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# Ergodic Theory and Maximal Abelian Subalgebras of the Hyperfinite Factor

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Let $T$ be a free ergodic measure-preserving action of an abelian group $G$ on $(X,\mu)$. The crossed product algebra $R_T = L^\infty(X,\mu) \rtimes G$ has two distinguished masas, the image $C_T$ of $L^\infty(X,\mu)$ and the algebra $S_T$ generated by the image of $G$. We conjecture that conjugacy of the singular masas $S_{T^{(1)}}$ and $S_{T^{(2)}}$ for weakly mixing actions $T^{(1)}$ and $T^{(2)}$ of different groups implies that the groups are isomorphic and the actions are conjugate with respect to this isomorphism. Our main result supporting this conjecture is that the conclusion is true under the additional assumption that the isomorphism $\gamma: R_{T^{(1)}} \to R_{T^{(2)}}$ such that $\gamma(S_{T^{(1)}}) = S_{T^{(2)}}$ has the property that the Cartan subalgebras $\gamma(C_{T^{(1)}})$ and $C_{T^{(2)}}$ of $R_{T^{(2)}}$ are inner conjugate. We discuss a stronger conjecture about the structure of the automorphism group $\mathrm{Aut}(R_T, S_T)$, and a weaker one about entropy as a conjugacy invariant. We study also the Pukanszky and some related invariants of $S_T$, and show that they have a simple interpretation in terms of the spectral theory of the action $T$. It follows that essentially all values of the Pukanszky invariant are realized by the masas $S_T$, and there exist non-conjugate singular masas with the same Pukanszky invariant. © 2002 Elsevier Science (USA)

## 1. INTRODUCTION

It is well known that if one has a Lebesgue space $(X,\mu)$ with a free ergodic measure-preserving action $T$ of an abelian group $G$, then the crossed product algebra $R_T = L^\infty(X,\mu) \rtimes G$ is the hyperfinite factor with two distinguished maximal abelian subalgebras (masas), the image $C_T$ of $L^\infty(X,\mu)$ and the masa $S_T$ generated by the canonical unitaries in $R_T$ implementing the action. It is the purpose of the present work to investigate

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how much information about the system $(X, \mu, T)$ can be extracted from properties of the masas $C_T$ and $S_T$.

In Section 2, we formulate our main conjecture that for weakly mixing actions the masas $S_T$ determine the actions up to an isomorphism of the groups. Here, we also give a short proof of the singularity of $S_T$, a result due to Nielsen [Ni], and more generally describe the normalizer of $S_T$ for arbitrary actions, which is a result of Packer [P1].

Apparently, the only conjugacy invariant of singular masas which has been effectively used over the years is the invariant of Pukanszky [P]. It arises as a spectral invariant of two commuting representations of a masa $A \subset M$ on $B(L^2(M))$ coming from the left and right actions of $A$ on $M$. It is not surprising that for the masas $S_T$ this invariant is closely related to spectral properties of the action $T$. This fact has two consequences. On the one hand, we have a lot of actions with different Pukanszky invariants. On the other hand, for most interesting systems such as Bernoullian systems, the invariant gives us nothing. This is described in Section 3.

In Section 4, we prove the main result supporting our conjecture. Namely, for weakly mixing actions the pair consisting of the masa $S_T$ and the inner conjugacy class of $C_T$ is an invariant of the action. In fact, if $\mathrm{Aut}(R_T, S_T)$ denotes the subgroup of $\gamma \in \mathrm{Aut}(R_T)$ such that $\gamma(S_T) = S_T$, we prove a stronger result describing the subgroup of $\mathrm{Aut}(R_T, S_T)$ consisting of automorphisms $\gamma$ such that $\gamma(C_T)$ and $C_T$ are inner conjugate. We conjecture that this subgroup is actually the whole group $\mathrm{Aut}(R_T, S_T)$. One test for our conjecture is to prove that this subgroup is closed, and we are able to do this under slightly stronger assumptions than weak mixing.

The group of inner automorphisms defined by unitaries in $S_T$ is not always closed, and this gives us the possibility of constructing non-conjugate singular masas with the same Pukanszky invariant.

Finally in Section 5, which is independent of the others, we consider a weaker conjecture stating that the entropy of the action is a conjugacy invariant for $S_T$. We prove that if $S_{T^{(1)}}$ and $S_{T^{(2)}}$ are conjugate and under this conjugacy the canonical generators of these algebras coincide on a small projection, then the entropies of the actions coincide. The proof is an application of the theory of non-commutative entropy.

## 2. PRELIMINARIES ON CROSSED PRODUCTS

Let $G$ be a countable abelian group, $g \mapsto T_g \in \mathrm{Aut}(X, \mu)$ a free ergodic measure-preserving action of $G$ on a Lebesgue space $(X, \mu)$. Consider the corresponding action $g \mapsto \alpha_g$ on $L^\infty(X, \mu)$, $\alpha_g(f) = f \circ T_{-g}$, and the crossed product algebra $L^\infty(X, \mu) \rtimes_\alpha G$, which will be denoted by $R_T$ throughout the paper. Let $g \mapsto v_g$ be the canonical homomorphism of $G$ into the unitary group of $R_T$. We denote by $S_T$ the abelian subalgebra of $R_T$ generated by $v_g,\ g \in G$. The algebra $L^\infty(X,\mu)$ considered as a subalgebra of $R_T$ will be denoted by $C_T$.

To fix notations, the unitary on $L^2(Y,\nu)$ associated with an invertible non-singular transformation $S$ of a measure space $(Y,\nu)$ will be denoted by $u_S,\ u_S f = (dS_*\nu/d\nu)^{1/2} f \circ S^{-1}$, and the corresponding automorphism of $L^\infty(Y,\nu)$ will be denoted by $\alpha_S,\ \alpha_S(f) = f \circ S^{-1}$. For a given action $T$, we shall usually suppress $T$ in such notations, so we write $u_g$ and $\alpha_g$ instead of $u_{T_g}$ and $\alpha_{T_g}$.

We shall usually consider $R_T$ in its standard representation on $L^2(X,\mu) \otimes L^2(\hat{G},\lambda)$, where $\hat{G}$ is the dual group and $\lambda$ is its Haar measure. The elements of the group $G$ considered as functions on $\hat{G}$ define two types of operators on $L^2(\hat{G})$, the operator $m_g$ of multiplication by $g$, $(m_g f)(\chi) = \langle \chi, g \rangle f(\chi)$, and the projection $e_g$ onto the one-dimensional space $\mathbb{C}g$. Then the representation $\pi$ of $R_T$ on $L^2(X) \otimes L^2(\hat{G})$ is given by

$$\pi(v_g) = 1 \otimes m_g, \qquad \pi(f) = \sum_g \alpha_g(f) \otimes e_{-g} \quad \text{for } f \in L^\infty(X).$$

Then $R_T$ is in its standard form with the tracial vector $\xi \equiv 1$. The modular involution $J$ is given by

$$J = \tilde{J} \sum_g u_g \otimes e_g = \left( \sum_g u_g \otimes e_{-g} \right) \tilde{J}, \tag{2.1}$$

where $\tilde{J}$ is the usual complex conjugation on $L^2(X \times \hat{G})$. Indeed, since $(1 \otimes e_g)\xi = 0$ for $g \neq 0$,

$$\begin{aligned} J\pi(v_g f)\xi &= J\pi(v_g)(f \otimes e_0)\xi = J(f \otimes m_g e_0)\xi = J(f \otimes e_g m_g)\xi = \tilde{J}(u_g f \otimes e_g m_g)\xi \\ &= \tilde{J}(\alpha_g(f) \otimes e_g m_g)\xi = (\alpha_g(\bar{f}) \otimes e_{-g} m_{-g})\xi = \pi(\bar{f} v_g^*)\xi. \end{aligned}$$

In particular,

$$J\pi(f)J = \bar{f} \otimes 1, \qquad J\pi(v_g)J = u_g \otimes m_g^*. \tag{2.2}$$

Hence $R_T$ is the fixed point subalgebra of $L^\infty(X) \otimes B(L^2(\hat{G}))$ for the action $g \mapsto \alpha_g \otimes \operatorname{Ad} m_g^*$ of $G$ (see [S, Corollary 19.13]).

Recall [D] that a maximal abelian subalgebra $A$, or masa, of a von Neumann algebra $M$ is called regular if its normalizer $N(A)$ consisting of unitaries $u \in M$ such that $uAu^* = A$ generates $M$ as a von Neumann algebra, and singular if the normalizer consists only of unitaries in $A$. If $A$ is regular and there exists a faithful normal conditional expectation of $M$ onto $A$ then $A$ is called Cartan [FM].

Since the action $T$ is free and ergodic, the algebras $C_T$ and $S_T$ are maximal abelian in $R_T$. The algebra $C_T$ is Cartan. Nielsen [Ni] was the first who noticed that if the action is weakly mixing (i.e. the only eigenfunctions are constants), then $S_T$ is singular (see [P2, SS] for different proofs). More generally, the normalizer $N(S_T)$ always depends only on the discrete part of the spectrum [P1] (see also [H]). We shall first give a short proof of this result.

THEOREM 2.1. *Let $L_0^\infty(X)$ be the subalgebra of $L^\infty(X)$ generated by the eigenfunctions of the action $\alpha$. Then the von Neumann algebra $\mathcal{N}(S_T)$ generated by $N(S_T)$ is $L_0^\infty(X) \rtimes_\alpha G$.*

*Proof.* If $u \in C_T$ is an eigenfunction, $\alpha_g(u) = \langle \chi, g \rangle u$ for some $\chi \in \hat{G}$, then since the action is ergodic, $u$ is a unitary. It is in the normalizer of $S_T$, $uv_g u^* = \langle \chi, -g \rangle v_g$. Thus $L_0^\infty(X) \rtimes_\alpha G \subset \mathcal{N}(S_T)$.

Conversely, let $u \in N(S_T)$. Then $\mathrm{Ad}\, u$ defines an automorphism of $S_T$ which corresponds to a measurable transformation $\sigma$ of $\hat{G}$. Consider $R_T$ in the Hilbert space $L^2(X) \otimes L^2(\hat{G})$ as above. Then the operator $v = u(1 \otimes u_\sigma^*)$ commutes with $1 \otimes L^\infty(\hat{G})$, hence it belongs to

$$(L^\infty(X) \otimes B(L^2(\hat{G}))) \cap (1 \otimes L^\infty(\hat{G}))' = L^\infty(X) \otimes L^\infty(\hat{G}).$$

Thus $v$ is given by a measurable family $\{v_\ell\}_{\ell \in \hat{G}}$ of unitaries in $L^\infty(X)$. Since $u \in R_T$ and $v$ commutes with $1 \otimes m_g$, we have

$$\begin{aligned} u = (\alpha_g \otimes \mathrm{Ad}\, m_g^*)(u) &= (\alpha_g \otimes \mathrm{Ad}\, m_g^*)(v)(\alpha_g \otimes \mathrm{Ad}\, m_g^*)(1 \otimes u_\sigma) \\ &= (\alpha_g \otimes 1)(v)(1 \otimes m_g^* u_\sigma m_g). \end{aligned}$$

Hence $v = (\alpha_g \otimes 1)(v)(1 \otimes m_g^* u_\sigma m_g u_\sigma^*)$. The operator $u_\sigma m_g u_\sigma^*$ is the operator of multiplication by the function $g \circ \sigma^{-1}$. Thus for almost all $\ell \in \hat{G}$

$$v_\ell = \langle \bar{\ell} \sigma^{-1}(\ell), g \rangle \alpha_g(v_\ell).$$

We see that for almost all $\ell$ the unitary $v_\ell$ lies in $L_0^\infty(X)$, which means that $v \in L_0^\infty(X) \otimes L^\infty(\hat{G})$. Thus,

$$\begin{aligned} u = v(1 \otimes u_\sigma) &\in (L_0^\infty(X) \otimes B(L^2(\hat{G}))) \cap R_T = (L_0^\infty(X) \otimes B(L^2(\hat{G})))^{\alpha \otimes \mathrm{Ad}\, m^*} \\ &= L_0^\infty(X) \rtimes_\alpha G. \quad \blacksquare \end{aligned}$$

*Remark.* The proof works without any modifications in the case when a locally compact separable abelian group acts ergodically on a von Neumann algebra with separable predual.

All the Cartan algebras $C_T$ are conjugate by a well-known result of Dye [Dy], so the position of $C_T$ inside $R_T$ does not contain any information about the original action. On the other hand, the relative position of $C_T$ and $S_T$ defines the action. More precisely, we have

PROPOSITION 2.2. *Let* $g \mapsto T_g^{(i)} \in \mathrm{Aut}(X_i, \mu_i)$ *be a free measure-preserving action of a countable abelian group* $G_i$, $i = 1, 2$. *Suppose there exists an isomorphism* $\gamma : R_{T^{(1)}} \to R_{T^{(2)}}$ *such that* $\gamma(S_{T^{(1)}}) = S_{T^{(2)}}$ *and* $\gamma(C_{T^{(1)}}) = C_{T^{(2)}}$. *Then there exist an isomorphism* $S : (X_1, \mu_1) \to (X_2, \mu_2)$ *of measure spaces and a group isomorphism* $\beta : G_2 \to G_1$ *such that* $T_g^{(2)} = ST_{\beta(g)}^{(1)}S^{-1}$ *for* $g \in G_2$.

*Proof.* The result follows easily from the fact that the only unitaries in $S_T$ which normalize $C_T$ are the scalar multiples of $v_g$, $g \in G$. Indeed, if $v \in S_T$ normalizes $C_T$ and $v = \sum_g a_g v_g$, $a_g \in \mathbb{C}$, is its Fourier series then for arbitrary $x \in C_T$ the equality $vx = \alpha(x)v$ for $x \in C_T$, where $\alpha = \mathrm{Ad}\, v$, implies $a_g\alpha_g(x) = a_g\alpha(x)$ for all $g \in G$. Thus $\alpha_g = \alpha$ if $a_g \neq 0$. Since the action is free, this means that $a_g \neq 0$ for a unique $g$, and $v = a_g v_g$. Hence, if we have an isomorphism $\gamma$ as in the formulation of the proposition, then there exist an isomorphism $\beta$ of $G_2$ onto $G_1$ and a character $\chi \in \hat{G}_2$ such that $\gamma(v_{\beta(g)}) = \langle \chi, g \rangle v_g$ for $g \in G_2$. Then for $x \in C_{T^{(1)}}$ and $g \in G_2$ we have $\gamma(\alpha_{\beta(g)}(x)) = \gamma(v_{\beta(g)} x v_{\beta(g)}^*) = v_g \gamma(x) v_g^* = \alpha_g(\gamma(x))$. So for $S$ we can take the transformation implementing the isomorphism $\gamma$ of $C_{T^{(1)}}$ onto $C_{T^{(2)}}$. ∎

This observation leads to the following question. How much information about the system is contained in the algebra $S_T$? If the spectrum is purely discrete, then $S_T$ is a Cartan subalgebra, so in this case we get no information.

*Conjecture.* For weakly mixing systems the algebra $S_T$ determines the system completely. In other words, the assumption $\gamma(C_{T^{(1)}}) = C_{T^{(2)}}$ in Proposition 2.2 is redundant.

## 3. SPECTRAL INVARIANTS

One approach to the problem of conjugacy of masas in a $\mathrm{II}_1$-factor, initiated in the work of Pukanszky [P] is to consider together with a masa $A \subset M$ its conjugate $JAJ$, where $J$ is the modular involution associated with a tracial vector $\xi$, and then to consider the conjugacy problem for such pairs in $B(L^2(M))$. We thus identify $A$ with an algebra $L^\infty(Y, \nu)$ and consider a direct integral decomposition of the representation $a \otimes b \mapsto aJb^*J$ of the C*-tensor product algebra $A \otimes A$. Thus, we obtain a measure class $[\eta]$ on

$Y \times Y$ and a measurable field of Hilbert spaces $\{H_{x,y}\}_{(x,y)\in Y\times Y}$ such that $[\eta]$ is invariant with respect to the flip $(x,y) \mapsto (y,x)$, its left (and right) projection onto $Y$ is $[v]$, and

$$L^2(M) = \int_{Y\times Y}^{\oplus} H_{x,y}\, d\eta(x,y),$$

see [FM] for details. Let $m(x,y) = \dim H_{x,y}$ be the multiplicity function. Note that $m(x,x) = 1$ and the subspace $\int_{Y\times Y}^{\oplus} H_{x,x}\, d\eta(x,x)$ is identified with $\overline{A\xi}$. Indeed, $\zeta \in L^2(M)$ lives on the diagonal $\Delta(Y) \subset Y \times Y$ if and only if $a\zeta = Ja^*J\zeta$ for all $a \in A$. Since $A$ is maximal abelian, this is equivalent to $\zeta \in \overline{A\xi}$. In particular, the projection $e_A = [A\xi]$ corresponds to the characteristic function of $\Delta(Y)$, so it belongs to $A \vee JAJ$ (see [Po1]).

The triple $(Y, [\eta], m)$ is a conjugacy invariant for the pair $(A, J)$ in the following sense. If $A \subset M$ and $B \subset N$ are masas, then a unitary $U : L^2(M) \to L^2(N)$ such that $UAU^* = B$ and $UJ_MU^* = J_N$ exists if and only if there exists an isomorphism $F : (Y_A, [v_A]) \to (Y_B, [v_B])$ such that $(F \times F)_*([\eta_A]) = [\eta_B]$ and $m_B \circ (F \times F) = m_A$. Indeed, the fact that $U$ defines $F$ follows by definition. Conversely, for given $F$ we can suppose without loss of generality that $\eta_A$ is invariant with respect to the flip and $(F \times F)_*(\eta_A) = \eta_B$. Then there exists a measurable field of unitaries $\tilde U_{x,y} : H^A_{x,y} \to H^B_{F(x),F(y)}$, and we can define the unitary $\tilde U = \int^{\oplus}_{Y_A\times Y_A} \tilde U_{x,y}\, d\eta_A(x,y)$. It has the property $\tilde U A \tilde U^* = B$. We want to modify $\tilde U$ in a way such that the condition $UJ_MU^* = J_N$ is also satisfied. Note that $J_M$ is given by a measurable field of anti-unitaries $J^A_{x,y} : H^A_{x,y} \to H^A_{y,x}$ such that $J^A_{y,x}J^A_{x,y} = 1$, and analogoulsy $J_N$ defines a measurable field $\{J^B_{x,y}\}_{x,y}$. We can easily arrange $\tilde U_{x,x}J^A_{x,x} = J^B_{F(x),F(x)}\tilde U_{x,x}$. Outside of the diagonal, we choose a measurable subset $Z \subset Y_A \times Y_A$ which meets every two-point set $\{(x,y),(y,x)\}$ only once. Then we define

$$U_{x,y} = \begin{cases} \tilde U_{x,y} & \text{if } (x,y) \in \Delta(Y_A) \cup Z, \\ J^B_{F(y),F(x)}\tilde U_{y,x}J^A_{x,y} & \text{otherwise.} \end{cases}$$

Then $U_{y,x}J^A_{x,y} = J^B_{F(x),F(y)}U_{x,y}$, so for $U = \int^{\oplus}_{Y_A\times Y_A} U_{x,y}\, d\eta_A(x,y)$ we have $U J_M = J_N U$.

A rougher invariant is the set $P(A) \subset \mathbb{N} \cup \{\infty\}$ of essential values of the multiplicity function $m$ on $(Y \times Y)\backslash\Delta(Y)$, which was introduced by Pukanszky [P] (we rather use the definition of Popa [Po1]). In other words, $P(A)$ is the set of $n$ such that the type I algebra $(A \vee JAJ)'(1 - e_A)$ has a non-zero component of type $\mathrm{I}_n$. This invariant solves a weaker conjugacy problem: $P(A) = P(B)$ if and only if there exists a unitary $U$ such that $U(A \vee J_MAJ_M)U^* = B \vee J_NBJ_N$ and $Ue_AU^* = e_B$.

Return to our masas $S_T$ in $R_T$. As above, consider $R_T$ acting on $L^2(X \times \hat G)$ with the modular involution given by (2.1) and (2.2). For the construction of the triple $(Y_T, [\eta_T], m_T)$ for the masa $S_T$ it is natural to take $Y_T = \hat{G}$. Let $\mu_T$ and $n_T$ be the spectral measure and the multiplicity function of the representation $g \mapsto u_g$, so that

$$L^2(X) = \int_{\hat{G}}^{\oplus} H_\ell \, d\mu_T(\ell).$$

Following [H], we have a direct integral decomposition

$$L^2(X \times \hat{G}) = \int_{\hat{G}\times\hat{G}}^{\oplus} H_{\ell_2} \, d\lambda(\ell_1) \, d\mu_T(\ell_2),$$

with respect to which $v_g = 1 \otimes m_g$ corresponds to the function $(\ell_1, \ell_2) \mapsto g(\ell_1)$, while $Jv_g^* J = u_{-g} \otimes m_g$ corresponds to $(\ell_1, \ell_2) \mapsto g(\ell_1 \overline{\ell_2})$. Hence if we define $\eta_T$ as the image of the measure $\lambda \times \mu_T$ under the map $\hat{G} \times \hat{G} \mapsto \hat{G} \times \hat{G}$, $(\ell_1, \ell_2) \mapsto (\ell_1, \ell_1 \overline{\ell_2})$, then with respect to the decomposition

$$L^2(X \times \hat{G}) = \int_{\hat{G}\times\hat{G}}^{\oplus} H_{\ell_1 \overline{\ell_2}} \, d\eta_T(\ell_1, \ell_2),$$

the operator $v_g$ corresponds to the function $(\ell_1, \ell_2) \mapsto g(\ell_1)$, while $Jv_g^* J$ corresponds to $(\ell_l, \ell_2) \mapsto g(\ell_2)$. This is the decomposition we are looking for. Thus we have proved the following (see also [H]).

Proposition 3.1. *The triple* $(Y_T, [\eta_T], m_T)$ *associated with the masa* $S_T$ *in* $R_T$ *is given by* $Y_T = G$, $\int f \, d\eta_T = \int f(\ell_1, \ell_1 \overline{\ell_2}) \, d\lambda(\ell_1) \, d\mu_T(\ell_2)$, $m_T(\ell_1, \ell_2) = n_T(\ell_1 \overline{\ell_2})$, *where* $\mu_T$ *and* $n_T$ *are the spectral measure and the multiplicity function for the representation* $g \mapsto u_g$ *of* $G$.

Corollary 3.2. *The Pukanszky invariant* $P(S_T)$ *is the set of essential values of the multiplicity function* $n_T$ *on* $\hat{G} \setminus \{e\}$.

This corollary is also obvious from

$$S_T \vee JS_T J = \{u_g \mid g \in G\}'' \otimes L^\infty(\hat{G}), \qquad e_{S_T} = p_1 \otimes 1,$$

where $p_1 \in B(L^2(X))$ is the projection onto the constants.

Pukanszky introduced his invariant to construct a countable family of non-conjugate singular masas in the hyperfinite $\mathrm{II}_1$-factor. For each $n \in \mathbb{N}$ he constructed a singular masa $A$ with $P(A) = \{n\}$. Thanks to advances in the spectral theory of dynamical systems [KL] we now know much more.

Corollary 3.3. *For any subset* $E$ *of* $\mathbb{N}$ *containing* 1 *there exists a weakly mixing automorphism* $T$ *such that* $P(S_T) = E$.

If the spectrum of the representation $g \mapsto u_g$ is Lebesgue, i.e. the spectral measure $\mu_T$ is equivalent to the Haar measure $\lambda$ on $\hat{G} \setminus \{e\}$, then $[\eta_T] = [\lambda \times \lambda]$ on $(\hat{G} \times \hat{G}) \setminus \Delta(\hat{G})$. Hence, if we have two such systems then any measurable isomorphism $F : (\hat{G}_1, [\lambda_1]) \to (\hat{G}_2, [\lambda_2])$ has the property $(F \times F)_*([\eta_{T^{(1)}}]) = [\eta_{T^{(2)}}]$. Thus we have

Corollary 3.4. *Let $g \mapsto T_g^{(i)} \in \mathrm{Aut}(X_i, \mu_i)$ be a free ergodic measure-preserving action of a countable abelian group $G_i$, $i = 1, 2$. Suppose these actions have homogeneous Lebesgue spectra of the same multiplicity. Then for any $*$-isomorphism $\gamma : S_{T^{(1)}} \to S_{T^{(2)}}$ there exists a unitary $U : L^2(R_{T^{(1)}}) \to L^2(R_{T^{(2)}})$ such that $UaU^* = \gamma(a)$ for $a \in S_{T^{(1)}}$ and $UJ_{T^{(1)}}U^* = J_{T^{(2)}}$.*

It is clear, however, that in order to be extended to an isomorphism of $R_{T^{(1)}}$ on $R_{T^{(2)}}$, $\gamma$ has to be at least trace-preserving. But even this is not always enough, see Section 5. Thus for such system as Bernoulli shifts, which have countably multiple Lebesgue spectra, the invariant $(Y_T, [\eta_T], m_T)$ does not contain any useful information.

## 4. THE ISOMORPHISM PROBLEM

As a partial result towards a proof of our conjecture we have

Theorem 4.1. *Let $g \mapsto T_g^{(i)} \in \mathrm{Aut}(X_i, \mu_i)$ be a weakly mixing free measure-preserving action of a countable abelian group $G_i$, $i = 1, 2$. Suppose there exists an isomorphism $\gamma : R_{T^{(1)}} \to R_{T^{(2)}}$ such that $\gamma(S_{T^{(1)}}) = S_{T^{(2)}}$ and such that the Cartan algebras $\gamma(C_{T^{(1)}})$ and $C_{T^{(2)}}$ are inner conjugate in $R_{T^{(2)}}$. Then there exist an isomorphism $S : (X_1, \mu_1) \to (X_2, \mu_2)$ of measure spaces and a group isomorphism $\beta : G_2 \to G_1$ such that $T_g^{(2)} = ST_{\beta(g)}^{(1)}S^{-1}$ for $g \in G_2$.*

We shall also describe explicitly all possible isomorphisms $\gamma$ as in the theorem. In other words, for a weakly mixing free measure-preserving action $T$ of a countable abelian group $G$ on $(X, \mu)$ we shall compute the group $\mathrm{Aut}(R_T, S_T \mid C_T)$ consisting of all automorphisms $\gamma$ of $R_T$ with the properties $\gamma(S_T) = S_T$, and the masas $\gamma(C_T)$ and $C_T$ are inner conjugate.

Recall (see [FM]) that any automorphism $S$ of the orbit equivalence relation defined by the action of $G$ extends canonically to an automorphism $\alpha_S$ of $R_T$. Such an automorphism leaves $S_T$ invariant if and only if there exists an automorphism $\beta$ of $G$ such that $T_g S = ST_{\beta(g)}$. Denote by $I(T)$ the group of all such transformations $S$. For $S \in I(T)$, $\alpha_S$ is defined by the equalities $\alpha_S(f) = f \circ S^{-1}$ for $f \in C_T = L^\infty(X, \mu)$, $\alpha_S(v_g) = v_{\beta^{-1}(g)}$ for $g \in G$. Consider also the dual action $\sigma$ of $\hat{G}$ on $R_T$, $\sigma_\chi(f) = f$ for $f \in C_T$, $\sigma_\chi(v_g) =$

$\langle\chi, -g\rangle v_g$. The group of automorphisms of the form $\sigma_\chi \circ \alpha_S$ ($\chi \in \hat{G}$ and $S \in I(T)$) is the intersection of the groups $\mathrm{Aut}(R_T, C_T)$ and $\mathrm{Aut}(R_T, S_T)$. It turns out that up to inner automorphisms defined by unitaries in $S_T$ such automorphisms exhaust the whole group $\mathrm{Aut}(R_T, S_T \mid C_T)$.

THEOREM 4.2. *The group* $\mathrm{Aut}(R_T, S_T \mid C_T)$ *of automorphisms* $\gamma$ *of* $R_T$ *for which* $\gamma(S_T) = S_T$, *and* $\gamma(C_T)$ *and* $C_T$ *are inner conjugate*, *consists of elements of the form* $\mathrm{Ad}\, w \circ \sigma_\chi \circ \alpha_S$, *where* $w \in S_T$, $\chi \in \hat{G}$, $S \in I(T)$.

We conjecture that in fact this theorem gives the description of the group $\mathrm{Aut}(R_T, S_T)$.

It is well known that all Cartan subalgebras of the hyperfinite $\mathrm{II}_1$-factor are conjugate [CFW], so they are approximately inner conjugate in an appropriate sense. It is known also that if the $L^2$-distance between the unit balls of two Cartan subalgebras is less than one, then they are inner conjugate [Po2, Po3]. However, there exists an uncountable family of Cartan subalgebras, no two of which are inner conjugate [P1].

We shall first prove that Theorem 4.1 follows from Theorem 4.2. Consider the group $G = G_1 \times G_2$ and its action $T$ on $(X, \mu) = (X_1 \times X_2, \mu_1 \times \mu_2)$, $T_{(g_1,g_2)} = T^{(1)}_{g_1} \times T^{(2)}_{g_2}$. Then $R_T$ can be identified with $R_{T^{(1)}} \otimes R_{T^{(2)}}$ in such a way that $C_T = C_{T^{(1)}} \otimes C_{T^{(2)}}$, $v_{(g_1,g_2)} = v_{g_1} \otimes v_{g_2}$. Consider the automorphism $\tilde{\gamma}$ of $R_T$,

$$\tilde{\gamma}(a \otimes b) = \gamma^{-1}(b) \otimes \gamma(a).$$

By Theorem 4.2, $\tilde{\gamma}$ must be of the form $\mathrm{Ad}\, w \circ \sigma_\chi \circ \alpha_{\tilde{S}}$ with $w \in S_T$, $\chi = (\chi_1, \chi_2) \in \hat{G}_1 \times \hat{G}_2$ and $\tilde{S} \in I(T)$. Let $\tilde{\beta} \in \mathrm{Aut}(G)$ be such that $T_g \tilde{S} = \tilde{S} T_{\tilde{\beta}(g)}$. Since $\tilde{\gamma}^2 = \mathrm{id}$, we have $\tilde{\beta}^2 = \mathrm{id}$. Define the homomorphism $\beta : G_2 \to G_1$ as the composition of the map $g_2 \mapsto \tilde{\beta}(0, g_2)$ with the projection $G_1 \times G_2 \to G_1$, and $\beta' : G_1 \to G_2$ as the composition of the map $g_1 \mapsto \tilde{\beta}(g_1, 0)$ with the projection $G_1 \times G_2 \to G_2$. Fix $g_2 \in G_2$. Then $\tilde{\beta}(0, g_2) = (\beta(g_2), h)$ for some $h \in G_2$. We have

$$\gamma^{-1}(v_{g_2}) \otimes 1 = \tilde{\gamma}(1 \otimes v_{g_2}) = \langle\chi_1, -\beta(g_2)\rangle\langle\chi_2, -h\rangle v_{\beta(g_2)} \otimes v_h.$$

It follows that $h = 0$, that is $\tilde{\beta}(0, g_2) = (\beta(g_2), 0)$. Analogously $\tilde{\beta}(g_1, 0) = (0, \beta'(g_1))$. Thus $\tilde{\beta}(g_1, g_2) = (\beta(g_2), \beta'(g_1))$. Since $\tilde{\beta}^2 = \mathrm{id}$, we conclude that $\beta' = \beta^{-1}$. Then the identity $T_g \tilde{S} = \tilde{S} T_{\tilde{\beta}(g)}$ is rewritten in terms of the actions on $L^\infty(X_1 \times X_2)$ as

$$(\alpha_{g_1} \otimes \alpha_{g_2}) \circ \alpha_{\tilde{S}} = \alpha_{\tilde{S}} \circ (\alpha_{\beta(g_2)} \otimes \alpha_{\beta^{-1}(g_1)}).$$

Letting $g_2 = 0$ we see that for $f \in L^\infty(X_1)$

$$((\alpha_{g_1} \otimes 1) \circ \alpha_{\tilde{S}})(f \otimes 1) = (\alpha_{\tilde{S}} \circ (1 \otimes \alpha_{\beta^{-1}(g_1)}))(f \otimes 1) = \alpha_{\tilde{S}}(f \otimes 1),$$

so that $\alpha_{\tilde{S}}(L^\infty(X_1) \otimes 1) \subset L^\infty(X_1 \times X_2)^{\alpha_{G_1} \otimes 1} = 1 \otimes L^\infty(X_2)$. Analogously $\alpha_{\tilde{S}}(1 \otimes L^\infty(X_2)) \subset L^\infty(X_1) \otimes 1$. It follows that

$$\alpha_{\tilde{S}}(L^\infty(X_1) \otimes 1) = 1 \otimes L^\infty(X_2) \qquad \text{and} \qquad \alpha_{\tilde{S}}(1 \otimes L^\infty(X_2)) = L^\infty(X_1) \otimes 1.$$

Hence there exist isomorphisms $S : (X_1, \mu_1) \to (X_2, \mu_2)$ and $S' : (X_2, \mu_2) \to (X_1, \mu_1)$ such that for almost all $(x_1, x_2)$ we have $\tilde{S}(x_1, x_2) = (S'x_2, Sx_1)$. The identity $(T^{(1)}_{g_1} \times T^{(2)}_{g_2})\tilde{S} = \tilde{S}(T^{(1)}_{\beta(g_2)} \times T^{(2)}_{\beta^{-1}(g_1)})$ implies that $T^{(2)}_{g_2} S = S T^{(1)}_{\beta(g_2)}$.

Now we turn to the proof of Theorem 4.2. The proof will be given in a series of lemmas. Let $\gamma \in \mathrm{Aut}(R_T, S_T \mid C_T)$.

LEMMA 4.3. *The automorphism $\gamma$ can be implemented by a unitary $U$ on $L^2(R_T)$ such that $UJC_TJU^* = JC_TJ$, where $J$ is the modular involution.*

*Proof.* Let $\tilde{U}$ be the canonical implementation of $\gamma$ commuting with $J$. From the assumption that $C_T$ and $\gamma(C_T)$ are inner conjugate we can choose $u \in R_T$ such that $uC_Tu^* = \gamma(C_T)$. Then we can take $U = Ju^*J\tilde{U}$. ∎

Representing $R_T$ on $L^2(X) \otimes L^2(\hat{G})$ as usual, so that $JC_TJ = L^\infty(X) \otimes 1$ and $S_T = 1 \otimes L^\infty(\hat{G})$ (see (2.2)), we conclude that Ad $U$ defines measure-preserving transformations $S_1$ of $X$ and $\sigma$ of $\hat{G}$. Then $W = U(u^*_{S_1} \otimes u^*_\sigma)$ commutes with $L^\infty(X) \otimes 1$ and $1 \otimes L^\infty(\hat{G})$, hence it is a unitary in $L^\infty(X \times \hat{G})$. For $\ell \in \hat{G}$ denote by $w_\ell$ the function in $L^\infty(X)$ defined by $w_\ell(x) = W(x, \ell)$.

Since $U$ defines an automorphism of $R_T$, for $f \in L^\infty(X)$ the element $U\pi(f)U^*$ must by (2.2) commute with $u_h \otimes m^*_h$.

LEMMA 4.4. *With the above notations, for $\zeta \in L^2(\hat{G}, L^2(X)) \cong L^2(X) \otimes L^2(\hat{G})$ we have*

$$(U\pi(f)U^*\zeta)(\ell) = \sum_{g \in G} (\alpha_{S_1} \circ \alpha_g)(f) \int_{\hat{G}} \langle \overline{\sigma^{-1}(\ell)}\sigma^{-1}(\ell_1), g\rangle w_\ell w^*_{\ell_1} \zeta(\ell_1)\, d\lambda(\ell_1),$$

$$\begin{aligned}((u_h \otimes m^*_h)&U\pi(f)U^*(u^*_h \otimes m_h)\zeta)(\ell)\\ &= \sum_{g \in G} (\alpha_h \circ \alpha_{S_1} \circ \alpha_g)(f) \int_{\hat{G}} \langle \bar{\ell}\ell_1, h\rangle \langle \overline{\sigma^{-1}(\ell)}\sigma^{-1}(\ell_1), g\rangle \alpha_h(w_\ell w^*_{\ell_1}) \zeta(\ell_1)\, d\lambda(\ell_1).\end{aligned}$$

*The above series are meaningless for fixed $\ell$ and should be considered as series of functions in $L^2(X \times \hat{G})$.*

*Proof*. Note that $(W\zeta)(\ell) = w_\ell \zeta(\ell)$, $((1 \otimes m_g)\zeta)(\ell) = \langle \ell, g\rangle \zeta(\ell)$. The operator $u_\sigma e_g u_\sigma^*$ is the projection onto the one-dimensional space spanned by the function $u_\sigma g \in L^2(\hat{G})$, so for $f \in L^2(\hat{G})$,

$$(u_\sigma e_g u_\sigma^* f)(\ell) = (u_\sigma g)(\ell) \cdot (f, u_\sigma g) = \int_{\hat{G}} \langle \sigma^{-1}(\ell)\overline{\sigma^{-1}(\ell_1)}, g\rangle f(\ell_1)\, d\lambda(\ell_1).$$

Hence $((1 \otimes u_\sigma e_g u_\sigma^*)\zeta)(\ell) = \int_{\hat{G}} \langle \sigma^{-1}(\ell)\overline{\sigma^{-1}(\ell_1)}, g\rangle \zeta(\ell_1)\, d\lambda(\ell_1)$. Now we compute:

$$\begin{aligned}
(U\pi(f)U^*\zeta)(\ell) &= \left( W(u_{S_1} \otimes u_\sigma)\left(\sum_g \alpha_g(f) \otimes e_{-g}\right)(u_{S_1}^* \otimes u_\sigma^*)W^*\zeta\right)(\ell) \\
&= \sum_g (W((\alpha_{S_1} \circ \alpha_g)(f) \otimes u_\sigma e_{-g} u_\sigma^*)W^*\zeta)(\ell) \\
&= \sum_g w_\ell(\alpha_{S_1} \circ \alpha_g)(f) \int_{\hat{G}} \langle \sigma^{-1}(\ell)\overline{\sigma^{-1}(\ell_1)}, -g\rangle (W^*\zeta)(\ell_1)\, d\lambda(\ell_1) \\
&= \sum_g (\alpha_{S_1} \circ \alpha_g)(f) \int_{\hat{G}} \langle \overline{\sigma^{-1}(\ell)}\sigma^{-1}(\ell_1), g\rangle w_\ell w_{\ell_1}^* \zeta(\ell_1)\, d\lambda(\ell_1)
\end{aligned}$$

and

$$\begin{aligned}
&((u_h \otimes m_h^*)U\pi(f)U^*(u_h^* \otimes m_h)\zeta)(\ell) \\
&\quad = \langle \bar{\ell}, h\rangle u_h (U\pi(f)U^*(u_h^* \otimes m_h)\zeta)(\ell) \\
&= \langle \bar{\ell}, h\rangle u_h \sum_g (\alpha_{S_1} \circ \alpha_g)(f) \int_{\hat{G}} \langle \overline{\sigma^{-1}(\ell)}\sigma^{-1}(\ell_1), g\rangle w_\ell w_{\ell_1}^* ((u_h^* \otimes m_h)\zeta)(\ell_1)\, d\lambda(\ell_1) \\
&= \langle \bar{\ell}, h\rangle u_h \sum_g (\alpha_{S_1} \circ \alpha_g)(f) \int_{\hat{G}} \langle \overline{\sigma^{-1}(\ell)}\sigma^{-1}(\ell_1), g\rangle w_\ell w_{\ell_1}^* u_h^* \langle \ell_1, h\rangle \zeta(\ell_1)\, d\lambda(\ell_1) \\
&= \sum_g (\alpha_h \circ \alpha_{S_1} \circ \alpha_g)(f) \int_{\hat{G}} \langle \bar{\ell}\ell_1, h\rangle \langle \overline{\sigma^{-1}(\ell)}\sigma^{-1}(\ell_1), g\rangle \alpha_h(w_\ell w_{\ell_1}^*)\zeta(\ell_1)\, d\lambda(\ell_1). \quad \blacksquare
\end{aligned}$$

LEMMA 4.5. *Let $g \mapsto P_g \in \mathrm{Aut}(X, \mu)$ be a free measure-preserving action of $G$, $Q \in \mathrm{Aut}(X, \mu)$, $H$ a Hilbert space, $a_g$ and $b_g$ maps from $X$ to $H$ such that*

(i) *the vectors $a_g(x)$, $g \in G$, are mutually orthogonal for almost all $x \in X$;*

(ii) $\sum_g \|a_g(x)\|^2$ *is finite and non-zero for almost all* $x$; *and the same conditions hold for* $\{b_g\}_g$. *Suppose for all* $f \in L^\infty(X)$ *and almost all* $x \in X$

$$\sum_g (\alpha_Q \circ \alpha_{P_g})(f)(x)a_g(x) = \sum_g \alpha_{P_g}(f)(x)b_g(x).$$

*Then* $Q$ *is in the full group generated by* $P_g$, $g \in G$, *and if* $g(x) \in G$ *is such that* $Q^{-1}x = P_{-g(x)}x$ *then,* $a_g(x) = b_{g+g(x)}(x)$ *for all* $g \in G$ *and almost all* $x \in X$.

*Proof.* Let $X_0 = \{x \in X \mid Q^{-1}x \notin P_G x, P_g x \neq x \text{ for } g \neq 0\}$. There exists a countable family $\{X_i\}_{i \in I}$ of measurable subsets of $X$ such that for arbitrary finite subset $F$ of $G$ and almost all $x \in X_0$ there exists $i \in I$ such that $x \in X_i$, the sets $P_g X_i$, $g \in F$, are mutually disjoint and $Q^{-1}x \notin \bigcup_{g \in F} P_g X_i$. Indeed, first note that choosing an arbitrary $Q$- and $P_g$-invariant norm-separable weakly dense C*-subalgebra $A$ of $L^\infty(X)$, we can identify the measure space $(X, \mu)$ with the spectrum of $A$. Thus without loss of generality, we can suppose that $X$ is a compact metric space and $Q$ and $P_g$ are homeomorphisms. Moreover, by regularity of the measure it is enough to prove the assertion for arbitrary compact subset $K$ of $X_0$. But then for fixed $F$ we can consider for each $x \in K$ a neighborhood $U_x$ such that $P_g U_x$, $g \in F$, are disjoint, $Q^{-1}U_x \cap P_g U_x = \emptyset$ for $g \in F$, and then choose a finite subcovering from $\{U_x\}_{x \in K}$.

Consider the countable set $\mathscr{F} \subset L^\infty(X)$ consisting of characteristic functions of the sets $X_i$, $i \in I$, and all their translations under the action of $G$. For almost all $x \in X_0$ and all $f \in \mathscr{F}$ the assumptions of the lemma are satisfied. Let $x \in X_0$ be such a point. Fix $h \in G$. For arbitrary finite subset $F$ of $G$, $h \in F$, there exists $f \in \mathscr{F}$ such that $\alpha_{P_h}(f)(x) = 1$, $\alpha_{P_g}(f)(x) = 0$ for $g \in F \setminus \{h\}$ and $(\alpha_Q \circ \alpha_{P_g})(f)(x) = 0$, for $g \in F$. Then,

$$\|b_h(x)\| = \left\| \sum_{g \notin F} (\alpha_Q \circ \alpha_{P_g})(f)(x)a_g(x) - \sum_{g \notin F} \alpha_{P_g}(f)(x)b_g(x) \right\|$$

$$\leqslant \left( \sum_{g \notin F} \|a_g(x)\|^2 \right)^{1/2} + \left( \sum_{g \notin F} \|b_g(x)\|^2 \right)^{1/2}.$$

It follows that $b_h(x) = 0$. But this contradicts the assumption $\sum_h \|b_h(x)\|^2 > 0$. Hence the set $X_0$ has zero measure. Thus $Q$ is indeed in the full group generated by $P_g$.

Let $Q^{-1}x = P_{-g(x)}x$. In the same way as above (or by referring to the Rokhlin lemma), we can find a countable collection $\mathscr{F}$ of characteristic functions such that for almost all $x \in X$ and arbitrary finite $F \subset G$, $0 \in F$, there exists $f \in \mathscr{F}$ such that $f(x) = 1$, $\alpha_{P_g}(f)(x) = 0$ for $g \in F \setminus \{0\}$. Then

$$a_{-g(x)}(x) - b_0(x) = \sum_{g \notin F} \alpha_{P_g}(f)(x) b_g(x) - \sum_{g \notin F - g(x)} (\alpha_Q \circ \alpha_{P_g})(f)(x) a_g(x),$$

and we conclude that $a_{-g(x)}(x) = b_0(x)$. Replacing $f$ by $\alpha_{P_h}(f)$ in the formulation of the lemma we see that its assumptions are also satisfied for the collections $\{a_{g-h}\}_g$ and $\{b_{g-h}\}_g$, so that $a_{-g(x)-h}(x) = b_{-h}(x)$. ∎

Fix $h \in G$ and apply Lemma 4.5 to $P_g = S_1 T_g S_1^{-1}$, $Q = T_h$, $H = L^2(\hat{G})$,

$$a_g(x)(\ell) = \int_{\hat{G}} \langle \bar{\ell}\ell_1, h \rangle \langle \overline{\sigma^{-1}(\ell)} \sigma^{-1}(\ell_1), g \rangle \alpha_h(w_\ell w_{\ell_1}^*)(x)\, d\lambda(\ell_1)$$

$$b_g(x)(\ell) = \int_{\hat{G}} \langle \overline{\sigma^{-1}(\ell)} \sigma^{-1}(\ell_1), g \rangle (w_\ell w_{\ell_1}^*)(x)\, d\lambda(\ell_1).$$

To see that the assumptions of the lemma are satisfied, note that up to the factor $\ell \mapsto \langle \bar{\ell}, h \rangle \alpha_h(w_\ell)(x)$ the series $\sum_g a_g(x)$ is the Fourier series of the function $\ell \mapsto \langle \ell, h \rangle \alpha_h(w_\ell^*)(x)$ with respect to the orthonormal basis $\{\overline{u_\sigma g}\}_{g \in G}$.

Thus by Lemmas 4.4 and 4.5, we conclude that there exists $g(h, x)$ such that $T_{-h}x = S_1 T_{-g(h,x)} S_1^{-1} x$ and $a_g(x) = b_{g+g(h,x)}(x)$, that is

$$\int_{\hat{G}} \langle \sigma^{-1}(\ell_1), g \rangle (\langle \bar{\ell}\ell_1, h \rangle \alpha_h(w_\ell w_{\ell_1}^*)(x) - \langle \overline{\sigma^{-1}(\ell)} \sigma^{-1}(\ell_1), g(h,x) \rangle (w_\ell w_{\ell_1}^*)(x))\, d\lambda(\ell_1) = 0.$$

Since the functions $u_\sigma g = (\ell_1 \mapsto \langle \sigma^{-1}(\ell_1), g \rangle)$, $g \in G$, form an orthonormal basis of $L^2(\hat{G})$, we conclude that for almost all $(x, \ell, \ell_1)$

$$\alpha_h(w_\ell w_{\ell_1}^*)(x) = \langle \ell \overline{\ell_1}, h \rangle \langle \overline{\sigma^{-1}(\ell)} \sigma^{-1}(\ell_1), g(h,x) \rangle (w_\ell w_{\ell_1}^*)\,(x). \tag{4.1}$$

Lemma 4.6. *There exists a continuous automorphism $\sigma_0$ of $\hat{G}$ and $\chi \in \hat{G}$ such that $\sigma(\ell) = \chi \sigma_0(\ell)$ for almost all $\ell$.*

*Proof.* Replace $\ell$ by $\sigma(\ell)$ and $\ell_1$ by $\sigma(\ell_1)$ in (4.1). Then we get

$$\alpha_h(w_{\sigma(\ell)} w_{\sigma(\ell_1)}^*)(x) = \langle \sigma(\ell) \overline{\sigma(\ell_1)}, h \rangle \langle \bar{\ell}\ell_1, g(h,x) \rangle (w_{\sigma(\ell)} w_{\sigma(\ell_1)}^*)(x). \tag{4.2}$$

Now substitute $\ell\ell_2$ for $\ell$ and $\ell_1\ell_2$ for $\ell_1$. We get

$$\alpha_h(w_{\sigma(\ell\ell_2)} w_{\sigma(\ell_1\ell_2)}^*)(x) = \langle \sigma(\ell\ell_2) \overline{\sigma(\ell_1\ell_2)}, h \rangle \langle \bar{\ell}\ell_1, g(h,x) \rangle (w_{\sigma(\ell\ell_2)} w_{\sigma(\ell_1\ell_2)}^*)(x). \tag{4.3}$$

Multiplying (4.2) by the equation conjugate to (4.3) we see that for almost all $(\ell, \ell_1, \ell_2)$, the element $w_{\sigma(\ell)} w_{\sigma(\ell_1)}^* w_{\sigma(\ell_1\ell_2)} w_{\sigma(\ell\ell_2)}^*$ is an eigenfunction with eigenvalue $\sigma(\ell)\overline{\sigma(\ell_1)}\sigma(\ell_1\ell_2)\overline{\sigma(\ell\ell_2)}$. Since the action is weakly mixing, we conclude that

$$\sigma(\ell\ell_2)\overline{\sigma(\ell)} = \sigma(\ell_1\ell_2)\overline{\sigma(\ell_1)}$$

(this is the only place where we use weak mixing instead of ergodicity). Hence, there exists a measurable map $\tilde{\sigma}_0$ of $\hat{G}$ onto itself such that $\tilde{\sigma}_0(\ell_2) = \sigma(\ell\ell_2)\overline{\sigma(\ell)}$ for almost all $(\ell_1, \ell_2)$. Then for almost all $(\ell_1, \ell_2)$

$$\tilde{\sigma}_0(\ell_1\ell_2) = \sigma(\ell\ell_1\ell_2)\overline{\sigma(\ell)} = \sigma(\ell\ell_1\ell_2)\overline{\sigma(\ell\ell_2)}\sigma(\ell\ell_2)\overline{\sigma(\ell)} = \tilde{\sigma}_0(\ell_1)\tilde{\sigma}_0(\ell_2).$$

So $\tilde{\sigma}_0$ is essentially a homomorphism, and since it is measurable, it coincides almost everywhere with a continuous homomorphism $\sigma_0$. Choose a character $\ell_1$ such that the equality $\sigma_0(\ell) = \sigma(\ell_1\ell)\overline{\sigma(\ell_1)}$ holds for almost all $\ell$. Set $\chi = \sigma(\ell_1)\overline{\sigma_0(\ell_1)}$. Then $\sigma(\ell) = \chi\sigma_0(\ell)$ for almost all $\ell$. Since $\sigma$ is an invertible measure-preserving transformation, $\sigma_0$ must be an automorphism. ∎

Now we can rewrite (4.1) as

$$\alpha_h(w_\ell w_{\ell_1}^*)(x) = \langle \ell_1\overline{\ell_1}, h\rangle\langle \sigma_0^{-1}(\bar{\ell}\ell_1), g(h,x)\rangle(w_\ell w_{\ell_1}^*)(x). \tag{4.4}$$

LEMMA 4.7. *Let $\ell_1$ be such that* (4.4) *holds for almost all* $(x,\ell) \in X \times \hat{G}$. *Then there exist a unitary $b$ in $L^\infty(\hat{G})$ and a measurable map $e: X \to G$ such that for almost all $(x,\ell)$ we have*

$$(w_{\ell\ell_1} w_{\ell_1}^*)(x) = b(\ell)\langle \ell, e(x)\rangle.$$

*For all $h \in G$ and almost all $x \in X$ we have*

$$g(h,x) = \beta(e(x)) - \beta(e(T_{-h}x)) + \beta(h),$$

*where $\beta$ is the automorphism of $G$ dual to $\sigma_0$*, i.e. $\langle \sigma_0(\ell), g\rangle = \langle \ell, \beta(g)\rangle$.

*Proof.* Denote $w_{\ell\ell_1}w_{\ell_1}^*$ by $v_\ell$. Then by (4.4)

$$\alpha_h(v_\ell)(x) = \langle \ell, h\rangle\langle \sigma_0^{-1}(\bar{\ell}), g(h,x)\rangle(v_\ell)(x). \tag{4.5}$$

Multiplying these identities for $v_\ell$, $v_{\ell_2}$ and $v_{\ell\ell_2}^*$ we see that the function $c(\ell,\ell_2) = v_\ell v_{\ell_2} v_{\ell\ell_2}^*$ is $G$-invariant, so it is a constant. Thus, we obtain a measurable symmetric (i.e. $c(\ell,\ell_2) = c(\ell_2,\ell)$) 2-cocycle on $G$ with values in $\mathbb{T}$. Since $G$ is abelian, any such a cocycle is a coboundary (see e.g. [M]), $c(\ell,\ell_2) = b(\ell)b(\ell_2)\overline{b(\ell\ell_2)}$. Then $\ell \mapsto \overline{b(\ell)}v_\ell$ is a measurable homomorphism of $\hat{G}$ into the unitary group of $L^\infty(X)$. By [M, Theorem 1] there exists a measurable map $e: X \to G$ such that $\overline{b(\ell)}v_\ell(x) = \langle \ell, e(x)\rangle$.

Equation (4.5) implies that

$$\langle \ell, e(T_{-h}x)\rangle = \langle \ell, h\rangle\langle \sigma_0^{-1}(\bar{\ell}), g(h,x)\rangle\langle \ell, e(x)\rangle,$$

equivalently,

$$\langle \ell, e(T_{-h}x) - h + \beta^{-1}(g(h,x)) - e(x)\rangle = 1,$$

from what the second assertion of the lemma follows. ∎

Recall that $S_1$ is the transformation of $X$ defined by Ad $U|_{L^\infty(X)}$.

Lemma 4.8. *Define a measurable map $S_2$ of $X$ onto itself by letting*

$$S_2 x = S_1 T_{-\beta(e(x))} S_1^{-1} x.$$

*Then $S_2$ is invertible and measure preserving. Its inverse is given by*

$$S_2^{-1}x = T_{e(x)}x.$$

*Proof.* Recall that $g(h,x)$ was defined by the equality $T_{-h}x = S_1 T_{-g(h,x)} S_1^{-1}x$. Since by Lemma 4.7, $g(-e(x),x) = -\beta(e(T_{e(x)}x))$, it follows that

$$S_2 T_{e(x)}x = S_1 T_{-\beta(e(T_{e(x)}x))} S_1^{-1} T_{e(x)}x = S_1 T_{-\beta(e(T_{e(x)}x)) - g(-e(x),x)} S_1^{-1}x = x.$$

Hence $S_2$ is essentially surjective. Since it is also one-to-one and measure preserving on the sets $e^{-1}(\{g\})$, we conclude that $S_2$ is invertible, measure preserving and its inverse is given by $S_2^{-1}x = T_{e(x)}x$. ∎

The final step is

Lemma 4.9. *The mapping $S = S_2^{-1}S_1$ has the property $T_g S = S T_{\beta(g)}$.*

*Proof.* We compute

$$\begin{aligned} S^{-1}T_{-h}x &= S_1^{-1}S_2 T_{-h}x \\ &= S_1^{-1}S_1 T_{-\beta(e(T_{-h}x))} S_1^{-1} T_{-h}x \\ &= T_{-\beta(e(T_{-h}x)) - g(h,x)} S_1^{-1}x \\ &= T_{-\beta(h) - \beta(e(x))} S_1^{-1}x \\ &= T_{-\beta(h)} S_1^{-1} S_2 x \\ &= T_{-\beta(h)} S^{-1}x, \end{aligned}$$

where in the fourth equality we used Lemma 4.7. ∎

Summarizing the results of Lemmas 4.6–4.9, we can decompose $U = W(u_{S_1} \otimes u_\sigma)$ as follows. First, by Lemma 4.7 for almost all $(x,\ell)$, $w_\ell(x) = \langle \ell\overline{\ell_1}, e(x)\rangle w_{\ell_1}(x) b(\ell\overline{\ell_1})$, so $W$ is the product of $u' \otimes 1$, $v$ and $1 \otimes w$, where $u' \in L^\infty(X)$, $u'(x) = \langle \overline{\ell_1},\ e(x)\rangle w_{\ell_1}(x)$, $v \in L^\infty(X \times \hat{G})$, $\quad v(x,\ell) = \langle \ell, e(x)\rangle$, and $w \in L^\infty(\hat{G})$, $w(\ell) = b(\ell\overline{\ell_1})$. By Lemmas 4.8 and 4.9, $u_{S_1} = u_{S_2} u_S$. Finally by Lemma 4.6, $u_\sigma = \lambda_\chi u_{\sigma_0}$, where $\lambda_\chi$ is the operator of the left regular representation of $\hat{G}$ on $L^2(\hat{G})$. Thus with $v' = v(u_{S_2} \otimes 1)$, we have

$$
\begin{aligned}
U &= (u' \otimes 1)v(1 \otimes w)(u_{S_2} u_S \otimes 1)(1 \otimes \lambda_\chi u_{\sigma_0}) \\
&= (u' \otimes 1)v'(1 \otimes w)(1 \otimes \lambda_\chi)(u_S \otimes u_{\sigma_0}).
\end{aligned}
$$

The unitaries $u' \otimes 1$ and $v'$ both lie in the commutant $R_T'$. This is obvious for $u' \otimes 1$ and follows for $v'$ from the formula

$$
v' = \sum_g (p_g \otimes 1)(u_g^* \otimes m_g),
$$

where $p_g$ is the characteristic function of the set $e^{-1}(\{g\})$. Indeed, if $x \in e^{-1}(\{g\})$ then $S_2^{-1}x = T_g x$ by Lemma 4.8, and hence for arbitrary $\zeta \in L^2(X \times \hat{G})$ we have

$$
((p_g \otimes 1)(u_g^* \otimes m_g)\zeta)(x,\ell) = \langle \ell, g\rangle \zeta(T_g x, \ell) = \langle \ell, e(x)\rangle \zeta(S_2^{-1}x, \ell) = (v'\zeta)(x,\ell).
$$

Thus the automorphism $\gamma$ is implemented by the unitary $(1 \otimes w)(1 \otimes \lambda_\chi)(u_S \otimes u_{\sigma_0})$, so $\gamma = \operatorname{Ad} w \circ \sigma_\chi \circ \alpha_S$, and the proof of Theorem 4.2 is complete.

From the definition of the group $\operatorname{Aut}(R_T, S_T \mid C_T)$ it is unclear whether it is a closed subgroup of $\operatorname{Aut}(R_T, S_T)$ (in the topology of point-wise strong convergence). But if our conjecture that this group coincides with $\operatorname{Aut}(R_T, S_T)$ (which is stronger than our main conjecture in Section 2) is true, then this group must be closed. We shall prove that it is closed under slightly stronger assumptions than weak mixing.

Recall that an action $T$ is called *rigid* if there exists a sequence $\{g_n\}_n$ such that $g_n \to \infty$ and $u_{g_n} \to 1$ strongly.

PROPOSITION 4.10. *Suppose $T$ is a weakly mixing action which is not rigid. Then the group* $\operatorname{Aut}(R_T, S_T \mid C_T)$ *is closed in* $\operatorname{Aut}(R_T)$.

*Proof.* Suppose a sequence $\{\alpha_n\}_n \subset \operatorname{Aut}(R_T, S_T \mid C_T)$ converges to an automorphism $\alpha$. By Theorem 4.2, $\alpha_n = \sigma_{\chi_n} \circ \operatorname{Ad} w_n \circ \alpha_{S_n}$. Passing to a subsequence we may suppose that the sequence $\{\chi_n\}_n$ converges to a character $\chi$. Then $\{\operatorname{Ad} w_n \circ \alpha_{S_n}\}_n$ converges to $\sigma_\chi^{-1} \circ \alpha$, so to simplify the notations we may suppose that all characters $\chi_n$ are trivial.

Let $f$ be a unitary generating $C_T$. Set

$$\delta = \inf_{g \neq 0} \|\alpha_g(f) - f\|_2.$$

Since the action is not rigid, $\delta > 0$. Suppose for some $n$ and $m$

$$\|(\mathrm{Ad}\, w_n \circ \alpha_{S_n})(f) - (\mathrm{Ad}\, w_m \circ \alpha_{S_m})(f)\|_2 < \varepsilon.$$

We assert that if $\varepsilon < \delta^2/4$ then there exist $g \in G$ and $c \in \mathbb{T}$ such that

$$\|w_n - c w_m v_g\|_2 < (2\varepsilon)^{1/2}, \tag{4.6}$$

where $v_g$, $g \in G$ are the canonical generators of $S_T$. Indeed, let $v = w_m^* w_n$, $v = \sum_g a_g v_g$, $a_g \in \mathbb{C}$. If $E : R_T \to C_T$ is the trace-preserving conditional expectation, then for arbitrary $x \in C_T$ we have $E(vxv^*) = \sum_g |a_g|^2 \alpha_g(x)$, whence

$$\begin{aligned}
\varepsilon^2 &> \|(\mathrm{Ad}\, w_n \circ \alpha_{S_n})(f) - (\mathrm{Ad}\, w_m \circ \alpha_{S_m})(f)\|_2^2 \\
&= 2(1 - \mathrm{Re}\, \tau(v \alpha_{S_n}(f) v^* \alpha_{S_m}(f^*))) \\
&= 2(1 - \mathrm{Re}\, \tau(E(v \alpha_{S_n}(f) v^*) \alpha_{S_m}(f^*))) \\
&= 2 \sum_g |a_g|^2 (1 - \mathrm{Re}\, \tau((\alpha_g \circ \alpha_{S_n})(f) \alpha_{S_m}(f^*))).
\end{aligned}$$

Set $Y = \{g \in G \mid 1 - \mathrm{Re}\, \tau((\alpha_g \circ \alpha_{S_n})(f)\alpha_{S_m}(f^*)) < \varepsilon/2\}$. If $g \in Y$ then $\|(\alpha_g \circ \alpha_{S_n})(f) - \alpha_{S_m}(f)\|_2 < \varepsilon^{1/2}$. Thus if $g_1 \neq g_2$ both lie in $Y$ then $\|(\alpha_{g_1} \circ \alpha_{S_n})(f) - (\alpha_{g_2} \circ \alpha_{S_n})(f)\|_2 < 2\varepsilon^{1/2}$. Since $\alpha_g \circ \alpha_{S_n} = \alpha_{S_n} \circ \alpha_{\beta(g)}$ for some automorphism $\beta$, we get a contradiction if $2\varepsilon^{1/2} < \delta$. Hence the set $Y$ consists of at most one point. On the other hand, we have

$$\varepsilon^2 > 2 \sum_g |a_g|^2 (1 - \mathrm{Re}\, \tau((\alpha_g \circ \alpha_{S_n})(f)\alpha_{S_m}(f^*))) \geqslant \varepsilon \sum_{g \notin Y} |a_g|^2,$$

so that $\sum_{g \notin Y} |a_g|^2 < \varepsilon$. If follows that $Y$ is non-empty. Hence it consists precisely of one point $g$, and $|a_g|^2 > 1 - \varepsilon$. Then with $c = a_g/|a_g|$, we have

$$\|w_n - c w_m v_g\|_2^2 = \|v - c v_g\|_2^2 = \sum_{h \neq g} |a_h|^2 + |a_g - c|^2 = 2(1 - |a_g|) < 2\varepsilon.$$

It follows that passing to a subsequence we may suppose that for each $n \geqslant 2$ there exist $g_n \in G$ and $c_n \in \mathbb{T}$ such that

$$\|c_n w_n v_{g_n} - w_{n-1}\|_2 < \frac{1}{2^n}.$$

Then replacing $w_n$ by $c_n \dots c_2 w_n v_{g_n+\dots+g_2}$ and $S_n$ by $T_{-g_2-\dots-g_n} S_n$ we still have $\alpha_n = \operatorname{Ad} w_n \circ \alpha_{S_n}$, but now the sequence $\{w_n\}_n$ converges strongly to a unitary $w \in S_T$. Then $\alpha(C_T) = (\operatorname{Ad} w)(C_T)$. ∎

Note that the group $\operatorname{Aut}(R_T, C_T \mid S_T)$ consisting of all automorphisms $\gamma \in \operatorname{Aut}(R_T, C_T)$ such that $\gamma(S_T)$ and $S_T$ are inner conjugate is never closed. Indeed, let $c \in Z^1(\mathcal{R}_T, \mathbb{T})$ be a $\mathbb{T}$-valued 1-cocycle on the orbit equivalence relation $\mathcal{R}_T$ defined by $T$, and $\sigma_c \in \operatorname{Aut}(R_T, C_T)$ the corresponding automorphism [FM]. Then $\sigma_c(S_T)$ and $S_T$ are inner conjugate if and only if $c$ is cohomologous to the cocycle $c_\chi$, $c_\chi(x, T_g x) = \langle \chi, g \rangle$, for some $\chi \in \hat{G}$ [P1] (this result was proved in [P1] for actions with purely discrete spectrum, but with minor changes the proof works for arbitrary ergodic actions; in our weakly mixing case using Theorem 4.2 and the fact that if $\gamma \in \operatorname{Aut}(R_T, C_T \mid S_T)$ then $\operatorname{Ad} u \circ \gamma \in \operatorname{Aut}(R_T, S_T \mid C_T)$ for some unitary $u$, it is easy to obtain a more precise result: the group $\operatorname{Aut}(R_T, C_T \mid S_T)$ consists of automorphisms of the form $\sigma_c \circ \alpha_S$, where $c$ is a cocycle cohomologous to $c_\chi$ and $S \in I(T)[T]$, where $[T]$ is the full group generated by $T_g$, $g \in G$). Since the equivalence relation is hyperfinite, any cocycle can be approximated by coboundaries, so all automorphisms $\sigma_c$ are in the closure of $\operatorname{Aut}(R_T, C_T \mid S_T)$. On the other hand, there always exist cocycles which are not cohomologous to cocycles $c_\chi$, because otherwise $Z^1(\mathcal{R}_T, \mathbb{T})$ would be a continuous isomorphic image of the group $\hat{G} \times I(X, \mathbb{T})$, where $I(X, \mathbb{T})$ is the factor of the unitary group of $L^\infty(X)$ by the scalars (note that since the action is weakly mixing, $c_\chi$ is not a coboundary for $\chi \in \hat{G} \setminus \{e\}$), hence $Z^1(\mathcal{R}_T, \mathbb{T})$ would be topologically isomorphic to $\hat{G} \times I(X, \mathbb{T})$, which would imply that the group of coboundaries is closed.

If the action is rigid, it is still possible that $\operatorname{Aut}(R_T, S_T \mid C_T)$ is closed. However, as the following result shows the group $\operatorname{Int}(S_T)$ consisting of inner automorphisms of $R_T$ defined by unitaries in $S_T$ is not closed in this case, which may indicate that we should consider systems satisfying stronger mixing properties than weak mixing. Note that if an action is mixing then it is not rigid.

PROPOSITION 4.11. *The following conditions are equivalent*:

(i) *the action $T$ is rigid*;
(ii) *there exist non-trivial central sequences in $S_T$*;
(iii) *the subgroup* $\operatorname{Int}(S_T)$ *of* $\operatorname{Aut}(R_T)$ *is not closed.*

*Proof.* The equivalence of (ii) and (iii) is well known [C]. The implication (i) ⇒ (ii) is obvious. Suppose that the action is not rigid. Let $\{u_n\}_n$ be a central sequence of unitaries in $S_T$. For fixed $n$ apply (4.6) to $w_n = u_n$, $w_m = 1$, $S_n = S_m = \mathrm{id}$. Then we conclude that there exist $c_n \in \mathbb{T}$ and $g_n \in G$ such that $\|u_n - c_n v_{g_n}\|_2 \to 0$ as $n \to \infty$. The sequence $\{v_{g_n}\}_n$ is central, which is equivalent to the strong convergence $u_{g_n} \to 1$ in $B(L^2(x))$. Since the action is not rigid, this implies that eventually $g_n = 0$, so the central sequence $\{u_n\}_n$ is trivial. Thus (ii) implies (i). ∎

The following corollary is not surprising in view of Proposition 3.1 but is worth mentioning.

COROLLARY 4.12. *There exist weakly mixing transformations $T^{(1)}$ and $T^{(2)}$ such that the singular masas $S_{T^{(1)}}$ and $S_{T^{(2)}}$ are not conjugate but their Pukanszky invariants coincide.*

*Proof.* The class of weakly mixing measure-preserving transformations with simple spectrum, i.e. of spectral multiplicity one, contains both rigid and non-rigid transformations (e.g. certain Gauss systems are rigid and have simple spectrum [CFS, Chap. 14], while Ornstein's rank-one transformations are mixing [Na, Chap. 16]). Since rigidity is a conjugacy invariant by Proposition 4.11, there exist transformations $T^{(1)}$ and $T^{(2)}$ such that $S_{T^{(1)}}$ and $S_{T^{(2)}}$ are not conjugate, while $P(S_{T^{(1)}}) = P(S_{T^{(2)}}) = \{1\}$. ∎

## 5. ENTROPY

A weak form of our conjecture would be to say that conjugacy of masas $S_T$ for actions of an abelian group $G$ implies coincidence of the entropies. In this form, the conjecture may hold without any assumptions on the spectrum, since systems with purely discrete spectrum have zero entropy. The main result of this section is a step towards the solution of this weaker problem. While in the previous section we proved that if the conjecture is false, then the isomorphism $\gamma : R_{T^{(1)}} \to R_{T^{(2)}}$ for non-isomorphic systems sends $C_{T^{(1)}}$ far from $C_{T^{(2)}}$, in this section we shall prove that if the entropies are distinct, the images $\gamma(v_g)$ of the canonical generators of $S_{T^{(1)}}$ cannot coincide with the generators of $S_{T^{(2)}}$ even on small projections.

We shall consider only the case $G = \mathbb{Z}$, since the theory of non-commutative entropy is not well developed for actions of general abelian (or amenable) groups, though in fact the result is true for arbitrary abelian $G$.

THEOREM 5.1. *Let $T^{(i)} \in \mathrm{Aut}(X_i, \mu_i)$ be a measure-preserving transformation, $i = 1, 2$. Denote by $v_i$ the canonical generator of $S_{T^{(i)}}$. Suppose there exists an isomorphism $\gamma : R_{T^{(1)}} \to R_{T^{(2)}}$ such that $\gamma(S_{T^{(1)}}) = S_{T^{(2)}}$, and the unitary $\gamma(v_1)v_2^*$ has an eigenvalue. Then $h(T^{(1)}) = h(T^{(2)})$.*

The result will follow from

PROPOSITION 5.2. *Let $T \in \mathrm{Aut}(X, \mu)$ be a measure-preserving transformation, $v \in S_T$ the canonical generator. Then for any non-zero projection $p \in S_T$ we have $H(\mathrm{Ad}\, v|_{pR_Tp}) = h(T)$, where $H(\mathrm{Ad}\, v|_{pR_Tp})$ is the entropy of Connes and Størmer* [CS] *of the inner automorphism* $\mathrm{Ad}\, v|_{pR_Tp}$ *computed with respect to the normalized trace $\tau_p = \tau(p)^{-1}\tau|_{pR_Tp}$.*

*Proof of Theorem* 5.1. By assumption, there exists $\theta \in \mathbb{T}$ such that the spectral projection $p$ of the unitary $\gamma(v_1)v_2^*$ corresponding to the set $\{\theta\}$ is non-zero. Then $\gamma(v_1)p = \theta v_2 p$. By Proposition 5.2, we get

$$h(T_1) = H(\mathrm{Ad}\, v_1|_{\gamma^{-1}(p)R_{T(1)}\gamma^{-1}(p)}) = H(\mathrm{Ad}\, \gamma(v_1)|_{pR_{T(2)}p})$$

$$= H(\mathrm{Ad}\, v_2|_{pR_{T(2)}p}) = h(T_2). \quad \blacksquare$$

To prove Proposition 5.2 consider the more general situation when we are given a finite injective von Neumann algebra $M$ with a fixed normal faithful trace $\tau$ and a $\tau$-preserving automorphism $\alpha$. For each projection $p$ in the fixed point algebra $M^\alpha$ we set

$$\tau_\alpha(p) = \tau(p)H(\alpha\,|_{pMp}).$$

PROPOSITION 5.3. *The mapping $p \mapsto \tau_\alpha(p)$ extends uniquely to a normal (possibly infinite) trace $\tau_\alpha$ on $M^\alpha$, which is invariant with respect to all $\tau$-preserving automorphisms in* $\mathrm{Aut}(M, M^\alpha)$ *commuting with $\alpha$.*

*Proof.* To prove that the mapping extends to a normal trace it is enough to check that the following three properties are satisfied: $\tau_\alpha(upu^*) = \tau_\alpha(p)$ for any unitary $u$ in $M^\alpha$, *if* $p_n \nearrow p$ then $\tau_\alpha(p_n) \nearrow \tau_\alpha(p)$, the mapping $p \mapsto \tau_\alpha(p)$ is finitely additive.

The first property is a particular case of the last statement of the proposition. If $\beta \in \mathrm{Aut}(M, M^\alpha)$ commutes with $\alpha$ and preserves the trace $\tau$, then it defines an isomorphism of the systems $(pMp, \tau_p, \alpha)$ and $(\beta(p)M\beta(p), \tau_{\beta(p)}, \alpha)$, so their entropies coincide.

The second property follows from the well-known continuity properties of entropy:

$$\tau_\alpha(p_n) = \tau(p_n)H(\alpha|_{p_nMp_n}) = \tau(p)H(\alpha|_{p_nMp_n+\mathbb{C}(p-p_n)}) \nearrow \tau(p)H(\alpha|_{pMp}) = \tau_\alpha(p).$$

To prove the third one consider a finite family $\{p_i\}_{i=1}^n$ of mutually orthogonal projections in $M^\alpha$ and set $p = \sum_i p_i$. Let

$$B = p_1Mp_1 + \cdots + p_nMp_n.$$

By affinity of entropy,

$$H(\alpha|_B) = \sum_i \frac{\tau(p_i)}{\tau(p)} H(\alpha|_{p_i M p_i}) = \tau(p)^{-1} \sum_i \tau_\alpha(p_i).$$

So in order to prove finite additivity it is enough to prove that $H(\alpha|_{pMp}) = H(\alpha|_B)$. The trace-preserving conditional expectation $E : pMp \to B$ has the form

$$E(x) = p_1 x p_1 + \cdots + p_n x p_n.$$

It commutes with $\alpha$ and is of finite index, $E(x) \geqslant \frac{1}{n} x$ for $x \in pMp$, $x \geqslant 0$. Indeed, if we consider $pMp$ acting on some Hilbert space, then for a vector $\xi$ we set $\xi_i = p_i \xi$ and get

$$(x\xi, \xi) = \sum_{i,j} (x^{1/2}\xi_i, x^{1/2}\xi_j) \leqslant \sum_{i,j} \|x^{1/2}\xi_i\| \cdot \|x^{1/2}\xi_j\| = \left( \sum_i \|x^{1/2}\xi_i\| \right)^2$$

$$\leqslant n \sum_i \|x^{1/2}\xi_i\|^2 = n \sum_i (x p_i \xi, p_i \xi) = n(E(x)\xi, \xi).$$

By [NS, Corollary 2], we conclude that $H(\alpha|_{pMp}) = H(\alpha|_B)$. ∎

*Proof of Proposition* 5.2. Consider the weight $\tau_{\mathrm{Ad}\, v}$ on $S_T$ corresponding to the automorphism $\mathrm{Ad}\, v$ of $R_T$. Then we have to prove that $\tau_{\mathrm{Ad}\, v} = h(T)\tau|_{S_T}$. By Proposition 5.3, the weight $\tau_{\mathrm{Ad}\, v}$ is invariant under the dual action. Since this action is ergodic on $S_T$, $\tau_{\mathrm{Ad}\, v}$ is a scalar multiple of $\tau|_{S_T}$, $\tau_{\mathrm{Ad}\, v} = c \cdot \tau|_{S_T}$ for some $c \in [0, +\infty]$. By definition of $T_{\mathrm{Ad}\, v}$ we have $c = H(\mathrm{Ad}\, v)$. But by [GN, Vo3], $H(\mathrm{Ad}\, v) = h(T)$, and the proof is complete. ∎

The definition of the weight above leads to the following interesting problem in entropy theory. Let $A$ be an abelian subalgebra of a finite algebra $M$. For each unitary $u \in A$ consider the weight $\tau_u$ on $A$, which is the restriction of the weight $\tau_{\mathrm{Ad}\, u}$ to $A$.

*Problem.* Find the connection between $\tau_u$ and $\tau_{\phi(u)}$, where $\phi$ is a Borel mapping from $\mathbb{T}$ onto itself.

Voiculescu's approach to entropy using norm of commutators [Vo1, Vo2] suggests that such a connection exists at least when $\phi$ is smooth. More interesting is the case when $u$ is a Haar unitary and $\phi$ is an invertible transformation preserving Lebesgue measure, so that $\phi(u)$ is again Haar and generates the same algebra. Note also some resemblance of this problem to the computation of entropy of Bogoliubov automorphisms [SV, N].

However, the correspondence $u \mapsto \tau_u$ does not have nice continuity properties which makes the problem more difficult.

Finally, note that the problems studied in the paper can also be considered for topological dynamical systems and $C^*$-crossed products. In this setting, isomorphism of crossed products already implies that the systems have a non-trivial relationship. For example, for minimal homeomorphisms of Cantor sets the crossed products are isomorphic if and only if the systems are strongly orbit equivalent [GPS]. Since rotations are the only measure- and orientation-preserving homeomorphisms of the circle, if $\gamma$ is an isomorphism of $C(X_1)\rtimes\mathbb{Z}$ onto $C(X_2)\rtimes\mathbb{Z}$ which maps $C^*(v_1)$ onto $C^*(v_2)$ then $\gamma(v_1)=\theta v_2^{\pm 1}$, so the homeomorphisms have the same topological entropy.

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## REFERENCES

[C] A. Connes, Almost periodic states and factors of type $\mathrm{III}_1$, *J. Funct. Anal.* **16** (1974), 415–445.

[CFW] A. Connes, J. Feldman, and B. Weiss, An amenable equivalence relation is generated by a single transformation, *Ergodic Theory Dynam. Systems* **1** (1981), 431–450.

[CS] A. Connes and E. Størmer, Entropy of automorphisms in $\mathrm{II}_1$ von Neumann algebras, *Acta Math.* **134** (1975), 289–306.

[CFS] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, "Ergodic Theory," Springer, New York, 1982.

[D] J. Dixmier, Sous-anneaux abéliens maximaux dans les facteurs de type fini, *Ann. of Math.* **59** (1954), 279–286.

[Dy] H. A. Dye, On groups of measure preserving transformations. II, *Amer. J. Math.* **85** (1963), 551–576.

[FM] J. Feldman and C. C. Moore, Ergodic equivalence relations, cohomology, and von Neumann algebras. II, *Trans. Amer. Math. Soc.* **234** (1977), 325–359.

[GPS] T. Giordano, I. F. Putnam, and C. F. Skau, Topological orbit equivalence and $C^*$-crossed products, *J. Reine Angew. Math.* **469** (1995), 51–111.

[GN] V. Ya. Golodets and S. V. Neshveyev, Entropy of automorphisms of $\mathrm{II}_1$-factors arising from the dynamical systems theory, *J. Funct. Anal.* **181** (2001), 14–28.

[H] P. Hahn, Reconstruction of a factor from measures on Takesaki's unitary equivalence relation, *J. Funct. Anal.* **31** (1979), 263–271.

[KL] J. Kwiatkowski and M. Lemanczyk, On the multiplicity function of ergodic group extensions. II, *Studia Math.* **116** (1995), 207–215.

[M] C. C. Moore, Group extensions and cohomology for locally compact groups. IV, *Trans. Amer. Math. Soc.* **221** (1976), 35–58.

[Na] M. G. Nadkarni, "Spectral Theory of Dynamical Systems," Birkhäuser, Basel, 1998.

[N] S. V. Neshveyev, Entropy of Bogoliubov automorphisms of CAR and CCR algebras with respect to quasi-free states, *Rev. Math. Phys.* **13** (2001), 29–50.

[NS] S. Neshveyev and E. Størmer, Entropy in type I algebras, *Pacific J. Math.* **201** (2001), 421–428.

[Ni] O. A. Nielsen, Maximal abelian subalgebras of hyperfinite factors. II, *J. Funct. Anal.* **6** (1970), 192–202.

[P1] J. A. Packer, Point spectrum of ergodic abelian group actions and the corresponding group-measure factors, *Pacific J. Math.* **119** (1985), 381–405.

[P2] J. A. Packer, On the embedding of subalgebras corresponding to quotient actions in group-measure factors, *Pacific J. Math.* **119** (1985), 407–443.

[Po1] S. Popa, Notes on Cartan subalgebras in type $\mathrm{II}_1$ factors, *Math. Scand.* **57** (1985), 171–188.

[Po2] S. Popa, An example of a property $\Gamma$ factor with countable fundamental group, preprint, 2001.

[Po3] S. Popa, On a class of type $\mathrm{II}_1$ factors with Betti numbers invariants, preprint, 2001.

[P] L. Pukanszky, On maximal abelian subrings of factors of type $\mathrm{II}_1$, *Canad. J. Math.* **12** (1960), 289–296.

[SS] A. Sinclair and R. Smith, Strongly singular masas in type $\mathrm{II}_1$ factors, preprint, 2001.

[S] S. Ştrătilă, "Modular Theory in Operator Algebras," Abacus Press, Tunbridge Wells, 1981.

[SV] E. Størmer and D. Voiculescu, Entropy of Bogoliubov automorphisms of the canonical anti-commutation relations, *Comm. Math. Phys.* **133** (1990), 521–542.

[Vo1] D. Voiculescu, Entropy of dynamical systems and perturbations of operators, *Ergodic Theory Dynam. Systems* **11** (1991), 779–786.

[Vo2] D. Voiculescu, Entropy of dynamical systems and perturbations of operators. II, *Houston J. Math.* **17** (1991), 651–661.

[Vo3] D. Voiculescu, Dynamical approximation entropies and topological entropy in operator algebras, *Comm. Math. Phys.* **170** (1995), 249–281.