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Bilinear Optimal Control of the Velocity Term in a Kirchhoff Plate Equation

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We consider a bilinear optimal control problem where the state equation is a Kirchhoff plate equation. The control acts as a multiplier of the velocity term. We prove the existence of an optimal control in a class $h \in U_M = \{h \in L^\infty(0,T); -M \le h(t) \le M\}$ and uniqueness of this optimal control for T sufficiently small.

Key Words: Kirchhoff plate; optimal control; bilinear control.

1. INTRODUCTION

We consider the problem of controlling the solution of a Kirchhoff plate equation. The motion with appropriate boundary conditions describes the motion of a thin plate which is clamped along one portion of its boundary and has free vibrations on the other portion of the boundary. We consider bilinear optimal control in which the control acts as a multiplier of a velocity term.

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Given control

$$h \in U_M = \{h \in L^\infty(0,T); -M \le h(t) \le M\},$$

the "displacement" solution $w = w(h)$ of our state equation satisfies

$$\begin{aligned}
&w_{tt} + \Delta^2 w = h(t) w_t \qquad \text{on } Q = \Omega \times (0,T) \\
&w(x,y,0) = w_0(x,y), \quad w_t(x,y,0) = w_1(x,y) \qquad \text{on } \Omega \\
&w = \frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \Sigma_0 = \Gamma_0 \times (0,T), \\
&\left.\begin{aligned} &\Delta w + (1-\mu) B_1 w = 0 \\ &\frac{\partial \Delta w}{\partial \nu} + (1-\mu) B_2 w = 0 \end{aligned}\right\} \qquad \text{on } \Sigma_1 = \Gamma_1 \times (0,T),
\end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ with C^2 boundary, $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma_0 \neq \emptyset$, $\nu = \langle n_1, n_2 \rangle$ is the outward unit normal vector on $\partial\Omega$, and

$$B_1 w = 2 n_1 n_2 w_{xy} - n_1^2 w_{yy} - n_2^2 w_{xx},$$

$$B_2 w = \frac{\partial}{\partial \tau}\left[\left(n_1^2 - n_2^2\right) w_{xy} + n_1 n_2 (w_{yy} - w_{xx})\right].$$

The direction τ in $B_2 w$ is the tangential direction along Γ_1. The plate is clamped along Γ_0 and has free vibrations along Γ_1. The constant μ, $0 < \mu < \frac{1}{2}$, represents Poisson's ratio.

We take as our objective functional

$$J(h) = \frac{1}{2}\left(\int_Q (w - z)^2 \, dQ + \beta \int_0^T h^2(t)\, dt\right),$$

where z in $L^\infty(Q)$ is the desired evolution for the plate and the quadratic term in h represents the cost of implementing the control. We seek to minimize the objective functional, i.e., characterize an optimal control $h^* \in U_M$ such that

$$J(h^*) = \min_{h \in U_M} J(h).$$

For background on plate models and control, see the books by Lagnese and Lions [16], Lagnese [14], Lagnese et al. [15], Kormornik [12], Li and Yong [20], and Lions [22]. The bilinear control case treated here does not fit into the Riccati framework [18]; even though the objective functional is quadratic, the state equation has a bilinear term, hw_t. See [4, 6, 9–11, 13, 17] for control papers involving Kirchhoff plates. Bilinear control problems similar to the problem here were introduced in three papers by Ball, Marsden, and Slemrod [1–3], and in Bradley and Lenhart [5] (with control on a coefficient of a zero-order term, hw).

In Section 2, we show well-posedness of our state problem. In Section 3, we show the existence of an optimal control by a minimizing sequence argument. In Section 4, we derive a characterization for optimal controls, in terms of the solutions of an optimality system. The optimality system consists of the state equation coupled with an adjoint equation, and it is derived by differentiating the objective functional and the map $h \to w(h)$ with respect to the control. The existence of the solution of the adjoint problem had to be handled in an unusual way due to the $-(hp)_t$ term. For T sufficiently small, the uniqueness of the optimal control is shown by the strict convexity of the functional $J(h)$ with respect to h.

2. WELL-POSEDNESS OF THE STATE EQUATION

We will begin by proving existence, uniqueness, and regularity results for the state equation. We first define our solution spaces,

$$H^2_{\Gamma_0}(\Omega) = \left\{ w \in H^2(\Omega) \mid w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_0 \right\}$$

and

$$\mathscr{H} = H^2_{\Gamma_0}(\Omega) \times L^2(\Omega).$$

Note that the bilinear form on $H^2_{\Gamma_0}(\Omega)$,

$$a(u,v) = \int_\Omega \left\{ \Delta w \Delta v + (1-\mu)\left(2 w_{xy} v_{xy} - w_{xx} v_{yy} - w_{yy} w_{xx}\right)\right\} d\Omega,$$

induces a norm on $H^2_{\Gamma_0}(\Omega)$ which is equivalent to the usual H^2 norm on $H^2_{\Gamma_0}(\Omega)$ (see [11]).

DEFINITION 1. Given $h \in U_M$, $\tilde{w} = \tilde{w}(h) = (w(h), w_t(h))$ is a weak solution to (1.1) if $\tilde{w} \in C([0,T];\mathscr{H})$, $\tilde{w}(0) = (w_0, w_1)$, and $\tilde{w}$ satisfies

$$\int_0^T \langle w_{tt}, \phi \rangle + \int_0^T a(w, \phi)(t)\, dt = \int_Q h w_t \phi \, d\Omega \, dt$$

for all $\phi \in H^2_{\Gamma_0}(\Omega)$, where $\langle \cdot , \cdot \rangle$ denotes the duality pairing between $[H^2_{\Gamma_0}(\Omega)]'$ and $H^2_{\Gamma_0}(\Omega)$.

THEOREM 2.1 (i). *Let* $\tilde{w}(0) = (w_0, w_1) \in \mathscr{H}$ *and* $h \in U_M$. *Then the system* (1.1) *has a unique weak solution* $\tilde{w} = \tilde{w}(h) = (w, w_t)$.

(ii) *In addition, if* $(w_0, w_1) \in D_0$, *where*

$$D_0 = \Big\{(w_0, w_1) \in \big(H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega)\big) \times H^2_{\Gamma_0}(\Omega):$$
$$\Delta w_0 + (1-\mu) B_1 w_0 = 0 \textit{ on } \Gamma_1,$$
$$\frac{\partial \Delta w_0}{\partial \nu} + (1-\mu) B_2 w_0 = 0 \textit{ on } \Gamma_1\Big\},$$

and if $h \in U_M \cap C^2(0,T)$, *then the weak solution satisfies*

$$\tilde{w} \in C\Big([0,T]; \big(H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega)\big) \times H^2_{\Gamma_0}(\Omega)\Big)$$

and

$$w_{tt} \in C\big([0,T]; L^2(\Omega)\big).$$

Furthermore, Eq. (1.1) *holds in the* L^2 *sense*.

Proof. (i) To write the system in semigroup form, we define the operator $\mathscr{A}$:

$$\mathscr{A} w = \Delta^2 w \text{ with domain}$$
$$\mathscr{D}(\mathscr{A}) = \Big\{w \in H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega) : \Delta w + (1-\mu) B_1 w = 0 \text{ on } \Gamma_1,$$
$$\frac{\partial \Delta w}{\partial \nu} + (1-\mu) B_2 w = 0 \text{ on } \Gamma_1\Big\}.$$

Then define operator A by

$$A\colon H^4(\Omega) \times H^2_{\Gamma_0}(\Omega) \to \mathscr{H}$$
$$A\tilde{w} = \begin{bmatrix} 0 & I \\ -\mathscr{A} & 0 \end{bmatrix} \tilde{w} \qquad \text{with } \mathscr{D}(A) = D_0.$$

Then the state equation (1.1) can be written as

$$\frac{d}{dt}\tilde{w}(t) = A\tilde{w}(t) + B\tilde{w}(t)$$
$$\tilde{w}(0) = \tilde{w}_0 = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix},$$

with

$$B\tilde{w}(t) = \begin{bmatrix} 0 \\ h(t)w_2(t) \end{bmatrix},$$

where $w_2(t) = w_t(x,t)$. Using skew-adjointness, the operator A generates a strongly continuous unitary group on $\mathscr{H}$. Since B is a bounded perturbation of A on $\mathscr{H}$, by standard semigroup theory [23], we have the conclusion of (i).

(ii) Assume that $\tilde{w}_0 \in D_0$ and $h \in U_M \cap C^2(0,T)$. From the variation of parameters [23] and (i),

$$\tilde{w}(t) = e^{At}\tilde{w}_0 + \int_0^t e^{A(t-\tau)}B(\tilde{w})(\tau)\,d\tau, \tag{2.1}$$

where e^{At} represents the semigroup generated by A. Proceeding to formally differentiate (2.1) in the t variable and defining a new variable $\tilde{v} = (v_1, v_2) = \frac{d\tilde{w}}{dt}$, we seek a solution of the form

$$\tilde{v}(t) = Ae^{At}\tilde{w}_0 + B\tilde{w}(t) + \int_0^t Ae^{A(t-\tau)}B\tilde{w}(\tau)\,d\tau.$$

Setting

$$Fv = Ae^{At}\tilde{w}_0 + B\tilde{w}(t) + \int_0^t Ae^{A(t-\tau)}B\tilde{w}(\tau)\,d\tau, \tag{2.2}$$

we seek a fixed point of F, i.e.,

$$F\tilde{v} = \tilde{v}$$

has a unique fixed point in $C([0,T];\mathscr{H})$. Note that

$$\begin{aligned}
&\int_0^t Ae^{A(t-\tau)}B\tilde{w}(\tau)\,d\tau \\
&\quad = -\int_0^t \frac{d}{d\tau}\left(e^{A(t-\tau)}B\tilde{w}(\tau)\right)d\tau + \int_0^t e^{A(t-\tau)}\frac{d}{d\tau}B\tilde{w}(\tau)\,d\tau \\
&\quad = -B\tilde{w}(t) + e^{At}\begin{bmatrix} 0 \\ h(0)w_1 \end{bmatrix} + \int_0^t e^{A(t-\tau)}\begin{bmatrix} 0 \\ h_\tau w_\tau + hw_{\tau\tau} \end{bmatrix} d\tau
\end{aligned}$$

Thus from (2.2), F can be rewritten as

$$F(\tilde{v}) = Ae^{At}\tilde{w}_0 + e^{At}\begin{bmatrix} 0 \\ h(0)w_1 \end{bmatrix} + \int_0^t e^{A(t-\tau)}\begin{bmatrix} 0 \\ h_\tau v_1 + hv_2 \end{bmatrix} d\tau.$$

Since $\tilde{w}_0 \in \mathscr{D}(A)$ and h is continuous at $t = 0$, $F\colon C([0,T];\mathscr{H}) \to C([0,T];\mathscr{H})$ is bounded. We now verify that F is a contraction on $C([0,T];\mathscr{H})$ for small T_0 for $0 \le t \le T_0$,

$$\begin{aligned}
&\|F\tilde{v}_1 - F\tilde{v}_2\|_{C([0,T_0];\mathscr{H})} \\
&\quad\le \left\| \int_0^t e^{A(t-\tau)}\begin{bmatrix} 0 \\ h_\tau(v_{11} - v_{21}) + h(v_{12} - v_{22}) \end{bmatrix} d\tau \right\|_{C([0,T_0];\mathscr{H})} \\
&\quad\le \sup_{0\le t\le T_0} \int_0^t \|h_\tau(v_{11} - v_{21})(\tau)\|_{L^2(\Omega)}\, d\tau \\
&\qquad + \sup_{0\le t\le T_0} \int_0^t \|h(v_{12} - v_{22})(\tau)\|_{L^2(\Omega)}\, d\tau \\
&\quad\le T_0 C\|\tilde{v}_1 - \tilde{v}_2\|_{C([0,T_0];\mathscr{H})},
\end{aligned}$$

where $C = \max(\|h_\tau\|_{C[0,T]}, \|h\|_{C[0,T]})$. Taking $T_0 < \frac{1}{C}$, we have that F is a contractive mapping on $C([0,T_0],\mathscr{H})$. To complete the proof, we set $\tilde{v}(T_0)$ (where $\tilde{v}$ is the fixed point) as the new initial data and repeat the argument to obtain F as a contraction on $C([T_0, 2T_0],\mathscr{H})$. Repeating this procedure yields the result on $[0,T]$.

We observe first that

$$(w_t, w_{tt}) \in C([0,T];\mathscr{H}),$$

and then $hw_t \in L^2(Q)$ with Eq. (1.1) gives

$$\Delta^2 w \in C\big([0,T]; L^2(\Omega)\big).$$

By standard elliptic theory,

$$w \in C\Big([0,T]; H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega)\Big).$$

∎

We now present an a priori estimate needed for the existence of an optimal control.

LEMMA 2.1 (A priori estimate). *Given $\tilde{w}_0 \in \mathscr{H}$ and $h \in U_M$, the weak solution to* (1.1) *satisfies*

$$\|\tilde{w}\|_{C([0,T];\mathscr{H})} \le C_1(1 + 2MTe^{2MT})^{1/2}, \tag{2.3}$$

with $C_1 = \|\tilde{w}_0\|_{\mathscr{H}}$.

Proof. Since D_0 is dense in $\mathscr{H}$, there exist sequences, $\{\tilde{w}_0^n\}$ in D_0 and $\{h^n\}$ in $C^2([0,T]) \cap U_M$, such that

$$\tilde{w}_0^n \to \tilde{w}_0 \qquad \text{strongly in } \mathscr{H}$$

and

$$h^n \to h \qquad \text{strongly in } L^2([0,T]).$$

Denoting by $\tilde{w}^n$ the solution of (1.1) with initial data $\tilde{w}_0^n$ and control h^n, $\tilde{w}^n$ has the additional regularity from Theorem 2.1(ii). Using w_t^n as a multiplier in (1.1), we obtain

$$\begin{aligned} 0 &= \int_0^s \int_\Omega \left(w_{tt}^n w_t^n + \Delta^2 w^n w_t^n - h^n (w_t^n)^2 \right) d\Omega\, dt \\ &= \int_0^s \int_\Omega \frac{1}{2} \frac{d}{dt} (w_t^n)^2 \, d\Omega\, dt + \int_0^s \frac{1}{2} \frac{d}{dt} a(w^n, w^n)\, dt - \int_0^s \int_\Omega h^n (w_t^n)^2 \, d\Omega\, dt. \end{aligned}$$

Consequently, we have

$$\begin{aligned} &\frac{1}{2} \int_\Omega (w_t^n)^2 (x,y,s)\, d\Omega + \frac{1}{2} a(w^n, w^n)(s) \\ &\qquad = \frac{1}{2} \|\tilde{w}_0^n\|_{\mathscr{H}}^2 + \int_0^s \int_\Omega h^n (w_t^n)^2 \, d\Omega\, dt \\ &\qquad \le \frac{1}{2} \|\tilde{w}_0^n\|_{\mathscr{H}}^2 + M \int_0^s \|\tilde{w}^n(t)\|_{\mathscr{H}}^2 \, dt. \end{aligned}$$

Gronwall's Inequality implies

$$\begin{aligned} &\sup_{0 \le s \le T} \left\{ \int_\Omega (w_t^n)^2 (x,y,s)\, d\Omega + a(w^n, w^n)(s) \right\} \\ &\qquad \le \|\tilde{w}_0^n\|_{\mathscr{H}}^2 (1 + 2MTe^{2MT}), \end{aligned} \tag{2.4}$$

which gives the desired result for smooth approximations. Now since (2.4) does not depend on the C^2-smoothness of the h^n, we can pass to the limit and obtain (2.3) for $\tilde{w}$. ∎

3. EXISTENCE OF OPTIMAL CONTROLS

We now prove the existence of an optimal control by a minimizing sequence argument.

THEOREM 3.1. *There exists an optimal control $h^* \in U_M$, which minimizes the objective functional $J(h)$ over h in U_M.*

Proof. Let $\{h^n\}$ be a minimizing sequence in U_M, i.e.,

$$\lim_{n\to\infty} J(h^n) = \inf_{h\in U_M} J(h).$$

By Lemma 2.1, for $\tilde{w}^n = \tilde{w}(h^n)$,

$$\|\tilde{w}^n\|_{C([0,T],\mathscr{H})} \le C_1 e^{C_2 MT}.$$

On a subsequence, we have

$$w^n \rightharpoonup w^* \qquad \text{weakly* in } L^\infty\big([0,T]; H^2_{\Gamma_0}(\Omega)\big)$$

$$w^n_t \rightharpoonup w^*_t \qquad \text{weakly* in } L^\infty([0,T]; L^2(\Omega))$$

$$w^n_{tt} \rightharpoonup w^*_{tt} \qquad \text{weakly* in } L^\infty\Big([0,T]; \big(H^2_{\Gamma_0}(\Omega)\big)'\Big)$$

and

$$h^n \rightharpoonup h^* \qquad \text{weakly in } L^2(0,T).$$

In weak form, w^n satisfies

$$\int_0^T [\langle w^n_{tt}, \phi\rangle + a(w^n, \phi)(t)]\, dt = \int_Q h^n w^n_t \phi\, dQ, \tag{3.1}$$

where $\phi \in H^2_{\Gamma_0}(\Omega)$. In the convergence as $n \to \infty$, the only difficult term is on the RHS of (3.1). We now show convergence of the RHS. Define the sequence of functions $v^n(t)$ by

$$v^n(t) = \int_\Omega w^n_t(x,y,t)\phi(x,y)\, d\Omega,$$

so that the RHS of (3.1) becomes

$$\int_0^T h^n(t) v^n(t)\, dt.$$

We note that $\{v^n\}$ is uniformly bounded, independent of n, by the *a priori* estimate (2.3). By the continuity of w^n_t in time into $L^2(\Omega)$, for each fixed t,

$$v^n(t) \to v(t) = \int_\Omega w^*_t(x,y,t)\phi(x,y)\, d\Omega \qquad \text{pointwise as } n \to \infty,$$

using the weak convergences above. By Egorof's Theorem [24], for any $\varepsilon > 0$, there exists a set $E \subset [0, T]$ such that $m(E) < \varepsilon$ and

$$v^n(t) \to v(t) \qquad \text{uniformly on } [0, T] \setminus E.$$

Then we obtain

$$\int_0^T |h^n v^n - h^* v| \, dt \le \int_0^T |(h^n v^n - h^* v)\chi_E| \, dt + \int_0^T |(h^n v^n - h^* v)\chi_{[0,T]\setminus E}| \, dt.$$

The integral term on $[0, T] \setminus E$ approaches 0 as $n \to \infty$ by the uniform convergence of $v^n \to v$ on $[0, T] \setminus E$. The integral term on E can be estimated,

$$\begin{aligned} \int_0^T |(h^n v^n - h^* v)\chi_E| \, dt &\le M \int_0^T (|v^n| + |v|)\chi_E \, dt \\ &\le M\big(\|v^n\|_{L^2(0,T)} + \|v\|_{L^2(0,T)}\big) m(E) \\ &\le C m(E), \end{aligned}$$

where C does not depend on n and $m(E) < \varepsilon$. Hence we obtain the desired convergence,

$$\int_Q h^n w_t^n \phi \, dQ \to \int_Q h^* w_t^* \phi \, dQ.$$

We can pass to the limit in the w^n PDE and obtain $\tilde{w}^* = \tilde{w}(h^*)$, the solution to (1.1).

Since the objective functional is lower semicontinuous with respect to weak convergence, we obtain

$$J(h^*) = \inf_{h \in U_M} J(h)$$

and h^* is an optimal control. ∎

4. NECESSARY CONDITIONS

We now derive necessary conditions that any optimal control must satisfy. To derive these necessary conditions, we must differentiate our functional $J(h)$ and $w = w(h)$ with respect to h. The differentiation of J and uniqueness result give a characterization of the unique optimal control in terms of the optimality system.

LEMMA 4.1. *The mapping*

$$h \in U_M \to \tilde{w}(h) \in C([0,T];\mathscr{H})$$

is differentiable in the sense

$$\frac{\tilde{w}(h+\varepsilon l) - \tilde{w}(h)}{\varepsilon} \rightharpoonup \tilde{\psi} \qquad \textit{weakly* in } L^\infty([0,T];\mathscr{H}),$$

as $\varepsilon \to 0$, *for any* $h, h+\varepsilon l \in U_M$. *Moreover, the limit* $\tilde{\psi} = (\psi, \psi_t)$ *is a weak solution to the system*

$$\begin{aligned}
\psi_{tt} + \Delta^2\psi - h\psi_t &= lw_t && \text{in } Q\\
\psi(x,y,0) = \psi_t(x,y,0) &= 0 && \text{in } \Omega\\
\psi = \frac{\partial\psi}{\partial\nu} &= 0 && \text{on } \Sigma_0\\
\Delta\psi + (1-\mu)B_1\psi &= 0 && \text{on } \Sigma_1\\
\frac{\partial\Delta\psi}{\partial\nu} + (1-\mu)B_2\psi &= 0 && \text{on } \Sigma_1.
\end{aligned} \tag{4.1}$$

Proof. Denote by $\tilde{w}^\varepsilon = \tilde{w}(h+\varepsilon l)$ and $\tilde{w} = \tilde{w}(h)$. By (1.1), $(\tilde{w}^\varepsilon - \tilde{w})/\varepsilon$ is a weak solution of

$$\left(\frac{w^\varepsilon - w}{\varepsilon}\right)_{tt} + \Delta^2\left(\frac{w^\varepsilon - w}{\varepsilon}\right) = h\left(\frac{w^\varepsilon - w}{\varepsilon}\right)_t + lw_t^\varepsilon \qquad \text{in } Q$$

with homogeneous initial and boundary conditions. Using the proof of Lemma 2.1 with source term lw_t^ε, we obtain

$$\left\|\frac{\tilde{w}^\varepsilon - \tilde{w}}{\varepsilon}\right\|_{C([0,T];\mathscr{H})} \le \|lw_t^\varepsilon\|_{L^2(Q)} e^{CMT}.$$

But we have *a priori* estimates on w_t^ε,

$$\|lw_t^\varepsilon\|_{L^2(Q)} \le T\|l\|_\infty \|\tilde{w}^\varepsilon\|_{C(0,T;\mathscr{H})} \le C_1,$$

using Lemma 2.1 on $\tilde{w}^\varepsilon$. Hence on a subsequence, as $\varepsilon \to 0$,

$$\frac{\tilde{w}^\varepsilon - \tilde{w}}{\varepsilon} \rightharpoonup \tilde{\psi} \qquad \text{weakly* in } L^\infty([0,T];\mathscr{H}).$$

Similar to the proof of Theorem 3.1, we can obtain that $\tilde{\psi}$ is a weak solution of (4.1). ∎

We obtain the existence of an adjoint solution and use it in the differentiation of the map $h \to J(h)$ to obtain our characterization of an optimal control.

THEOREM 4.1. *Given an optimal control h^* in U_M and corresponding state solution $\tilde{w}^* = \tilde{w}(h^*)$ to* (1.1), *there exists a unique weak solution,*

$$\tilde{p} = (p, p_t) \in L^\infty([0,T];\mathscr{H}),$$

to the adjoint problem

$$\begin{aligned} p_{tt} + \Delta^2 p + (hp)_t &= w^* - z \qquad \text{in } Q \\ p = \frac{\partial p}{\partial \nu} &= 0 \qquad \text{on } \Sigma_0 \\ \Delta p + (1-\mu) B_1 p &= 0 \qquad \text{on } \Sigma_1 \\ \frac{\partial \Delta p}{\partial \nu} + (1-\mu) B_2 p &= 0 \qquad \text{on } \Sigma_1 \\ p(x,y,T) = p_t(x,y,T) &= 0 \ (\text{transversality condition}) \\ &\qquad \text{in } \Omega \times \{T\}, \end{aligned} \tag{4.2}$$

where the solution is distributionally defined with respect to t. Furthermore,

$$h^*(t) = \max\left(-M, \min\left(-\frac{1}{\beta}\int_\Omega w_t^* p(x,y,t)\, d\Omega, M\right)\right). \tag{4.3}$$

Proof. First, we prove the existence of the solution to the adjoint equation. This proof differs from the existence of the solution of the state equation due to the $(hp)_t$ term with h not necessarily weakly differentiable and the source term $w - z$. In the system formulation, the solution to the adjoint equation formally becomes

$$\tilde{p}(t) = -\int_t^T e^{A(t-s)} \begin{bmatrix} 0 \\ (hp)_s + w - z \end{bmatrix} ds,$$

where A is as in Theorem 2.1. However, this solution is only formal, since $h(t)$ is only in $L^\infty(0,T)$. It has been shown (see [7] and [19]) that the solvability of the system (4.2) is equivalent to showing that there exists $\tilde{p} \in L^\infty([0,T],\mathscr{H})$ of the form

$$\tilde{p}(t) = \int_t^T \left(-Ae^{A(t-s)} \begin{bmatrix} 0 \\ hp \end{bmatrix} + e^{A(t-s)} \begin{bmatrix} 0 \\ w - z \end{bmatrix} \right) ds + \begin{bmatrix} 0 \\ hp(t) \end{bmatrix}$$
$$\text{in } [D(A)]'. \tag{4.4}$$

Initially, we must understand this equation only in the sense of duality (i.e., in $[D(A)]'$). However, from our fixed-point argument below, we will achieve the stronger regularity needed for the well-posedness of (4.2). To find a solution $\tilde{p}$ satisfying (4.4), we prove the existence of a unique fixed point in $L^\infty([0,T];\mathscr{H})$ of the map

$$F\tilde{p}(t) = \int_t^T \left(-Ae^{A(t-s)} \begin{bmatrix} 0 \\ hp \end{bmatrix} + e^{A(t-s)} \begin{bmatrix} 0 \\ w-z \end{bmatrix} \right) ds + \begin{bmatrix} 0 \\ hp(t) \end{bmatrix}.$$

One first step is to prove the fixed-point result on $[T-T_0, T]$ for T_0 sufficiently small. We show that

$$F\colon L^\infty([T-T_0,T],\mathscr{H}) \to L^\infty([T-T_0,T];\mathscr{H})$$

is bounded and contractive.

To show boundedness, consider the $hp(t)$ term,

$$\begin{aligned}
\left\| \begin{bmatrix} 0 \\ hp \end{bmatrix} \right\|_{L^\infty([T-T_0,T];\mathscr{H})} &= \|hp\|_{L^\infty([T-T_0,T],\,L^2(\Omega))} \\
&\le M \operatorname{ess\,sup}_t \left(\int_\Omega (p(x,t))^2\,dx \right)^{1/2} \\
&= M \operatorname{ess\,sup}_t \left(\int_\Omega \left(-\int_t^T p_t(x,s)\,ds \right)^2 dx \right)^{1/2} \\
&\le M \operatorname{ess\,sup}_t \left(\int_\Omega T_0 \int_t^T p_t^2(x,s)\,ds\,dx \right)^{1/2} \\
&\le M T_0^{1/2} \left(\int_{T-T_0}^T \int_\Omega p_t^2(x,s)\,dx\,ds \right)^{1/2} \\
&\le M T_0^{1/2} \|p_t\|_{L^\infty([T-T_0,T];\,L^2(\Omega))}.
\end{aligned}$$

To complete the boundedness property,

$$\begin{aligned}
\|F\tilde{p}\|_{L^\infty([T-T_0],\mathscr{H})} \le{}& M T_0^{1/2} \|p_t\|_{L^\infty([T-T_0,T];\,L^2(\Omega))} \\
&+ \operatorname{ess\,sup}_t \int_t^T \left\| A \begin{pmatrix} 0 \\ hp \end{pmatrix} \right\|_{\mathscr{H}} ds + T_0 \|w-z\|_{L^\infty([T-T_0,T];\,L^2(\Omega))} \\
\le{}& \operatorname{ess\,sup}_t \int_t^T \|hp\|_{H^2(\Omega)}\,ds + M T_0^{1/2} \|p_t\|_{L^\infty([T-T_0,T];\,L^2(\Omega))} \\
&+ T_0 \|w-z\|_{L^\infty[T-T_0,T];\,L^2(\Omega))} \\
\le{}& 2 M T_0^{1/2} \|\tilde{p}\|_{L^\infty([T-T_0,T],\mathscr{H})} + T_0 \|w-z\|_{L^\infty([T-T_0,T];\mathscr{H})},
\end{aligned}$$

the last inequality holding since $h = h(t)$. For the contraction property, we have

$$\|F\tilde{p}_1 - F\tilde{p}_2\|_{L^\infty([T-T_0,T],\mathscr{H})} \le 2MT_0^{1/2}\|\tilde{p}_1 - \tilde{p}_2\|_{L^\infty([T-T_0,T];\mathscr{H})},$$

which gives the contraction for T_0 small. Thus we obtain our unique fixed point on $[T - T_0, T]$ and then apply the argument on $[T - 2T_0, T - T_0]$. Continuing gives the existence of a unique solution to the adjoint equation.

Let $h^* + \varepsilon l$ be another control in U_M and let $\tilde{w}^\varepsilon = \tilde{w}(h^* + \varepsilon l)$ be the corresponding solution to the state equation. Then since J achieves its minimum at h^*, we have

$$\begin{aligned} 0 &\le \lim_{\varepsilon\to 0^+} \frac{J(h^* + \varepsilon l) - J(h^*)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0^+} \int_Q \left(\frac{w^\varepsilon - w^*}{\varepsilon}\right)\left(\frac{w^\varepsilon + w^* - 2z}{2}\right) dQ + \frac{\beta}{2}\int_0^T (2lh^* + \varepsilon l^2)\, dt \\ &= \int_Q \psi(w^* - z)\, dQ + \beta\int_0^T h^* l\, dt. \end{aligned}$$

Substituting in from the adjoint equation (4.2) for $w^* - z$ and then using ψ PDE (4.1), we obtain

$$\begin{aligned} 0 &\le \int_0^T \langle \psi, p_{tt}\rangle\, dt + \int_0^T a(\psi, p)\, dt \\ &\quad + \int_Q \psi_t h^* p\, dQ + \beta\int_0^T h^* l\, dt \\ &= \int_0^T \langle \psi_{tt}, p\rangle\, dt + \int_0^T a(\psi, p)\, dt \\ &\quad + \int_Q \psi_t h^* p dQ + \beta\int_0^T h^* l\, dt \\ &= \int_0^T l\left(\beta h^* + \int_\Omega (w_t^* p)\, d\Omega\right) dt. \end{aligned}$$

Using a standard control argument based on the choices for the variation $l(t)$, we obtain the desired characterization for h^*:

$$h^*(t) = \max\left(-M, \min\left(-\frac{1}{\beta}\int_\Omega w_t^* p(x, y, t)\, d\Omega, M\right)\right).$$

∎

Last, we prove a uniqueness result for the optimal control.

THEOREM 4.2. *For T sufficiently small, there is a unique optimal control.*

Proof. We show uniqueness by showing strict convexity of the following map:

$$h \in U_M \to J(h).$$

This convexity follows from showing for all $h, l \in U_M$, $0 < \epsilon < 1$,

$$g''(\epsilon) > 0,$$

where $g(\epsilon) = J(\epsilon h + (1-\epsilon)l) = J(l + \epsilon(h-l))$.

To calculate

$$g'(\epsilon) = \lim_{\delta \to 0} \frac{J(l + (\epsilon+\delta)(h-l)) - J(l + \epsilon(h-l))}{\delta} \tag{4.5}$$

denote

$$\tilde{w}^{\epsilon} = \tilde{w}(l + \epsilon(h-l))$$
$$\tilde{w}^{\epsilon,\delta} = \tilde{w}(l + (\epsilon+\delta)(h-l)).$$

By an argument like that in Lemma 4.1,

$$\frac{\tilde{w}^{\epsilon,\delta} - \tilde{w}^{\epsilon}}{\delta} \rightharpoonup \tilde{\psi}^{\epsilon} \qquad \text{weak* in } L^{\infty}([0,T],\mathscr{H})$$

as $\delta \to 0$, and ψ^{ϵ} satisfies

$$\psi^{\epsilon}_{tt} + \Delta^2\psi^{\epsilon} = (l + \epsilon(h-l))\psi^{\epsilon}_t + (h-l)w^{\epsilon}_t$$

with zero initial and boundary conditions. Estimating as in Lemma 2.1, for $0 < s < T$, we obtain

$$\int_{\Omega}(\psi^{\epsilon}_t)^2(s)\,d\Omega + a(\psi^{\epsilon},\psi^{\epsilon})(s)$$
$$\le \int_0^s\int_{\Omega}(l + \epsilon(h-l))(\psi^{\epsilon}_t)^2\,d\Omega\,dt + \int_0^s\int_{\Omega}(h-l)w^{\epsilon}_t\psi^{\epsilon}_t\,d\Omega\,dt$$
$$\le C_M\int_0^s\int_{\Omega}(\psi^{\epsilon}_t)^2\,d\Omega\,dt + \int_0^s(h-l)^2\int_{\Omega}(w^{\epsilon}_t)^2\,d\Omega\,dt$$
$$\le C_M\int_0^s\int_{\Omega}(\psi^{\epsilon}_t)^2\,d\Omega\,dt + C_1(1 + 2MTe^{2MT})\int_0^T(h-l)^2\,dt,$$

where C_M depends on M and $C_1(1 + 2MTe^{2MT})$ is from Lemma 2.1. Using Gronwall's Inequality, we obtain

$$\int_\Omega (\psi_t^\epsilon)^2(s)\, d\Omega \le C_2 \int_0^T (h-l)^2\, dt, \tag{4.6}$$

with $C_2 = C_1(1 + 2MTe^{2MT})(1 + C_M Te^{C_M T})$.

To calculate $g''(\epsilon)$, we need a second derivative of w with respect to the control. Similar *a priori* estimates imply that

$$\frac{\tilde{\psi}^{\epsilon+\eta} - \tilde{\psi}^{\epsilon}}{\eta} \rightharpoonup \tilde{\sigma}^\epsilon \qquad \text{weak* in } L^\infty([0,T],\mathscr{H})$$

as $\eta \to 0$, and σ^ϵ satisfies

$$\sigma_{tt}^\epsilon + \Delta^2 \sigma^\epsilon = (l + \epsilon(h-l))\sigma_t^\epsilon + 2(h-l)\psi_t^\epsilon,$$

with zero initial and boundary conditions. Estimating in this case gives

$$\begin{aligned}
&\int_\Omega (\sigma_t^\epsilon)^2(s)\, d\Omega + a(\sigma^\epsilon, \sigma^\epsilon)(s) \\
&\quad \le \int_0^s \int_\Omega |l + \epsilon(h-l)| (\sigma_t^\epsilon)^2\, d\Omega\, dt + \int_0^s \int_\Omega 2(h-l)\psi_t^\epsilon \sigma_t^\epsilon\, d\Omega\, dt \\
&\quad \le C_M \int_0^s \int_\Omega (\sigma_t^\epsilon)^2\, d\Omega\, dt + \int_0^s (h-l)^2 \int_\Omega (\psi_t^\epsilon)^2\, d\Omega\, dt \\
&\quad \le C_M \int_0^s \int_\Omega (\sigma_t^\epsilon)^2\, d\Omega\, dt + C_2 \left(\int_0^T (h-l)^2\, dt, \right)^2
\end{aligned}$$

using (4.6) in the last inequality. Using Gronwall's inequality to estimate

$$\int_\Omega \sigma_t^\epsilon(s)^2\, d\Omega + a(\sigma^\epsilon, \sigma^\epsilon)(t) \le C_3 \left(\int_0^T (h-l)^2\, dt \right)^2$$

with $C_3 = C_2(1 + C_M Te^{C_M T})$, Poincaré's Inequality gives

$$\int_Q (\sigma^\epsilon)^2\, dQ \le TC_4 C_3 \left(\left(\int_0^T (h-l)^2\, dt \right)^2, \right.$$

where C_4 is from Poincaré's Inequality.

Continuing from (4.5), we are ready to calculate derivatives of g. We have

$$g'(\epsilon) = \int_Q \psi^\epsilon (w^\epsilon - z)\, dQ + \beta \int_0^T \left[(h-l)l + \epsilon(h-l)^2 \right] dt.$$

For the second derivative, we have

$$g''(\epsilon) = \lim_{\eta \to 0} \frac{g'(\epsilon+\eta) - g'(\epsilon)}{\eta}$$

$$= \lim_{\eta \to 0} \int_Q [\psi^{\epsilon+\eta}(w^{\epsilon+\eta} - z) - \psi^{\epsilon}(w^{\epsilon} - z)] \left(\frac{1}{\eta}\right) dQ + \beta \int_0^T (h-l)^2 d$$

$$\geq -\left(\int_Q (\sigma^{\epsilon})^2 \, dQ\right)^{1/2} \left(\int_Q (w^{\epsilon} - z)^2 \, dQ\right)^{1/2} + \beta \int_0^T (h-l)^2 \, dt$$

$$\geq (\beta - TC_4 C_3) \int_0^T (h-l)^2 \, dt$$

which gives the desired result for T sufficiently small. ▮

Remark. One can also obtain the strict convexity of the objective functional and the resulting uniqueness of the optimal control if one assumes that β is sufficiently large (as opposed to assuming T is sufficiently small). Note that the condition that T be sufficiently small also occurs in the uniqueness of optimal controls for solutions of the optimality systems in wave equations [21], parabolic equations [8], and plate equations [5].

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