).

Let (U, V) be a bitableau of shape a, possibly skew, Young diagram then $[U \mid \boxed{V}] := \sum_{V' \sim_c V} [U \mid V']$, and $[U \mid V]$ shall be called a *symmetrised bipermanent*.

By $[\boxed{U} \mid V]$ we shall mean $\sum_{\sigma \in Q} \operatorname{sgn}(\sigma) D_L({}^{\sigma}U, \overset{*}{T}_{\nu\setminus\lambda})[\overset{*}{T}_{\nu\setminus\lambda} \mid V]$, see Theorem 2.2(g), where $\beta = (\tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_{\lambda_1}, 0^{\nu_1 - \lambda_1})$, $Q = \prod_{i=1}^{\nu_1} S(\{\beta_i + 1, \beta_i + 2, ..., \tilde{\nu}_i\})$ and $({}^{\sigma}U)_{i,j} = U_{\sigma_j(i),j}$ for all i, j. Here $S(P)$ for a set P is the symmetric group on P. And $[\boxed{U} \mid T]$ shall be called an *alternated bipermanent*.

The results in the exterior letter place algebras we need are (compare with Theorem 1.1):

THEOREM 2.2. *Let $r, s \in \mathbb{N}$, ν, and λ be Young diagrams with $\nu \supsetneqq \lambda$. Let $\gamma \models n$, $\delta \models n$, and $(S, U) \in BT^{\nu\setminus\lambda}(\gamma, \delta)$.*

(a) (*Straightening, implicitly in* [1], *see below*).

(1) *Suppose one is in the situation of Theorem* 1.1(a) *with respect to U then*

$$\sum_{\sigma \in R} \operatorname{sgn}(\sigma)[S \mid U^{\sigma}] \in \langle [V \mid W] \mid (V \mid W] \mid (V, W) \in BT^{\rho}(\gamma, \delta) \text{ for some } \rho \vdash n$$

with $\rho > \nu >_{Z \cdot 1_A}$, where R and U^{σ} are as in Theorem 1.1(a).

(2) *Suppose* $\lambda=(0)$ *and* $\tilde{\nu}_1 \geq 2$ *and let* $i \in \{1, 2, ..., \tilde{\nu}_1 - 1\}$, $j \in \{1, 2, ..., \nu_{i+1}\}$ *and* $k \in \{0, 1, ..., \nu_{i+1} - j\}$ *and write the ith and* $(i+1)$*th row of S like*

$$\begin{matrix} a_1 & a_2 & \cdots & a_{j-1} & c_{j+k+1} & c_{j+k+2} & & \cdots & & c_f \\ c_1 & c_2 & \cdots & c_{j-1} & c_j & c_{j+1} & \cdots & c_{j+k} b_1 b_2 \cdots b_d, \end{matrix}$$

where $d = \nu_{i+1} - j - k$ *and* $f = \nu_i + k + 1$. *Suppose one has*

$$\begin{matrix} a_1 \leq a_2 \leq \cdots \leq a_{j-1} \leq c_{j+k+1} \leq c_{j+k+2} \leq \cdots \leq c_f \\ \wedge \quad \wedge \quad \cdots \quad \wedge \quad \vee\!\! / \\ c_1 \leq c_2 \leq \cdots \leq c_{j-1} \leq \quad c_j \quad = c_{j+1} = \cdots = c_{j+k} < b_1 \leq b_2 \leq \cdots \leq b_d, \end{matrix}$$

then $\sum_{\sigma \in Q} r_\sigma \cdot [{}^\sigma S \mid U] \in \langle [V \mid W] \mid (V, W) \in BT^\rho(\gamma, \delta)$ *for some* $\rho \vdash n$ *with* $\rho > \nu \rangle_{\mathbb{Z} \cdot 1_A}$, *where Q and* r_σ *are as in Theorem* 1.1(f) *except that the* c_i*'s are now in the ith and* $(i+1)$*th and* $(i+1)$*th row. Observe that* ${}^\sigma S <_r S$ *for* $\sigma \notin S_\varepsilon$, *using the notation of Theorem* 1.1(f). *In case* $U = T_\nu$ *the span is zero.*

(b) (*Implicitly in* [1], *see below*)

$$[S \mid U] \in \langle [S \mid W] \mid W \in ST^\nu(\delta),\ W \leq_r U \rangle_{\mathbb{Z} \cdot 1_A} + B_\nu(\gamma, \delta),$$

where $B_\nu(\gamma, \delta) = \langle [V \mid W] \mid (V, W) \in BT^\rho(\gamma, \delta)$ *for some* $\rho \vdash n$ *with* $\rho > \nu$, *and W and* V^{tr} *are standard*$\rangle_{\mathbb{Z} \cdot 1_A}$.

Moreover, $[S \mid U] \in \langle [V \mid U] \mid V \in T^\nu(\delta),\ V \leq_r S,\ V^{\mathrm{tr}}$ *is standard*$\rangle_{\mathbb{Z} \cdot 1_A} + B_\nu(\gamma, \delta)$. *When* $U = T_\nu$, $B_\nu(\gamma, \delta) = 0$.

(c) (*Compare* [12].) *Suppose U is standard. If* $\lambda = (0)$ *or the entries of U are mutually distinct then*

$$C_P(U)[S \mid U] = [S \mid T_{\nu \backslash \lambda}].$$

If $\lambda = (0)$ *and* $U' \in T^\nu(\delta)$ *with* $U' >_c U$ *then* $C_P(U)[S \mid U'] = 0$.

(d) (*Implicitly in* [1], *see below.*)

$$\{[S \mid T] \mid (S, T) \in \bigcup_{\nu \vdash n} BT^\nu(\leq r, \leq s),\ S^{\mathrm{tr}} \text{ and } T \text{ are standard}\}$$

is an A-independent system.

(e) (*Implicitly in* [1], *see below.*) *Put* $\alpha = (\lambda_1, \lambda_2, ..., \lambda_{\tilde{\lambda}_1}, 0^{\tilde{\nu}_1 - \tilde{\lambda}_1})$ *and* $\beta = \nu \backslash \lambda$, *as partitions. Suppose there is an* $i \in \{1, 2, ..., \tilde{\nu}_1 - 1\}$ *such that* $\beta_i \neq 0$, $\beta_{i+1} \neq 0$ *and* $\nu_{i+1} > \alpha_i$ (*i.e., the ith and* $(i+1($*th row meet.*) *Let* $j \leq \beta_{i+1}$ *with* $j > \alpha_i - \alpha_{i+1}$ *and let* $k \in \{0, 1, ..., \beta_{i+1} - j\}$ *and write the ith and* $(i+1)$*th-row of S like.*

$$\begin{matrix} & & a_1 & a_2 & \cdots & a_e & c_{j+k+1} & c_{j+k+2} & \cdots & c_f \\ c_1 c_2 \cdots c_{j-e} & c_{j-e+1} \cdots c_{j-1} & & & & & c_j & c_{j+1} \cdots c_{j+k} b_1 b_2 \cdots b_d, & & & \end{matrix}$$

so the transpose of the situation in Theorem 1.1(f). *Suppose there are equalities and inequalities as in Theorem* 1.1(f) *then* $\sum_{\sigma \in Q} r_\sigma \cdot [{}^\sigma S \mid T_{\nu \backslash \lambda}] = 0$, *where* Q *and* r_σ *are as in Theorem* 1.1(f) *and* ${}^\sigma S$ *has a similar definition as* ${}^\sigma U$ *in Theorem* 1.1(f) *except that the* c_i*'s are in the* i*th and* $(i+1)$*th row. Observe that* ${}^\sigma S \leqslant_r S$ *when* $\sigma \notin S_\varepsilon$, *where* ε *is as in Theorem* 1.1(f). *Moreover*

$$[S \mid T_{\nu \backslash \lambda}] \in \langle [V \mid T_{\nu \backslash \lambda}] \mid V \in T^{\nu \backslash \lambda}(\gamma),\ V \leqslant_r S,\ V^{\mathrm{tr}} \text{ is standard} \rangle_{\mathbb{Z} \cdot 1_A},$$

$$\{ [S \mid T_{\nu \backslash \lambda}] \mid S \in T^{\nu \backslash \lambda}\ (\leqslant r),\ S^{\mathrm{tr}} \text{ is standard} \} \text{ is an } A\text{-independent system.}$$

(f) (1) *Suppose one is in the situation of Theorem* 1.1(f) *with respect to* U. *Then* $\sum_{\sigma \in Q} [S \mid \boxed{{}^\sigma U}\,] = 0$, *where* Q *and* ${}^\sigma U$ *are as in Theorem* 1.1(f). *And*

$$[S \mid \boxed{U}\,] \in \langle [S \mid \boxed{V}\,] \mid V \in ST^{\nu \backslash \lambda}(\delta),\ V \leqslant_c U \rangle_{\mathbb{Z} \cdot 1_A},$$

$$\{ [(T_{\tilde{\nu} \backslash \tilde{\lambda}})^{\mathrm{tr}} \mid \boxed{T}\,] \mid T \in ST^{\nu \backslash \lambda}\ (\leqslant s) \} \text{ is an } A\text{-independent system.}$$

(2) *Put* $\alpha = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_{\lambda_1}, 0^{\nu_1 - \lambda_1})$ *and* $\beta = \nu \backslash \lambda$, *as partitions. Suppose there is an* $i \in \{1, 2, \ldots, \nu_1 - 1\}$ *such that* $\beta_i \neq 0$, $\beta_{i+1} \neq 0$ *and* $\tilde{\nu}_{i+1} > \alpha_i$, (*i.e., the* i*th and* $(i+1)$*th-column meet*). *Let* $j \in \{1, 2, \ldots, \beta_{i+1}\}$ *with* $j \geqslant \tilde{\nu}_{i+1} - \alpha_i$ *and write the* i*th and* $(i+1)$*th-column of* S *like*

$$\begin{array}{cc}
 & c_1 \\
 & \vdots \\
a_1 & c_{j-e+1} \\
a_2 & c_{j-e+2} \\
\vdots & \vdots \\
a_e & c_{j-1} \\
c_{j+1} & c_j \\
 & b_1 \\
\cdot & b_2 \\
\cdot & \vdots \\
\cdot & b_d \\
c_f &
\end{array}$$

the transpose of the situation in Theorem 1.1(e).

Suppose there are equalities and inequalities as in Theorem 1.1(e). *Then* $\sum_{\sigma \in R} [\,\boxed{S^\sigma} \mid U] = 0$, *where* R *is as in Theorem* 1.1(e) *and* S^σ *has a similar* *definition as U^σ in Theorem* 1.1(e) *except that the c_i's are in the ith and $(i+1)$th-column. Observe that $S^\sigma <_c S$ for $\sigma \neq 1$. Moreover,*

$$[\boxed{S} \mid U] \in \langle [\boxed{V} \mid U] \mid V \in T^{\nu\setminus\lambda}(\gamma),\ V \leqslant_c S,\ V^{\mathrm{tr}} \textit{ is standard}\rangle_{\mathbb{Z}\cdot 1_A},$$

$$[\boxed{S} \mid U] \in \langle [\boxed{S} \mid W] \mid W \in ST^{\nu\setminus\lambda}(\delta),\ W \leqslant_r U\rangle_{\mathbb{Z}\cdot 1_A}.$$

(g) (*Compare* [7].) $D_P(U, \overset{*}{T}_{\nu\setminus\lambda})[S \mid \overset{*}{T}_{\nu\setminus\lambda}] = [S \mid U]$.

$$D_P(U, T_{\nu\setminus\lambda})[S \mid T_{\nu\setminus\lambda}] = [S \mid \boxed{U}\,].$$

$D_L(S, \overset{*}{T}_{\nu\setminus\lambda})[\overset{*}{T}_{\nu\setminus\lambda} \mid U] = H(S)\cdot[S \mid U]$, *where $H(S)$ is the product of the orders of the stabilisors of the rows of S (see the definition of one-rowed bipermanents).*

Proof. (a) (1) Suppose we have proved the result with $\overset{*}{T}_\nu$ in the place of S (and $\lambda = (1^n)$). Then by applying $H(S)^{-1}\cdot D_L(S, \overset{*}{T}_\nu)$, we see that we are done over $\mathbb{Z}$, hence over A, where $H(S)$ is the product of the orders of the stabilisors of the rows of S (see the definition of one-rowed bipermanents). So let $S = \overset{*}{T}_\nu$, and $\lambda = (1^n)$, then $[\overset{*}{T}_\nu \mid U^\sigma]$ and all the $[V \mid W]$ in the span can be seen as bideterminants, see the remarks about one-rowed bipermanents. The assertion about $\Sigma\,\mathrm{sgn}(\sigma)[\overset{*}{T}_\nu \mid U^\sigma]$ is a consequence of the Laplace expansion for a determinant corresponding to a division of columns in two groups (see [7] in case of doubt). The anti-commutativity of the exterior letter place algebras causes no problems.

(2) As in the proof of (a)(1) we shall prove the straightening for a special case and derive the general case from it.

Replace all the c_i's by $c := c_j$ and let $\hat{S}$ be the resulting tableau on the letter side and let $\hat{\gamma}$ the content of $\hat{S}$. Let us denote the ith and $(i+1)$th-row of U by $(x_1 x_2 \cdots x_{\nu_i})$ respectively $(y_1 y_2 \cdots y_{\nu_{i+1}})$ then

$$\left[\begin{array}{ccccccc|cccc} a_1 a_2 & \cdots & a_{j-1} c & \cdots & \cdots & \cdots & c & x_1 x_2 & \cdots & x_{\nu_i} \\ c\ c & \cdots & c & c & \cdots & c b_1 b_d & \cdots\ b_d & y_1 y_2 & \cdots & y_{\alpha_{i+1}} \end{array}\right]$$

equals

$$\pm \sum_{\sigma_1,\sigma_2} \mathrm{sgn}(\sigma_1)\,\mathrm{sgn}(\sigma_2)$$

$$\times\left[\begin{array}{cccc|ccc} c\ c & \cdots & c & & y_{\sigma_2(1)} y_{\sigma_2(2)} & \cdots\ y_{\sigma_2(j+k)} x_{\sigma_1(j)} & \cdots\ x_{\sigma_1(\nu_i)} \\ a_1 a_2 & \cdots & a_{j-1} & & x_{\sigma_1(1)} x_{\sigma_1(2)} & \cdots\ x_{\sigma_1(j-1)} & \\ b_1 b_2 & \cdots & b_d & & y_{\sigma_2(j+k+1)} y_{\sigma_2(j+k+2)} & \cdots\ y_{\sigma_2(\nu_{i+1})} & \end{array}\right],$$

where the summation is over all $\sigma_1 \in \{\sigma \in S_{\nu_i} \mid \sigma(1) < \sigma(2) < \cdots < \sigma(j-1),\ \sigma(j) < \sigma(j+1) < \cdots < \sigma(\nu_i)\}$ and all $\sigma_2 \in \{\sigma \in S_{\nu_{i+1}} \mid \sigma(1) < \sigma(2) < \cdots < \sigma(j+k),\ \sigma(j+k+1) < \sigma(j+k+2) < \cdots < \sigma(\nu_{i+1})\}$.

Hence, $[\hat{S}|U] \in \langle [V|W] | (V, W) \in BT^{\rho}(\hat{\gamma}, \delta)$ for some $\rho \vdash n$ with $\rho > \nu\rangle_{\mathbb{Z}\cdot 1_A}$. By applying $\prod_{l \neq c} D_L^{\varepsilon_l}(l, c)$ one finds the assertion for $\sum_{\sigma \in Q} [{}^{\sigma}S | U]$, where ε is as in Theorem 1.1(f). In case $U = T_{\nu}$ the span is zero since each $W \in BT^{\rho}(\tilde{\nu})$ with $\rho > \nu$ has two equal entries in one of its rows.

(b) Follows from (a), see the remark below Theorem 1.1.

(c) Can be proved in a similar (straightforward) way as Theorem 1.1(c).

(d) Let us denote the system by I and suppose there is a non-trivial relation $\sum_{(S,T) \in I} a_{S,T}[S|T] = 0$. Let T_0 be the column lexicographic smallest element of $\{T \in \bigcup_{\mu \vdash n} ST^{\mu}\ (\leqslant s) | a_{S,T} \neq 0$ for some $S\}$.

Then by (c), $0 = C_P(T_0)(\sum a_{S,T}[S|T]) = \sum_S a_{S,T_0}(S|T_{\mu})$, where μ is the shape of T_0.

Claim. $J := \{[S|T_{\mu}] | S \in T^{\mu}\ (\leqslant r)$, S^{tr} is standard$\}$ is an A independent system.

Proof of Claim. Set $K = \{U \in T^{\mu}\ (\leqslant r) |$ in each column of U the elements are weakly increasing from top to bottom$\}$. And let for $U \in K$, $[U]$ denote the "monomial":

$$[U_{1,1}|1] \cdot [U_{2,1}|1] \cdot \cdots \cdot [U_{\tilde{\mu}_1,1}|1] \\ \cdot [U_{1,2}|2] \cdot \cdots \cdot [U_{\tilde{\mu}_2,2}|2] \cdot \cdots \cdot [U_{\tilde{\mu}_{\mu_1},\mu_1}|\mu_1].$$

Then $L := \{[U] | U \in K\}$ is an A-independent system. And the column lexicographic order on $T^{\mu}\ (\leqslant r)$ induces a total order on L. Now for $S \in T^{\mu}$ $(\leqslant r)$ with S^{tr} standard write $[S|T_{\mu}]$ as a linear combination of elements of L, then the smallest element of L which occur with a non-zero coefficient is $[S]$, and its coefficient is in fact 1. The independence of J now follows.

By the claim, $a_{S,T_0} = 0$ for all S, which implies a contradiction. So I is independent, as was to be proved.

(e) The straightening can be proved in a similar way as that in (a)(2). The result for $[S|T_{\nu\backslash\lambda}]$ follows from the first assertion in (e), see the remark below Theorem 1.1. And the independence system can be proved as that for J in the proof of (d).

(f) (1) Concerning the first two assertions we may assume, as in the proof of (a)(1), that $S = \overset{*}{T}_{\nu\backslash\lambda}$.

Now the second map in Lemma 2.1 maps $(\overset{*}{T}_{\nu\backslash\lambda} | \boxed{T})$ to $[\overset{*}{T}_{\nu\backslash\lambda} | \boxed{T}]$ for all $T \in T^{\nu\backslash\lambda}\ (\leqslant S)$. Hence the first two assertions follow from Theorem 1.1(f). The independence of the system can be proved in a similar way as that for S in the proof of (d).

(2) Concerning the first two assertions we may asume $U = \overset{*}{T}_{\nu\backslash\lambda}$ by (g). Now the first map in Lemma 2.1 maps $(V^{\mathrm{tr}} \,|\, \boxed{(\overset{*}{T}_{\nu\backslash\lambda})^{\mathrm{tr}}})$ to $[\boxed{V} \,|\, \overset{*}{T}_{\nu\backslash\lambda}]$ for all $V \in T^{\nu\backslash\lambda}$ $(\leqslant r)$. Now by Theorem 1.1(g), $(V^{\mathrm{tr}} \,|\, \boxed{(\overset{*}{T}_{\nu\backslash\lambda})^{\mathrm{tr}}}) = D_p((\overset{*}{T}_{\nu\backslash\lambda})^{\mathrm{tr}}, T_{\tilde{\nu}\backslash\tilde{\lambda}})(V^{\mathrm{tr}} \,|\, T_{\tilde{\nu}\backslash\tilde{\lambda}})$, for all $V \in T^{\nu\backslash\lambda}$ $(\leqslant r)$. So the first two assertions follow from Theorem 1.1(e). Concerning the last assertion observe that the second map in Lema 2.1 maps $(\boxed{\overset{*}{T}_{\nu\backslash\lambda}} \,|\, V)$ to $[\boxed{\overset{*}{T}_{\nu\backslash\lambda}} \,|\, V]$ for all $V \in T^{\nu\backslash\lambda}$ $(\leqslant s)$. By Theorem 1.1(g), $(\boxed{T_{\nu\backslash\lambda}} \,|\, V) = D_L(\overset{*}{T}_{\nu\backslash\lambda}, T_{\nu\backslash\lambda})(T_{\nu\backslash\lambda} \,|\, V)$ for all $V \in T^{\nu\backslash\lambda}$ $(\leqslant s)$. Hence, the last assertion follows from Theorem 1.1(e). ∎

Remark. The "missing" parts with respect to Theorem 1.1 are, in (e), results for $[(T_{\tilde{\nu}\backslash\tilde{\lambda}})^{\mathrm{tr}} \,|\, U]$ and, in (f)(2), and independence result for $\{[\boxed{S} \,|\, \overset{*}{T}_{\nu\backslash\lambda}] \,|\, S \in T^{\nu\backslash\lambda}\ (\leqslant r)\}$.

Now, starting with the later, the independence is not true in general but it is true when A is $\mathbb{Z}$-torsion free, which shall be proved in the second paper.

Concerning the elements $[(T_{\tilde{\nu}\backslash\tilde{\lambda}})^{\mathrm{tr}} \,|\, U]$, observe that there is an A-isomorphism

$$\langle [(T_{\tilde{\nu}\backslash\tilde{\lambda}})^{\mathrm{tr}} \,|\, U] \,|\, U \in T^{\nu\backslash\lambda}\ (\leqslant s) \rangle_A \cong \bigotimes_{i=1}^{\tilde{\nu}_1} \Lambda^{\alpha_i}(A^s), \qquad \textit{where} \quad \alpha = \nu\backslash\lambda.$$

Now see Proposition 2.3.

We shall now make the connection between [1] and exterior letter place algebras. The connection is similar as for ordinary letter place algebras. Let $\nu \vdash n$, $a \in \mathbb{N}_0$, $\lambda \vdash a$ and suppose $\nu \supset \lambda$. Put $\alpha = \tilde{\nu}\backslash\tilde{\lambda}$ and $\beta = \nu\backslash\lambda$, as partitions. Define, as in [1], natural transformations $\bigotimes_{j=1}^{\tilde{\nu}_1} D^{\beta_j} \to (-)^{\otimes n-a} \to \bigotimes_{i=1}^{\tilde{\nu}_1} \Lambda^{\alpha_i}$ as follows.

Define the first one by ($m_{i,j} \in M$ all i, j),

$$\bigotimes_{i=1}^{\tilde{\nu}_1} (m_{i,1}{}^{(n_{i,1})} \cdot m_{i,2}{}^{(n_{i,2})} \cdot \cdots \cdot m_{i,\beta_i}{}^{(n_i,\beta_i)})$$

$$\mapsto \sum_{\sigma} \bigotimes_{i} (m_{i,\sigma_i(1)} \otimes m_{i,\sigma_i(2)} \otimes \cdots \otimes m_{i,\sigma_i(\beta_i)}),$$

where $n_i := (n_{i,1}, n_{i,2}, \ldots, n_{i,\beta_i}) \models \beta_i$ all i and the summation is over all $\sigma \in \prod_{i=1}^{\nu_1} R_i$ with R_i a transversal for S_{β_i}/S_{n_i} for all i. And $\bigotimes_i x_i$, for $x_i \in M^{\otimes \beta_i}$ all $i \leqslant \tilde{\nu}_1$, means the image of $x_1 \otimes x_2 \otimes \cdots \otimes x_{\tilde{\nu}_1}$ in $M^{\otimes (n-a)}$.

Define the second transformation by ($m_i \in M$ all i),

$$m_1 \otimes m_2 \otimes \cdots \otimes m_n \mapsto \bigotimes_{j=1}^{\tilde{\nu}_1} (m_{1,j} \wedge m_{2,j} \wedge \cdots \wedge m_{\alpha_j,j}), \quad \text{where, for all } (i,j),$$

$m_{i,j}=m_{l(i,j)}$ with $l(i,j)=\sum_{k=1}^{n(i,j)-1}\beta_k+j-(\nu_{n(i,j)}-\beta_{n(i,j)})$, where $n(i,j)=\tilde{\nu}_j-a_j+i$. This latter transformation can be described in a similar way as the corresponding one in the definition of $d_{\tilde{\nu}\backslash\tilde{\lambda}}$.

Let us denote the composition of the above transformations by $d^0_{\nu\backslash\lambda}$ then by Theorem 2.2(c) there is a commutative diagram for each $r\in\mathbb{N}_0$:

$$\begin{array}{ccc} D^{\beta_1}(A^r)\otimes D^{\beta_2}(A^r)\otimes\cdots\otimes D^{\beta_{\tilde{\nu}_1}}(A^r) & \xrightarrow{d^0_{\tilde{\nu}\backslash\tilde{\lambda}}} & \Lambda^{\alpha_1}(A^r)\otimes\Lambda^{\alpha_2}(A^r)\otimes\cdots\otimes\Lambda^{\alpha_{\nu_1}}(A^r) \\ \downarrow\cong & & \cong\downarrow \\ \langle[S\mid\overset{*}{T}_{\nu\backslash\lambda}]\mid S\in T^{\nu\backslash\lambda}(\leqslant r)\rangle_A & \xrightarrow{C_p(\overset{*}{T}_{\nu\backslash\lambda})} & \langle[S\mid T]\mid(S,T)\in BT(\leqslant r,\alpha)\rangle_A. \end{array}$$

Here the vertical maps have a similar definition as the corresponding ones for $d_{\tilde{\nu}\backslash\tilde{\lambda}}(A^r)$.

By Theorem 2.2(e), $\operatorname{Im} C_p(\overset{*}{T}_{\nu\backslash\lambda})=\langle(S\mid T_{\nu\backslash\lambda})=\langle(S\mid T_{\nu\backslash\lambda})\mid S\in T^{\tilde{\nu}\backslash\tilde{\lambda}}(\leqslant r)$, S^{tr} is standard$\rangle_A$ is universally free. Slightly generalising [1]:

DEFINITION. Let ν and λ be Young diagrams with $\nu\supset\lambda$ then $W_A^{\nu\backslash\lambda}:=\operatorname{Im}(d^0_{\nu\backslash\lambda}: A\text{-mod}\to A\text{-mod})$, and $W_A^{\nu\backslash\lambda}$ shall be called the (*skew*) *Weyl functor* for the (skew) Young diagram $\nu\backslash\lambda$ over A. Observe that the image of $W_A^{\nu\backslash\lambda}$ is indeed in A-mod because $W_A^{\nu\backslash\lambda}$ is a functor and $W_A^{\nu\backslash\lambda}(M)$ free of finite rank when M is. Also $B\otimes W_A^{\nu\backslash\lambda}\cong W_B^{\nu\backslash\lambda}(B\otimes -)$ for any commutative unitary A-algebra B. (We shall drop the suffix "A" in $W_A^{\nu\backslash\lambda}$ when there is no danger of confusion.) Clearly, $W^{\lambda}(A^m)=0$ when $\tilde{\lambda}_1>m$.

EXAMPLES. $W^{(n)}\cong D^n$ and $W^{(1^n)}=\Lambda^n$ (the "extreme" cases).

Remarks. (1) $W^{\nu\backslash\lambda}(A^r)$ is the module $K_{\nu\backslash\lambda}(A^r)$ in [1], and it corresponds to $coschur_A(\nu\backslash\lambda)$ in [3], and, for $\lambda=(0)$, to $\bar{V}^{\tilde{\nu}}$ in [6], (because of Theorem 1.1(f) and Theorem 2.2(e)), and to $\mathscr{W}_r^{\nu}(A)$ in [7]. Also $W=\bigvee_\nu$ in the notation of [20]. Now $W^{\nu\backslash\lambda}(A^r)$ is the skew Weyl module for $\nu\backslash\lambda$ in the representation theory of $Gl(r,A)$ (where A^r is the natural representation space of $Gl(r,A)$), which explains our notation and namegiving.

(2) By Theorem 2.2, parts (e) and (f)(1),

$$W^{\nu\backslash\lambda}(A^r)\cong\langle[(T_{\nu\backslash\lambda})^{\mathrm{tr}}\mid T]\mid T\in ST^{\tilde{\nu}\backslash\tilde{\lambda}}(\leqslant r)\rangle_A \quad \text{as } A[\operatorname{End}_A(A^r)]\text{-modules.}$$

It is easily seen that $d^0_{\tilde{\nu}\backslash\tilde{\lambda}}(M)$ is the composition

$$\bigotimes_{j=1}^{\tilde{\nu}_1} D^{\beta_j}(M)\simeq\bigotimes_j(S^{\beta_j}(M^*)^*)\simeq\Big(\bigotimes_j S^{\beta_j}(M^*)\Big)^*\xrightarrow{d_{\tilde{\nu}\backslash\tilde{\lambda}}(M^*)^*}\Big(\bigotimes_{i=1}^{\nu_1}\Lambda^{\alpha_i}(M^*)\Big)^*$$

$$\simeq\bigotimes_i(\Lambda^{\alpha_i}(M^*)^*)\simeq\bigotimes_i\Lambda^{\alpha_i}(M),$$

where $(-)^*$ denotes the functor $\operatorname{Hom}_A(-,A)$, the first two isomorphisms are the obvious ones and the last two are the inverses of the obvious ones. All maps are natural so they define natural transformations, and hence $S^{\nu\backslash\lambda}((-)^*)^* \cong W_A^{\nu\backslash\lambda}$ as endofunctors on A-mod.

DEFINITION. Let F: A-mod $\to$ A-mod be a functor then the *contravariant dual* of F is the endofunctor on A-mod defined by $M \mapsto F(M^*)^*$. The *contravariant dual* of f is denoted by F^0. Especially $(S_A^{\nu\backslash\lambda})^0 \cong W_A^{\nu\backslash\lambda}$ for all Young diagrams ν and λ with $\nu \supset \lambda$. Now $S^{\nu\backslash\lambda}((A^r)^*)^*$, seen as representation of $Gl(r, A)$, is usually denoted by $S^{\nu\backslash\lambda}(A^r)^0$ and is called the *contravariant dual* of $S^{\nu\backslash\lambda}(A^r)$, see, for example, [13], which explains our notations and namegiving. The duality between Schur and Weyl functors goes back to [20].

Consider the diagram for $d^0_{\tilde{\nu}\backslash\tilde{\lambda}}(A^r)$ for $r = n - a$.

Clearly $\langle [S \mid T] \mid (S, T) \in BT(1^{n-a}, \alpha)\rangle_A \cong A[S_{n-a}] \otimes_{A[S_\alpha]^{\mathrm{sgn}}} A$ as $A[S_{n-a}]$-module, via the vertical map on the right in fact using the convention about the inclusion $A[S_{n-a}] \otimes_{A[S_\alpha]^{\mathrm{sgn}}} A$ in $\bigotimes_i A^{\alpha_i}(A^{n-a})$.

Also $\langle [S \mid \overset{*}{T}_{\nu\backslash\lambda}] \mid S \in T^{\nu\backslash\lambda}(1^{n-a})\rangle_A \cong A[S_{n-a}] \otimes_{A[S_\beta]^{\mathrm{triv}}} A$ as $A[S_{n-a}]$-module.

CONVENTION (Notation as Above). We shall identify $A[S_{n-a}] \otimes_{A[S_\beta]^{\mathrm{triv}}} A$ with the subspace of $\bigotimes_j D^{\beta_j}(A^{n-a})$ corresponding to $\langle [S \mid \overset{*}{T}_{\nu\backslash\lambda}] \mid S \in T^{\nu\backslash\lambda}(1^{n-a})\rangle_A$, via the vertical map on the left in the diagram for $d^0_{\tilde{\nu}\backslash\tilde{\lambda}}(A^{n-a})$.

So there is a commutative diagram

$$\begin{array}{ccc} A[S_{n-a}] \otimes_{A[S_\beta]^{\mathrm{triv}}} A & \xrightarrow{e^*_{\tilde{\nu}\backslash\lambda}} & A[S_{n-a}] \otimes_{A[S_\alpha]^{\mathrm{sgn}}} A \\ \cong\downarrow & & \cong\downarrow \\ \langle [S \mid \overset{*}{T}_{\nu\backslash\lambda}] \mid S \in T^{\nu\backslash\lambda}(1^{n-a})\rangle_A & \xrightarrow{C_P(\overset{*}{T}_{\nu\backslash\lambda})} & \langle [S \mid T] \mid (S, T) \in BT(1^{n-a}, \alpha)\rangle_A. \end{array}$$

By Theorem 2.2(e), $\operatorname{Im} C_P(\overset{*}{T}_{\nu\backslash\lambda}) = \langle [S \mid T_{\nu\backslash\lambda}] \mid S \in T^{\nu\backslash\lambda}(1^{n-a}),\ S^{\mathrm{tr}}$ is standard$\rangle_A$ which is universally free.

DEFINITION. $\mathcal{S}^{\nu\backslash\lambda}(A) := \operatorname{Im}(e^*_{\tilde{\nu}\backslash\tilde{\lambda}})$, and $\mathcal{S}^{\nu\backslash\lambda}(A)$ is called the (*skew*) *dual Specht module* of S_{n-a} for the (skew) diagram $\nu\backslash\lambda$ over A. When $\lambda = (0)$ it corresponds to $\mathcal{S}^\nu(A)$ in [7], because of Theorem 2.2(e) and theorem 1.1(f). The isomorphism $S^{\nu\backslash\lambda}((A^{n-a})^*)^* \cong W_A^{\nu\backslash\lambda}(A^{n-a})$ yields an isomorphism $\mathcal{S}_{\nu\backslash\lambda}(A)^* \cong \mathcal{S}^{\nu\backslash\lambda}(A)$, where S_{n-a} acts on $\mathcal{S}_{\nu\backslash\lambda}(A)^*$ in the usual way: $\sigma \cdot f(x) = f(\sigma^{-1}x)$ for $f \in \operatorname{Hom}_A(\mathcal{S}_{\nu\backslash\lambda}(A), A)$, $\sigma \in S_{n-a}$, and $x \in \mathcal{S}_{\nu\backslash\lambda}(A)$.

The kernel of $d^0_{\tilde{\nu}\backslash\tilde{\lambda}}$, compare Proposition 1.2, is described in:

PROPOSITION 2.3 (Compare [20]). *Let ν and λ be Young diagrams with $\nu \supsetneqq \lambda$. Put $\alpha = (\lambda_1, \lambda_2, ..., \lambda_{\tilde{\lambda}_1}, 0^{\tilde{\nu}_1 - \tilde{\lambda}_1})$ and $\beta = \nu \backslash \lambda$ as partitions. Then* $\ker d^0_{\tilde{\nu}\backslash\tilde{\lambda}}(M)$ *is generated by all the elements:*

$$\sum_{(l,h)} \{x_1 \otimes x_2 \otimes \cdots \otimes x_{i-1} \otimes (a_1{}^{(m_1)} \cdot a_2{}^{(m_2)} \cdot \cdots \cdot a_e{}^{(m_e)} \cdot c_1{}^{(l_1)} \cdot c_2{}^{(l_2)} \cdot \cdots \cdot c_s{}^{(l_s)})$$
$$\otimes (c_1{}^{(h_1)} \cdot c_2{}^{(h_2)} \cdot \cdots \cdot c_s{}^{(h_s)} \cdot b_1{}^{(g_1)} \cdot b_2{}^{(g_2)} \cdot \cdots \cdot b_d{}^{(g_d)}) \otimes x_{i+2} \otimes \cdots \otimes x_{\nu_1}\},$$

where i, j, k, d, e, and f are as in the Theorem 2.2(e)), $x_r \in D^{\beta_r}(M)$ all r. And $s \in \{1, 2, ..., f-k\}$, $t \in \{1, 2, ..., r\}$, $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_s) \models f$ such that $\sum_{i=1}^{t-1} \varepsilon_i \leqslant j-1$ and $\sum_{i=1}^{t} \varepsilon_i \geqslant j+k$.

And $a_1, a_2, ..., a_e$, $c_1, c_2, ..., c_s$, $b_1, b_2, ..., b_d \in M$, $(m_1, m_2, ..., m_e) \models e$, $(g_1, g_2, ..., g_d) \models d$ and the summation is over $\{(l, h) \in (\mathbb{N}_0^s \times \mathbb{N}_0^s)^f \mid l \models \beta_i - e,\ h \models \beta_{i+1} - d,\ l_i + h_i = \varepsilon_i \text{ all } i\}$.

(This can be proved in a similar way as Proposition 1.2 using Theorem 2.2(e).) ∎

II.2. *The Filtrations*

Here is the result from [1] in which implicitly the straightening on the exterior letter place algebras was involved. It also shows how Weyl functors turn up in studying exterior powers.

PROPOSITION 2.4 (Slight Refinement of [1]). *The functor* $\Lambda^n(-\otimes-)$: A-mod $\times$ A-mod $\to$ A-mod *admits an explicit filtration by subfunctors*

$$\Lambda^n(-\otimes-) = L^{\mu_1} \supset L^{\mu_2} \supset \cdots \supset L^{\mu^{l+1}} = 0$$

with $l \in \mathbb{N}$, $\mu^1 < \mu^2 < \cdots < \mu^l$ are all Young diagrams for n and for all $i \in \underline{l}$ there is an equivalence

$$W^{\mu^i} \otimes S^{\tilde{\mu}^i} \simeq L^{\mu^i}/L^{\mu^{i+1}}.$$

Moreover when $\nu \vdash n$, $\bigotimes_{i=1}^{\tilde{\nu}_1} \Lambda^{\nu_i}$: A-mod $\to$ A-mod admits an explicit filtration by subfunctors

$$\bigotimes_i \Lambda^{\nu_i} = K^{\lambda^1} \supset K^{\lambda^2} \supset \cdots \supset K^{\lambda^{m+1}} = 0$$

with $m \in \mathbb{N}$, $\lambda^1 < \lambda^2 < \cdots < \lambda^m$ are all elements of $\{\lambda \vdash n \mid \tilde{\lambda} \leqslant \nu\}$ and for all $i \in \underline{m}$ there is an equivalence $W^{\lambda^i} \otimes F_i \simeq K^{\lambda^i}/K^{\lambda^{i+1}}$, where F_i is the constant functor with value theb free A-module of rank $\# ST^{\lambda^i}(\nu)$.

In case $\nu = (1^n)$ then S_n acts on $\bigotimes_i \Lambda^{\nu^i} = (1)^{\otimes n}$ as follows: for $\sigma \in S_n$, σ acts via $\operatorname{sgn}(\sigma) \cdot G_\sigma$, *where G_σ is the usual permutation action on the nth tensor power given by σ. In this case the K^{λ^i} can be chosen to be S_n-invariant* *and such that the action on $K^{\lambda^i}/K^{\lambda^{i+1}}$ corresponds to an action on F_i turning the value into $\mathscr{S}^{\tilde{\lambda}^i}(A)$.*

Proof. The proof below is quite similar to the proof of Proposition 1.3. Let $\mu \vdash n$ and $\varphi_\mu: \bigotimes_{i=1}^{\tilde{\mu}_1} D^{\mu_i}(M) \otimes \bigotimes_{i=1}^{\tilde{\mu}_1} \Lambda^{\mu_i}(N) \to \Lambda^n(M \otimes N)$ be the natural A- homomorphism defined by

$$\bigotimes_i (m_{i,1}{}^{(l_{i,1})} \cdot m_{i,2}{}^{(l_{i,2})} \cdot \cdots \cdot m_{i,\mu_i}{}^{(l_{i,\mu_i})}) \otimes \bigotimes_i (n_{i,1} \wedge n_{i,2} \wedge \cdots \wedge n_{i,\mu_i})$$

$$\mapsto \sum_\sigma \wedge_i ((m_{i,\sigma_i(1)} \otimes n_{i,1}) \wedge (m_{i,\sigma_i(2)} \otimes n_{i,2}) \wedge \cdots \wedge (m_{i,\sigma_i(\mu_i)} \otimes n_{i,\mu_i})),$$

where $l_i := (l_{i,1}, l_{i,2}, ..., l_{i,\mu_i}) \models \mu_i$ all i and the summation is over all $\sigma \in \prod_i R_i$ with R_i a transversal for S_{μ_i}/S_{l_i} for all i. Here $\bigwedge_i x_i$, for $x_i \in \Lambda^{\mu_i}(M \otimes N)$ all i, means the image of $x_1 \otimes x_2 \otimes \cdots \otimes x_{\tilde{\mu}_1}$ in $\Lambda^n(M \otimes N)$ via the product map for the exterior power algebra of $M \otimes N$.

When M and N are free A-modules of rank r respectively s, then $\langle [(T_{\tilde{\mu}})^{\text{tr}} | T] \mid T \in T^\mu\ (\leqslant s) \rangle_A \simeq \bigotimes_i \Lambda^{\mu_i}(N)$ via the isomorphism defined by $[(T_{\tilde{\mu}})^{\text{tr}} | T] \to \bigotimes_i (e_{T_{i,1}} \wedge e_{T_{i,2}} \wedge \cdots \wedge e_{T_{i,\mu_i}})$, where $(e_1, e_2, ..., e_s)$ is the natural basis of $N = A^s$. Hence, φ_μ corresponds to the A-linear map (see above the definition of $W^{\nu \backslash \lambda}$): $\langle [S | \mathring{T}_\mu] \mid S \in T^\mu\ (\leqslant r) \rangle_A \otimes \langle [(T_{\tilde{\mu}})^{\text{tr}} | T] \mid T \in T^\mu\ (\leqslant s) \rangle_A \to \langle [S | T] \mid (S, T) \in \bigcup_{\mu \vdash n} BT^\mu\ (\leqslant r, \leqslant s) \rangle_A$ defined by $[S | \mathring{T}_\mu] \otimes [(T_{\tilde{\mu}})^{\text{tr}} | T] \mapsto [S | T]$.

Moreover by specialising Theorem 2.2(a) we find that the quotient homomorphism $\bar{\varphi}_\mu$: $\bigotimes_i D^{\mu_i}(M) \otimes \bigotimes_i \Lambda^{\mu_i}(N) \to \Lambda^n(M \otimes N)$ factorizes through a natural A-homomorphism,

$$c_\mu: W^\mu(M) \otimes S^{\tilde{\mu}}(N) \to \Lambda^n(M \otimes N) \Big/ \sum_{\tau > \mu} \operatorname{Im} \varphi_\tau.$$

When M and N are free A-modules of rank r respectively s then c_μ corresponds by Theorems 2.2(b) to the A-linear map,

$$\langle [S^{\text{tr}} | T_\mu] \mid S \in ST^{\tilde{\mu}}\ (\leqslant r) \rangle_A \otimes \langle [(T_{\tilde{\mu}})^{\text{tr}} | T] \mid T \in ST^\mu\ (\leqslant s) \rangle_A$$

$$\to \langle [S^{\text{tr}} | T] \mid (S, T) \in \bigcup_{\mu \vdash n} (ST^{\tilde{\mu}}\ (\leqslant r) \times ST^\mu\ (\leqslant s)) \rangle_A,$$

$$\text{defined by} \quad [S^{\text{tr}} | T_\mu] \otimes [(T_{\tilde{\mu}})^{\text{tr}} | T] \mapsto [S^{\text{tr}} | T].$$

By Theorem 2.2(d) this map is an isomorphism onto its image.

Observe that $\varphi_{(1^n)}$ is surjective hence $\sum_\mu \operatorname{Im} \varphi_\mu = \Lambda^n(M \otimes N)$.

Observe that φ_μ generalises to a natural transformation Φ_μ: $\bigotimes_i D^{\mu_i}(-) \otimes \bigotimes_i \Lambda^{\mu_i}(-) \to \Lambda^n(- \otimes -)$. Hence c_μ generalises to a natural transformation

$C_\mu\colon W^\mu \otimes S^{\tilde\mu} \to \Lambda^n(-\otimes-)/\sum_{\tau>\mu} \operatorname{Im} \Phi_\tau$, which is an equivalence onto its image.

Hence, put $L^\mu = \sum_{\tau \geqslant \mu} \operatorname{Im} \Phi_\tau$ then the L^μ make up the desired filtration. Concerning $\bigotimes_i \Lambda^{\nu_i}$, observe that there is an equivalence $\bigoplus_{(\alpha_1, \alpha_2, ..., \alpha_n) \models n} \bigotimes_{i=1}^n \Lambda^{\alpha_i} \simeq \Lambda^n(-\otimes A^n)$, defined on the summand for $\alpha \models n$ by

$$\bigotimes_i (m_{i,1} \wedge m_{i,2} \wedge \cdots \wedge m_{i,\alpha_i})$$
$$\mapsto \bigwedge_i ((m_{i,1} \otimes e_i) \wedge (m_{i,2} \otimes e_i) \wedge \cdots \wedge (m_{i,\alpha_i} \otimes e_i)),$$

where $m_{i,j} \in M$ all i, j and $(e_1, e_2, ..., e_n)$ is the natural basis of A^n. Now the filtration for $\bigotimes_i \Lambda^{\nu_i}$ is the summand of the one for $\Lambda^n(-\otimes A^n)$ corresponding to $\alpha = \nu$. The dimension of the value of F_i follows from Theorem 2.2(d). The filtration for $\bigotimes_i \Lambda^{\nu_i}$ with $\nu = (1^n)$ has the desired properties. ▌

Remark. As a direct summand of the filtration for $\bigotimes_i \Lambda^{\nu_i}(A^n)$ there is a filtration for $A[S_n] \otimes_{A[S_\nu]^{\mathrm{sgn}}} A$, corresponding to letter content (1^n). Compare [8].

Skew Weyl functors arise as follows:

THEOREM 2.5. *Let* $\nu \vdash n$ *and* $L \in A$-mod.

(a) *There are explicit subfunctors* F_k, *for* $k \in \{0, 1, ..., n\}$, *of* $W^\nu(-\oplus-)\colon A\text{-mod} \times A\text{-mod} \to A\text{-mod}$ *such that* $W^\nu(-\oplus-) = \bigoplus_{k=0}^n F_k$. *And such that, for each k,* F_k *admits an explicit filtration by subfunctors*

$$F_k = K^{\lambda^1} \supset K^{\lambda^2} \supset \cdots \supset K^{\lambda^{l(k)+1}} = 0$$

with $l(k) \in \mathbb{N}$, $\lambda^1 < \lambda^2 < \cdots < \lambda^{l(k)}$ *are all elements of* $\{\lambda \vdash k \mid \nu \supset \lambda\}$ *and for all* $i \in \underline{l(k)}$ *there is an equivalence*

$$W^{\lambda^i} \otimes W^{\nu\backslash\lambda^i} \simeq K^{\lambda^i}/K^{\lambda^{i+1}}.$$

(b) *Suppose there is an exact sequence* $0 \to N \to M \to L \to 0$ *in* A-mod *then* $W^\nu(M)$ *admits an explicit natural filtration*:

$$W^\nu(M) = N^{\mu^1} \supset N^{\mu^2} \supset \cdots \supset N^{\mu^{l+1}} = 0 \qquad \textit{with} \quad l \in \mathbb{N}, \quad \mu^1 < \mu^2 < \cdots < \mu^l$$

are all elements of $I = \bigcup_{k=0}^n \{\mu \vdash k \mid \nu \supset \mu\}$ *and for all* $i \in \underline{l}$ *there is a natural isomorphism* $W^{\mu^i}(N) \otimes W^{\nu\backslash\lambda^i}(L) \simeq N^{\mu^i}/N^{\mu^{i+1}}$. *By "natural" we mean the same as in Theorem* 1.4(b).

Proof. A proof similar to that of Theorem 1.4 can be given, therefore we only give the framework.

(a) Let $k \in \{0, 1, ..., n\}$ and put $P(k) = \{\alpha \in (\mathbb{N}_0)^{\tilde{\nu}_1} \mid \sum_{i=1}^{\tilde{\nu}_1} \alpha_i = k, a_i \leqslant \nu_i$ all $i\}$. Let $\alpha \in P(k)$ then one defines a natural A-linear map

$$b_\alpha \colon \bigotimes_{i=1}^{\tilde{\nu}_1} D^{\alpha_i}(N) \otimes \bigotimes_{i=1}^{\tilde{\nu}_1} D^{\beta_i}(L) \to \bigotimes_{i=1}^{\tilde{\nu}_1} D^{\nu_i}(N \oplus L),, \qquad \text{where}$$

$$\beta_i = \nu_i - \alpha_i \text{ all } i,$$

by

$$\bigotimes_i \left(\prod_{j=1}^{\alpha_i} n_{i,j}{}^{(m_{i,j})} \right) \otimes \bigotimes_i \left(\prod_{h=1}^{\beta_i} l_{i,h}{}^{(g_{i,h})} \right) \mapsto \bigotimes_i \left(\prod_j n_{i,j}{}^{(m_{i,j})} \cdot \prod_h l_{i,h}{}^{(g_{i,h})} \right),$$

where $n_{i,j} \in N$ and $l_{i,j} \in L$ all i, j, h,

and $(m_{i,1}, m_{i,2}, ..., m_{i,\alpha_i}) \models \alpha_i$

and $(l_{i,1}, l_{i,2}, ..., l_{i,\beta_i}) \models \beta_i$ all i.

Put $db_\alpha = d_{\tilde{\nu}}^0(N \oplus L) \circ b_\alpha \colon \bigotimes_i D^{\alpha_i}(N) \otimes \bigotimes_i D^{\beta_i}(L) \to W^\nu(N \oplus L)$.

Put $K_k = \sum_{\alpha \in P(k)} \operatorname{Im} b_\alpha$, then K_k is a natural submodule of $W^\nu(N \oplus L)$ and $\sum_{k=0}^n K_k = W^\nu(N \oplus L)$.

Let $k \in \{0, 1, ..., n\}$ and let $\lambda \vdash k$ be such that $\nu \supset \lambda$ then db_λ shall mean db_α with $\alpha = (\lambda_1, \lambda_2, ..., \lambda_{\tilde{\lambda}_1}, 0^{\tilde{\nu}_1 - \tilde{\lambda}_1})$.

Then the following claims can be proved, by Theorem 2.2 and Proposition 2.3, in a similar way as the corresponding claims in the proof of Theorem 1.4:

Claim 1. For $\alpha \in P(k)$, $\operatorname{Im} db_\alpha \subset \sum_{\tau \vdash k, \tau \geqslant \alpha} \operatorname{Im} db_\tau$, especially $K_k = \sum_{\lambda \vdash k} \operatorname{Im} db_\lambda$. Moreover $\sum_{k=0}^n K_k = \bigoplus_{k=0}^n K_k$.

Claim 2. Let $\lambda \vdash k$ be such that $\nu \supset \lambda$, and put $\beta = \nu \backslash \lambda$.
Then $db_\lambda(x) \in \sum_{\tau \vdash j, \tau > \lambda} \operatorname{Im} db_\tau$ for all $x \in [\operatorname{Ker} d_{\tilde{\lambda}}^0(N) \otimes \bigotimes_{i=1}^{\tilde{\nu}_1} D^{\beta_i}(L)) \cup (\bigotimes_{l=1}^{\tilde{\lambda}_1} D^{\lambda_l}(N) \otimes \operatorname{Ker} d_{\tilde{\nu} \backslash \tilde{\lambda}}^0(L))]$.

By these two claims the quotient map $\overline{db}_\lambda \colon \bigotimes_l D^{\lambda_l}(N) \otimes \bigotimes_i D^{\beta_i}(L) \to K_k / \sum_{\tau \vdash k, \tau > \lambda} \operatorname{Im} db_\tau$ factories through a natural map $c_\lambda \colon W^\lambda(N) \otimes W^{\nu \backslash \lambda}(L) \to K_k / \sum_{\tau \vdash k, \tau > \lambda} \operatorname{Im} db_\tau$.

Claim 3. For all $\lambda \vdash k$ with $\nu \supset \lambda$, c_λ is an isomorphism onto its image.

Now the maps db_α generalise to natural transformations $DB_\alpha \colon \bigotimes_i D^{\alpha_i}(-) \otimes \bigotimes_i D^{\beta_i}(-) \to W^\nu(- \oplus -)$. And each K_k generalises to a functor F_k, with $\bigoplus_{k=0}^{n} F_k = W^{\nu}(-\oplus-)$. And c_λ generalises a natural transformation

$$C_\lambda\colon W^\lambda \otimes W^{\nu\backslash\lambda} \to F_k \Big/ \sum_{\tau \vdash k,\, \tau > \lambda} \operatorname{Im} DB_\tau$$

which is an equivalence onto its image. Hence, the desired filtration follows.

(b) Similar to the proof of Theorem 1.4(b). ∎

Remarks. (1) A slightly weaker version of (a) was claimed in [1].

(2) Theorem 2.5 generalises the well-known results for exterior and divided powers (the cases $\nu=(1^n)$ respectively $\nu=(n)$). Similar observations as in Remark (2) below Theorem 1.4 can be made concerning Weyl functors.

THEOREM 2.6. *Let $a \in \mathbb{N}$, $\lambda \vdash a$, $\nu \vdash n$, suppose $\nu \supsetneq \lambda$ and put $\beta = \tilde{\nu}\backslash\tilde{\lambda}$ as partition. Then $W^{\nu\backslash\lambda}$ admits an explicit filtration by subfunctors*

$$W^{\nu\backslash\lambda} = N^{T^1} \supset N^{T^2} \supset \cdots \supset N^{T^{l+1}} = 0$$

with $l \in \mathbb{N}$, $T^1 <_c T^2 <_c \cdots <_c T^l$ are all elements of $L = \{T \in ST^\mu(\beta) \mid \mu \vdash (n-a),\ \varphi^{\mu}_{\tilde{\nu}\backslash\tilde{\lambda}}(\tilde{\mu})\}$ and for all i there is an equivalence $W^{\mu^i} \xrightarrow{\sim} N^{T^i}/N^{T^{i+1}}$, where μ^i is the shape of T^i. Here $\varphi^{\mu}_{\tilde{\nu}\backslash\tilde{\lambda}}$ is defined as in Theorem 1.5.

Proof. A proof similar to the one of Theorem 1.5 can be given. Therefore we shall only give a framework.

Let $T \in L$ and let μ be its shape. Define the set P and the tableau T^ρ as in the proof of Theorem 1.5 with the only difference that $\tilde{\nu}$ and $\tilde{\lambda}$ are replaced by ν respectively λ. Define a natural A-homomorphism $b_T\colon \bigotimes_{i=1}^{\tilde{\mu}_1} D^{\mu_i}(M) \to \Lambda^{n-a}(M \otimes A^n)$ by

$$x \mapsto \sum_{\rho \in P} \operatorname{sgn}(\rho) \cdot \varphi_\mu \left(x \otimes \bigotimes_i \left(e_{(T^\rho)_{i,1}} \wedge e_{(T^\rho)_{i,2}} \wedge \cdots \wedge e_{(T^\rho)_{i,\mu_i}} \right) \right),$$

where $(e_1, e_2, ..., e_n)$ is the natural basis of A^n and φ_μ is the map in the proof of Proposition 2.4.

There is a natural injective A-homomorphism $\bigotimes_{j=1}^{\nu_1} \Lambda^{\beta_j}(M) \to \Lambda^{n-a}(M \otimes A^n)$ defined by $\bigotimes_j (m_{j,1} \wedge m_{j,2} \wedge \cdots \wedge m_{j,\beta_j}) \mapsto \bigwedge_j ((m_{j,1} \otimes e_j) \wedge (m_{j,2} \otimes e_j) \wedge \cdots \wedge (m_{j,\beta_j} \otimes e_j))$, where $m_{j,k} \in M$ all j, k. The image contains $\operatorname{Im}(b_T)$ since the content of T is β.

When M is free of rank r then "b_T": $\bigotimes_i D^{\mu_i}(M) \to \bigotimes_j \Lambda^{\beta_j}(M)$ corresponds to the A-linear map,

$\langle [U \mid \overset{*}{T}_\mu] \mid U \in T^\mu (\leqslant r) \rangle_A \to \langle [U \mid V] \mid (U, V) \in BT(\leqslant r, \beta) \rangle_A$ defined by

$$[U \mid \overset{*}{T}_\mu] \mapsto \sum_{\rho \in P} \operatorname{sgn}(\rho) [U \mid T^\rho].$$

Claim. (a) For all $U \in T^{\mu}$ ($\leqslant r$), $\sum_{\rho \in P} \operatorname{sgn}(\rho)[U \mid T^{\rho}]$ is an element of $\langle [S^{\text{tr}} \mid T_{\nu\setminus\lambda}] \mid S \in ST^{\tilde{\nu}\setminus\tilde{\lambda}}(\leqslant r)\rangle_A$.

(b) If $U \in ST^{\mu}$ ($\leqslant r$) then $\sum_{\rho \in P} \operatorname{sgn}(\rho)[U \mid T^{\rho}] = [U \mid T] + \sum b_{V,W}[V \mid W]$ for certain $b_{V,W} \in A$, where the summation is over $I(U, T) = \{(V, W) \in BT\ (\leqslant r, \beta) \mid V^{\text{tr}}$ and W are standard, $[W >_c T$ or $(W = T$ and $V >_r U)]\}$.

(c) Let $0 \neq X \in \langle [S^{\text{tr}} \mid T_{\nu\setminus\lambda}] \mid S \in ST^{\tilde{\nu}\setminus\tilde{\lambda}}(\leqslant r)\rangle_A$ and write $X = a_{U',T'}[U' \mid T'] + \sum a_{V,W}[V \mid W]$ for $b_{U',T'} \in A\setminus\{0\}$ and $a_{V,W} \in A$ all (V, W), where $(U')^{\text{tr}}$ and T' are standard and the summation is over $I(U', T')$, see (b), (This is possible because of Theorem 2.2(b) and the expression is unique by Theorem 2.2(d).)

Then $T' \in L$.

(This can be proved in a similar way as the Claim 1 in the proof of Theorem 1.5, by Theorem 2.2. See concerning turning over "shuffles" the proof of Theorem 2.2(a)(1)).

By this claim $\sum_{T \in L} \operatorname{Im} b_T = W^{\nu\setminus\lambda}(M)$ and, using Theorem 2.2 as well, the quotient map $\bar{b}_T\colon \bigotimes_i D^{\mu_i}(M) \to W^{\nu\setminus\lambda}(M)/\sum_{V >_c T} \operatorname{Im} b_V$ factorises through a natural map $c_T\colon W^{\mu}(M) \to W^{\nu\setminus\lambda}(M)/\sum_{V >_c T} \operatorname{Im} b_V$ which is an isomorphism onto its image.

Now b_T generalises to a natural transformation $B_T\colon \bigotimes_i D^{\mu_i} \to W^{\nu\setminus\lambda}$, hence c_T generalises to a natural transformation $C_T\colon W^{\mu} \to W^{\nu\setminus\lambda}/\sum_{V >_c T} \operatorname{Im} B_V$ which is an equivalence onto its image. The desired filtration follows. ∎

Remark. Remarks similar to those in Remark 2 below Theorem 1.5 can be made concerning Weyl functors and dual Specht modules.

COROLLARY 2.6 (Generalisation of [4]). *Let* $m \in \mathbb{N}$, $\mu \vdash\!\!- m$, $\lambda \vdash\!\!- n$. *Then* $W^{\lambda} \otimes W^{\mu}$: A-mod $\to$ A-mod *admits an explicit filtration by subfunctors*

$$W^{\lambda} \otimes W^{\mu} = L^{T^1} \supset L^{T^2} \supset \cdots \supset L^{T^{l+1}} = 0$$

with $l \in \mathbb{N}$, $T^1 <_c T^2 <_c \cdots <_c T^l$ *are all elements of* $\{T \in ST^{\nu}(\tilde{\omega}\setminus\tilde{\rho}) \mid \nu \vdash\!\!- (n+m),\ \varphi^{\nu}_{\tilde{\omega}\setminus\tilde{\varphi}}(T) \in ST^{\omega\setminus\tau}(\tilde{\nu})\}$ *and for all* $i \in \underline{l}$ *there is an equivalence*

$$W^{\nu^i} \simeq L^{T^i}/L^{T^{i+1}} \qquad \text{where } \nu^i \text{ is the shape of } T^i.$$

(The proof is similar to that of Corollary 1.5). ∎

Remark. Remarks similar to those in Remark 2 below Corollary 1.5 can be made.

The contravariant dual of the symmetric power is the divided power. In Proposition 1.3 there is given a filtration for a smmetric power applied on a tensor product with tensor products of Schur functors as subquotients.

Hence, there must be a corresponding filtration for the the divided power applied to a tensor product with tensor products of Weyl functors (the contravariant duals of Schur functors) as subquotients. In order to construct such a filtration one could hope for a divided letter place algebra which would take care of the combinatorics. In fact we will sketch the construction of such a divided letter place algebra below. However, the Weyl modules cannot be expected to be submodules in the way Schur modules are embedded in the (ordinary) letter place algebra. This follows from the definition of Weyl functors: they are *quotients* of products of divided powers. Hence, Weyl modules ought to be subquotients of the divided power algebra.

As algebra the divided letter place algebra for $(\underline{m}, \underline{n})$ $(m, n \in \mathbb{N}_0)$ is the divided power algebra of the free A-module with basis $((i \| j) \mid i \in \underline{m}, j \in \underline{n})$. The i on the left in $(i \| j)$ can be seen as a letter and the j on the right as a place. Clearly, the divided power algebra for $(\underline{m}, \underline{n})$ is isomorphic to $D(A^m \otimes A^n)$ when we let $(i \| j)$ correspond to $e_i \otimes f_j$ as usual. Via this isomorphism the divided letter place algebra becomes an $A[\mathrm{End}_A(A^m) \times \mathrm{End}_A(A^n)]$-algebra.

In order to produce an A-basis suitable for filtrations we define a kind of bipermanent:

Let $n, m, k \in \mathbb{N}_0$, $\alpha \models k$ and let $(U, V) \in BT^\alpha$ $(\leqslant m, \leqslant n)$. When $\alpha = (k)$ put $(U \| V) = \sum_{\sigma \in R} \{U_{\sigma(1)} U_{\sigma(2)} \cdots U_{\sigma(k)} \| V_1 V_2 \cdots V_k\}$, where R is a transversal for $H_2 \backslash S_k / H_1$ with H_1 the row stabiliser of U and H_2 that for V (see the definition of one-rowed bipermanents). And, for $(W, X) \in BT^{(k)}(\leqslant m, \leqslant n)$,

$$\{W \| X\} = \prod_{i=1}^{m} \prod_{j=1}^{n} (i \| j)^{(m_{i,j})} \qquad \text{with } m_{i,j} = \#\{l \in \underline{k} \mid (W_l, X_l) = (i, j)\} \text{ all } i, j.$$

In general, put $(U \| V) = \prod_{i=1}^{b} (U_{i,1} U_{i,2} \cdots U_{i,\alpha_i} \| V_{i,1} V_{i,2} \cdots V_{i,\alpha_i})$, where b is the number of coordinates of α.

The straightening is easily seen to be seen as in Theorem 2.2(a)(2) for letters *and* places.

Hence, when $\lambda \vdash n$, $\alpha \models n$, $\beta \models n$ and $(U, V) \in BT^\lambda(\alpha, \beta)$ then $(U \| V) \in \langle (S \| T) \mid (S, T) \in BT^\mu(\alpha, \beta)$ for some $\mu \vdash n$ with $\mu \geqslant \lambda$, $S \leqslant_r U$, $T \leqslant_r V$, S^{tr} and T^{tr} are standard$\rangle_A$.

The independence of $((S^{\mathrm{tr}} \| T^{\mathrm{tr}}) \mid (S, T) \in \bigcup_{\mu \vdash n} SBT^\mu$ $(\leqslant m, \leqslant n))$ follows from Theorem 1.1(d), by a dimension argument. And hence, as usual, a filtration $D^n(- \otimes -) = L^{\mu_1} \supset L^{\mu_2} \supset \cdots \supset L^{\mu^{l+1}} = 0$ with $l \in \mathbb{N}$, $\mu_1 < \mu_2 < \cdots < \mu_l$ are all Young diagrams for n and for $i \in \underline{l}$ an equivalence $W^{\mu^i} \otimes W^{\mu^i} \xrightarrow{\sim} L^{\mu^i}/L^{\mu^{i+1}}$. As a summand, for example, a filtration for $\bigotimes_{i=1}^{\tilde{\lambda}_1} D^{\lambda_i}$. With respect to the bases of standard tableaux observe that the exterior letter place algebra is a mixture of the ordinary and the divided letter place algebra.

Finally, one can define $(\boxed{U}\, UV\rangle$ and $\langle U \| \boxed{V})$ in a similar way as alternated bipermanents. The straightening on both sides, is as for alternated bipermanents, and the independence for standard tableaux is as for alternated bipermanents.

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