JOURNAL OF COMBINATORIAL THEORY, Series A 29, 257–260 (1980)

Note

On Characterizing Subspaces*

A. A. BRUEN

University of Western Ontario, London, Ontario, Canada

B. L. ROTHSCHILD

University of California, Los Angeles, California 90032

AND

J. H. VAN LINT

Technische Hogeschool, Eindhoven, The Netherlands

Communicated by the Managing Editors

Received July 1, 1979

In this paper we correct and extend the results of an earlier paper of Rothschild and van Lint [4]. There, higher dimensional analogues of the following question are discussed: Let S be a set of points in a projective n-space P_n over $GF(q)$, and let S have $(q^{k+1}-1)/(q-1)$ points, the same number as in a k-subspace. Suppose that for every hyperplane $P_{n-1} \subseteq P_n$, $S \cap P_{n-1}$ has either $(q^{k+1}-1)/(q-1)$ or $(q^k-1)/(q-1)$ points. Then must S be a k-subspace of P_n? We show how the results of [4] can be strengthened by weakening the hypotheses of the theorems. Also, we point out an error and provide a correct proof of Theorem 5 in [4].

The notation is from [4]. $\{{P_n \atop r}\}$ denotes the set of r-subspaces of P_n (over $GF(q)$), and $\{{n \atop r}\}$ its cardinality. Let $S' = \{{P_k \atop r}\}$ for a k-subspace P_k of P_n. Then a set $S \subset \{{P_n \atop r}\}$ has property $P = P(n, q;\ k, r, j)$, $r \leqslant k \leqslant n$, if (a) $|S| = |S'|$ and if (b) $\{|S \cap \{{P_{n-j} \atop r}\}|:\ P_{n-j} \subseteq P_n\} \subseteq \{|S' \cap \{{P_{n-j} \atop r}\}|:\ P_{n-j} \subseteq P_n\}$. The analogous notions for affine spaces are used to define $A = A(n, q{:}\ k, r, j)$. P (resp. A) *characterizes* subspaces if the only S satisfying it are of the form $\{{P_k \atop r}\}$. In [4] the question is considered: Does P (resp. A) characterize subspaces? The answer was shown to be affirmative if $r = 0$ or if $j = 1$. (The case $r = 0$ and $j = 1$ was originally settled in [2].) If we weaken condition (b) above by requiring only that $|S \cap \{{P_{n-j} \atop r}\}| = \{{l \atop r}\}$, the l

0097-3165/80/050257–04$02.00/0

depending on the choice of P_{n-j} but not limited in value as in condition (b), then we call the new condition $P' = P'(n, q; k, r, j)$. The affine condition $A' = A'(n, q; k, r, j)$ is defined in an analogous manner. We show below how to modify the proofs in [4] in order to obtain all the results there even when the hypotheses P, A are replaced by the weaker assumptions P', A', respectively. The new, strengthened results are now as follows:

THEOREM 1'. *A' characterizes subspaces for $r=0, j=1$.*

THEOREM 2'. *A' characterizes subspaces for $j=1$, $r \geqslant 1$.*

THEOREM 3'. *P' characterizes subspaces for $j=1$.*

THEOREM 4'. *A' characterizes subspaces for $r=0$, $j \geqslant 1$ except when $q=2$ and $j=n-1$.*

THEOREM 5'. *P' characterizes subspaces for $r=0, j \geqslant 1$.*

Let us give a brief summary of their proofs. In [4] the hypotheses P, A guarantee certain equalities. In place of these equalities, the weaker assumptions P', A' now yield only inequalities. However, these inequalities are in the right direction to allow the arguments in [4] to go through. We list at the end of this note the key changes which are needed.

In the case of Theorem 5 in [4], the only changes needed to convert to Theorem 5' occur by p. 108, line 14; they are listed at the end of this note. However, there is a gap in the proof of Theorem 5 in [4]. In that proof "maximality of y" is invoked on p. 109, line 8. Regrettably, if $y=0$, we only get an inequality instead of an equality on line 9. The ensuing argument, based on the integrity of α, then fails. We now describe how this oversight can be remedied:

Page 108, after line 14, insert: "*Comment*: Since the right hand side of (18) is a strict inequality unless $n=k+1$, the assumption

$$|S_n \cap P_{n-j-1}| \leqslant \begin{Bmatrix} k-j \\ 0 \end{Bmatrix} \qquad \text{and} \qquad n \neq k+1$$

also leads to a contradiction."

Page 108, line 15: replace "$0 \leqslant y < j-1$" by "$0 \leqslant y \leqslant j$."

Page 108, after line 16, insert: "If $y=j$, then $S \subseteq P_{n-j-1}$, and we are done by induction. So we assume that $y \leqslant j-1$."

Page 108, line 17, after "maximal" insert: "If $y=0$, then by the comment above we must have $n=k+1$. But whenever $n-1=k+y$, $S \supseteq P'_{n-j-1}$. By $P(n, q, k, 0, j)$ (actually by $P'(n, q, k, 0, j)$) any P_{n-j-1} in S and any other point of S generate a P_{n-j} contained in S. This implies that, for any two points in S, all the points on the line joining them are also in S. Thus S is a subspace. It follows that we can assume $n-1 \neq k+y$. Since $n=k+1$ we have in particular that $y>0$."

Page 109, line 1. Replace "It" by "Since $n-1 \neq k+y$, it..."

Page 109, line 3, replace the second inequality by a strict inequality.

Page 109, after line 6, insert: "$T \neq \varnothing$, then $S \subseteq P'_{n-j-2}$ and we are done by induction. If $T \neq \varnothing$ let..."

Page 109, line 7, replace the comma at the end of line by a period.

Page 109, line 8, replace "then" by "Then".

Delete from p. 109, line 14 to p. 110, line 3, and replace by: "which is not an integer. This completes the proof."

This completes the correction and strengthening of the results [4]. It was mentioned there [4, p. 99] that for the higher dimensions things seem to be more difficult. Indeed, for any $r>0$ there exists j so that $P(n, q; k, r, j)$ does not characterize subspaces. For example, in $PG(4,2)$, any set S of 7 skew lines satisfies $P(4, 2; 2, 1, 2)$ and yet S is not the set of lines of a plane.

Of course S is also a set satisfying $P'(4, 2; 2, 1, 2)$. Apart from P', A' various other weakenings of hypotheses can be considered. For example, the condition that $|S| = {k \brace r}$ might be weakened. This was done for the analogous case of sets in [1]. (There, the counterexamples arise only when the condition on $|S|$ is relaxed and only under certain special conditions on r and j.) In the case of spaces, the condition $|S| = {k \brace r}$ cannot be relaxed even for $r=0$. An interesting counterexample is provided by the points of an ovoid in $PG(3, q)$. In fact, for $r=0$ this is the only "counterexample" to Theorem 1 when $|S| \neq {k \brace r}$. (The details will appear elsewhere.) Thus the results in [4] are sharp in that the theorems there will not hold in general without some restrictions on r, j or $|S|$.

ADJUSTMENTS TO PROOFS IN [4]

We list these by page and line numbers.

Theorem 1. (100, 11): replace "$\{0, {k-1 \brack 0}, {k \brack 0}\}$" by "$\{0, 1, {1 \brack 0}, \ldots, {k \brack 0}\}$".

(100, 13): replace "$=0$ or q^{k-1}" by "$\leqslant q^{k-1}$" and "$=0$" by "$< q^{k-1}$."

(100, 16): replace "$q^k = |S| =$" by "$q^k - q^{k-1} < |S| - |S \cap A_{n-1}| =$".

Theorem 2. (101, 11): replace "$=0$ or ${k-1 \brack 0}$" by "$\leqslant {k-1 \brack r}$.

(101, 15): replace "$= {k-1 \brack r}$" by ">0".

(101, 22): replace "a ${k-1 \brack r}$" by "at most a ${k-1 \brack r}$".

(102, 1): replace "Equating" by "Comparing."
(102, 2): replace "=" by "$\geqslant$".
(102, 4): replace equation by "$(q^{n-r}-1)/(q-1)\cdot(q^k-1)/(q^{k-r}-1)q \leqslant a \leqslant (q^n-1)/(q-1)q$".
(102, 10): replace "$P(n, q\colon k, r, 1)$" by "$P'(n, q; k, r, 1)$".

Theorem 3. (102, 14): should read "$| S \cap \{{P_{n-1} \atop r}\}| = \{{k \atop r}\} \cdots$".
(102, 16): "=" should be "$\neq$".
(These two were simply misprints.)
(102, 18): replace "$\{{k-1 \atop r}\}$" by "at most $\{{k-1 \atop r}\}$".
(102, 19): replace "=" by "$\geqslant$".
(102, 20): replace "=" by "$\leqslant$".

Theorem 4. (103, 2), (104, 17), (104, 31), replace "$A(n, q; k, 0, j)$" by "$A'(n, q; k, 0, j)$".
(104, 31): replace "$= 0$ or q^{k-1}" by "$\leqslant q^{k-1}$".

Theorem 5. (106, 12), (106, 23), (106, 27), (108, 9): replace "$P(n, q; k, 0, j)$" by "$P'(n, q; k, 0, j)$".
(108, 10): replace "=" by "$\leqslant$".
(108, 13): replace "=" by "$\geqslant$".
(108, 14): replace "=" by "$\leqslant$".

References

1. P. Erdős, N. M. Singhi, and B. L. Rothschild, Characterizing cliques in graphs, *Ars. Combinatoria* **4** (1977), 81–118.
2. F. J. MacWilliams, Error correcting codes for multiple-level transmissions, *Bell System Tech. J.* **40** (1961), 281–308.
3. B. L. Rothschild and N. M. Singhi, Characterizing k-flats in geometric designs, *J. Combinatorial Theory Ser. A* **20** (1976), 398–403.
4. B. L. Rothschild and J. H. van Lint, Characterizing finite subspaces, *J. Combinatorial Theory Ser. A* **16** (1974), 97–110.