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Hawking temperature of Kerr–Newman–AdS black hole from tunneling

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ABSTRACT

Using the null-geodesic tunneling method of Parikh and Wilczek, we derive the Hawking temperature of a general four-dimensional rotating black hole. In order to eliminate the motion of ϕ degree of freedom of a tunneling particle, we have chosen a reference system that is co-rotating with the black hole horizon. Then we give the explicit result for the Hawking temperature of the Kerr–Newman–AdS black hole from the tunneling approach.

1. Introduction

Since the original discovery of black hole radiation by Hawking [1], the studies on this topic have not terminated. There are many different methods for the derivation of Hawking radiation [2–9]. In [10,11], a semiclassical method for the derivation of Hawking radiation was formulated by Parikh and Wilczek based on the quantum tunneling picture. In such a method, the radiated particles of a black hole are treated as s-waves.[1] When a particle is radiated from the black hole horizon, it tunnels through a barrier that is made by the tunneling particle itself due to the horizon's contraction [10,11]. To use the WKB approximation, the tunneling rate of an s-wave from inside to outside the black hole horizon is given by

$$\Gamma = \Gamma_0 \exp(-2\,\mathrm{Im}\,\mathcal{I}). \tag{1}$$

Here, $\mathcal{I}$ is the action of the tunneling particle, Γ_0 is a normalization factor. On the other hand, a black hole's radiation satisfies the law of Boltzmann distribution classically, thus the emission rate of a particle of energy E from a black hole horizon can be expressed by

$$\Gamma = \Gamma_0 \exp(-\beta E), \tag{2}$$

where $\beta = 2\pi/\kappa$, κ is the surface gravity of the horizon. To compare (2) with (1), the Hawking temperature of a black hole can be derived.

After the original work of Parikh and Wilczek, many developments on this topic have been carried out [12–15], and many applications of this method for the derivation of Hawking radiation of different types of black holes have been done [16–27]. In this Letter, we study the Hawking radiation of general four-dimensional rotating black holes from the tunneling approach. We use the null-geodesic method of Parikh and Wilczek [10,11] to calculate the action of a tunneling particle. In [21], Hawking temperature of Kerr and Kerr–Newman black holes have been derived from tunneling approach using dragging coordinate systems. In such a kind of coordinate system, the spacetime of a four-dimensional rotating black hole has been contracted to a three-dimensional slice. Thus the topology of the spacetime of a rotating black hole has been changed to use the method of [21]. In order to keep the spacetime topology of a rotating black hole, we choose a reference system that is co-rotating with the event horizon to eliminate the motion of ϕ degree of freedom of a tunneling particle. We obtain that for a general four-dimensional rotating black hole, its thermal radiation temperature derived from the tunneling approach is in accordance with its Hawking temperature derived from black hole thermodynamics. These contents are given in Section 2. In Section 3, we give the explicit result of the Hawking temperature of the Kerr–Newman–AdS black hole from the tunneling approach. In Section 4, we discuss some of the problems.

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[1] This is reasonable because for an observer at infinity, the radiation of a black hole is spherically symmetric, no matter whether the black hole is rotating or not.

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2. Hawking temperature of four-dimensional rotating black holes from tunneling

The metric of a four-dimensional spherically symmetric black hole can be expressed as

$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2\,d\Omega^2 \tag{3}$$

generally. Following [10,11], the imaginary part of the action of a tunneling particle in terms of an s-wave can be calculated from

$$\operatorname{Im}\mathcal{I} = \operatorname{Im}\int_{r_h(M)}^{r_h(M-E)} p_r\,dr = \operatorname{Im}\int_{r_h(M)}^{r_h(M-E)}\int_0^{p_r} dp_r'\,dr, \tag{4}$$

where r_h is the radius of the outer horizon, M is the total mass of the black hole, $M-E$ is the total mass of the black hole after the particle is emitted, E is the energy of the tunneling particle. To make use of the Hamilton's equation

$$\dot r = \frac{dH}{dp_r} = \frac{d(M-\omega)}{dp_r}, \tag{5}$$

we can write

$$dp_r = \frac{d(M-\omega)}{\dot r}. \tag{6}$$

To substitute (6) into (4), we have

$$\operatorname{Im}\mathcal{I} = \operatorname{Im}\int_{r_h(M)}^{r_h(M-E)}\int_0^{E}\frac{d(M-\omega)}{\dot r}\,dr = \operatorname{Im}\int_0^{E}\int_{r_h(M-E)}^{r_h(M)}\frac{dr}{\dot r}\,d\omega. \tag{7}$$

Usually, the Hawking temperature of a black hole is very small, zero-mass particles will possess the main part of the whole radiation spectrum. For a tunneling particle of zero-mass in terms of an s-wave, it moves in a radial null geodesic. To transform the metric (3) to the Painlevé form, $\dot r$ can be obtained from $ds^2 = 0$ [10,11].

The metric of a four-dimensional rotating black hole can be cast in the form

$$\begin{aligned} ds^2 &= -g_{tt}(r,\theta)\,dt^2 + g_{rr}(r,\theta)\,dr^2 + g_{\theta\theta}(r,\theta)\,d\theta^2 \\ &\quad + g_{\phi\phi}(r,\theta)\,d\phi^2 - 2g_{t\phi}(r,\theta)\,dt\,d\phi \end{aligned} \tag{8}$$

generally. For the tunneling of a rotating black hole, we can still use the s-wave approximation, this is because for an observer at infinity, the radiation of a rotating black hole is still spherically symmetric. However, when a particle is tunneling through the horizon of a rotating black hole, it will be dragged by the rotation of the black hole. Thus, a tunneling particle will have motion in the ϕ degree of freedom, i.e. $d\phi \neq 0$, which means that in the calculation of the action of a tunneling particle in formula (4), we need also to consider the contribution to the action that comes from the motion on the ϕ degree of freedom, as we can see in [21]. Meanwhile, in the equation of the null geodesic, we cannot set $d\phi = 0$, thus, $\dot r$ cannot be obtained from $ds^2 = 0$ conveniently.

In order to eliminate the motion of ϕ degree of freedom of a tunneling particle, we can choose a reference system that is co-rotating with the black hole horizon. This can be realized through the rotating coordinate transformation

$$\phi' = \phi - \Omega_h t \quad \text{or} \quad \phi = \phi' + \Omega_h t, \tag{9}$$

where Ω_h is the angular velocity of the event horizon of a rotating black hole, which is a constant and is defined by

$$\Omega_h = \frac{g_{t\phi}}{g_{\phi\phi}}\bigg|_{r=r_h}. \tag{10}$$

In (10) and in the following, we use r_h to represent the radius of the event horizon of a rotating black hole. In such a co-rotating reference system, the observers located at the horizon cannot observe the rotation of the black hole, they will find that the angular velocity Ω_h' of the black hole is zero. Because the tunneling of a particle takes place at the horizon, it will not be dragged by the rotation of the black hole to observe from such a co-rotating reference system. Therefore we have $d\phi' = 0$ for a tunneling particle, i.e., a tunneling particle has no motion in the ϕ' degree of freedom. This makes us be able to use Eq. (4) to calculate the action. Meanwhile, in obtaining the expression of $\dot r$ from the null-geodesic method, we can set $d\phi' = 0$.

Under the coordinate transformation (9), the metric (8) turns to the form

$$\begin{aligned} ds^2 &= -G_{tt}(r,\theta)\,dt^2 + g_{rr}(r,\theta)\,dr^2 + g_{\theta\theta}(r,\theta)\,d\theta^2 \\ &\quad + g_{\phi\phi}(r,\theta)\,d\phi'^2 - 2g_{t\phi}'(r,\theta)\,dt\,d\phi', \end{aligned} \tag{11}$$

where

$$G_{tt} = g_{tt} + 2g_{t\phi}\Omega_h - g_{\phi\phi}\Omega_h^2, \tag{12}$$

$$g_{t\phi}' = g_{t\phi} - \Omega_h g_{\phi\phi}. \tag{13}$$

Because of (10), we have

$$g_{t\phi}'\big|_{r=r_h} = 0. \tag{14}$$

This also indicates $\Omega_h' = g_{t\phi}'/g_{\phi\phi}|_{r=r_h} = 0$. On the other hand, according to (A.5), we have

$$G_{tt}\big|_{r=r_h} = 0. \tag{15}$$

The horizon's radius of the metric (11) is determined by $g^{rr}|_{r=r_h} = g_{rr}^{-1}|_{r=r_h} = 0$, which is the same equation of the horizon's radius of the metric (8), thus, the horizon's radius of a rotating black hole will not be changed under the coordinate transformation (9). On the other hand, because of (15), the horizon's radius for the metric (11) is also determined by (15).

Because in metric (11), g_{rr} is singular on the horizon, in order to calculate the action of a tunneling particle, we need to eliminate such a coordinate singularity first. This can be realized through the Painlevé coordinate transformation [10,11]. We use T to represent the Painlevé time coordinate and make a coordinate transformation

$$dt = dT - \sqrt{\frac{g_{rr}(r,\theta_0) - 1}{G_{tt}(r,\theta_0)}}\,dr \tag{16}$$

to the metric (11). In (16), like that in [15] in studying the tunneling from Kerr–Newman black hole, we have set θ to be a constant in order to make the coordinate transformation integrable. Such a manipulation is reasonable because for a tunneling particle in terms of an s-wave, it satisfies $d\theta = 0$, therefore we can consider the tunneling of a particle at a constant angle θ_0. At last we can obtain that the physical result does not depend on the angle θ_0. However, the explicit integral of (16) is not needed to be given here. Under the above coordinate transformation, for the metric (11), we have

$$\begin{aligned} ds^2 &= -G_{tt}(r,\theta_0)\,dT^2 + 2\sqrt{G_{tt}(r,\theta_0)}\sqrt{g_{rr}(r,\theta_0) - 1}\,dr\,dT \\ &\quad + dr^2 + g_{\phi\phi}(r,\theta_0)\,d\phi'^2 \\ &\quad - 2g_{t\phi}'(r,\theta_0)\,d\phi'\left(dT - \sqrt{\frac{g_{rr}(r,\theta_0) - 1}{G_{tt}(r,\theta_0)}}\,dr\right). \end{aligned} \tag{17}$$

The horizon's radius for the metric (17) is determined by $G_{tt}|_{r=r_h} = 0$, thus, the horizon's radius for the metric (11) is not changed after the coordinate transformation (16). As mentioned above, for a tunneling particle in the co-rotating reference system, it satisfies

$d\phi' = 0$. Thus we have

$$ds^2 = -G_{tt}(r,\theta_0)\,dT^2 + 2\sqrt{G_{tt}(r,\theta_0)}\sqrt{g_{rr}(r,\theta_0)-1}\,dr\,dT + dr^2. \tag{18}$$

To suppose that the mass of the tunneling particle is zero, then its motion is determined by the null-geodesic equation $ds^2 = 0$. To solve this equation, we obtain

$$\dot r = \sqrt{G_{tt}(r,\theta_0)\cdot g_{rr}(r,\theta_0)}\left(\pm 1 - \sqrt{1 - \frac{1}{g_{rr}(r,\theta_0)}}\right). \tag{19}$$

Because $G_{tt}|_{r=r_h} = 0$, $g_{rr}^{-1}|_{r=r_h} = 0$, r_h is a simple zero point of G_{tt} and g_{rr}^{-1}, $G_{tt}\cdot g_{rr}$ should be regular at the horizon. The plus and minus signs in (19) correspond to outgoing and ingoing radial null geodesics respectively. For an outgoing tunneling particle, $\dot r$ is positive, we have

$$\dot r = \sqrt{G_{tt}(r,\theta_0)\cdot g_{rr}(r,\theta_0)}\left(1 - \sqrt{1 - \frac{1}{g_{rr}(r,\theta_0)}}\right). \tag{20}$$

To substitute (20) into (7), we obtain

$$\operatorname{Im}\mathcal{I} = \operatorname{Im}\int_0^E \int_{r_h(M-E)}^{r_h(M)} \frac{dr}{\sqrt{G_{tt}(r,\theta_0)\cdot g_{rr}(r,\theta_0)}\left(1 - \sqrt{1 - \frac{1}{g_{rr}(r,\theta_0)}}\right)}\,d\omega. \tag{21}$$

To multiply $1 + \sqrt{1 - \frac{1}{g_{rr}(r,\theta_0)}}$ in the numerator and denominator of the integrand at the same time, we obtain

$$\operatorname{Im}\mathcal{I} = \operatorname{Im}\int_0^E \int_{r_h(M-E)}^{r_h(M)} \frac{1 + \sqrt{1 - \frac{1}{g_{rr}(r,\theta_0)}}}{\sqrt{G_{tt}(r,\theta_0)\cdot g_{rr}(r,\theta_0)}\,\frac{1}{g_{rr}(r,\theta_0)}}\,dr\,d\omega. \tag{22}$$

For the metric of a four-dimensional rotating black hole, because g_{rr} is singular on the horizon, generally, we can write g_{rr} in the form

$$g_{rr}(r,\theta) = \frac{C(r,\theta)}{r - r_h}, \tag{23}$$

where $C(r,\theta)$ is a function regular on the horizon. To substitute (23) into (22), we have

$$\operatorname{Im}\mathcal{I} = \operatorname{Im}\int_0^E \int_{r_h(M-E)}^{r_h(M)} \frac{1 + \sqrt{1 - \frac{r-r_h}{C(r,\theta_0)}}}{\sqrt{G_{tt}(r,\theta_0)\cdot g_{rr}(r,\theta_0)}\,\frac{r-r_h}{C(r,\theta_0)}}\,dr\,d\omega. \tag{24}$$

In (24), r_h is a simple pole of the integrand. To add a small imaginary part to the variable r, and to let the integral path round the pole in a semicircle, the integral of dr can be evaluated which results

$$\operatorname{Im}\mathcal{I} = 2\pi\int_0^E \frac{C(r_h,\theta_0)}{\sqrt{G_{tt}(r_h,\theta_0)\cdot g_{rr}(r_h,\theta_0)}}\,d\omega. \tag{25}$$

It is reasonable to suppose that the energy E of the tunneling particle is far less than the total mass M of the black hole, i.e. $E \ll M$, thus, in (25), the integrand can be treated as a constant. Therefore we obtain

$$\operatorname{Im}\mathcal{I} = 2\pi E\frac{C(r_h,\theta_0)\sqrt{g^{rr}(r_h,\theta_0)}}{\sqrt{G_{tt}(r_h,\theta_0)}}. \tag{26}$$

Because $G_{tt}(r_h,\theta) = 0$, $g^{rr}(r_h,\theta) = 0$, near the horizon, we can expand $G_{tt}(r,\theta_0)$ and $g^{rr}(r,\theta_0)$ in the form

$$G_{tt}(r,\theta_0) = G'_{tt}(r_h,\theta_0)(r - r_h) + \cdots, \tag{27}$$

$$g^{rr}(r,\theta_0) = g^{rr\prime}(r_h,\theta_0)(r - r_h) + \cdots, \tag{28}$$

where in (27) and (28), "$\cdots$" represents higher-order terms of $(r - r_h)$. From (23), we have

$$g^{rr\prime}(r_h,\theta_0) = \frac{1}{C(r_h,\theta_0)}. \tag{29}$$

To substitute (27)–(29) into (26), we obtain

$$\operatorname{Im}\mathcal{I} = \frac{2\pi E}{\sqrt{G'_{tt}(r_h,\theta_0)g^{rr\prime}(r_h,\theta_0)}}. \tag{30}$$

To substitute (30) into (1), we can see that the tunneling rate can be cast in the form of (2), which is the Boltzmann distribution, and we obtain

$$\operatorname{Im}\mathcal{I} = \frac{\pi E}{\kappa(r_h)}. \tag{31}$$

To compare (31) with (30), we obtain

$$\kappa(r_h) = \frac{\sqrt{G'_{tt}(r_h,\theta_0)g^{rr\prime}(r_h,\theta_0)}}{2}. \tag{32}$$

Thus, we obtain the thermal temperature of a four-dimensional rotating black hole

$$T_H = \frac{\sqrt{G'_{tt}(r_h,\theta_0)g^{rr\prime}(r_h,\theta_0)}}{4\pi}. \tag{33}$$

Eq. (33) is derived from the tunneling approach. On the other hand, in Appendix A, we have derived a formula (A.12) for the surface gravity of a four-dimensional rotating black hole from black hole thermodynamics which is given by

$$\kappa(r_h) = \lim_{r\to r_h}\frac{\partial_r\sqrt{G_{tt}}}{\sqrt{g_{rr}}} = \lim_{r\to r_h}\frac{\partial_r G_{tt}}{2\sqrt{G_{tt}\cdot g_{rr}}}. \tag{34}$$

From black hole thermodynamics [28,29], we know that on the horizon, $\kappa(r_h)$ is a constant, therefore we can evaluate it at an arbitrary angle θ_0. To substitute (27) and (28) into (34), we obtain

$$\kappa(r_h) = \frac{\sqrt{G'_{tt}(r_h,\theta_0)g^{rr\prime}(r_h,\theta_0)}}{2}. \tag{35}$$

To compare (32) with (35), we can see that they are equivalent. Because $\kappa(r_h)$ is a constant on the horizon, the explicit result for the surface gravity of a rotating black hole obtained from (35) will not depend on the parameter θ_0. This means that in (32) and (33), the explicit results for the surface gravity and Hawking temperature of a rotating black hole will not depend on the parameter θ_0 either.

3. Hawking temperature of Kerr–Newman–AdS black hole

In this section, we derive the Hawking temperature of the Kerr–Newman–AdS black hole using (33) of Section 2. In the Boyer–Lindquist coordinates, the metric of the Kerr–Newman–AdS is given by [30]

$$\begin{aligned} ds^2 = {} & -\frac{1}{\Sigma}\left[\Delta_r - \Delta_\theta a^2\sin^2\theta\right]dt^2 + \frac{\Sigma}{\Delta_r}dr^2 + \frac{\Sigma}{\Delta_\theta}d\theta^2 \\ & + \frac{1}{\Sigma\Xi^2}\left[\Delta_\theta\left(r^2+a^2\right)^2 - \Delta_r a^2\sin^2\theta\right]\sin^2\theta\,d\phi^2 \\ & - \frac{2a}{\Sigma\Xi}\left[\Delta_\theta(r^2+a^2) - \Delta_r\right]\sin^2\theta\,dt\,d\phi, \end{aligned} \tag{36}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Xi = 1 + \frac{1}{3}\Lambda a^2, \tag{37}$$

$$\Delta_\theta = 1 + \frac{1}{3}\Lambda a^2\cos^2\theta,$$

$$\Delta_r = \left(r^2+a^2\right)\left(1 - \frac{1}{3}\Lambda r^2\right) - 2Mr + Q^2, \tag{38}$$

Λ is the cosmological constant, $\Lambda < 0$. The horizons of the metric (36) are determined by

$$\Delta_r = (r^2 + a^2)\left(1 - \frac{1}{3}\Lambda r^2\right) - 2Mr + Q^2 = -\frac{1}{3}\Lambda\left[r^4 - \left(\frac{3}{\Lambda} - a^2\right)r^2 + \frac{6M}{\Lambda}r - \frac{3}{\Lambda}(a^2 + Q^2)\right] = -\frac{1}{3}\Lambda(r - r_{++})(r - r_{--})(r - r_+)(r - r_-) = 0. \quad (39)$$

The equation $\Delta_r = 0$ has four roots [31], where r_{++} and r_{--} are a pair of complex conjugate roots, r_+ and r_- are two real positive roots, and we suppose $r_+ > r_-$. Thus, $r = r_+$ is the event horizon. We first calculate the Hawking temperature at a special value of θ_0 and we choose $\theta_0 = 0$. To expand $g^{rr}(r, \theta_0 = 0)$ near the event horizon $r_h = r_+$, we obtain

$$g^{rr}(r, \theta_0 = 0) = \frac{-\frac{1}{3}\Lambda(r_+ - r_{++})(r_+ - r_{--})(r_+ - r_-)}{r_+^2 + a^2}(r - r_+) + \cdots, \quad (40)$$

where "$\cdots$" are higher-order terms of $(r - r_+)$. G_{tt} is defined by (12). We can rewrite it in the form

$$G_{tt} = g_{tt} + g_{t\phi}\Omega_h + (g_{t\phi} - \Omega_h g_{\phi\phi})\Omega_h = g_{tt} + g_{t\phi}\Omega_h + g'_{t\phi}\Omega_h \quad (41)$$

generally. According to (14), $g'_{t\phi}$ is zero on the horizon, thus the last term of (41) does not need to be considered when we expand $G_{tt}(r, \theta_0)$ near the horizon. The angular velocity of the Kerr–Newman–AdS black hole defined by (10) is $\Omega_h = \frac{a\Xi}{r_+^2 + a^2}$. At $\theta_0 = 0$, $G_{tt}(r, \theta_0 = 0)$ can be expanded as

$$G_{tt}(r, \theta_0 = 0) = \frac{-\frac{1}{3}\Lambda(r_+ - r_{++})(r_+ - r_{--})(r_+ - r_-)}{r_+^2 + a^2}(r - r_+) + \cdots, \quad (42)$$

where "$\cdots$" are higher-order terms of $(r - r_+)$. To compare (42) and (40) with (27) and (28), we can obtain $G'_{tt}(r_+, \theta_0 = 0)$ and $g^{rr\prime}(r_+, \theta_0 = 0)$. To substitute $G'_{tt}(r_+, \theta_0 = 0)$ and $g^{rr\prime}(r_+, \theta_0 = 0)$ into (33), we obtain, for the Kerr–Newman–AdS black hole,

$$T_H = -\frac{\Lambda}{12\pi(r_+^2 + a^2)}(r_+ - r_{++})(r_+ - r_{--})(r_+ - r_-). \quad (43)$$

Because $\Lambda < 0$, r_+ and r_- are positive, $r_+ > r_-$, r_{++} and r_{--} are complex conjugate, these make sure that T_H is positive. At an arbitrary value of θ_0, through explicit calculation, $g^{rr}(r, \theta_0)$ and $G_{tt}(r, \theta_0)$ can be expanded as

$$g^{rr}(r, \theta_0) = \frac{-\frac{1}{3}\Lambda(r_+ - r_{++})(r_+ - r_{--})(r_+ - r_-)}{r_+^2 + a^2\cos^2\theta_0} \times (r - r_+) + \cdots, \quad (44)$$

$$G_{tt}(r, \theta_0) = \frac{-\frac{1}{3}\Lambda(r_+ - r_{++})(r_+ - r_{--})(r_+ - r_-)(r_+^2 + a^2\cos^2\theta_0)}{(r_+^2 + a^2)^2} \times (r - r_+) + \cdots. \quad (45)$$

To compare (45) and (44) with (27) and (28), we can obtain $G'_{tt}(r_+, \theta_0)$ and $g^{rr\prime}(r_+, \theta_0)$. To substitute $G'_{tt}(r_+, \theta_0)$ and $g^{rr\prime}(r_+, \theta_0)$ into (33), we obtain again

$$T_H = -\frac{\Lambda}{12\pi(r_+^2 + a^2)}(r_+ - r_{++})(r_+ - r_{--})(r_+ - r_-). \quad (46)$$

In [32], another expression for the Hawking temperature of the Kerr–Newman–AdS black hole has been obtained which is given by

$$T_H = \frac{3r_+^4 + (a^2 + l^2)r_+^2 - l^2(a^2 + Q^2)}{4\pi l^2 r_+(r_+^2 + a^2)}, \quad (47)$$

where $\Lambda = -3/l^2$. It is not difficult to verify that these two expressions of T_H for the Kerr–Newman–AdS black hole are equivalent. The result of (46) is also equal to that obtained from (A.13) and (A.14). From this example, we can also see that the explicit result of the Hawking temperature given by (33) does not depend on the parameter θ_0.

In the case $\Lambda = 0$, the metric (36) degenerates to the metric of four-dimensional Kerr–Newman black hole. If the charge is zero, the metric will be the Kerr black hole. Following the same approach as above, we can also obtain their Hawking temperature from tunneling.

4. Discussion

In this paper, we have studied the Hawking radiation of general four-dimensional rotating black holes using the tunneling method of Parikh and Wilczek [10,11]. We obtain that the tunneling rate of a zero-mass particle is given by

$$\Gamma = \Gamma_0 \exp(-\beta E) = \Gamma_0 \exp(-E/T_H), \quad (48)$$

which is just the Boltzmann distribution. The thermal temperature T_H of a four-dimensional rotating black hole is given by (33), which is in accordance with the Hawking temperature derived from black hole thermodynamics. And we have given the explicit result for the Hawking temperature of the Kerr–Newman–AdS black hole from the tunneling approach. In order to eliminate the motion of ϕ degree of freedom of a tunneling particle from a rotating black hole, we choose a reference system that is co-rotating with the black hole horizon. In such a co-rotating reference system, we avoided the dimension degeneration in the method of dragging coordinate system adopted in [21] for the tunneling of a rotating black hole.

It is necessary to point out that if we use the method of [21] to calculate the action of a tunneling particle directly for the Kerr–Newman–AdS black hole, then we need to consider the action that comes from the motion on the ϕ degree of freedom, the calculation will be rather complicated in this case. In order to simplify the calculation, we have made a rotating coordinate transformation first. At the same time, the method provided in this paper is general for a general four-dimensional rotating black hole. And then we applied our result to the special case of the Kerr–Newman–AdS black hole. Another point needed to point out here is that there are some overlaps between the approach of this paper and the manipulation of the tunneling from the Kerr–Newman black hole in [15] using the null-geodesic method. The difference lies in that in [15] the rotating coordinate transformation for the tunneling of a rotating black hole was not proposed clearly, and it has not been used to a general four-dimensional rotating black hole. While in this Letter, we have studied the Hawking temperature of a general four-dimensional rotating black hole from tunneling using the rotating coordinate transformation clearly, and then we applied our result to the special case of the Kerr–Newman–AdS black hole. An alternative method for the calculation of the action of a tunneling particle was proposed in [14] from the Hamilton–Jacobi equation approach. Such a method was applied to the tunneling of some rotating black holes in [14,15]. For the tunneling of the Kerr–Newman–AdS black hole, to use the Hamilton–Jacobi equation method of [14], we can also make a rotating coordinate transformation first to simplify the calculation. The same results of (33) and (46) will be obtained at last. However, limited by the length of this Letter, we will not give such a derivation further in this Letter.

The tunneling rate (48) and Hawking temperature (33) for a rotating black hole are obtained in the reference system co-rotating with the black hole horizon. However, because the obtained tunneling rate and Hawking temperature of a black hole are scalars, they will not change for an observer static relatively to infinity. Thus, we can deduce that for an observer static relatively to infinity, the tunneling rate and Hawking temperature of a four-dimensional rotating black hole are still given by (48) and (33). The difference lies in that, for a tunneling particle, or an observer, the angular velocity of a rotating black hole is zero in the co-rotating reference system, while it is Ω_h in the static reference system. To combine the first law of black hole thermodynamics, we can generalize the tunneling rate (48) to a particle with non-zero angular momentum and non-zero charge.

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Appendix A. Hawking temperature of four-dimensional rotating black holes from black hole thermodynamics

In this appendix, we give an expression for the Hawking temperature of a four-dimensional rotating black hole from black hole thermodynamics. The metric of a four-dimensional rotating black hole is given by (8) generally. For the metric (8), there exists the Killing field

$$\xi^\mu = \frac{\partial}{\partial t} + \Omega_h \frac{\partial}{\partial \phi}, \tag{A.1}$$

where Ω_h is the angular velocity of the horizon which is a constant. Here, we mean that the horizon is the outer horizon for a rotating black hole. Because the horizon is a null surface and ξ^μ is normal to the horizon, we have on the horizon [28]

$$\xi^\mu \xi_\mu \big|_{r=r_h} = 0. \tag{A.2}$$

For the metric (8), we have

$$\xi^\mu \xi_\mu = g_{tt} + 2g_{t\phi}\Omega_h - g_{\phi\phi}\Omega_h^2. \tag{A.3}$$

Here, we have defined that the square of the norm of the Killing field is positive outside the horizon, at least for the case $\Omega_h = 0$. As in (12), we define

$$G_{tt} = g_{tt} + 2g_{t\phi}\Omega_h - g_{\phi\phi}\Omega_h^2. \tag{A.4}$$

Thus we have

$$G_{tt}\big|_{r=r_h} = 0. \tag{A.5}$$

Following [28] we write

$$\xi^\mu \xi_\mu = -\lambda^2, \tag{A.6}$$

where λ is a scalar function, and it is a constant on the horizon. According to (A.3), we have $\lambda^2 = -G_{tt}$ for the metric (8) of a four-dimensional rotating black hole. Let ∇^μ represent the covariant derivative operator, thus $\nabla^\mu(\xi^\nu\xi_\nu)$ is also normal to the horizon. Then, according to [28,29], there exists a function κ satisfying the equation

$$\nabla^\mu\left(-\lambda^2\right) = -2\kappa\xi^\mu, \tag{A.7}$$

where on the horizon $\kappa(r_h)$ is a constant and is just the horizon's surface gravity.

Similarly, we have the lower index equation

$$\nabla_\mu\left(-\lambda^2\right) = -2\kappa\xi_\mu. \tag{A.8}$$

Thus, from (A.7) and (A.8) we have

$$\nabla^\mu\left(-\lambda^2\right)\nabla_\mu\left(-\lambda^2\right) = -4\kappa^2\lambda^2. \tag{A.9}$$

Because λ^2 is a scalar function, κ^2 is also a scalar function. Therefore the surface gravity of a black hole horizon is invariant under general coordinate transformations, including the rotation of (9). From (A.3), (A.4), (A.6), (A.9), and the axial symmetry of the metric, we obtain

$$4\kappa^2 G_{tt} = g^{rr}(\partial_r G_{tt})^2 + g^{\theta\theta}(\partial_\theta G_{tt})^2. \tag{A.10}$$

Because of (A.5), we have

$$\lim_{r\to r_h} \partial_\theta G_{tt} = 0. \tag{A.11}$$

Therefore, to take the limit $r \to r_h$ in both sides of (A.10) yields

$$\kappa(r_h) = \lim_{r\to r_h} \frac{\partial_r\sqrt{G_{tt}}}{\sqrt{g_{rr}}}. \tag{A.12}$$

In (A.12), because G_{tt} is zero on the horizon, the partial derivative is taken before the limit. Thus, the Hawking temperature of the metric (8) is given by

$$T_H = \frac{\kappa(r_h)}{2\pi} = \lim_{r\to r_h} \frac{\partial_r\sqrt{G_{tt}}}{2\pi\sqrt{g_{rr}}}. \tag{A.13}$$

Because $\kappa(r_h)$ is a constant on the horizon [28,29], it can be evaluated at an arbitrary θ. For convenience, it can be evaluated at $\theta = 0$ usually. For the metrics of many four-dimensional rotating black holes, we can see that usually they satisfy $G_{tt}|_{\theta=0} = g_{tt}|_{\theta=0}$. Thus we can write

$$T_H = \frac{\kappa(r_h)}{2\pi} = \lim_{\theta=0, r\to r_h} \frac{\partial_r\sqrt{g_{tt}}}{2\pi\sqrt{g_{rr}}}. \tag{A.14}$$

On the other hand, because $\kappa(r_h)$ is a constant on the horizon, this means that, in formula (A.13), the dependence of T_H on the variable θ is only apparent.

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