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Computation of Affine Covariants of Quadratic Bivariate Differential Systems

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This paper deals with affine covariants of autonomous differential systems. The main result is the construction of a minimal system of generators of the algebra of affine covariants of quadratic bivariate differential systems which is helpful in qualitative and numerical study. To this end, we establish a theorem (true for general systems of dimension n and degree m) which provides a procedure of construction of systems of generators for affine covariants from those of center-affine

Key Words: nonlinear differential systems; transformations of differential systems; classical invariant theory; classical groups; covariants.

1. MOTIVATIONS

Let k be a field of characteristic zero, V the k-vector space k^n, and $\mathscr{A}(n, m)$ the set of autonomous differential systems

$$\frac{dx^j}{dt} = \sum_{k=0}^{m} \sum_{\alpha_1=1}^{n} \cdots \sum_{\alpha_k=1}^{n} a^j_{\alpha_1\alpha_2\cdots\alpha_k} x^{\alpha_1} x^{\alpha_2} \cdots x^{\alpha_k}, \qquad j \in \{1, ..., n\}, \tag{1}$$

where $x = {}^T(x^1, x^2, ..., x^n) \in V$ is represented with superscript indices and coefficients $a^j_{\alpha_1\alpha_2\cdots\alpha_k}$ belonging to k. The time t may be complex or real. The right hand side of (1) is an n-tuple of polynomials of degree at most m.

This paper is motivated by different applications of classical invariant theory to the study of the differential systems (1) [9, 13] such as the characterization of each group orbit (i.e., normal forms), the description of particular trajectories (singularities, closed trajectories, etc.), the computation of particular and first algebraic integrals, and the existence and the number of the symmetry axes of vector fields.

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The idea to systematically use the classical invariant theory in qualitative theory of autonomous differential systems is due to C. S. Sibirskii [8, 9].

Many works [8–10] have been devoted to the invariants of differential systems with respect to the group of invertible matrices called the center-affine group. Systems of generators of invariants are constructed and used with great success.

From the point of view of symbolic computation, this approach is fruitful for the following reason: it allows geometric properties of trajectories to be expressed with the help of algebraic and/or semialgebraic relations depending on the coefficients of entries (given systems) and therefore the computations are exact.

Other motivations for this work could be related to invariant theory. It is well known that if a group is reductive, the algebras of invariants of its representations are finitely generated. Several people try to determine the upper bound of degrees of generators of a given algebra of invariants: see [2] for a complete and interesting description of this problem. However, until now, there is no constructive method that gives a good bound [2]. Only some families of algebraic forms and matrices provide good examples of exact upper bounds of degrees of generators. The two-dimensional quadratic systems constitute another example for the center-affine and affine groups (see Section 3 for more details).

2. INTRODUCTION AND NOTATIONS

The space of algebraic forms of degree k can be looked upon as the quotient space of $V^{*\otimes k}$ by the subspace generated by all elements

$$x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \cdots \otimes x_k - x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \cdots \otimes x_k.$$

We denote this space by $\mathscr{S}_k$. The homogeneous part of degree k of the polynomial vector field (1) could be identified with the vector space $S_k \otimes V^*$ denoted by $\mathscr{S}^1_k$. For example, $\mathscr{S}^1_1$ is the linear part of (1). In tensorial language, $\mathscr{S}^1_k$ is the space of tensors once contravariant and k times covariant which are symmetric with respect to the subscript indices.

Consequently,

$$\mathscr{A}(n, m) \simeq S^1_0 \oplus S^1_1 \oplus \cdots \oplus S^1_m.$$

Let G be a linear group acting rationally on a finite-dimensional vector space $\mathscr{W}$, $GL(\mathscr{W})$ the group of automorphisms of $\mathscr{W}$, and

$$\rho\colon G \mapsto GL(\mathscr{W})$$

the corresponding rational representation. Let $k[\mathscr{W}]$ be the algebra of polynomials whose indeterminates are the coordinates of a generic vector of $\mathscr{W}$.

DEFINITION 1. A polynomial function $K \in k[\mathscr{W}]$ is said to be a G-invariant of $\mathscr{W}$ if there exists a character of the group G, denoted λ, such that

$$\forall g \in G, \qquad K \circ \rho(g) = \lambda(g).K.$$

Here, the character of the group G is a rational (commutative) morphism of the group G into $k^\star$ where $k^\star$ is the multiplicative group of the field k. If $\lambda(g) \equiv 1$, then the invariant is said to be absolute. Otherwise, it is said to be relative.

In our situation, $\mathscr{W}$ should be $\mathscr{A}(n, m)$ or $\mathscr{A}(n, m) \times V$ and the group G should be one of the following classical groups:

1. $Gl(n)$: a group of center-affine transformations or invertible matrices,
2. $T(n)$: a group of translations,
3. $Aff(n) = T(n) \ltimes Gl(n)$: a semidirect group of affine transformations.

In this work, the notion of a covariant is taken in the following precise sense:

DEFINITION 2. A G-covariant of $\mathscr{A}(n, m)$ is a G-invariant of $\mathscr{A}(n, m) \times V$.

For example, $\det(x, Ax, A^2x, ..., A^{n-1}x)$ where A is the linear part of (1) is a relative $Gl(n)$-covariant of $\mathscr{A}(n, m)$.

Because we have a great deal with tensors, through this paper we shall use the Einstein summation convention (for instance, $a^j_{\alpha_1\alpha_2} a^{\alpha_1}_{j\alpha_2}$ stands for the same as $\sum_{j=1}^n \sum_{\alpha_1=1}^n a^j_{\alpha_1\alpha_2} a^{\alpha_1}_{j\alpha_2}$).

Using this notation, the system (1) becomes

$$\frac{dx^j}{dt} = a^j + a^j_{\alpha_1} x^{\alpha_1} + a^j_{\alpha_1\alpha_2} x^{\alpha_1} x^{\alpha_2} + \cdots + a^j_{\alpha_1\alpha_2\cdots\alpha_m} x^{\alpha_1} x^{\alpha_2} \cdots x^{\alpha_m}, \tag{2}$$

$$j, \alpha_1, \alpha_2, ..., \alpha_m \in \{1, ..., n\}.$$

The transformations $(x^i) \mapsto (Q^i_j x^j)$, $(x^i) \mapsto (x^i - p^i)$, $(x^i) \mapsto (Q^i_j x^j - p^i)$ of the above groups $Gl(n)$ of such systems. $T(n)$ and $Aff(n)$ transform each system (1) of $\mathscr{A}(n, m)$ into the system of $\mathscr{A}(n, m)$ defined by the formula:

$$[\rho_1(P)(a)]^j_{\alpha_1\alpha_2\cdots\alpha_k} = Q^j_i P^{\beta_1}_{\alpha_1} P^{\beta_2}_{\alpha_2} \cdots P^{\beta_k}_{\alpha_k} a^i_{\beta_1\beta_2\cdots\beta_k}, \tag{3}$$

$$[\rho_2(p)(a)]^j_{\alpha_1\alpha_2\cdots\alpha_k} = \sum_{i=0}^{m-k} \binom{k+i}{i} a^j_{\alpha_1\alpha_2\cdots\alpha_k\beta_1\beta_2\cdots\beta_i} p^{\beta_1} p^{\beta_2} \cdots p^{\beta_i}, \tag{4}$$

$$[\rho_3(P,p)(a)]^j_{\alpha_1\alpha_2\cdots\alpha_k} = \sum_{i=0}^{m-k} \binom{k+i}{i} Q^j_l P^{\gamma_1}_{\alpha_1} P^{\gamma_2}_{\alpha_2} \cdots P^{\gamma_k}_{\alpha_k}$$
$$a^l_{\gamma_1\gamma_2\cdots\gamma_k\beta_1\beta_2\cdots\beta_i} p^{\beta_1} p^{\beta_2} \cdots p^{\beta_i}, \tag{5}$$

where P is the inverse of the matrix Q (or the linear part of g) of the corresponding transformation and $\binom{k+i}{i}$ is the binomial coefficient $\frac{(k+i)!}{k!i!}$.

In the above definition, when G is one of the groups $Gl(n)$, $T(n)$, or $Aff(n)$, it is well known [3, 9] that the character λ is equal to $\det(g)^{-\kappa}$ where the integer κ is called the weight of the invariant K. Clearly, every relative invariant of $Gl(n)$ is an absolute invariant of $Sl(n)$. The converse is also true (for homogeneous polynomials). This allows us to consider the set of $Gl(n)$-covariants (respectively invariants) of $\mathscr{A}(n,m)$ as a k-algebra, denoted by $\mathscr{K}(n,m)$ (respectively $\mathscr{I}(n,m)$). Similarly, the set of $Aff(n)$-covariants (respectively $Aff(n)$-invariants) of $\mathscr{A}(n,m)$ is a k-algebra, denoted $\mathscr{Q}(n,m)$ (respectively $\mathscr{J}(n,m)$). It is obvious that the inclusion $Gl(n) \subset Aff(n)$ implies the relations

$$\mathscr{J}(n,m) \subset \mathscr{I}(n,m) \qquad \text{and} \qquad \mathscr{Q}(n,m) \subset \mathscr{K}(n,m).$$

Let us return to the general case of a group G and a vector space $\mathscr{W}$. Denote by $k[\mathscr{W}]^G$ the algebra of $Gl(n)$-invariants of $\mathscr{W}$. According to V. L. Popov [6], the main problem of the classical theory of invariants is to describe explicitly the algebras $k[\mathscr{W}]^G$. The idea of the description is as follows:

1. See whether $k[\mathscr{W}]^G$ has a finite system of generators;

2. if it does, give a constructive method for finding a minimal system (ideally, find it explicitly) of generators of $\mathscr{C}[\mathscr{W}]^G$.

For the reductive groups G (such as $Gl(n)$ [2], this fact is known from Hilbert) the first problem is positively solved. However, this is not the case for the affine group.

The present work contains three parts. In the first one (Section 3) we recall some general facts about the invariant theory and the difficulties arising in the knowledge of the algebras of the invariants (or covariants). In the second part (Section 4), we prove a result (2) which provides an isomorphism between the algebras $\mathscr{I}(n,m)$ and $\mathscr{Q}(n,m)$ and a systematic method to construct systems of generators of $\mathscr{Q}(n, m)$ from those of $\mathscr{I}(n, m)$. We apply it to the case $n=m=2$ because a minimal system of generators of center-affine invariant already exists. The goal of the third part (Sections 5 and 6) is to express the generators of $\mathscr{Q}(n, m)$ with the help of the generators of $\mathscr{K}(n, m)$. To this aim, we need to compute a minimal system of generators of the center-affine covariants and after that, we will use the previous theorem. All algorithms constitute the package SIB.

3. RECALLS ABOUT CENTER-AFFINE INVARIANTS

In this section we are interested in center-affine invariants and covariants of $\mathscr{A}(n, m)$. As a $Gl(n)$-module, $\mathscr{A}(n, m)$ is a direct sum of the subspaces S_k^1 with $k \in \{0, 1, 2, ..., m\}$.

The algebra $\mathscr{K}(n, m)$ is multigraded,

$$\mathscr{K}(n, m) = \bigoplus_{n_0, n_1, n_2, ..., n_m, p \in \mathbb{N}} K(n_0, n_1, ..., n_m, p),$$

where $K(n_0, n_1, ..., n_m, p)$ is the subalgebra of multihomogeneous covariants of multidegree $(n_0, n_1, ..., n_m, p)$, i.e., homogeneous of degree n_k with respect to coordinates of $(a^j_{i_1, i_2, ..., i_k})$ and of degree p with respect to coordinates of $x \in V$.

In the same way, we decompose $\mathscr{I}(n, m)$ as follows:

$$\mathscr{I}(n, m) = \bigoplus_{n_0, n_1, n_2, ..., n_m \in \mathbb{N}} I(n_0, n_1, ..., n_m).$$

Before giving the fundamental theorem of center-affine invariants, let us recall the definitions of two fundamental operations over tensors which are the contraction and the alternation.

In the following definitions, we identify each element of a vector space with its coordinates in a selected basis.

Definition 3. A contraction over the tensorial space $V^{\otimes p} \otimes V^{*\otimes q}$ is the linear map

$$\varphi: V^{\otimes p} \otimes V^{*\otimes q} \mapsto V^{\otimes (p-1)} \otimes V^{*\otimes (q-1)}$$

defined by

$$\varphi(v)^{j_1\cdots j/_l\cdots j_p}_{i_1\cdots i/_k\cdots i_q}=\sum_{h=1}^{n} v^{j_1\cdots j_{l-1}hj_{l+1}\cdots j_p}_{i_1\cdots i_{k-1}hi_{k+1}\cdots i_q}.$$

If $p=q$, a sequence of contractions

$$V^{\otimes p}\otimes V^{*\otimes p}\overset{\varphi_1}{\mapsto} V^{\otimes(p-1)}\otimes V^{*\otimes(p-1)}\overset{\varphi_2}{\mapsto}$$
$$\cdots\ \varphi_p\mapsto V\otimes V^{*}\xmapsto{\varphi_{p+1}} k[V^{\otimes p}\otimes V^{*\otimes q}]$$

is called a complete contraction.

For each pair of natural numbers (l, l') with $1\leqslant l\leqslant p$, $1\leqslant l''\leqslant q$ we get one contraction.

DEFINITION 4. A contravariant (respectively covariant) n-vector is an element of $V^{\otimes n}$ (respectively $V^{*\otimes n}$) whose coordinates $(\varepsilon^{i_1i_2\cdots i_n})$ $(\varepsilon_{i_1i_2\cdots i_n})$ are defined by

$$\varepsilon^{i_1i_2\cdots i_n}=\varepsilon_{i_1i_2\cdots i_n}=\begin{cases}1 & \text{if}\quad (i_1i_2\cdots i_n)\text{ is an even permutation of }(12\cdots n)\\ -1 & \text{if}\quad (i_1i_2\cdots i_n)\text{ is an odd permutation of }(12\cdots n)\\ 0 & \text{otherwise.}\end{cases}$$

DEFINITION 5. A contravariant (covariant) alternation over the tensorial space $V^{\otimes p}\otimes V^{*\otimes q}$ is the map

$$\varphi\colon V^{\otimes p}\otimes V^{*\otimes q}\mapsto V^{\otimes(p)}\otimes V^{*\otimes(q-n)}$$
$$(\varphi\colon V^{\otimes p}\otimes V^{*\otimes q}\mapsto V^{\otimes(p-n)}\otimes V^{*\otimes(q)})$$

that is defined by

$$\varphi(v)^{j_1\cdots j_p}_{i_1\cdots h/_1\cdots h/_2\cdots h/_n\cdots i_q}=\sum_{h_1=1}^{n}\cdots\sum_{h_n=1}^{n} v^{j_1\cdots j_p}_{i_1\cdots h_1\cdots h_2\cdots h_n\cdots i_q}\varepsilon^{h_1\cdots h_n}$$
$$\left(\varphi(v)^{j_1\cdots h/_1\cdots h/_2\cdots h/_n\cdots j_p}_{i_1\cdots i_q}=\sum_{h_1=1}^{n}\cdots\sum_{h_n=1}^{n} v^{j_1\cdots h_1\cdots h_2\cdots h_n\cdots j_p}_{i_{\text{of such systems}.1}\cdots i_q}\varepsilon_{h_1\cdots h_n}\right).$$

Now, we are able to give the fundamental theorem of the classical theory of invariants:

THEOREM 1 [4, pp. 188–189]. *The expressions obtained with the help of successive alternations and complete contraction over the tensorial products*

$$(S_0^1)^{\otimes n_0} \otimes (S_1^1)^{\otimes n_1} \otimes \cdots \otimes (S_m^1)^{\otimes n_m} \otimes V^{\otimes r} \subset V^{*\otimes p} \otimes V^{\otimes q}$$

with $p = (n_0 + n_1 + \cdots + n_m + r)$ *and* $q = (n_1 + 2n_2 + \cdots + mn_m)$ *form a system of generators of* $K(n_0, n_1, ..., n_m, r)$.

Such polynomials are called basic covariants.

EXAMPLE. For instance, if $n = m = 2$, the polynomials $a_p^\alpha a_{aq}^\beta a_{\beta\gamma}^\gamma \varepsilon^{pq}$ and $a_\gamma^\alpha a_\delta^\beta a_{\alpha\beta}^\gamma x^\delta$ belong respectively to $I(0, 1, 2, 0)$ and $K(0, 2, 1, 1)$. We shall come back to this case.

In order to have a complete contraction, the exponents $(n_0, n_2, ..., n_m, r)$ have to verify the relation

$$n_0 + r - (n_2 + 2n_3 + \cdots (m-1) n_m) = sn \qquad \text{with} \qquad s \qquad \text{integer.}$$

The above theorem is known as the fundamental theorem of the classical theory of invariants [4]. It can be presented with the contractions and determinants (alternations) [3, 12] of contravariant and covariant vectors obtained from the symbolic decomposition of the tensors of S_k^1 and V. It defines a process of construction of covariants of $\mathscr{A}(n, m)$, degree by degree, and gives for each suitable choice of alternations and contractions an element of $k[\mathscr{A}(n, m) \times V]$.

To get a minimal system of generators we need to know an upper bound of degrees of these generators which we denote by $\beta(n, m)$. This bound has been calculated by V. Popov [7] and recently improved by H. Derksen [2]. From our point of view, it is still too large for concrete examples. Indeed, let $\sigma(2, 2)$ be the smallest integer d with the following property: if $a \in \mathscr{A}(2, 2)$ and 0 does not lie in the Zariski closure of the $GL(2, k)$-orbit of a, then there exists a nonconstant homogeneous invariant I of degree $\leqslant d$ such that $I(a) \neq 0$. In the case of planar quadratic differential systems (the list of a minimal system of generators of $\mathscr{I}(2, 2)$ is given in Section 5 and the system of generators of the ideal of syzygies is given in [9]), $\sigma(2, 2) = 6$ and $\dim(\mathscr{I}(2, 2)) = 9$. Following Popov,

$$\beta(2, 2) \leqslant 12.LCM(1, 2, ..., \sigma(2, 2)) = 12 \times 60 = 720,$$

where LCM is the least common multiple. Following Derksen,

$$\beta(2, 2) \leqslant \max(\sigma(2, 2), \tfrac{3}{8} \dim(\mathscr{I}(2, 2))\, \sigma(2, 2)^2) = \tfrac{243}{2}.$$

We should not forget that the dimension of the linear space $\mathscr{A}(2, 2)$ is 12 and so computations over polynomials depending on 12 indeterminates quickly become complicated.

Truly, $\beta(2, 2) = 7$.

Let us return to the computation of center-affine covariants. Actually, it is sufficient to consider the basic covariants with either the contravariant n-vectors or the covariant n-vectors. Indeed, by following relations between the coordinates of n-vectors and those of Kronecker's symbols

$$\varepsilon^{i_1 i_2 \cdots i_n} \varepsilon_{j_1 j_2 \cdots j_n} = \begin{vmatrix} \delta^{i_1}_{j_1} & \delta^{i_1}_{j_2} & \cdots & \delta^{i_1}_{j_n} \\ \delta^{i_2}_{j_1} & \delta^{i_2}_{j_2} & \cdots & \delta^{i_2}_{j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{i_n}_{j_1} & \delta^{i_n}_{j_2} & \cdots & \delta^{i_n}_{j_n} \end{vmatrix}, \tag{6}$$

any basic covariant can be reduced to the sum of basic covariants that contain n-vectors of one kind.

EXAMPLES. Suppose $m = 1$ and $n = 2$. Systems (2) become

$$\frac{dx}{dt} = a + Ax,$$

where a is an element of V and A is a 2×2 matrix.

1. The polynomial $A^p_r A^q_s \varepsilon_{pq} \varepsilon^{rs}$ (half of the determinant of the matrix A) is nothing but $A^p_p A^r_s - A^p_r A^r_p = (Trace(A))^2 - Trace(A^2)$.

2. The center-affine invariants of (3) containing the vector a and the matrix A are necessarily the multiples of $\det(a, Aa)$.

Remark 1. Taking into account the Einstein symbolic notation, we remark that a total contraction over $V^{\otimes p} \otimes V^{*\otimes p}$ is a polynomial function $\varphi\colon V^{\otimes p} \otimes V^{*\otimes p} \mapsto k$ with $\varphi(v) = v^{j_1 \cdots j_p}_{j_{\sigma(1)} \cdots j_{\sigma(p)}}$ where σ is a permutation of $(1, 2, ..., p)$.

It follows from this remark that any basic center-affine covariant belonging to $K(n_0, n_1, ..., n_m, p)$ can be represented by the form

$$\begin{aligned} &a^{i_1} \cdots a^{i_{n_0}} a^{i_{n_0+1}}_{j_1} \cdots a^{i_{n_0+n_1}}_{j_{n_1}} \cdots \cdots a^{i_{l_{m-1}+1}}_{j_{k_{m-1}+1} \cdots j_{k_{m-1}+m}} \cdots a^{i_{l_{m-1}+n_m}}_{j_{k_m-m} \cdots j_{k_m}} \\ &\quad \times x^{i_{l_m+1}} \cdots x^{i_{l_m+p}} \varepsilon^{i_{l_m+p+1} \cdots i_{l_m+p+n}} \cdots \varepsilon^{i_{l_m+p+(n-1)q+1} \cdots i_{l_m+p+nq}} \end{aligned} \tag{7}$$

or by the form

$$a^{i_1}\cdots a^{i_{n_0}}\,a^{i_{n_0+1}}_{j_1}\cdots a^{i_{n_0+n_1}}_{j_{n_1}}\cdots\quad\cdots a^{i_{l_{m-1}+1}}_{j_{k_{m-1}+1}\cdots j_{k_{m-1}+m}}\quad\cdots a^{i_{l_{m-1}+n_m}}_{j_{k_m-m}\cdots j_{k_m}}$$
$$\times x^{i_{l_m+1}}\cdots x^{i_{l_m+p}}\,\varepsilon_{j_{k_m+1}\cdots j_{k_m+n}}\cdots\varepsilon_{j_{k_m+p+(n-1)q'+1}\cdots j_{k_m+nq'}}, \tag{8}$$

where $(j_1, ..., j_{k_m})$ and $(j_1, ..., j_{k_m+nq'})$ represent all possible permutations of $(i_1, ..., i_{l_m+p+nq})$ and $l_s=\sum_{r=0}^{s} n_r$, $k_s=\sum_{r=1}^{1} n_r$.

Of course, we have $l_m+p+nq=k_m+nq'$.

Equations (7) and (8) are used to expand a center-affine basic covariant.

4. AFFINE COVARIANTS

In this section we show that the knowledge of a system of generators of the algebra of center-affine invariants $\mathscr{I}(n, m)$ induces to one of a system of generators the algebra of affine covariants. Actually, this correspondence is an isomorphism. In order to construct it we need the following lemma.

LEMMA 1. *The group $T(n)$ induces a polynomial representation on the vector space $\mathscr{A}(n, m)$.*

Proof. It suffices to prove that

$$\forall p, q \in T(n),\ \rho_2(p+q)=\rho_2(p)\,\rho_2(q). \tag{9}$$

By formula 4, we have

$$[\rho_2(p+q)(a)]^j_{\alpha_1\alpha_2\cdots\alpha_k}$$
$$=\sum_{r=0}^{m-k}\sum_{s=0}^{r}\binom{k+r}{r}\binom{r}{s}a^j_{\alpha_1\alpha_2\cdots\alpha_k\beta_1\beta_2\cdots\beta_s\gamma_1\gamma_2\cdots\gamma_{r-s}}\,p^{\beta_1}\cdots p^{\beta_s}q^{\gamma}_1\cdots q^{\gamma_{r-s}},$$

$$[\rho_2(p)\,\rho_2(q)(a)]^j_{\alpha_1\alpha_2\cdots\alpha_k}$$
$$=\sum_{i=0}^{m-k}\binom{k+i}{i}\sum_{l=0}^{m-k-i}\binom{k+i+l}{l}a^j_{\alpha_1\alpha_2\cdots\alpha_k\beta_1\beta_2\cdots\beta_i\gamma_1\gamma_2\cdots\gamma_l}\,p^{\beta_1}$$
$$\cdots p^{\beta_i}q^{\gamma_1}\cdots q^{\gamma_l}.$$

We remark that the coefficients of algebraic form

$$a^j_{\alpha_1\alpha_2\cdots\alpha_k\beta_1\beta_2\cdots\beta_i\gamma_1\gamma_2\cdots\gamma_l}\,p^{\beta_1}\cdots p^{\beta_i}q^{\gamma_1}\cdots q^{\gamma_l}$$

in both expressions are equal to $\frac{(k+i+l)!}{k!i!}$. Hence, the equality (9).

Recall that the actions of the groups $Gl(n)$ and $T(n)$ over V are respectively defined by $(P, x) \mapsto P^{-1}x$ and $(p, x) \mapsto x - p$.

THEOREM 2. *A polynomial $Q \in k[\mathscr{A}(n, m) \times V]$ is an $Aff(n)$-covariant if and only if there exists a $Gl(n)$-invariant $I \in \mathscr{I}(n, m)$ such that*

$$Q(a, x) = I(\rho_2(x)(a)), \forall (a, x) \in \mathscr{A}(n, m) \times V.$$

Proof. Let us consider a $Gl(n)$-invariant I of $\mathscr{A}(n, m)$ and let Q be the polynomial $Q(a, x) = I(\rho_2(x)(a))$. We can suppose that I is a basic invariant. Following (4), (7), (8), and Theorem 1 the polynomial $Q(a, x)$ is a sum of basic center-affine covariants. It remains to prove that $Q(a, x)$ is a $T(n)$-invariant. By Lemma 1,

$$Q(\rho_2(p)(a), x - p) = I(\rho_2(x - p)\, \rho_2(p)(a)) = Q(a, x).$$

Let us suppose now that $Q(a, x)$ is an $Aff(n)$-covariant and put $I(a) = Q(a, 0)$. Then $I(a)$ is a $Gl(n)$-invariant. So, substituting p by x in the relation

$$Q(\rho_2(p)(a), x - p) = Q(a, x), \forall p \in T(n), \forall (a, x) \in \mathscr{A}(n, m) \times k^n,$$

we obtain

$$I(\rho_2(x)(a)) = Q(\rho_2(x)(a), 0) = Q(a, x). \forall x \in T(n), \forall a \in \mathscr{A}(n, m) \times k^n.$$

Thus, the correspondence $I(a) \leftrightarrow I(\rho_2(x)(a)) = Q(a, x)$ gives an isomorphism which we denote Φ, between the k-algebras $\mathscr{I}(n, m)$ and $\mathscr{Q}(n, m)$. The affine covariant $\Phi(I)$, where $I \in \mathscr{I}(n, m)$, denoted $I(x)$ is sometimes called the "translated" of I.

In the next section, we compute effectively a minimal system of generators of $\mathscr{Q}(2, 2)$.

DEFINITION 6. A basic affine covariant is an image by Φ of a basic center-affine invariant.

EXAMPLE. The polynomial

$$a^{\alpha}_{\alpha} + 2a^{\alpha}_{\alpha\beta}x^{\beta} + \cdots + \binom{m+1}{1} a^{\alpha}_{\alpha\beta_1\cdots\beta_{m-1}} x^{\beta_1} \cdots x^{\beta_{m-1}}$$

is a basic affine covariant. It corresponds to the divergence of the vector field defined by the system.

Remark 2. Let I be a center-affine invariant of $\mathscr{A}(n, m)$. If it is multi-homogeneous of multidegree $(d_0, d_1, ..., d_m)$ then the affine covariant $Q(a, x) = I(x.a)$ has the expansion

$$Q(a, x) = I(a) + Q_1(a, x) + \cdots + Q_s(a, x),$$

where the polynomial $Q_i(a, x)$ is homogeneous of degree i with respect to coordinates x and $s = \sum_{j=0}^{m} (m-j)\, d_i$.

COROLLARY 1. *A family $\mathscr{F}$ of $Gl(n)$-invariants is algebraically dependent if and only if $\Phi(\mathscr{F})$ is algebraically dependent.*

It is clear that the previous theorem remains true if we substitute the group $GL(n)$ by any of its subgroups such as the orthogonal one, $O(n)$. In the particular case where this subgroup contains only identity, we have the following corollary:

COROLLARY 2. *The family of polynomials*

$$\{[\rho_2(x)(a)]^j_{\alpha_1\alpha_2 \cdots \alpha_k}, 1 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_k \leqslant n, k \in \{0, ..., m\}, j \in \{0, ..., n\}\}$$

forms a minimal polynomial system of generators of $T(n)$-covariants of $\mathscr{A}(n, m)$.

5. THE CASE OF QUADRATIC BIVARIATE DIFFERENTIAL SYSTEMS

Many works are devoted to the study of center-affine invariants of systems

$$\frac{dx^j}{dt} = a^j + a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta (j, \alpha, \beta = 1, 2). \tag{10}$$

They are based on the classical approach, called Aronhold's symbolic method [4], and resumed in [9]. In particular, a minimal system of generators of $\mathscr{I}(2, 2)$ was obtained:

Minimal System of Generators of $\mathscr{I}(2, 2)$

$$J_1 = a^\alpha_\alpha, \quad J_2 = a^\alpha_\beta a^\beta_\alpha, \quad J_3 = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\gamma} \varepsilon^{pq}, \quad J_4 = a^\alpha_p a^\beta_{\beta q} a^\gamma_{\alpha\gamma} \varepsilon^{pq},$$

$$J_5 = a^\alpha_p a^\beta_{\gamma q} a^\gamma_{\alpha\beta} \varepsilon^{pq}, \quad J_6 = a^\alpha_p a^\beta_\gamma a^\gamma_{\alpha q} a^\delta_{\beta\delta} \varepsilon^{pq}, \quad J_7 = a^\alpha_{pr} a^\beta_{\alpha q} a^\gamma_{\beta s} a^\delta_{\gamma\delta} \varepsilon^{pq} \varepsilon^{rs},$$

$$J_8 = a^\alpha_{pr} a^\beta_{\alpha q} a^\gamma_{\delta s} a^\delta_{\beta\gamma} \varepsilon^{pq} \varepsilon^{rs}, \quad J_9 = a^\alpha_{pr} a^\beta_{\beta q} a^\gamma_{\gamma s} a^\delta_{\alpha\delta} \varepsilon^{pq} \varepsilon^{rs}, \quad J_{10} = a^\alpha_p a^\beta_\delta a^\gamma_\mu a^\delta_{\alpha q} a^\mu_{\beta\gamma} \varepsilon^{pq}$$

$$J_{11} = a^\alpha_p a^\beta_{qr} a^\gamma_{\beta s} a^\delta_{\alpha\gamma} a^\mu_{\delta\mu} \varepsilon^{pq} \varepsilon^{rs}, \quad J_{12} = a^\alpha_p a^\beta_{qr} a^\gamma_{\beta s} a^\delta_{\alpha\delta} a^\mu_{\gamma\mu} \varepsilon^{pq} \varepsilon^{rs},$$

$$J_{13} = a^\alpha_p a^\beta_{qr} a^\gamma_{\gamma s} a^\delta_{\alpha\beta} a^\mu_{\delta\mu} \varepsilon^{pq} \varepsilon^{rs}, \quad J_{14} = a^\alpha_p a^\beta_r a^\gamma_{\alpha q} a^\delta_{\beta s} a^\mu_{\gamma\delta} a^\nu_{\mu\nu} \varepsilon^{pq} \varepsilon^{rs},$$

$$J_{15} = a^\alpha_{pr} a^\beta_{qk} a^\gamma_{\alpha s} a^\delta_{\delta l} a^\mu_{\beta\gamma} a^\nu_{\mu\nu} \varepsilon^{pq} \varepsilon^{rs} \varepsilon^{kl}, \quad J_{16} = a^\alpha_p a^\beta_r a^\gamma_\delta a^\delta_{\alpha q} a^\mu_{\beta s} a^\nu_{\gamma\tau} a^\tau_{\mu\nu} \varepsilon^{pq} \varepsilon^{rs},$$

$$J_{17} = a^\alpha_{\alpha\beta} a^\beta, \quad J_{18} = a^p_\alpha a^\alpha a^q \varepsilon_{pq}, \quad J_{19} = a^\alpha_\beta a^\beta_{\alpha\gamma} a^\gamma, \quad J_{20} = a^\alpha_\gamma a^\beta_{\alpha\beta} a^\gamma,$$

$$J_{21} = a^p_{\alpha\beta} a^\alpha a^\beta a^q \varepsilon_{pq}, \quad J_{22} = a^\alpha_{\alpha\beta} a^\beta_{\gamma\delta} a^\gamma a^\delta, \quad J_{23} = a^\alpha_{\beta\gamma} a^\beta_{\alpha\delta} a^\gamma a^\delta,$$

$$J_{24} = a^\alpha_\gamma a^\beta_\delta a^\gamma_{\alpha\beta} a^\delta, \quad J_{25} = a^\alpha_{\alpha p} a^\beta_{\gamma q} a^\gamma_{\beta\delta} a^\delta \varepsilon^{pq}, \quad J_{26} = a^\alpha_{\alpha p} a^\beta_{\delta q} a^\gamma_{\beta\gamma} a^\delta \varepsilon^{pq},$$

$$J_{27} = a^p_\alpha a^\alpha_{\beta\gamma} a^\beta a^\gamma a^q \varepsilon_{pq}, \quad J_{28} = a^\alpha_\beta a^\beta_{\alpha\gamma} a^\gamma_{\delta\mu} a^\delta a^\mu, \quad J_{29} = a^\alpha_\gamma a^\beta_{\alpha\beta} a^\gamma_{\delta\mu} a^\delta a^\mu,$$

$$J_{30} = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\delta} a^\delta_{\gamma\mu} a^\mu \varepsilon^{pq}, \quad J_{31} = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\mu} a^\delta_{\gamma\delta} a^\mu \varepsilon^{pq},$$

$$J_{32} = a^\alpha_p a^\beta_{\beta q} a^\gamma_{\alpha\mu} a^\delta_{\gamma\delta} a^\mu \varepsilon^{pq}, \quad J_{33} = a^\alpha_{\beta\nu} a^\beta_{\alpha\gamma} a^\gamma_{\delta\mu} a^\delta a^\mu a^\nu,$$

$$J_{34} = a^\alpha_{\mu p} a^\beta_{\alpha q} a^\gamma_{\beta\nu} a^\delta_{\gamma\delta} a^\mu a^\nu \varepsilon^{pq}, \quad J_{35} = a^\alpha_p a^\beta_\nu a^\gamma_{\alpha q} a^\delta_{\beta\mu} a^\mu_{\gamma\delta} a^\nu \varepsilon^{pq},$$

$$J_{36} = a^\alpha_{pr} a^\beta_{\nu q} a^\gamma_{\alpha s} a^\delta_{\beta\gamma} a^\mu_{\delta\mu} a^\nu \varepsilon^{pq} \varepsilon^{rs}.$$

With the help of transvectants it was established by N. Vulpe [10] that any element of $\mathscr{I}(2, 2)$ is a polynomial of $J_1, ..., J_{36}$. In [8], a minimal system of generators of the ideal of the syzygies of $\mathscr{I}(2, 2)$ was given.

Minimal System of Generators of $\mathscr{K}(2, 2)$

Now, let us show that we can deduce from this family of center-affine invariants a minimal system of generators of $\mathscr{K}(2, 2)$. Obviously, the proposed method can be applied for any degree m.

It is clear that if we substitute a^j by x^j in the invariants $J_{17}, J_{18}, ..., J_{36}$, we obtain center-affine covariants that we note respectively $K_1, K_2, ..., K_{20}$. Actually, we can generalize this procedure of substitution to get all other covariants.

We now introduce the new center-affine covariant

$$K_{21} = \det(a^j, x^j) = a^1 x^2 - a^2 x^1.$$

Consider a basic center-affine invariant $I(a^i, a^i_j, ...)$ whose notation is (7) or (8). This is a function of a^i and other coordinates $a^j_{\alpha_1 \alpha_2, ..., \alpha_k}$. There are different ways to replace p times the vector (a^j) by (x^j). For example, if $I = a^p_{\alpha\beta} a^q a^\alpha a^\beta \varepsilon_{pq}$ and $p = 1$, the substitution may give $a^p_{\alpha\beta} x^q a^\alpha a^\beta \varepsilon_{pq}$ or

$a^p_{\alpha\beta}a^q a^\alpha x^\beta \varepsilon_{pq}$. We shall now show that, in general, these two covariants are the same modulo K_{21}.

Let us denote by $K'(a^i, x^i, ...)$ and $K''(a^i, x^i, ...)$ two basic center-affine homogeneous covariants of multidegree $(n_0 - p, n_1, ..., n_m, p)$ obtained from the tensorial notation (7) (or (8) of I by replacing "p" $a = (a^j)$ by "p" $x = (x^j)$. Then

$$K'(a^i, a^i, ...) - K''(a^i, a^i, ...) = I(a^i, ...) - I(a^i, ...) = 0.$$

Because of multihomogeneity with respect to coordinates a^j and x^j, we deduce from this fact that the difference $K'(a^i, x^i, ...) - K''(a^i, x^i, ...)$ contains as a factor the covariant $a^1x^2 - a^2x^1 = K_{21}$. Consequently, we have the following result

LEMMA 2. *Let I be a multihomogeneous basic center-affine invariant of multidegree $(n_0, n_1, ..., n_m)$ and p a natural number such that $p \leqslant n_0$. For any two center-affine covariants K', K'' obtained from the invariant I by replacing in its tensorial notation "p" vectors (a^j) by "p" vectors $x = (x^j)$, the difference $K' - K''$ is a multiple of K_{21}.*

That is to say that each center-invariant $J_1, ..., J_{36}$ of multidegree $(n_0, n_1, n_2, 0)$ gives rise, for all $p \in \{1, ..., n_1 - 1\}$, to one and only one covariant of multidegree $(n_0 - p, n_1, n_2, p)$ that cannot be expressed as a polynomial of covariants of lower total degree.

Applying this lemma to the family $\{J_i, i = 1 \cdots 36\}$ we obtain a polynomially independent family of covariants $\mathscr{K}(2, 2)$:

$$K_1 = a^\alpha_{\alpha\beta}x^\beta, \qquad K_2 = a^p_\alpha x^\alpha x^q \varepsilon_{pq}, \qquad K_3 = a^\alpha_\beta a^\beta_{\alpha\gamma}x^\gamma, \qquad K_4 = a^\alpha_\gamma a^\beta_{\alpha\beta}x^\gamma,$$

$$K_5 = a^p_{\alpha\beta}x^\alpha x^\beta x^q \varepsilon_{pq}, \qquad K_6 = a^\alpha_{\alpha\beta}a^\beta_{\gamma\delta}x^\gamma x^\delta, \qquad K_7 = a^\alpha_{\beta\gamma}a^\beta_{\alpha\delta}x^\gamma x^\delta,$$

$$K_8 = a^\alpha_\gamma a^\beta_\delta a^\gamma_{\alpha\beta}x^\delta, \qquad K_9 = a^\alpha_{\alpha p}a^\beta_{\gamma q}a^\gamma_{\beta\delta}x^\delta \varepsilon^{pq}, \qquad K_{10} = a^\alpha_{\alpha p}a^\beta_{\delta q}a^\gamma_{\beta\gamma}x^\delta \varepsilon^{pq},$$

$$K_{11} = a^p_\alpha a^\alpha_{\beta\gamma}x^\beta x^\gamma x^q \varepsilon_{pq}, \qquad K_{12} = a^\alpha_\beta a^\beta_{\alpha\gamma}a^\gamma_{\delta\mu}x^\delta x^\mu, \qquad K_{13} = a^\alpha_\gamma a^\beta_{\alpha\beta}a^\gamma_{\delta\mu}x^\delta x^\mu,$$

$$K_{14} = a^\alpha_p a^\beta_{\alpha q}a^\gamma_{\beta\delta}a^\delta_{\gamma\mu}a^\mu \varepsilon^{pq}, \qquad K_{15} = a^\alpha_p a^\beta_{\alpha q}a^\gamma_{\beta\mu}a^\delta_{\gamma\delta}x^\mu \varepsilon^{pq},$$

$$K_{16} = a^\alpha_p a^\beta_{\beta q}a^\gamma_{\alpha\mu}a^\delta_{\gamma\delta}\alpha^\mu \varepsilon^{pq}, \qquad K_{17} = a^\alpha_{\beta\nu}a^\beta_{\alpha\gamma}a^\gamma_{\delta\mu}x^\delta x^\mu x^\nu,$$

$$K_{18} = a^\alpha_{\mu p}a^\beta_{\alpha q}a^\gamma_{\beta\nu}a^\delta_{\gamma\delta}x^\mu x^\nu \varepsilon^{pq}, \qquad K_{19} = a^\alpha_p a^\beta_\nu a^\gamma_{\alpha q}a^\delta_{\beta\mu}a^\mu_{\gamma\delta}x^\nu \varepsilon^{pq},$$

$$K_{20} = a^\alpha_{pr}a^\beta_{\nu q}a^\gamma_{\alpha s}a^\delta_{\beta\gamma}a^\mu_{\delta\mu}x^\nu \varepsilon^{pq}\varepsilon^{rs}, \qquad K_{21} = a^p x^q \varepsilon_{pq}, \qquad K_{22} = a^p_\alpha a^\alpha x^q \varepsilon_{pq},$$

$$K_{23} = a^\alpha a^\beta a^p_{\alpha\beta}x^q \varepsilon_{pq}, \qquad K_{24} = a^\alpha a^\beta_{\alpha\beta}a^\gamma_{\gamma\delta}x^\delta, \qquad K_{25} = a^\alpha a^\beta_{\alpha\gamma}a^\gamma_{\beta\delta}x^\delta,$$

$$K_{26} = a^\alpha a^\beta a^p_\gamma a^\gamma_{\alpha\beta}x^q \varepsilon_{pq}, \quad K_{27} = a^\alpha a^\beta_\gamma a^\gamma_{\beta\delta}a^\delta_{\alpha\mu}x^\mu \leqslant \alpha_2, \quad K_{28} = a^\alpha a^\beta_\delta a^\gamma_{\beta\gamma}a^\delta_{\alpha\mu}x^\mu,$$

$$K_{29} = a^\alpha a^\beta a^\gamma_{\delta\nu}a^\delta_{\gamma\mu}a^\mu_{\alpha\beta}x^\nu, \quad K_{30} = a^\alpha a^\beta_{\alpha p}a^\gamma_{\beta q}a^\delta_{\gamma\nu}a^\mu_{\delta\mu}x^\nu \varepsilon^{pq}, \quad K_{31} = a^p a^q_{\alpha\beta}x^\alpha x^\beta \varepsilon_{pq},$$

$$K_{32} = a^p a^q_\alpha a^\alpha_{\beta\gamma}x^\beta x^\gamma \varepsilon_{pq}, \qquad K_{33} = a^\alpha a^\beta_{\alpha\gamma}a^\gamma_{\beta\delta}a^\delta_{\mu\nu}x^\mu x^\nu.$$

THEOREM 3. *The set of covariants*

$$\{J_i; i=1, ..., 36\} \cup \{K_j; j=1, ..., 33\}$$

forms a minimal system of generators of the ring $\mathscr{K}(2, 2)$.

Proof. We have seen that this family is algebraically independent (as a consequence of the previous lemma and the procedure of construction of the covariants $\{K_i; i=1, ..., 33\}$). It is easy to show that it generates the algebra $\mathscr{K}(2, 2)$. Let K be a center-affine covariant of multidegree (n_0, n_1, n_2, p). Following Theorem 1, we can suppose that $K=K(x^i, a^i, ...)$ is a basic covariant, i.e., its form is as in (7) or (8). As the family $\{J_1, J_2, ..., J_{36}\}$ generates the algebra of center-affine invariants $\mathscr{I}(2, 2)$, the invariant $I=K(a^i, a^i, ...)$ is a polynomial expression of these invariants: $I=\mathscr{P}(J_1, J_2, ..., J_{36})=\lambda_1 I_1+\lambda_2 I_2+ \cdots +\lambda_l I_l$ where λ_i are constant coefficients and I_j are basic invariants of multidegree $(n_0+p, n_1, n_2, 0)$. Each term of I is a product of invariants J_i. After substituting $p(a^i)$ by $p(x^i)$, we obtain a center-affine covariant of multidegree (n_0, n_1, n_2, p). Of course, we can choose this substitution in such a way that the obtained covariant is a product of $K_1, K_2, ..., K_{33}$. Let us denote this covariant K'. It has the same multidegree as K. The difference $K-K'$ is a multiple of K_{21}. Let $\tilde{K}$ be the quotient of $K-K'$ by K_{21}. Its multidegree is $(n_0-1, n_1, n_2, p-1)$. We repeat this procedure. At the end, we obtain a polynomial expression of K with respect to $J_1, J_2, ..., J_{36}, K_1, K_2, ..., K_{33}$.

This list of center-affine covariants is already obtained (with some mistakes) in the report [1].

Remark 3. In the literature, we often find quadratic systems without free terms:

$$\frac{dx^j}{dt}=a^j_\alpha x^\alpha+a^j_{\alpha\beta}x^\alpha x^\beta (j, \alpha, \beta=1, 2). \tag{11}$$

The family

$$\{J_i; i=1, ..., 16\} \cup \{K_j; j=1, ..., 20\}$$

is a minimal system of covariants of these systems. This explains the indexing of covariants suggested in the previous theorem.

Remark 4. Concerning the affine invariants, it turns out that they are nothing more than the polynomial expressions of affine covariants which do not depend on the vector x. They form a subalgebra of $\mathscr{Q}(n, m)$.

EXAMPLES.

$$
\begin{aligned}
&J_1(x) = a^{\alpha}_{\alpha} + x^{\alpha} a^{\beta}_{\alpha\beta} = J_1 + 2K_1,\\
&J_3(x) = a^{\alpha}_{p} a^{\beta}_{\alpha q} a^{\gamma}_{\beta\gamma} \epsilon^{pq} + 2a^{\alpha}_{p\delta} a^{\beta}_{\beta q} a^{\gamma}_{\beta\gamma} x^{\delta} \epsilon^{pq} = J_3 + 2(K_9 - K_{10}),\\
&J_5(x) = a^{\alpha}_{p} a^{\beta}_{\gamma q} a^{\gamma}_{\alpha\beta} \epsilon^{pq} + 2a^{\alpha}_{p\delta} a^{\beta}_{\gamma q} a^{\gamma}_{\alpha\beta} x^{\delta} \epsilon^{pq} = J_5 + 2(K_9 - K_{10}), \qquad (12)\\
&\qquad \cdots\\
&J_{13}(x) = a^{\alpha}_{p} a^{\beta}_{qr} a^{\gamma}_{\gamma s} a^{\delta}_{\alpha\beta} a^{\mu}_{\delta\mu} \epsilon^{pq} \epsilon^{rs}\\
&\qquad 2a^{\alpha}_{pu} a^{\beta}_{qr} a^{\gamma}_{\gamma s} a^{\delta}_{\alpha\beta} a^{\mu}_{\delta\mu} x^{u} \epsilon^{pq} \epsilon^{rs} = J_{13} = 2J_7 K_1.
\end{aligned}
$$

Taking into account the relation $J_7(x) = J_7$, it is obvious that the polynomials $J_3 - J_5$ and $J_1 J_7 + J_{13}$ are affine invariants.

6. EFFECTIVE COMPUTATION OF A SYSTEM OF GENERATORS OF $\mathcal{Q}(2, 2)$

In this section we compute effectively a system of generators of $\mathcal{Q}(2, 2)$ from that of $\mathcal{I}(2, 2)$ (see the previous section). Some of the algorithms presented here can be generalized to differential systems of any degree m and any dimension n.

6.1. *Computing in* $\mathcal{K}(2, 2)$

The algebra of center-affine covariants of quadratic differential systems (10), $\mathcal{K}(2, 2)$ is a multigraded algebra,

$$
\mathcal{K}(2, 2) = \bigoplus_{l \in \mathbb{N}} \; \bigoplus_{n_0 + n_1 + n_2 + p = l} K(n_0, n_1, n_2, p),
$$

where $K(n_0, n_1, n_2, p)$ is the subalgebra of multihomogeneous covariants of multidegree (n_0, n_1, n_2, p), i.e., homogeneous of degree n_k with respect to coordinates of $(a^{j}_{i_1, i_2, \ldots, i_k})$ and of degree p with respect to coordinates x^j.

As it is proved previously, the family of covariants

$$
\mathcal{F} = \{J_1, \ldots, J_{36}, K_1, \ldots, K_{33}\}
$$

is a minimal system of generators of $\mathcal{K}(2, 2)$. The main task which arises here is to decompose any center-affine covariant (invariant) in the basis $\mathcal{F}$. To this end, we need to introduce the notion of type of a basic center-affine covariant: this is the sextuple (p, q, r, s, τ, v) where (p, q, r, s) is its multidegree with respect to coordinates $(a^i, a^i_j, a^i_{j,k}, x^i)$, τ is the number of contravariant 2-vectors ε^{jk}, and v, according to Remark 1, is the permutation of upper indices of the covariant.

Step 1.1. Expansion of a basic covariant of type (p, q, r, s, τ, v).
We take into account three facts:

1. the coordinates (a^i_{jk}) are symmetric with respect to subscripts,
2. if $n=2$, $\varepsilon^{ij}=\varepsilon_{ij}=j-i$,
3. the expansion of a basic covariant is obtained by successive contractions.

Step 1.2. Decomposition of the multidegree m w.r.t. the list of multidegrees of $\mathscr{F}$. This list, denoted t, has 49 elements:

$$t := [[0,1,0,0],[1,0,1,0],[0,2,0,0],[2,1,0,0],[1,1,1,0],[0,1,2,0],\\ [3,0,1,0],[2,0,2,0],[1,2,1,0],[1,0,3,0],[0,2,2,0],[0,0,4,0],\\ [3,1,1,0],[2,1,2,0],[1,3,1,0],[0,3,2,0],[0,1,4,0],[3,0,3,0],\\ [2,0,4,0],[1,2,3,0],[1,0,5,0],[0,2,4,0],[0,0,6,0],[0,3,4,0],\\ [1,0,0,1],[0,0,1,1],[1,1,0,1],[0,1,1,1],[2,0,1,1],[1,0,2,1],\\ [0,2,1,1],[0,0,3,1],[2,1,1,1],[1,1,2,1],[0,1,3,1],[2,0,3,1],\\ [1,0,4,1],[0,2,3,1],[0,0,5,1],[0,1,0,2],[1,0,1,2],[0,0,2,2],\\ [1,1,1,2],[0,1,2,2],[1,0,3,2],[0,0,4,2],[0,0,1,3],[0,1,1,3],\\ [0,0,3,3]].$$

The resulting procedure gives all possible decompositions $[p, q, r, s] = \sum_1^{49} \lambda_i t[i]$. For instance, all the possible decompositions of $[0, 0, 4, 0]$ are

$$[0,2,4,0] = 2[0,1,0,0] + [0,0,4,0] = [0,1,0,0] + [0,1,4,0]\\ = [0,2,0,0] + [0,0,4,0] = 2[0,1,2,0] = [0,2,2,0].$$

Step 1.3. Construction of a generating family of $K(p, q, r, s)$ (as a linear space), starting from $\mathscr{F}$.

With the help of the previous procedure, we get the different decompositions of the 4-tuple $K(p, q, r, s)$ w.r.t. the family $\mathscr{F}$. As we know that the dimension of spaces

$$K(t[1]), K(t[2]), K(t[3]), K(t[4]), K(t[7]), K(t[9]), K(t[11]), K(t[13]),$$
$$K(t[16]), K(t[18]), K(t[19]), K(t[20]), K(t[21]), K(t[22]), K(t[23]),$$
$$K(t[24]), K(t[25]), K(t[26]), K(t[27]), K(t[29]), K(t[31]), K(t[33]),$$
$$K'(t[36]), K'(t[37])$$

is one, that one of

$$K(t[5]), K(t[8]), K(t[10]), K(t[14]), K(t[28]), K(t[30]), K(t[32]),$$
$$K(t[34]), K(t[42]), K(t[44])$$

is two, and the dimension of spaces

$$K(t[6]), K(t[12]), K(t[15]), K(t[17]), K(t[35])$$

is three, for each term of the decomposition $[p, q, r, s] = \sum_1^{49} \lambda_i t[i]$ we take the linear basis of $K(t[i])$ and thus we form the generating family of the linear space $K(p, q, r, s)$ (w.r.t. to the family $\mathscr{F}$).

For instance (continuing the previous example), the linear space $K(0, 2, 4, 0)$ is generating by $J[1]^2\, J[7]$, $J[1]^2\, J[8]$, $J[1]^2\, J[9]$ (which correspond to the first decomposition $[0, 2, 4, 0] = 2[0, 1, 0, 0] + [0, 0, 4, 0]$), $J[1]\, J[11]$, $J[1]\, J[12]$, $J[1]\, J[13]$ (which correspond to the second decomposition), $J[2]\, J[7]$, $J[2]\, J[8]$, $J[2]\, J[9]$, $J[3]^2$, $J[3]\, J[4]$, $J[3]\, J[5]$, $J[4]^2$, $J[4]\, J[5]$, $J[5]^2$, and $J[14]$ (for the last decomposition). These 16 covariants (here, invariants) form a generating family of $K(0, 2, 4, 0)$.

Step 1.4. Expression of any covariant in the basis $\mathscr{F}$.

This task is reduced to solving a linear system. Before doing it, we can operate main simplifications. According to [2, 11] any center-affine covariant K of any tensorial representation of the group $GL(2, k)$ can be written as a homogeneous polynomial in $k[\mathscr{A}(2, 2)][x^1, x^2]$ of degree p:

$$K = A_{11\ldots 11}(x^1)^p + A_{11\ldots 12}(x^1)^{p-1}\, x^2 + \cdots + A_{22\ldots 22}(x^2)^p.$$

With the help of some operator, N. Vulpe shows that K, as a polynomial in $[x^1, x^2]$, vanishes if and only if the leading coefficient $A_{11\ldots 11}$ (called elsewhere the semi-invariant) vanishes. That means that the leading coefficient defines uniquely the whole covariant. This fact is also true for any homogeneous relation between covariants. Consequently, the polynomial relations between the covariants are equivalent to the same relations of their leading terms.

This result improves the complexity of the computations over $\mathscr{K}(2, 2)$. The corresponding procedure gives in particular the syzygies of multidegree (p, q, r, s).

6.2. *Computing a Minimal System of Generators of* $\mathscr{Q}(2, 2)$

The goal of this section is to compute a minimal system of generators of $\mathscr{Q}(2, 2)$ using the minimal system of generators $\mathscr{F}$ of $\mathscr{I}(2, 2)$ which is given in the previous section. Following Theorem 2, the image $\mathscr{G} = \phi(\mathscr{F})$ is a minimal system of generators of $\mathscr{Q}(2, 2)$. To get their expanded expressions, it suffices to make the substitutions

$$a^i \leftarrow a^i + a^i_\alpha x^\alpha + a^i_{\alpha\beta} x^\alpha x^\beta,$$

$$a^i_j \leftarrow a^i_j + 2a^i_{j\alpha} x^\alpha,$$

$$a^i_{jk} \leftarrow a^i_{jk}$$

in $\mathscr{F}$. The minimal system of generators of $\mathscr{Q}(2, 2)$ is then known.

It remains to express these covariants in $\mathscr{F}$. This problem is the same as the determination of their multihomogeneous parts.

Let J be a center-affine invariant. We decompose the corresponding affine covariant $\Phi(J)$ into multihomogeneous polynomials

$$\Phi(J) = L_1 + \cdots + L_k.$$

The different steps for computing $\Phi(J)$ are the following:

1. Determine the multidegrees $r_1, r_2, ..., r_k$ of $\Phi(J)$ w.r.t. the list of variables

$$varh = [\{a^1, a^2\}, \{a^1_1, a^1_2, a^2_1, a^2_2\}, \{a^1_{11}, a^1_{12}, a^1_{22}, a^2_{11}, a^2_{12}, a^2_{22}\}, \{x^1, x^2\}].$$

For finding these multidegrees, it suffices to expand the univariate polynomial

$$pp := (a + b \times x + c \times x^2)^p \times (b + 2 \times c \times x)^q \times c^r$$

and to take the list of elements

$$(degree(pp, a), degree(pp, b), degree(pp, c), degree(pp, x)).$$

2. Collect all the terms of the multihomogeneous part L_i of $\Phi(J)$ corresponding to a given multidegree r_i.

3. Decompose each multihomogeneous part L_i of $\Phi(J)$ w.r.t. $\mathscr{F}$.

All computations are made with Maple V.5.

APPENDIX: MINIMAL SYSTEM OF GENERATORS OF THE ALGEBRA $\mathscr{Q}(2, 2)$

$$Q_1 = J_1 + 2_K$$

$$Q_2 = J_2 + 4K_3 + 4K_7,$$

$$Q_3 = J_3 - 2K_{10} + 2K_9,$$

$$Q_4 = J_4 - 2K_{10},$$

$$Q_5 = J_5 - 2K_{10} + 2K_9,$$

$$Q_6 = (J_5 - 2J_3 + J_4)\, K_1 + J_6 - 2K_{16} + 4K_{15} - 2J_1 K_{10} + J_1 K_9 + 4K_{18},$$

$$Q_7 = J_7,$$

$$Q_8 = J_8,$$

$$Q_9 = J_9,$$

$$\begin{aligned} Q_{10} = {} & (2J_1 J_5 + 8K_{18} + 2J_1 K_{10} - J_1 J_3)\, K_1 + (2J_9 - 6J_8 + 2J_7)\, K_2 + \\ & (J_4 - 2J_3 + 5J_5 - 4K_{10})\, K_3 + (J_4 - 7J_5)\, K_4 - 4J_8 K_5 + \\ & (-8K_9 + 4K_{10} - 2J_4 - 4J_5 + 4J_3)\, K_6 + (-2J_3 + 8J_5 + 4K_9)\, K_7 + \\ & (-J_1^2 + 3J_2)\, K_9 + (-J_1^2 - J_2)\, K_{10} + J_{10} + 6K_{19} - 3J_1 K_{14} + \\ & 4J_1 K_{18} + 4J_1 K_{15} - 2J_1 K_{16}, \end{aligned}$$

$$Q_{11} = J_{11} - 2K_{20},$$

$$Q_{12} = (J_8 - J_7)\, K_1 + J_{12} - 2K_{20},$$

$$Q_{13} = J_{13} - 2J_7 K_1,$$

$$\begin{aligned} Q_{14} = {} & (J_1 J_9 - J_1 J_8 - 2J_{12})\, K_1 + (-2J_9 - 2J_7)\, K_3 + 2J_8 K_4 + \\ & (2J_7 + 2J_8)\, K_6 - 4J_7 K_7 + (4J_3 - 2J_5)\, K_9 + J_{14} - 2J_4 K_{10}, \end{aligned}$$

$$Q_{15} = J_{15},$$

$$Q_{16} = -4J_7K_1^3 + (10J_{12} - 4K_{20})\,K_1^2 + (\tfrac{3}{2}J_1J_{12} - J_3^2 - 16K_9K_{10} - 8J_9K_7 + 20J_7K_4 + 8J_8K_6 - \tfrac{3}{2}J_5^2 + \tfrac{1}{2}J_2J_8 - 8J_8K_3 - \tfrac{5}{2}J_1J_{11} + \tfrac{3}{2}J_4J_5)\,K_1 + (-8J_{12} + 4K_{20} - \tfrac{5}{2}J_1J_8 + J_{13} + 7J_{11})\,K_3 + (\tfrac{1}{2}J_1J_8 - 5J_{11} + 5J_{12} - J_{13})\,K_4 - 8J_{15}K_5 + (8J_{13} + 4J_{12} + 4K_{20} - 4J_{11})\,K_6 + (-16K_{20} - J_1J_8 + J_1J_9 - 10J_{12} + 8J_{11} - 8J_{13})\,K_7 + (3J_8 - 3J_7)\,K_8 + 6J_1K_9^2 + (-24K_{16} + \tfrac{3}{2}J_1J_4 + \tfrac{1}{2}J_1J_3 - J_1J_5 - 12K_{18} + 8K_{14} - 28J_1K_{10})\,K_9 + 20J_1K_{10}^2 + (20K_{16} + \tfrac{1}{2}J_1J_5 - 8K_{14} + 4K_{15} - \tfrac{3}{2}J_1J_3 - \tfrac{3}{2}J_1J_4 + 24K_{18})\,K_{10} + (6J_8 - 2J_9)\,K_{12} + (-2J_8 - 10J_7)\,K_{13} + (3J_3 + 3J_5 - 5J_4)\,K_{14} + 5J_4K_{15} + (J_5 - 4J_3)\,K_{16} + (4J_8 + 16J_9)\,K_{17} + 4J_5K_{18} + (-J_2 + \tfrac{1}{2}J_1^2)\,K_{20} + J_{16},$$

$$Q_{17} = J_{17} + K_4 + K_6,$$

$$Q_{18} = (-\tfrac{3}{2}K_2 - 2K_{21} - K_5)\,K_1^2 + (-2J_1K_5 - 2J_1K_2 - 4K_{22} + K_{11} + 2K_{31})\,K_1 + (\tfrac{1}{2}J_2 + 2J_{17} + 2K_3 + \tfrac{3}{2}K_7 - \tfrac{1}{2}J_1^2)\,K_2 + (K_5 + 4K_{21})\,K_3 - 2K_{21}K_4 + (-J_1^2 + K_7)\,K_5 + 2K_{21}K_7 + J_1K_{11} + (J_2 + 2J_{17})\,K_{21} + J_{18} - 2K_{23} - 2K_{32} - J_1K_{22} + 3J_1K_{31},$$

$$Q_{19} = J_{19} + 2K_{25} + K_8 - J_1K_1^2 + J_1K_7 + 2K_1K_4 - 2K_{13} + 3K_{12} + 2K_{17},$$

$$Q_{20} = -K_1^3 - 2J_1K_1^2 + (\tfrac{1}{2}J_2 + \tfrac{1}{2}J_1^2 + 2K_3 + K_7 + 2K_4 + 2K_6)\,K_1 + J_{20} + 2K_{24} + J_1K_4 - K_{13} + 2J_1K_6,$$

$$Q_{21} = (-\tfrac{3}{2}J_1K_5 - \tfrac{3}{2}J_1K_2)\,K_1^2 + ((-\tfrac{1}{2}J_1^2 - 3J_{17} - \tfrac{1}{2}J_2 - \tfrac{3}{2}K_6)\,K_2 + (\tfrac{3}{2}K_5 - 3K_21)\,K_4 + (-\tfrac{3}{2}J_{17} - K_6 - 2J_1^2)\,K_5 - \tfrac{3}{2}K_{21}K_6 + 2J_1K_{11} + (-3J_{17} - 3J_2)\,K_{21} + 2J_{18} + 2J_1K_{31} - 2K_{32})\,K_1 + (-3J_1J_{17} + \tfrac{3}{2}J_1K_7 - J_1K_4 + 3K_{12} - 3K_{13} + K_8 + 3J_{20} + 6K_{25} + J_1K_3 + 6K_{25} + J_1K_3 - 2J_1K_6)\,K_2 + 3K_{22}K_3 + (2K_{31} + J_1K_{21} + 2K_{11} - 4K_{22})\,K_4(\tfrac{3}{2}K_{25} - 3K_{13} + 3K_{12} + K_{17} - J_1J_2 - J_1K_6)\,K_5 + (K_{31} - 2J_1K_{21} + K_{11})\,K_6(\tfrac{3}{2}K_{11} - 3K_{22} + 3J_1K_{21})\,K_7 + 3K_{21}K_8 + (\tfrac{1}{2}J_2 + \tfrac{1}{2}J_1^2)\,K_{11} + 2K_{21}K_{13} + (2J_1J_{17} - 2J_{20} + 3K_{25})\,K_{21} - 3J_1K_{23} + 3K_{26} + (2J_{17} + \tfrac{3}{2}J_2 + \tfrac{3}{2}J_1^2)\,K_{31} + J_{21} + \tfrac{3}{2}K_{21}K_{17} - 3J_1K_{32},$$

$$Q_{22} = -J_1K_1^3 + (-J_{17} - J_1^2 + K_3 - \tfrac{1}{2}K_6)\,K_1^2 + (2J_{19} + K_{25} + J_1K_3 + J_1K_4 + \tfrac{1}{2}K_{17} - 2J_1J_{17} + J_1K_6)\,K_1 + (-J_3 + K_{10} - 2K_9)\,K_2 + (2J_{17} + K_7)\,K_4 + (-J_4 - \tfrac{1}{2}K_9)\,K_5 + K_6^2 + (2J_{17} + \tfrac{1}{2}J_2 + \tfrac{1}{2}J_1^2)\,K_6 + J_{22} - 2K_{28} + 2J_1K_{24} - K_{21}K_9 - J_1K_{13},$$

$$Q_{23} = -J_1K_1^3 + (-\tfrac{1}{2}J_2 + 2K_4 - \tfrac{1}{2}K_7)\,K_1^2 + (2K_{12} + 2K_{17} - 2K_{13} - J_1J_{17} + J_{20})\,K_1 + (-J_5 + J_4 + K_{10})\,K_2 + K_3K_7 + K_4^2 + (J_{17} - K_7)\,K_4 + (-J_5 + K_{10})\,K_5 + \tfrac{1}{2}K_4^2 + 2K_{33} - 2K_{28} + 2K_{27} + J_1K_{12} + J_{23} - J_1K_{13} - J_5K_{21} - K_6K_7 + J_1K_{25} + \tfrac{1}{2}J_2K_7 + J_1K_{17},$$

$$Q_{24} = -2K_1^4 + (4K_6 - \tfrac{5}{4}J_2 - K_3)\,K_1^2 + (4K_{17} - J_1 J_{17} - 3J_1 K_7 - J_1 K_3 + 2K_{12} - \tfrac{1}{2}J_1 K_4 + 2J_{20} + 4K_8 + 4K_{24})\,K_1 + (-2K_9 + 3J_3 - \tfrac{1}{2}J_5 - K_{10})\,K_2 + K_3^2 + (\tfrac{1}{2}J_2 + 4K_7 - 2K_4 - \tfrac{1}{2}J_1^2)\,K_3 + \tfrac{5}{2}K_4^2 + 3K_4K_7 + (4K_{10} - J_5)\,K_5 - 4K_6^2 + (\tfrac{1}{2}J_1^2 - 4J_{17} - \tfrac{1}{2}J_2)\,K_6 + 2K_7^2 + (-\tfrac{1}{4}J_1^2 + \tfrac{3}{2}J_2)\,K_7 + J_1 K_8 + 4K_{21}K_9 + \tfrac{7}{2}J_1 K_{12} - \tfrac{5}{2}J_1 K_{13} + 3J_1 K_{17} + (-J_4 + J_5)\,K_{21} + J_{24} + 4K_{27} + 4K_{33} + J_1 K_{25} - 2K_{28},$$

$$Q_{25} = (J_3 - \tfrac{1}{2}J_4 + \tfrac{1}{2}J_5 - K_{10} + 2K_9)\,K_1 + J_{25} - K_{15} + \tfrac{1}{2}J_1 K_9 - K_{18},$$

$$Q_{26} = (J_3 + K_9)\,K_1 + J_{26} - K_{16},$$

$$Q_{27} = J_{27} + (11K_3^2 + (\tfrac{1}{2}J_1^2 - \tfrac{5}{2}K_4 - \tfrac{1}{2}J_2)\,K_3 - 2K_4^2 + (\tfrac{7}{2}K_7 + \tfrac{1}{2}J_2 - \tfrac{1}{2}J_1^2 - K_6)\,K_4 + (-7J_3 + 3J_5)\,K_5 + \tfrac{5}{2}K_6^2 + (-\tfrac{9}{2}J_1^2 + 7J_2)\,K_6 - \tfrac{5}{4}K_7^2 + (\tfrac{7}{2}J_1^2 - \tfrac{27}{4}J_2)\,K_7 + \tfrac{9}{2}J_1 K_{13} + 6J_1 K_{17} + (2J_5 - 2J_4)\,K_{21} - 3J_{24} + 2K_{28} - 2K_{27} - \tfrac{3}{2}J_{23} + \tfrac{9}{2}J_{17}^2 - 4J_1 K_{25})\,K_2 + ((-\tfrac{5}{2}J_1^2 - \tfrac{5}{2}K_6 + \tfrac{5}{2}K_7)\,K_5 + 2J_1 K_{11} + (\tfrac{3}{2}J_2 - \tfrac{3}{2}J_1^2 - 9J_{17})\,K_{21} - 6K_{32})\,K_4 - \tfrac{13}{2}K_4^2 K_5 + (-7J_1 K_{11} + (-2J_2 - 6J_{17})\,K_{21} + \tfrac{9}{2}K_{32} + 3J_1 K_{22} + 3K_{23})\,K_7 + ((J_1^2 + K_6 - \tfrac{11}{2}J_2 - \tfrac{1}{2}K_7)\,K_5 - 4J_1 K_{31} + 4K_{32} + \tfrac{9}{2}J_1 K_{11})\,K_3 + \tfrac{1}{2}K_1^4 K_5 - 2K_{10}K_5^2 + (-2K_{31} - 12K_{11} + 2K_{22} + J_1 K_{21})\,K_8 + (12K_{24} + \tfrac{1}{2}J_1^3 - 6J_{20} + 4K_{25} + 6J_1 J_{17} - 4K_{12} - 5K_{13} - \tfrac{1}{2}J_1 J_2 + K_{17})\,K_{11} - 3K_{21}^2 K_{10} + 4K_{25}K_{31} - 12K_{24}K_{31} + (-10K_{31} + 6J_1 K_{21})\,K_{12} + (3K_{31} - \tfrac{7}{2}J_1 K_2 - 3J_1 K_5)\,K_1^3 + (-2J_1 K_{21} + 20K_{31})\,K_{13} + ((-\tfrac{5}{2}K_6 + \tfrac{5}{4}K_7 + \tfrac{5}{4}J_2 + \tfrac{7}{2}K_3)\,K_2 + (3K_5 + K_{21})\,K_3 - 5J_1 K_{22} - 6K_{21}K_6 + \tfrac{7}{2}K_{32} - 2J_2 K_{21} - \tfrac{3}{2}J_1 K_{11} - 3K_6 K_5 + K_{21}K_4)\,K_1^2 + (-3K_{22} - J_1 K_{21} - K_{31})\,K_{17} - 6J_4 K_{21}^2 + 3K_{21}K_6^2 + (-K_{25} + 6K_{24} - 2J_{19})\,K_{22} + (J_1 K_{25} + 3K_{28} + J_{17}^2 + 2J_1 J_{19} - 3K_{27} - J_{23})\,K_{21} + (-\tfrac{3}{2}J_1^2 + \tfrac{3}{2}J_2)\,K_{23} + (3J_{17} + \tfrac{3}{2}J_2 - \tfrac{3}{2}J_1^2)\,K_{32} + (\tfrac{3}{2}J_1^3 - J_{19} + 6J_{20} - \tfrac{3}{2}J_1 J_2)\,K_{31} + ((-K_{24} - \tfrac{1}{2}J_1^3 + \tfrac{1}{2}J_1 J_2 - K_{13} + \tfrac{9}{2}K_8 - 9J_1 K_3 + 2J_{20})\,K_2 + (6K_{11} + 6K_{22})\,K_3 - 4K_{22}K_4 + (-K_8 + \tfrac{3}{2}J_1 K_6 + 12J_9 + \tfrac{1}{2}J_1 J_2 + \tfrac{1}{2}J_1 K_7 - 8J_1 J_{17} + 3K_{25} - \tfrac{5}{2}K_{13})\,K_5 + 3J_1 K_{21}K_6 - J_1 K_{21}K_7 + 6K_{21}K_8 - \tfrac{1}{2}J_2 K_{11} - 21K_{21}K_{13} + 6J_{19}K_{21} + (\tfrac{3}{2}J_2 - 6J_{17} - \tfrac{3}{2}J_1^2)\,K_{22} + 2J_{21} + 3J_{17}K_{31})\,K_1 + (\tfrac{13}{2}J_5 + \tfrac{7}{2}K_{10} - \tfrac{17}{2}J_3)\,K_2^2 + (2K_6^2 + (5J_2 - 5J_1^2 + K_7)\,K_6 - \tfrac{1}{2}K_7^2 + (-\tfrac{9}{4}J_2 + \tfrac{19}{4}J_1^2)\,K_7 + 8J_1 K_{13} + \tfrac{1}{4}J_2^2 + 2J_1 K_{24} - 5J_1 K_{12} + 12K_{28} - 4J_{24} - J_1 K_{17} - 6J_1^2 J_{17} - 3K_{21}K_9 - 12K_{27} - \tfrac{1}{4}J_1^4 - 3K_{33} + \tfrac{11}{2}J_1 K_8)\,K_5 + (3K_{21}K_7 + 6J_1 K_{11} + (15J_{17} + 6J_2)\,K_{21} + 3K_{32} - 12K_{23} + 2J_1 K_{31})\,K_6,$$

$$\begin{aligned}Q_{28} = {} & J_{28} + 2K_{29} + J_1 K_{27} + 2J_1 K_{33} - J_2 K_1^3 - 4K_{21} K_{14} + \\ & 2J_5 K_{31} + (4J_1 K_6 + J_1 K_4 + J_1 K_7 - 2K_{12} + 6K_{25})\, K_3 - J_1 K_{28} + \\ & (2J_1 K_{10} - 2J_{26} - 2K_{14} - 4K_{15} - \tfrac{1}{1} J_1 J_5 - J_6 + 2K_{16} + \tfrac{1}{2} J_1 J_4)\, K_2 - \\ & 2J_5 K_{11} + (-2J_1 K_6 + J_{19} + 5K_{17} + \tfrac{1}{2} J_1 J_{17})\, K_4 + (-J_1 K_7 - \\ & \tfrac{1}{4} J_1^3 - 3J_1 K_3)\, K_1^2 + (-2J_6 + K_{14} - 2J_1 J_4)\, K_5 + (-\tfrac{1}{2} J_1 J_2 + \\ & \tfrac{1}{2} J_1^3 - 4K_8 + K_{12} + 2K_{17} - 2J_1 J_{17})\, K_6 + (2J_{20} + K_{17} + K_8 + 2K_{25} - \\ & 3J_{19} + 2J_1 J_{17} + \tfrac{1}{2} J_1 J_2 - \tfrac{1}{4} J_1^3 + \tfrac{5}{2} K_{12} - \tfrac{1}{2} J_1^2)\, K_{25} + (2J_{17} + \\ & \tfrac{1}{2} J_2)\, K_{12} + (J_3 - J_5)\, K_{22} + ((3J_4 - 2K_9 + J_3)\, K_2 + 4K_3^2 + \\ & (K_4 - \tfrac{1}{2} K_6 - J_{17})\, K_3 + 2K_4^2 + (-K_9 - \tfrac{1}{2} J_3 + \tfrac{1}{2} J_4)\, K_5 - \\ & K_6 K_7 + (-J_2 - 2J_{17})\, K_7 - 2K_{21} K_9 + 2J_1 K_{12} + (-J_3 + J_4)\, K_{21} - \\ & \tfrac{1}{2} J_1 J_{20} - 4J_1 K_{25} - \tfrac{1}{2} J_1^2 J_{17} + 2J_1 K_{24} + J_{24})\, K_1 + \\ & (4J_{17} + 3J_2 - J_1^2)\, K_{17} - \tfrac{1}{2} J_1^2 K_{13} + (\tfrac{1}{2} J_1 J_5 - J_6)\, K_{21} + J_1 K_7^2,\end{aligned}$$

$$\begin{aligned}Q_{29} = {} & -K_1^3 K_6 + (\tfrac{3}{2} K_{13} - J_1 J_{17} - 2K_{24} - 7K_8)\, K_1^2 + \\ & ((-6J_5 + 5J_3 + J_4)\, K_2 + 3K_3^2 + 2K_4^2 + (-J_{17} - K_6 - J_1^2)\, K_4 - \\ & K_{10} K_5 + K_6^2 + (-3J_1^2 + K_7)\, K_6 - J_{17}^2 - J_{23} + 3J_1^2 K_7 - \\ & 2K_{27} - 2J_1 J_{20} + 2J_{22} + J_1 K_{25} + 2K_{28})\, K_1 + (-J_6 + J_1 J_4 - \\ & 17K_{15} + J_{25} + 5K_{16} + 12K_{14} - 4J_{26})\, K_2 + (9J_1 K_6 - 12K_{12} + K_{17} + \\ & J_1 K_4 + 11K_{13} - 6J_1 K_7)\, K_3 + J_1 K_4^2 + (K_{25} + 2J_{20} - 2K_{13} + \\ & 2J_{19} - 5J_1 K_7 + 2J_1 K_6 + 4K_{24})\, K_4 + (4J_{25} - K_{14} - 8J_{26} - K_{18} - \\ & 6J_1 J_3 + 3J_1 J_4 - 4K_{15} - 2J_6)\, K_5 + (5K_{13} + 4K_{24} - \tfrac{1}{2} J_1 J_2 + \\ & \tfrac{1}{2} J_1^3 - 3K_8 + K_{17} + 2K_{25})\, K_6 + (-K_{12} + 12K_8 + \tfrac{5}{2} K_{13})\, K_7 + \\ & (K_{22} + 4K_{31})\, K_9 + (-4K_{11} - 4K_{31})\, K_{10} + 4J_{17} K_{12} + (\tfrac{1}{2} J_2 - \\ & \tfrac{1}{2} J_1^2 - 2J_{17})\, K_{13} + 2K_{21} K_{18} + 2J_4 K_{22} + (-J_2 + J_1^2)\, K_{24} + \\ & 2J_{17} K_{25} + J_{29},\end{aligned}$$

$$\begin{aligned}Q_{30} = {} & (3K_9 - 2K_{10} - \tfrac{1}{2} J_4)\, K_1^2 + (-J_1 K_{10} + \tfrac{1}{2} J_1 K_9 - J_{25} + J_{26} - \\ & 2K_{18})\, K_1 + (-\tfrac{3}{2} J_8 - \tfrac{1}{2} J_7)\, K_2 + (-2K_9 + 2K_{10})\, K_3 + K_4 K_9 + \\ & (-2J_7 + J_9 - J_8)\, K_5 + (J_4 + J_5 - J_3)\, K_6 + (-K_9 + \tfrac{1}{2} J_5)\, K_7 + \\ & J_1 K_{18} + J_{17} K_9 + K_{19} + 2K_{30} - J_8 K_{21} - J_{17} K_{10},\end{aligned}$$

$$\begin{aligned}Q_{31} = {} & (-2K_{10} - \tfrac{1}{2} J_3 + 3K_9)\, K_1^2 + (\tfrac{1}{4} J_1 J_4 - \tfrac{1}{2} J_1 J_3 - 2K_{18} - \tfrac{1}{4} J_1 J_5 + \\ & K_{14} + \tfrac{1}{2} J_6 - 2J_{25})\, K_1 - J_7 K_2 - \tfrac{1}{2} J_4 K_3 + (\tfrac{1}{2} J_5 + \tfrac{1}{2} J_3)\, K_4 + \\ & (-3J_7 + J_9)\, K_5 + (J_3 + J_5)\, K_6 + (-K_9 - \tfrac{3}{2} J_3)\, K_7 + (2J_{17} - \tfrac{1}{2} J_2 + \\ & \tfrac{1}{4} J_1^2)\, K_9 + J_1 K_{18} + 2K_{30} - 2J_9 K_{21} + \tfrac{1}{2} J_1 K_{15},\end{aligned}$$

$$\begin{aligned}Q_{32} = {} & (\tfrac{1}{2} J_4 - 2K_9 g - 2J_3)\, K_1^2 + (-J_1 J_3 + J_6 - J_1 K_9 - J_{25} + 2K_{15})\, K_1 - \\ & J_4 K_3 + (-2K_{10} + J_4 + K_9)\, K_4 + (-J_7 - J_9)\, K_5 + J_4 K_6 - \tfrac{1}{2} J_4 K_7 + \\ & J_{17} K_9 + (\tfrac{1}{2} J_1^2 - 2J_{17} - \tfrac{1}{2} J_2)\, K_{10} J_1 K_{16},\end{aligned}$$

$$\begin{aligned}Q_{33} = {} & 2J_{17} J_5 K_{21} + K_{17}^2 + J_{33} - \tfrac{1}{4} K_7^3 - K_{12}^2 + 9K_{25}^2 + 4K_3^3 + \\ & 9K_{13}^2 - 3K_{21} K_{19} + (6J_1 K_8 - 5K_{21} K_9 + \tfrac{3}{4} J_1 K_{17} + (\tfrac{3}{2} J_5 + \\ & \tfrac{3}{2} J_4)\, K_{21} - 10K_{27} + \tfrac{3}{2} J_1 K_{25} + \tfrac{3}{2} J_1 J_{19} - 6J_{24} + 4K_{28} - \\ & \tfrac{3}{2} K_{33} - \tfrac{15}{2} J_{23} - \tfrac{7}{2} J_1^2 J_{17} + \tfrac{11}{2} J_1 J_{20})\, K_7 + \\ & (-\tfrac{1}{2} J_{20} - J_{19})\, K_{24} + (-6J_{19} + \tfrac{3}{2} K_{17} - 8K_{13} + 10K_{24} + 4J_1 J_{17})\, K_{12} +\end{aligned}$$

$$(\tfrac{3}{2}J_1J_5+J_{25}+2J_{26}-2J_1J_3)\,K_{31}+2K_{16}K_{31}-28K_{15}K_{31}+ (10K_{13}+6K_{25}+3K_{12}+K_{17})\,K_8+(-\tfrac{3}{2}J_{20}+3J_{19})\,K_{25}-J_7K_5^2- 3J_8K_{21}^2+((\tfrac{3}{2}J_{17}+2J_1^2)\,K_{21}-\tfrac{11}{2}J_{18}-10J_1K_{31}+12K_{32})\,K_9+ (\tfrac{3}{2}J_2+3J_{17})\,K_{33}+(-3J_1K_{22}+13J_1K_{11}+2J_{18}-\tfrac{1}{2}J_1^2K_{21}- 6K_{32}+3K_{23})\,K_{10}+(4K_{18}+11J_6-2K_{14}+\tfrac{1}{2}J_1J_5+16K_{16}-8K_{15}- 3J_1J_3+\tfrac{3}{2}J_1J_4)\,K_{11}+(\tfrac{3}{2}K_{17}-8K_{25}+6J_{19}-6K_{24})\,K_{13}- 6K_3^2K_4+((2J_1J_3-6J_1K_{10}-12J_{26}+9J_1K_9-\tfrac{29}{4}K_{18}+ 6K_{14})\,K_2+(2J_{19}+6K_{12}-J_1K_4)\,K_3+3K_4K_{24}+(-2K_{16}+2J_{26}- 6K_{15}-3K_{14}-J_1J_5)\,K_5+(-J_1^3+3K_{17}+2K_8)\,K_6-K_7K_{17}- 7J_{17}K_8+13K_{22}K_9+2J_1K_{21}K_{10}+\tfrac{1}{2}J_2K_{12}+(J_2+ 2J_1^2)\,K_{13}+\tfrac{7}{2}J_{17}K_{17}-6J_{26}K_{21}+(8J_4-7J_3-J_5)\,K_{22}+ 2J_1^2K_{24}-\tfrac{3}{2}J_{17}K_{25}-6J_1K_{27}+(-11J_4+20J_3)\,K_{31}+ 5J_{29}-4J_{28}-\tfrac{3}{2}J_1J_{22})\,K_1+(13K_{31}-3K_{22})\,K_{14}-6J_7K_2^2+ 6K_4^2K_6+\tfrac{3}{4}J_2K_7^2+((9J_5-6J_4+12K_{10}-4J_3)\,K_3+ (-15K_9+4J_4-J_3-10J_5)\,K_4+(-\tfrac{59}{4}J_7+\tfrac{3}{4}J_8)\,K_5+(17K_{10}- \tfrac{51}{4}K_9)\,K_6+\tfrac{3}{2}K_9K_7+(7J_2+3J_{17}-6J_1^2)\,K_9+(-\tfrac{15}{4}J_2+ 2J_{17})\,K_{10}+6K_{19}-10K_{30}+\tfrac{11}{2}J_1J_{26}-3J_1K_{16}-5J_1J_{25}- 2J_1K_{15}+2J_1K_{14})\,K_2-5K_{31}K_{18}+(\tfrac{1}{4}J_1J_2+6J_{19}-7K_{24}+ 3K_{25})\,K_{17}+(-K_4^2+9J_{17}K_4-6_4K_5+4J_1^2K_6-\tfrac{21}{4}J_2K_7- 7J_1K_{12}+J_1K_{17}+(3J_4+2J_3+3J_5)\,K_{21}-8K_{28}-\tfrac{3}{2}J_{23}+7K_{33}- \tfrac{3}{2}J_{17}^2+6K_{27})\,K_3+(-\tfrac{1}{2}K_{17}+\tfrac{3}{4}J_1J_2-\tfrac{3}{2}J_1J_{17})\,K_1^3+ (2K_{10}K_6+(\tfrac{1}{2}J_1^2-\tfrac{3}{2}J_2)\,K_9-3K_{30}-4J_{10}-\tfrac{1}{4}J_2J_5+J_1K_{14}- 3K_{19}-J_1K_{18}-\tfrac{1}{2}J_1^2J_5+3J_2J_3+6J_1K_{15}-4J_2J_4+ 2J_1J_{25}-\tfrac{9}{2}J_2K_{10}-4J_1J_6)\,K_5+(-5K_9K_5-\tfrac{3}{4}J_2K_7- 4K_{21}K_{10}-2J_1K_{13}+(-J_3+J_4)\,K_{21}+\tfrac{3}{2}J_{17}^2+7K_{33})\,K_4+ (-\tfrac{29}{4}K_{10}K_2+\tfrac{1}{2}J_1^2K_3+(-\tfrac{1}{2}K_{10}+8J_3-3J_4-\tfrac{1}{2}K_9+3J_5)\,K_5+ (-3J_2-\tfrac{5}{2}J_1^2)\,K_6+J_1K_{13}+2J_3K_{21}+3K_{27}+\tfrac{1}{4}K_7^2-2J_1J_{20}- \tfrac{3}{4}J_1K_{17}-\tfrac{3}{4}J_2K_7)\,K_1^2+(\tfrac{17}{2}J_1^2-3J_2-\tfrac{5}{2}K_7)\,K_6^2+(J_3- \tfrac{3}{2}J_5-3J_4)\,K_{23}+(-20J_1K_{13}-7K_{21}K_{10}+10K_{33}+\tfrac{27}{2}K_{21}K_9+K_7^2- 10K_{27}-2J_{24}-5J_{17}K_7-J_3K_{21}+6J_1K_{12}-7J_1K_8)\,K_6-\tfrac{7}{2}J_{25}K_{22}+ 8J_{17}K_{27}-\tfrac{9}{2}J_{17}K_{28}+\tfrac{3}{2}J_1K_{29},$$

$$Q_{34}=J_1K_{30}+2K_{21}K_{20}+J_{34}+(-7J_3-4J_4)\,K_{12}+\tfrac{1}{2}J_2K_{18}+(J_5+J_4+ 5J_3)\,K_{24}+(2J_8K_2+(-2K_{10}-2J_3+4J_5)\,K_3+(2J_3-2K_{10}- 6J_5)\,K_4+(-\tfrac{1}{2}J_8-\tfrac{1}{2}J_7)\,K_5+(3J_5-J_4)\,K_6+\tfrac{3}{2}K_9K_7+ (J_{17}+J_1^2)\,K_9-4J_1K_{14}-J_1^2K_{10}-J_{17}J_3+2K_{19}-\tfrac{3}{2}J_1K_{16}+ 4J_1K_{15})\,K_1-\tfrac{1}{2}K_1^3K_9+4K_8K_9+J_{17}K_{14}-5J_{17}K_{15}-2J_3K_{25}- J_9K_{22}-2J_{13}K_{21}-J_7K_{11}+(2K_{15}-5J_{26}+10J_1K_{10}+10K_{16}+ J_{25})\,K_3-3K_{20}K_2+(7J_{26}-4J_{25}+10K_{14}-8K_{16}-\tfrac{17}{2}J_1K_{10}- 4J_1K_9-K_{18})\,K_4+(\tfrac{1}{2}K_{18}-3K_{14}+K_{16}+\tfrac{5}{2}J_1K_9+J_1J_5)\,K_1^2- J_9K_{31}+(4J_{13}+K_{20})\,K_5+(-J_1K_{10}+\tfrac{1}{2}J_1J_5)\,K_6+(5J_4- 3J_3)\,K_{13}+(-2K_{24}-J_1J_{17}+6K_{13})\,K_{10}+(3J_{20}+2K_{24}-K_{17}- 2K_{13}-K_{12})\,K_9+(-\tfrac{1}{2}K_{18}-5J_6-\tfrac{1}{2}J_1K_9-J_{25})\,K_7,$$

$$Q_{35} = (-11K_{12} - 2K_{25} - 2K_{24} + 17K_{13} + 4K_{17})\,K_9 + 2J_8K_{31} + (6K_{25} - 7K_{12} - J_{20} - 2K_{24} - 20K_{17})\,K_{10} + (-3J_8 + 12J_9 - 6J_7)\,K_{11} + (\tfrac{1}{2}J_5 + 4J_3)\,K_{12} + (\tfrac{3}{2}J_5 + J_4 - 4J_3)\,K_{13} + (-\tfrac{1}{2}J_1^2 + \tfrac{1}{2}J_2 + J_{17})\,K_{14} + J_1K_{30} + J_5K_{24} + (15J_4 + 4J_5 + J_3)\,K_{17} + J_1K_{19} + (-J_1^2 + 3J_2)\,K_{18} + (-\tfrac{1}{2}J_1J_7 + \tfrac{1}{2}J_1J_8)\,K_{21} + (-\tfrac{3}{2}J_5 + 2J_3)\,K_1^3 + (J_9 - 2J_8)\,K_{22} - 6K_{21}K_{20} + (\tfrac{5}{4}J_1J_5 - 4J_6 - 3J_1J_3 + J_1J_4)\,K_1^2 + (-J_4 + J_5)\,K_{25} + (J_7K_2 + 15K_3K_9 + (10J_3 - 6J_4 - 11K_9)\,K_4 + 8J_9K_5 + (26J_3 - 12K_{10} - 3J_5 - 15J_4 + 22K_9)\,K_6 + (6K_{10} + \tfrac{3}{2}J_4)\,K_7 + 6K_{30} + J_1J_{25} - J_2K_{10} - 2J_7K_{21})\,K_1 + (-\tfrac{7}{4}J_1J_8 - 5K_{20} - \tfrac{5}{4}J_1J_7 + \tfrac{1}{2}J_{12} - \tfrac{3}{2}J_{13} + 5J_{11})\,K_2 + (-3K_{14} + J_6 + 6K_{18} - \tfrac{3}{2}J_1K_9 + K_{15} + \tfrac{3}{4}J_1J_5)\,K_7 + (-8K_9 + 7K_{10})\,K_8 + (3J_1K_9 + 6J_1K_{10} + 7K_{16} + J_{25} - J_{26} - K_{14} - 8K_{15})\,K_4 + (3J_{11} - 4J_1J_9 - 9J_{12})\,K_5 + (6J_6 - 26K_{15} - 20K_{18})\,K_6 + (2J_{26} - 3J_{25} + K_{14} - 5J_1K_{10} - 6K_{15} + J^1J_9)\,K_3 + J_{35},$$

$$Q_{36} = -\tfrac{1}{2}J_8K_1^2 + (2K_{20} - \tfrac{1}{4}J_1J_8 - \tfrac{1}{4}J_1J_9)\,K_1 + (\tfrac{1}{2}J_9 + \tfrac{1}{2}J_7)\,K_3 + (\tfrac{1}{2}J_8 - + (\tfrac{1}{2}J_8 - \tfrac{1}{2}J_7)\,K_6 - \tfrac{1}{2}J_7K_7 + (J_3 - K_{10})\,K_9 + K_{10}^2 + (-\tfrac{1}{2}J_5 + \tfrac{1}{2}J_4 - J_3)\,K_{10} + J_{36} + \tfrac{1}{2}J_1K_{20}.$$

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