A class of graphs arising from the action of $\text{PSL}(2, q^2)$ on cosets of $\text{PGL}(2, q)$

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Abstract

Using elementary linear algebra techniques a detailed description of the suborbits of the action of $\text{PSL}(2, q^2)$ on the cosets of $\text{PGL}(2, q)$ which allows a construction of the corresponding orbital graphs is given.

1. Introduction

There has recently been a considerable amount of activity in the area of vertex-transitive graphs, i.e. graphs whose automorphism group is transitive on the vertex set. Among them a subclass of Cayley graphs is defined in the following way. For a group $G$ and a subset $M \subseteq G\setminus\{1\}$ closed under inversion, the Cayley graph $\mathcal{C}(G, M)$ of $G$ relative to $M$ has vertex set $G$ and two vertices $x, y \in G$ are adjacent if and only if $xy^{-1} \in M$. Cayley graphs are exactly those graphs which admit a regular group of automorphisms. However, not all vertex-transitive graphs are Cayley graphs. In fact, a useful method for generating non-Cayley vertex-transitive graphs is to take some family of transitive permutation groups without regular subgroups and to construct the corresponding orbital graphs. Namely, a transitive permutation group $G$ on a set $V$ gives rise to an induced action of $G$ on the set $V \times V$. Let $v \in V$ and $U$ be a suborbit of $G$ relative to $v$, i.e. an orbit of the stabilizer of $v$ in $G$. If $A$ is the corresponding orbit of the induced action of $G$ on $V \times V$, then the orbital graph $\mathcal{X}(G, U)$ of $G$ with respect to $U$ is the digraph with vertex set $V$ and arc set $A$. In particular, $A$ contains each of the arcs $(v, u), (u \in U)$. The suborbit $U$ is said to be self-paired if $A$ is a symmetric relation on $V$. In this case the orbital graph $\mathcal{X}(G, U)$ is an undirected graph. Moreover, if the

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group $G$ is primitive, then all non-trivial suborbits give rise to connected graphs [1, page 81]. In view of these remarks, we may look for possible examples of non-Cayley graphs among orbital graphs arising from the self-paired suborbits of primitive permutation groups without regular subgroups. Such is the case with orbital graphs of order $q(q^2 + 1)/2$ associated with $\text{PSL}(2, q^2)$, acting by right multiplication on the right cosets of a subgroup isomorphic to $\text{PGL}(2, q)$, i.e. this group is maximal in $\text{PSL}(2, q^2)$ (for example see [4, page 239]). Further, the automorphism group of a corresponding orbital graph must be contained in $\text{Aut} (\text{PSL} (2, q^2))$ and hence does not have a regular subgroup of degree $q(q^2 + 1)/2$. The study of these graphs is relevant for the classification of vertex-transitive graphs whose order is a product of two primes [6]. In fact, as it came to our notice while preparing the final version of this paper, these graphs were independently found by Praeger and Xu [7] who base their analysis on a rather elegant use of certain 4-dimensional vector spaces. Also, subdegrees of all primitive permutation representations of $\text{PSL}(2, p)$ were calculated in [8]. This thesis is quite nonavailable, but some extractions appeared in [2].

The approach we propose here gives an enumeration of the suborbits of the action of $\text{PSL}(2, q^2)$ on cosets of $\text{PGL}(2, q)$. A detailed description of these suborbits is thus obtained, allowing one to construct the corresponding orbital graphs and study their structural properties. In particular, it might be possible to decide if they are hamiltonian or not, giving a partial contribution to the long standing problem on the existence of hamilton paths in connected vertex-transitive graphs.

The aim of this paper is to prove the following result.

**Theorem 1.1.** Let $G = \text{PSL}(2, q^2)$ act on the cosets of an isomorphic copy of $\text{PGL}(2, q)$ inside $G$. If $q \equiv 1 \pmod{4}$, then $G$ has

(i) 1 suborbit of length $q(q - 1)/2$,
(ii) 1 suborbit of length $(q^2 - 1)$,
(iii) $(q - 5)/4$ suborbits of length $q(q - 1)$,
(iv) $(q - 1)/4$ suborbits of length $q(q + 1)$.

If $q \equiv 3 \pmod{4}$, then $G$ has

(v) 1 suborbit of length $q(q + 1)/2$,
(vi) 1 suborbit of length $(q^2 - 1)$,
(vii) $(q - 3)/4$ suborbits of length $q(q - 1)$,
(viii) $(q - 3)/4$ suborbits of length $q(q + 1)$.

Furthermore, all suborbits are self-paired. A more detailed description of these suborbits is given in Table 2.

Also, orbital graphs arising in the smallest admissible case $q = 3$ are constructed after the proof of Theorem 1.1 at the end of Section 4.

Throughout this paper we adopt the following notation. Let $q$ be a power of an odd prime and let $F = \text{GF}(q)$. Fixing a nonsquare $\beta \in F$, let $x$ be a solution of the equation $x^2 = \beta$. Then $V = F(x)$ is isomorphic to $\text{GF}(q^2)$. It will be convenient to think of $V$ as of a vector space over $F$. For $v, w \in V$, denote by $L(v)$ and $L(v, w)$ the subspaces of
Table 2

\[ \begin{array}{cccccc}
\text{Suborbit} & F' & \text{Length} & \frac{q(q-1)}{2} & \frac{q(q+1)}{2} & \frac{q(q-1)}{2} & \frac{q(q+1)}{2} \\
\text{Number if } q=1 & 1 & 1 & 0 & \frac{(q-5)}{4} & \frac{(q-1)}{4} \\
\text{Number if } q=3 & 1 & 1 & 0 & \frac{(q-3)}{4} & \frac{(q-3)}{4} \\
\text{Range of } \omega & - & \{0\} & \left\{ \frac{\beta^{-1}}{4} \right\} & \left\{ \frac{\beta^{-1}}{4} \right\} & \Omega - \left\{ \frac{\beta^{-1}}{4} \right\} & \Omega^+ \left\{ \frac{\beta^{-1}}{4} \right\}
\end{array} \]

For \( V \) generated by \( \{v\} \) and \( \{v, w\} \), respectively. Let \( R \) be a subset of \( V \) with the property that for all \( v \in V^* \) there is a unique \( r \in R \) such that \( v \in R(r) \). Without loss of generality we may put \( R = \{1, \alpha, r_1, r_1^{-1}, \ldots, r_m, r_m^{-1}\} \), where \( m = \frac{(q-1)}{2} \).

With \( S \) denoting the set of all squares of \( F \), let \( S^* = S \cap F^* = S \setminus \{0\} \), \( N = F \setminus S^* \), and \( N = F \setminus S^* \). Moreover let \( G = \text{PSL}(2, V) \) and \( K = \text{PSL}(2, F) \). With

\[
\hat{\alpha} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix},
\]

let \( H \) denote the subgroup of \( G \) generated by \( K \) and \( \hat{\alpha} \). It can be shown that \( H \) is isomorphic to \( \text{PGL}(2, F) \). Define \( \mathcal{H} \) and \( \mathcal{K} \) to be the sets of right cosets of \( K \) in \( G \) and of \( H \) in \( G \), respectively. A \( K \)-suborbit is a suborbit of the action of \( G \) on \( \mathcal{H} \) relative to the stabilizer \( K \). Similarly, an \( H \)-suborbit is a suborbit of the action of \( G \) on \( \mathcal{H} \) relative to the stabilizer \( H \).

Finally, we shall be sloppy and shall refer to the elements of \( G \) as matrices. The symbol \( * \) in place of an entry of a matrix of \( G \) will stand for a suitable element of \( V \).

2. A representation for cosets of \( K \)

We first obtain a representation for the right cosets of \( K \) in \( G \).

Let

\[
g = \begin{bmatrix} t & v \\ u & w \end{bmatrix}
\]

be an element of \( G \). If \( g' \in Kg \) then there are \( a, b, c, d \in F \) such that

\[
g' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} t & v \\ u & w \end{bmatrix} = \begin{bmatrix} at + bu & av + bw \\ ct + du & cv + dw \end{bmatrix}.
\]

Since \( \mathcal{L}(t, u) = \mathcal{L}(at + bu, ct + du) \), the vector subspace of \( V \) generated by the entry in the first column of \( g \) depends solely on the coset \( Kg \). The same is true for the second column. Therefore we can say that the coset \( Kg \) is of type \((i|j)\) if \( i \) is the dimension of \( \mathcal{L}(t, u) \) and \( j \) is the dimension of \( \mathcal{L}(v, w) \). Note that if \( \mathcal{F} \) is a \( K \)-suborbit then a right
multiplication by
\[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\]
establishes a bijection from the set of cosets of type \((2|1)\) in \(S\) to the set of cosets of type \((1|2)\) in \(S\).

**Lemma 2.1.** Let
\[
Kg=\begin{bmatrix}
t & v \\
u & w
\end{bmatrix}
\]
be a coset of \(K\) in \(G\). Then:

(i) if \(Kg\) is of type \((1|1)\) then \(g\) may be chosen to be of the form
\[
\begin{bmatrix}
r & 0 \\
0 & r^{-1}
\end{bmatrix}, \quad r \in R;
\]

(ii) if \(Kg\) is of type \((1|2)\), then \(g\) may be chosen to be of the form
\[
\begin{bmatrix}
x+y\alpha & -1 \\
\alpha & -2\alpha
\end{bmatrix}, \quad x, y \in F, \quad z \in F^*.
\]

(iii) if \(Kg\) is of type \((2|1)\) or \((2|2)\), then \(g\) may be chosen to be of the form
\[
\begin{bmatrix}
1 & x+y\alpha \\
z\alpha & \alpha
\end{bmatrix}, \quad x, y \in F, \quad z \in F^*.
\]

Moreover, in each of cases (i)–(iii) the representative of the given form is unique. The number of cosets of types \((1|1)\), \((1|2)\), \((2|1)\) and \((2|2)\) is \(q+1\), \(q^2-1\), \(q^2-1\) and \(q^3-2q^2+1\) respectively.

**Proof.** First we show that each column \([t \ u]^T\), such that \(S(t, u)\) has dimension 1, can be transformed into a column of the form \([v \ 0]^T\). If \(t=0\) then the matrix
\[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\]
takes \([0 \ u]^T\) into \([u \ 0]^T\). Assume thus that \(t \neq 0\). The column \([t \ zt]^T\) is taken into \([t \ 0]^T\) by the matrix
\[
\begin{bmatrix}
1 & 0 \\
-z & 1
\end{bmatrix}
\]
Also, \([t \ 0]^T\) can be transformed into any column of the form \([zt \ 0]^T\). Therefore, any coset of type \((1|1)\) has a representative with the first column \([r \ 0]^T, \ r \in R\), and so
of the form
\[
\begin{bmatrix}
  r & v \\
  0 & r^{-1}
\end{bmatrix}.
\]
For a suitable \(z \in F\), the matrix
\[
\begin{bmatrix}
  1 & z \\
  0 & 1
\end{bmatrix}
\]
takes the above element into
\[
\begin{bmatrix}
  r & 0 \\
  0 & r^{-1}
\end{bmatrix}.
\]
This proves (i).

By the remark preceding Lemma 2.1, part (ii) will follow immediately when part (iii) is proved.

Let us now suppose that \(Kg\) is of type \((2|j)\) and let \([t \ u]^T\) be the first column of an element of \(Kg\). We must prove that for some \(z \in F^*\) an element of \(K\) transforms the above column into \([1 \ z z]^T\). In fact, such is the matrix
\[
\begin{bmatrix}
  a & b \\
  cz & dz
\end{bmatrix}
\]
where \(z\) is the inverse of \(ad - bc\) and
\[
\begin{bmatrix}
  a & c \\
  b & d
\end{bmatrix}
\]
is the matrix of the change of bases \(t, u \to 1, z\). This proves (iii). The uniqueness of representatives of the given form is clear for cosets in (i) and (iii) and so for all cosets. The number of cosets of each type is computed using this uniqueness. We first count the cosets of type \((2|1)\). Letting \(v = x + yz\), the representative for such a coset given in (iii) satisfies either \(v = 0\) or \(1 + zxy \in \mathcal{L}(v)\). The latter implies \(v = (e - zz)^{-1}\) for some \(e \in F\). Hence we have \(q - 1\) choices for \(z\) and \(q\) choices for \(e\). Adding the \(q - 1\) cosets with \(v = 0\), we obtain \((q - 1)(q + 1)\) different representatives. Since the index of \(K\) in \(G\) is \(q^2(q + 1)\) and \(|R| = q + 1\) and there are as many cosets of type \((2|1)\) as of type \((1|2)\), the assertion follows.

Let \(Kg\) be a coset of type \((2|j)\), \((j = 1, 2)\). Then the unique matrix of the form given in part (iii) of Lemma 2.1 is said to be the canonical representative of \(Kg\). If
\[
g = \begin{bmatrix}
  1 & x + yz \\
  zxy & *
\end{bmatrix}
\]
is such a representative, then \(\chi(g) = \chi(Kg) = yz\) is called the character of \(g\) and of \(Kg\), and \(z(g) = z(Kg) = z\) is called the index of \(g\) and of \(Kg\).
Lemma 2.2. Let \( g, g' \) be the canonical representatives of cosets of \( K \). Then \( Kg \) and \( Kg' \) belong to the same \( K \)-suborbit if and only if one of the following holds:

(i) \( \chi(g) = \chi(g') \neq (0, -\beta^{-1}) \);
(ii) \( \chi(g) = \chi(g') \in \{0, -\beta^{-1}\} \) and \( z(g)z(g') \in S^* \).

Proof. Since

\[
\begin{bmatrix}
1 & x + y\alpha \\
z\alpha & *
\end{bmatrix} \begin{bmatrix}
1 & -x \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & y\alpha \\
z\alpha & *
\end{bmatrix},
\]

it follows that a coset of the form

\[
\begin{bmatrix}
1 & x + y\alpha \\
z\alpha & *
\end{bmatrix}
\]

lies in the \( K \)-suborbit containing

\[
\begin{bmatrix}
1 & y\alpha \\
z\alpha & *
\end{bmatrix}.
\]

So without loss of generality we let

\[
g = \begin{bmatrix}
1 & y\alpha \\
z\alpha & *
\end{bmatrix} \quad \text{and} \quad g' = \begin{bmatrix}
1 & y'\alpha \\
z\alpha & *
\end{bmatrix}.
\]

Also let \( \chi(g) = \chi \) and \( \chi(g') = \chi' \).

For \( Kg \) and \( Kg' \) to be in the same suborbit, we must have

\[
\begin{bmatrix}
a' & b' \\
c' & d'
\end{bmatrix}g' = g \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]

for suitable \( a, b, c, d, a', b', c', d' \in F \). This leads to the following two linear systems

\[
\begin{bmatrix}
a' & b' \\
c' & d'
\end{bmatrix} \begin{bmatrix}
1 & y'\alpha \\
z'\alpha & 1 + \chi'\beta
\end{bmatrix} = \begin{bmatrix}
a + cy\alpha & b + dy\alpha \\
a + cy\alpha & b + dy\alpha
\end{bmatrix},
\]

\[
\begin{bmatrix}
a' & b' \\
c' & d'
\end{bmatrix} \begin{bmatrix}
1 & y'\alpha \\
z'\alpha & 1 + \chi'\beta
\end{bmatrix} = \begin{bmatrix}
aza + c(1 + \chi\beta) & bza + d(1 + \chi\beta) \\
aza + c(1 + \chi\beta) & bza + d(1 + \chi\beta)
\end{bmatrix}.
\]

which we solve for \( a', b', c', d' \) by Cramer's rule as follows:

\[
a' = \frac{\begin{vmatrix}
a + cy\alpha & b + dy\alpha \\
z'\alpha & 1 + \chi'\beta
\end{vmatrix}}{\begin{vmatrix}
a + cy\alpha & b + dy\alpha \\
1 + \chi'\beta & 1 + \chi'\beta
\end{vmatrix}},
\]

\[
b' = \frac{\begin{vmatrix}
1 & y'\alpha \\
a + cy\alpha & b + dy\alpha
\end{vmatrix}}{\begin{vmatrix}
z'\alpha & 1 + \chi'\beta \\
z'\alpha & 1 + \chi'\beta
\end{vmatrix}},
\]

\[
c' = \frac{\begin{vmatrix}
aza + c(1 + \chi\beta) & bza + d(1 + \chi\beta) \\
z'\alpha & 1 + \chi'\beta
\end{vmatrix}}{\begin{vmatrix}
aza + c(1 + \chi\beta) & bza + d(1 + \chi\beta) \\
z'\alpha & 1 + \chi'\beta
\end{vmatrix}},
\]

\[
d' = \frac{\begin{vmatrix}
1 & y'\alpha \\
aza + c(1 + \chi\beta) & bza + d(1 + \chi\beta)
\end{vmatrix}}{\begin{vmatrix}
aza + c(1 + \chi\beta) & bza + d(1 + \chi\beta) \\
z'\alpha & 1 + \chi'\beta
\end{vmatrix}}.
\]
In order to \(a', b', c', d'\) to belong to \(F\), their components with respect to \(x\) in the basis \(1, x\) must be zero. We obtain thus the system

\[
\begin{align*}
-bz' + cy(1 + \chi' \beta) &= 0, \\
-ay' + dy &= 0, \\
az(1 + \chi' \beta) - dz'(1 + \chi \beta) &= 0, \\
bz - cy'(1 + \chi \beta) &= 0,
\end{align*}
\]

which splits into

\[
\begin{bmatrix}
-y' \\
z(1 + \chi' \beta) \\
-z'(1 + \chi \beta)
\end{bmatrix}
\begin{bmatrix}
a \\
d
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\] (1)

and

\[
\begin{bmatrix}
-z' \\
y(1 + \chi' \beta) \\
z \\
y'(1 + \chi \beta)
\end{bmatrix}
\begin{bmatrix}
b \\
c \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\] (2)

The determinants of the coefficient matrices of (1) and (2) are equal to \(\chi' - \chi\). Hence \(\chi' = \chi\) if \(Kg\) and \(Kg'\) are to be in the same suborbit. Therefore the respective solutions for (1) and (2) are given by ordered pairs \((\lambda z', \lambda z)\) and \((\mu y(1 + \chi \beta), \mu z')\), where \(\lambda, \mu \in F^*\). Thus the matrix

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]

must be of the form

\[
\begin{bmatrix}
\lambda z & \mu y(1 + \chi \beta) \\
\mu z' & \lambda z'
\end{bmatrix}
\]

Its determinant is \(\Delta = \lambda^2 zz' - \mu^2 yz'(1 + \chi \beta)\). If \(\chi\) and \(1 + \chi \beta\) are both nonzero, we can clearly find \(\lambda, \mu\) such that \(\Delta = 1\). This proves (i). Otherwise \(\Delta = \lambda^2 zz'\) which is possible if and only if \(zz'\) is a square. This proves (ii). \(\square\)

3. Stabilizers and \(K\)-suborbits

For the rest of the paper we let \(\Omega\) be the set of all elements \(x \in F\) such that \(x = y^2 \beta + y\) for some \(y \in F\). Furthermore, define \(\Omega^+ = \Omega \cap S^*\) and \(\Omega^- = \Omega \cap N^*\). Finally, if \(\chi\) is a character of a coset, put \(\omega = \chi^2 \beta + \chi\). In order to compute the length of the \(K\)-suborbits we need to know, for all \(Kg\), the size of the intersection \(\Sigma_g\) of \(K\) with the stabilizer of \(Kg\) in \(G\). This is the content of the next lemma.
Lemma 3.1. Let
\[ g = \begin{bmatrix} 1 & yz \\ zx & \ast \end{bmatrix}, \]
with \( z \in F^* \) and \( y \in F \), and let \( \chi = \chi(g) \). If \( \Sigma = \Sigma_g \) then
\[ |\Sigma| = \begin{cases} q & \text{if } \omega = 0, \\ (q+1)/2 & \text{if } \omega \in \Omega^- , \\ (q-1)/2 & \text{if } \omega \in \Omega^+. \end{cases} \] (3)

Proof. Let
\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Sigma. \]
To obtain the form of this matrix let \( y' = y \) and \( z' = z \) in systems (1) and (2). It follows that \( a = d \) and \( b = c(y^2 \beta + yz^{-1}) \). The condition \( ad - bc = 1 \) is then equivalent to
\[ a^2 - c^2(y^2 \beta + yz^{-1}) - 1 = 0. \] (4)
Letting \( \delta(c) = 1 + c^2(y^2 \beta + yz^{-1}) \) and \( \Delta = \{ c \in F \mid \delta(c) \in S \} \), the solutions of (4) are given by
\[ a_{1,2} = \pm \sqrt{\delta(c)} \] (5)
and lie in \( F \) if and only if \( c \in \Delta \). Since \( z^2(y^2 \beta + yz^{-1}) = \omega \), the quantities
\[ y^2 \beta + yz^{-1} \] and \( \omega \) are simultaneously in \( S^*, N^* \) or \( \{0\} \). (6)
Note that if \( q \equiv 1 \pmod{4} \) then \( |S \cap S+1| = |S \cap N+1| = (q+1)/2 \). Moreover if \( q \equiv 3 \pmod{4} \) then \( |S \cap S+1| = (q-1)/2 \) and \( |S \cap N+1| = (q+3)/2 \). This implies the following formula for the size of \( \Delta \).
\[ |\Delta| = \begin{cases} (q+1)/2 & \text{if } q \equiv 1 \pmod{4} \text{ and } \omega \neq 0, \\ (q+3)/2 & \text{if } q \equiv 3 \pmod{4} \text{ and } \omega \in \Omega^-, \\ (q-1)/2 & \text{if } q \equiv 3 \pmod{4} \text{ and } \omega \in \Omega^+. \end{cases} \] (7)

Let us count the number of distinct elements of \( \Sigma \) corresponding to elements \( c \) in \( \Delta \). For each \( i = 1, 2 \), with \( a_i \) as in (5), let
\[ M_i(c) = \begin{bmatrix} a_i & c(y^2 \beta + yz^{-1}) \\ c & a_i \end{bmatrix} \]
It is easily checked that
\[ M_1(-c) = M_2(c) \] (8)
and
\[ M_1(c) = M_2(c) \iff c = 0 \text{ or } \delta(c) = 0. \] (9)
Table 1

<table>
<thead>
<tr>
<th>Suborbit length</th>
<th>Number of suborbits</th>
<th>range of ( \omega )</th>
<th>Number of cosets of type ... in each</th>
<th>Range of ( z ) for cosets of type (2( l )j)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0)</td>
<td>( \Omega^* )</td>
<td>( \Omega^* )</td>
<td>( S^* )</td>
</tr>
<tr>
<td>(q( -1 ))/2</td>
<td>2</td>
<td>( \Omega^* )</td>
<td>( F^* )</td>
<td>( F^* )</td>
</tr>
<tr>
<td>(q( -1 ))/2</td>
<td>2</td>
<td>( \Omega^* )</td>
<td>( F^* )</td>
<td>( F^* )</td>
</tr>
</tbody>
</table>

Hence a pair \( \{c, -c\} \) gives rise to two different elements of \( \Sigma \), unless \( c = 0 \) or \( \delta(c) = 0 \) when only one such element is obtained. If \( \omega = 0 \) then \( \delta(c) = 1 \) for all \( c \in F \) and so \( \delta = F \).

It follows from (8) and (9) that the cardinality of \( \Sigma \) is \( q \), which proves the first line in (3).

(Note that the mapping \( c \mapsto M_1(c) \) is an isomorphism from \( (F, +) \) to \( \Sigma \).

Proposition 3.2. The structure of \( K \)-suborbits is described in Table 1.

Proof. The first column is self-evident, because the cosets represented by the identity and by \( \bar{z} \) form the unique two suborbits of length 1.

The second and third column correspond to the case where \( |\Sigma| = q \), i.e. \( \omega = 0 \) in view of Lemma 3.1. Let \( \mathcal{S} \) be one of the corresponding suborbits of length \( (q^2 - 1)/2 \). By Lemma 2.2, the cosets of type \( (2lj), (j = 1, 2) \) in \( \mathcal{S} \) have the same character \( \chi \in \{0, -\beta^{-1}\} \). Moreover, the corresponding values of the \( z \) are either all in \( S^* \) or all in \( N^* \). The total number of cosets of type \( (2lj), (j = 1, 2) \) in \( \mathcal{S} \) is then given by all possible choices for \( x, y \) and \( z \) satisfying the above conditions. These choices are \( q \) for \( x \) and \( (q - 1)/2 \) for \( z \), while \( y \) is uniquely determined by the equation \( yz = \chi \). Therefore the total number of these cosets is \( q(q - 1)/2 \). Among them, those of type \( (2l1) \) correspond to \( x = 0 \). So their number is \( (q - 1)/2 \) and thus the number of cosets of type \( (2l2) \) is \( (q - 1)^2/2 \). Also, the number of cosets of type \( (1l2) \) is \( (q - 1)/2 \). Furthermore, no coset of type \( (1l1) \) can lie in a suborbit of length \( (q^2 - 1)/2 \).

The fourth column corresponds to the case where \( |\Sigma| = (q + 1)/2 \), i.e. \( \omega \in \Omega^* \) in view of Lemma 3.1. Let \( \mathcal{S} \) be one of the corresponding suborbits of length \( q(q - 1) \). We first show that all the cosets in \( \mathcal{S} \) are of type \( (2l2) \). It suffices to see that \( \mathcal{S} \) contains no coset of type \( (2l1) \). In fact, the coset

\[
\begin{bmatrix}
1 & u \\
\alpha & 1 + zav
\end{bmatrix}
\]
is of type \((2|1)\) if and only if \(z \neq 0\) and \(\omega = x^2z^2\), which is a square. In view of Lemma 2.2 we may deduce that the suborbits are in 1–1 correspondence with characters \(\chi\) satisfying \(\omega = \chi^2 \beta + \chi \in \Omega^*\). By (6) their number is precisely \(|y^2 \beta + F^* \cap N^*| = |N^* \setminus \{y^2 \beta\}|\) which equals \((q - 3)/2\) because \(y^2 \beta \notin N^*\).

The last column is obtained from the rest of Table 1 in view of Lemmas 2.1 and 2.2. □

4. The \(H\)-suborbits

The next lemma is crucial for the proof of Theorem 1.1.

**Lemma 4.1.** Let \(Kg \in \mathcal{K}\) have type \((2|j)\), \((j = 1, 2)\) and be contained in \(\mathcal{S}\). Let \(\chi = \chi(Kg)\) and \(z = z(Kg)\). Then

(i) \(\chi(K\tilde{g}) = -\chi - \beta^{-1}\), \(z(K\tilde{g}) = -z\), \(\chi(Kg\tilde{z}) = \chi\) and \(z(Kg\tilde{z}) = z\beta^{-1}\).

(ii) \(\mathcal{S}\) contains also \(K\tilde{g}\) if and only if \(\omega = -(4\beta)^{-1}\). Its length is \(q(q - 1)\) if \(q \equiv 1 (mod 4)\) and \(q(q + 1)\) if \(q \equiv 3 (mod 4)\).

(iii) \(\mathcal{S}\) does not contain \(K\tilde{g}\) if and only if \(\omega = 0\).

**Proof.** If

\[
g = \begin{bmatrix} 1 & v \\ z\alpha & 1 + z\alpha v \end{bmatrix},
\]

then

\[
\tilde{g} = \begin{bmatrix} \alpha & zv \\ z & z^{-1} + zv \end{bmatrix}.
\]

Multiplying on the left by

\[
\begin{bmatrix} 0 & z^{-1} \\ -z & 0 \end{bmatrix}
\]

we have that

\[
\begin{bmatrix} 1 & z^{-1} \alpha^{-1} + v \\ -z\alpha & z\alpha v \end{bmatrix}
\]

belongs to \(K\tilde{g}\). Therefore \(\chi(K\tilde{g}) = -\chi - \beta^{-1}\) and \(z(K\tilde{g}) = -z\). Similarly, it can be seen that the canonical representative of \(Kg\tilde{z}\) is

\[
\begin{bmatrix} 1 & \beta^{-1}(v + (z\beta^{-1}\alpha)) \\ -z\beta\alpha & * \end{bmatrix}
\]

Therefore \(\chi(Kg\tilde{z}) = -\chi - \beta^{-1}\) and \(z(Kg\tilde{z}) = -z\beta^{-1}\). This proves (i).

If \(K\tilde{g}\) is in \(\mathcal{S}\) then we must have \(-\chi - \beta^{-1} = \chi\). Therefore \(\chi = -(2\beta)^{-1}\) and \(\omega = -(4\beta)^{-1}\). By Lemma 2.2 \(\mathcal{S}\) is uniquely determined. Part (ii) follows from Table 1 in view of the fact that \(-(4\beta)^{-1}\) is a square if and only if \(q \equiv 3 (mod 4)\).
In view of (i), $\chi(Kg\alpha) = \chi$ and $z(Kg\alpha) = z\beta^{-1}$. Therefore the product of the indices of $Kg$ and $Kg\alpha$ is a nonsquare and (iii) follows from Lemma 2.2. □

For each $\omega \in \Omega$ let $T_\omega$ be the set of all elements of $H$ containing a canonical representative $g$ whose character $\chi$ satisfies $\chi^2 \beta + \chi = \omega$.

**Proof of Theorem 1.1.** Each $K$-suborbit $S$ gives rise to an $H$-suborbit $T$ which consists of all cosets of the forms $C \cup \alpha C$ and $(C \cup \alpha C) \alpha$, $(C \in S)$. We observe that the length of $T$ is half the length of $S$ if $\alpha C \in S$ and is twice the length of $S$ if $\alpha C \notin S$. In all other cases lengths of $S$ and $T$ coincide.

Note that $\chi(-\chi^{-1} \omega) = -\beta^{-1} \omega$ and so in view of Lemma 4.1 the value of $\omega$ is invariant on each $H$-suborbit. It follows that $T = T_\omega$ for some $\omega \in \Omega$. The first column of Table 2 needs no explanation. The second column is a direct consequence of part (iii) in Lemma 4.1. Similarly, the third and fourth columns follow from part (ii) of Lemma 4.1. Finally, the last two columns follow from statements (ii) and (iii) of Lemma 4.1.

It remains to show that all of these suborbits are self-paired. This is clearly the case for suborbits $T_0$ and $T_{-(4\beta)^{-1}}$. Let $T = T_\omega$, with $\omega \notin \{0, -(4\beta)^{-1}\}$ and let $g$ be a canonical representative contained in an element of $T$. Then it can be seen that $\chi(Kg^{-1}) = \chi(Kg)$. Hence both $Hg$ and $Hg^{-1}$ belong to $T$ which is therefore self-paired. □

With the explicit description of $H$-suborbits $T$, the construction of the corresponding orbital graphs $X(G, T)$ is relatively simple. Of course, they must be undirected because of the self-pairedness of suborbits $T$. The structure of these graphs is best understood via a factorization modulo a cyclic group of order $(q^2 + 1)/2$. To illustrate this approach consider the smallest admissible case $q = 3$. Here we have two nontrivial $H$-suborbits. Namely, $T_1$ of length 6 containing cosets with representatives

$\begin{bmatrix} 1 + \alpha & 0 \\ 0 & -1 + \alpha \end{bmatrix}$, $\begin{bmatrix} -1 + \alpha & 1 \\ 1 - \alpha & \alpha \end{bmatrix}$, $\begin{bmatrix} -1 - \alpha & 1 \\ 1 + \alpha & -\alpha \end{bmatrix}$.

and $T_0$ of length 8 containing cosets with representatives

$\begin{bmatrix} 0 & 1 \\ -1 & -\alpha \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ \alpha & 1 + \gamma \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ \alpha & 1 - \alpha \end{bmatrix}$.

$\begin{bmatrix} 0 & 1 \\ -1 & \alpha \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ -\alpha & 1 - \alpha \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ -\alpha & 1 + \alpha \end{bmatrix}$.

We obtain two complementary graphs with 15 vertices. The graph $X(G, T_1)$ is depicted in Fig. 1, where the notation of Frucht [3] is used to emphasize the three
orbits of the cyclic group of order 5. (Note that the group $\text{PSL}(2,9)$ is isomorphic to $A_6$ and the graph in Fig. 1 is in fact the complement of the line graph of $K_6$).

References