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Generalized Euler-Type Partition Identities

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Let A and S be subsets of the natural numbers. Let $A'(n)$ be the number of partitions of n where each part appears exactly m times for some $m \in A$. Let $S(n)$ be the number of partitions of n into parts which are elements of S.

DEFINITION. A is *admissible* if there exists an S such that $A'(n) = S(n)$ for every n. If S exists, then S *corresponds* to A.

THEOREMS. (1) *A corresponds to itself if and only if $A = \{ic\}_{i=1}^{\infty}$ for some positive integer c.* (2) *Let k and d be positive integers. Then A is admissible if:*

$$A = \{m \mid m \neq (2i-1)k + j,\ 1 \leqslant i,\ 0 \leqslant j \leqslant k-1\}$$

or $$A = \{m \mid m \neq (2i-1)k + j,\ 1 \leqslant i \leqslant d,\ 0 \leqslant j \leqslant k-1\}$$

or $$A = \{m \mid m \neq (2i-1)k + j,\ 1 \leqslant i \leqslant d,\ 0 \leqslant j \leqslant k-1, \text{ and } m < (2d+1)k\}.$$

These results provide generalizations of theorems by Euler, Glaisher, and Andrews. The proofs extend the method used to prove Glaisher's theorem and are based on showing that the corresponding generating functions are identical.

1. INTRODUCTION

Let A and S be subsets of the natural numbers. Let $A'(n)$ be the number of partitions of n where each part appears exactly m times for some $m \in A$. For example, if $1, 2 \in A$, then $4 + 3 + 3$ is a partition of 10 enumerated by A'. Let $S(n)$ be the number of partitions of n into parts which are elements of S.

This paper investigates the problem: Given a set A, does there exist a set S such that $A'(n) = S(n)$ for every n? If such a set exists, we will say that A is *admissible* and that S *corresponds* to A.

Two results of this type state $A = \{1\}$ and $A = \{i\}_{i=2}^{\infty}$ are admissible sets. More specifically, the following two theorems are known [3, 2].

THEOREM (Euler). *The number of partitions of n, where each part appears exactly once, is equal to the number of partitions of n into odd parts.*

THEOREM (Andrews). *The number of partitions of n, where each part appears at least twice, is equal to the number of partitions of n into parts which are* $\not\equiv \pm 1 \pmod 6$.

The main purpose of this paper is to generalize these two results, i.e., to describe two classes of admissible sets. Subsequent papers will give other classes of admissible sets and some classes of inadmissible sets.

2. RESULTS

The two main theorems (Theorems 1 and 2) state that A is admissible if it is of one of the following two forms:

$$A = \{m \mid m \neq (2i-1)k + j,\ 1 \leqslant i \leqslant d,\ 0 \leqslant j \leqslant k-1, \text{ and } m < (2d+1)k\},$$

or

$$A = \{m \mid m \neq (2i-1)k + j,\ 1 \leqslant i \leqslant d,\ 0 \leqslant j \leqslant k-1\}.$$

THEOREM 1. *Let $k \geqslant 1$ and $d \geqslant 0$ be integers. Then the number of partitions of n, where no part appears exactly $(2i-1)k + j$ times with $1 \leqslant i \leqslant d$, $0 \leqslant j \leqslant k-1$ or more than $(2d+1)k - 1$ times, is equal to the number of partitions of n into parts which are $\not\equiv 0, (2i-1)k \pmod{(2d+2)k}$ with $1 \leqslant i \leqslant d+1$.*

For $k = 2$ and $d = 0$, this reduces to Euler's theorem. In fact, for $d = 0$ and $k > 1$, this reduces to Glaisher's theorem [4].

THEOREM (Glaisher). *The number of partitions of n, where no part appears more than $k - 1$ times, is equal to the number of partitions of n into parts which are $\not\equiv 0 \pmod k$.*

Letting $k = 1$ in Theorem 1, we obtain the following result.

COROLLARY 1. *The number of partitions of n, where parts appear 2, 4,..., or $2d$ times, is equal to the number of partitions of n into parts which are $\equiv 2, 4,\ldots, 2d \pmod{2d+2}$.*

THEOREM 2. *Let k and d be positive integers. Then the number of partitions of n, where no part appears exactly $(2i-1)k + j$ times with $1 \leqslant i \leqslant d$, $0 \leqslant j \leqslant k-1$, is equal to the number of partitions of n into parts which are $\not\equiv \pm(2i-1)k \pmod{(4d+2)k}$ with $1 \leqslant i \leqslant d$.*

For $k = 1$ and $d = 1$, this reduces to Andrews' theorem. For $k = 1$ and $d \geqslant 1$, we obtain a result similar to Corollary 1. For $d = 1$ and $k \geqslant 1$, we get the following result.

COROLLARY 2. *The number of partitions of n, where no part appears exactly k, $k+1,\ldots,$ or $2k-1$ times, is equal to the number of partitions of n into parts which are* $\not\equiv \pm k \pmod{6k}$.

Removing the restriction that d be finite in Theorem 2, we obtain an additional result.

THEOREM 3. *Let k be a positive integer. Then the number of partitions of n, where no part appears exactly $(2i-1)k+j$ times with $i \geqslant 1$, $0 \leqslant j \leqslant k-1$, is equal to the number of partitions of n into parts which are* $\not\equiv k \pmod{2k}$.

Letting $k = 1$ in Theorem 3, we note that if A is the set of even natural numbers, then A is not only admissible, but the corresponding set is A itself. In fact, this last result may be generalized independently to yield the following: A set A corresponds to itself if and only if it consists of all multiples of some positive integer.

THEOREM 4. *The number of partitions of n, where each part appears exactly m times for some $m \in A$, is equal to the number of partitions of n into parts which are elements of A if and only if $A = \{ic\}_{i=1}^{\infty}$ for some positive integer c.*

3. Proofs of Theorems

Proof of Theorem 1. For $d = 0$, this is Glaisher's theorem, so we may assume $d \geqslant 1$. In this case, the generating function for $A'(n)$ is given by

$$\prod_{n=1}^{\infty}\left[\frac{1}{1-x^n} - \sum_{i=1}^{d}\sum_{j=0}^{k-1} x^{((2i-1)k+j)n} - \sum_{j=(2d+1)k}^{\infty} x^{jn}\right]$$

$$= \prod_{n=1}^{\infty}\left[\frac{1}{1-x^n} - \left(\sum_{i=1}^{d} x^{(2i-1)kn}\right)\left(\sum_{j=0}^{k-1} x^{jn}\right) - \sum_{j=(2d+1)k}^{\infty} x^{jn}\right]$$

$$= \prod_{n=1}^{\infty}\left[\frac{1}{1-x^n} - \left(\frac{x^{kn} - x^{(2d+1)kn}}{1-x^{2kn}}\right)\left(\frac{1-x^{kn}}{1-x^n}\right) - \frac{x^{(2d+1)kn}}{1-x^n}\right]$$

$$= \prod_{n=1}^{\infty}\left(\frac{1}{1-x^n}\right)\left[1 - x^{kn}\left(\frac{1-x^{2dkn}}{1+x^{kn}}\right) - x^{(2d+1)kn}\right]$$

$$= \prod_{n=1}^{\infty} \left(\frac{1}{1-x^n}\right)\left(\frac{1}{1+x^{kn}}\right)$$
$$\times [(1+x^{kn}) - x^{kn}(1-x^{2dkn}) - x^{(2d+1)kn}(1+x^{kn})]$$
$$= \prod_{n=1}^{\infty} \left(\frac{1}{1-x^n}\right)\left(\frac{1}{1+x^{kn}}\right)(1-x^{(2d+2)kn})$$
$$= \prod_{n=1}^{\infty} \left(\frac{1}{1-x^n}\right)\left(\frac{1-x^{kn}}{1-x^{2kn}}\right)(1-x^{(2d+2)kn})$$
$$= \prod_{n=1}^{\infty} \left(\frac{1}{1-x^n}\right)(1-x^{k(2n-1)})(1-x^{(2d+1)kn}),$$

which is the generating function for the number of partitions into parts which are $\not\equiv k \pmod{2k}$ and $\not\equiv 0 \pmod{(2d+2)k}$, i.e., $\not\equiv 0, (2i-1)k \pmod{(2d+2)k}$ with $1 \leqslant i \leqslant d+1$.

Proof of Theorem 2. The generating function for $A'(n)$ is given by

$$\prod_{n=1}^{\infty} \left[\frac{1}{1-x^n} - \sum_{i=1}^{d}\sum_{j=0}^{k-1} x^{((2i-1)k+j)n}\right]$$
$$= \prod_{n=1}^{\infty} \left[\frac{1}{1-x^n} - \left(\sum_{i=1}^{d} x^{(2i-1)kn}\right)\left(\sum_{j=0}^{k-1} x^{jn}\right)\right]$$
$$= \prod_{n=1}^{\infty} \left[\frac{1}{1-x^n} - \left(\frac{x^{kn}-x^{(2d+1)kn}}{1-x^{2kn}}\right)\left(\frac{1-x^{kn}}{1-x^n}\right)\right]$$
$$= \prod_{n=1}^{\infty} \left(\frac{1}{1-x^n}\right)\left(\frac{1}{1+x^{kn}}\right)[(1+x^{kn}) - x^{kn}(1-x^{2dkn})]$$
$$= \prod_{n=1}^{\infty} \left(\frac{1}{1-x^n}\right)\left(\frac{1}{1+x^{kn}}\right)(1+x^{(2d+1)kn})$$
$$= \prod_{n=1}^{\infty} \left(\frac{1}{1-x^n}\right)\left(\frac{1-x^{kn}}{1-x^{2kn}}\right)\left(\frac{1-x^{(2d+1)2kn}}{1-x^{(2d+1)kn}}\right)$$
$$= \prod_{n=1}^{\infty} \left(\frac{1}{1-x^n}\right)\left(\frac{1-x^{k(2n-1)}}{1-x^{(2d+1)k(2n-1)}}\right),$$

which is the generating function for the number of partitions into parts which are $\not\equiv k \pmod{2k}$ or are $\equiv (2d+1)k \pmod{(4d+2)k}$; i.e., $\not\equiv \pm(2i-1)k \pmod{(4d+2)k}$, with $1 \leqslant i \leqslant d$.

Proof of Theorem 3. The generating function for $A'(n)$ is given by

$$\prod_{n=1}^{\infty}\left[\frac{1}{1-x^n}-\sum_{i=1}^{\infty}\sum_{j=0}^{k-1}x^{((2i-1)k+j)n}\right]$$

$$=\prod_{n=1}^{\infty}\left[\frac{1}{1-x^n}-\left(\sum_{i=1}^{\infty}x^{(2i-1)kn}\right)\left(\sum_{j=0}^{k-1}x^{jn}\right)\right]$$

$$=\prod_{n=1}^{\infty}\left[\frac{1}{1-x^n}-\left(\frac{x^{kn}}{1-x^{2kn}}\right)\left(\frac{1-x^{kn}}{1-x^n}\right)\right]$$

$$=\prod_{n=1}^{\infty}\left(\frac{1}{1-x^n}\right)\left(1-\frac{x^{kn}}{1+x^{kn}}\right)$$

$$=\prod_{n=1}^{\infty}\left(\frac{1}{1-x^n}\right)\left(\frac{1}{1+x^{kn}}\right)$$

$$=\prod_{n=1}^{\infty}\left(\frac{1}{1-x^n}\right)\left(\frac{1-x^{kn}}{1-x^{2kn}}\right)$$

$$=\prod_{n=1}^{\infty}\left(\frac{1}{1-x^n}\right)(1-x^{k(2n-1)}),$$

which is the generating function for the number of partitions into parts $\not\equiv k \pmod{2k}$.

Proof of Theorem 4. First we show that if A has the given form, then it corresponds to itself. Let $P(n)$ be the number of partitions of n, and let $A=\{ic\}_{i=1}^{\infty}$.

If $c \nmid n$, then $A(n)=0=A'(n)$. If $c \mid n$, then $A(n)=P(n/c)=A'(n)$, since there are obvious 1–1 correspondences between partitions counted by $A(n)$ and partitions of n/c, and those of the latter with those counted by $A'(n)$. Thus, $A'(n)=A(n)$ for every n.

Now assume that we have some set A such that $A'(n)=A(n)$ for every n. We must show that A has the desired form, i.e., if $A=\{c_i\}_{i=1}^{s}$ with $c_1<c_2<\ldots$, we must show that $s=\infty$ and that there is some c such that $c_i=ic$ for every i. The proof will be by induction.

Let $P_m(n)$ be the number of partitions of n into parts which are less than or equal to m. Let $Q_m(n)$ be the number of partitions of n into at most m parts. Then $P_m(n)=Q_m(n)$ for every m and n.

Let $c=c_1$ and assume $c_i=ic$ for $1\leqslant i\leqslant m<s$. We need to show that $c_{m+1}=(m+1)c$. Consider $n=c_1+c_{m+1}$. If $c\nmid c_{m+1}$, then $A(n)=1$ or 2 (2 if $n\in A$), since $c\nmid n$ implies that $n=c_1+c_{m+1}$ is the only such partition of n using more than one part. But $A'(n)=0$ or 1 (1 if $n\in A$),

since $c \nmid n$ implies that some part must appear at least c_{m+1} times. Such a part would have to be 1, in which case the partition would consist of all 1's, and $n \in A$. Therefore, if $c \nmid c_{m+1}$, then $A(n) > A'(n)$. Thus, $c \mid c_{m+1}$, so let $c_{m+1} = kc$, where $k > m$.

Now $A(c_{m+1}) = 1 + P_m(k)$, since $c_{m+1} = c_{m+1}$ is the only such partition of c_{m+1} which is not of the form $c_{m+1} = \sum_{i=1}^{m} t_i(c_i)$, and $kc = \sum_{i=1}^{m} t_i(c_i)$ if and only if $k = \sum_{i=1}^{m} t_i(i)$, since $c_i = ic$. $A'(c_{m+1}) \geqslant 1 + Q_m(k)$, since $c_{m+1} = 1 + 1 + \cdots + 1$, and if $k = \sum_j t_j(e_j)$, where $\sum_j t_j \leqslant m$, then $t_j \leqslant m$ for each j, so that $c_{m+1} = ck = \sum_j ct_j(e_j) = \sum_j c_{t_j}(e_j)$, which is a partition of c_{m+1} of the type enumerated by $A'(c_{m+1})$. Additionally, if $k > m + 1$, then $k = (k - m) + 1 + 1 + \cdots + 1$, which has $m + 1$ parts. And $c_{m+1} = (k - m) + (k - m) + \cdots + (k - m) + 1 + \cdots + 1$, where $(k - m)$ appears c_1 times and 1 appears c_m times. Therefore, if $k > m + 1$, then $A'(c_{m+1}) > 1 + Q_m(k) = 1 + P_m(k) = A(c_{m+1})$. Thus, $k = m + 1$ and $c_{m+1} = (m + 1)c$.

Now suppose $s < \infty$, i.e., $A = \{c, 2c, \ldots, sc\}$. Consider $n = (s + 2)c$. Here $A(n) = P_s(s + 2)$. But $A'(n) \geqslant 1 + Q_s(s + 2)$, since if $s + 2 = \sum_j t_j(e_j)$, where $\sum_j t_j \leqslant s$, then $t_j \leqslant s$ for each j, so that $n = (s + 2)c = \sum_j ct_j(e_j) = \sum_j c_{t_j}(e_j)$, which is a partition of the type enumerated by $A'(n)$. Additionally, $s + 2 = 2 + 1 + 1 + \cdots + 1$, which has $s + 1$ parts. So $n = (s + 2)c = 2 + 2 + \cdots + 2 + 1 + 1 + \cdots + 1$, where 2 appears c_1 times and 1 appears c_s times. Therefore, if $s < \infty$, then $A'(n) > A(n)$. So $s = \infty$ and $A = \{ic\}_{i=1}^{\infty}$.

References

1. H. L. Alder, Partition identities—from Euler to the present, *Amer. Math. Monthly* **76** (1969), 733–746.
2. G. E. Andrews, "Number Theory," Saunders, Philadelphia, PA, 1971.
3. L. Euler, Introductio analysin infinitorium, *Lausanne* **1** (1748), 253–275.
4. J. W. L. Glaisher, *Messenger of Mathematics* **12** (1883), 158–170.