Note

On Characterizing Subspaces*

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In this paper we correct and extend the results of an earlier paper of Rothschild and van Lint [4]. There, higher dimensional analogues of the following question are discussed: Let S be a set of points in a projective n-space P_n over GF(q), and let S have \((q^{k+1} - 1)/(q - 1)\) points, the same number as in a k-subspace. Suppose that for every hyperplane \(P_{n-1} \subseteq P_n\), \(S \cap P_{n-1}\) has either \((q^{k+1} - 1)/(q - 1)\) or \((q^k - 1)/(q - 1)\) points. Then must S be a k-subspace of \(P_n\)? We show how the results of [4] can be strengthened by weakening the hypotheses of the theorems. Also, we point out an error and provide a correct proof of Theorem 5 in [4].

The notation is from [4]. \(\binom{P_n}{r}\) denotes the set of r-subspaces of \(P_n\) (over GF(q)), and \(\binom{n}{r}\) its cardinality. Let \(S' = \binom{P_n}{r}\) for a k-subspace \(P_k\) of \(P_n\). Then a set \(S \subseteq \binom{P_n}{r}\) has property \(P = P(n, q; k, r, j)\), \(r \leq k \leq n\), if (a) \(|S| = |S'|\) and if (b) \(|S \cap \binom{P_n}{r-j}| = |S' \cap \binom{P_n}{r-j}|\); \(P_{n-j} \subseteq P_n\). The analogous notions for affine spaces are used to define \(A = A(n, q; k, r, j)\). P (resp. A) characterizes subspaces if the only S satisfying it are of the form \(\binom{P_k}{r}\). In [4] the question is considered: Does P (resp. A) characterize subspaces? The answer was shown to be affirmative if \(r = 0\) or if \(j = 1\). (The case \(r = 0\) and \(j = 1\) was originally settled in [2] .) If we weaken condition (b) above by requiring only that \(|S \cap \binom{P_n}{r-j}| = \binom{l}{r}\), the l
depending on the choice of \( P_{n-j} \) but not limited in value as in condition (b),
then we call the new condition \( \bar{P}' = P'(n, q; k, r, j) \). The affine condition \( A' = A'(n, q; k, r, j) \) is defined in an analogous manner. We show below how to
modify the proofs in [4] in order to obtain all the results there even when the
hypotheses \( P, A \) are replaced by the weaker assumptions \( P', A' \), respectively.
The new, strengthened results are now as follows:

**Theorem 1'**. \( A' \) characterizes subspaces for \( r = 0, j = 1 \).

**Theorem 2'**. \( A' \) characterizes subspaces for \( j = 1, r \geq 1 \).

**Theorem 3'**. \( P' \) characterizes subspaces for \( j = 1 \).

**Theorem 4'**. \( A' \) characterizes subspaces for \( r = 0, j > 1 \) except when
\( q = 2 \) and \( j = n - 1 \).

**Theorem 5'**. \( P' \) characterizes subspaces for \( r = 0, j \geq 1 \).

Let us give a brief summary of their proofs. In [4] the hypotheses \( P, A \)
guarantee certain equalities. In place of these equalities, the weaker assump-
tions \( P', A' \) now yield only inequalities. However, these inequalities are in
the right direction to allow the arguments in [4] to go through. We list at the
end of this note the key changes which are needed.

In the case of Theorem 5 in [4], the only changes needed to convert to
Theorem 5' occur by p. 108, line 14; they are listed at the end of this note.
However, there is a gap in the proof of Theorem 5 in [4]. In that proof
“maximality of \( y \)” is invoked on p. 109, line 8. Regrettably, if \( y = 0 \), we only
get an inequality instead of an equality on line 9. The ensuing argument,
based on the integrity of \( a \), then fails. We now describe how this oversight
can be remedied:

Page 108, after line 14, insert: “Comment: Since the right hand side of
(18) is a strict inequality unless \( n = k + 1 \), the assumption
\[
| S_n \cap P_{n-j-1} | \leq \left\{ \begin{array}{l} k-j \\ 0 \end{array} \right\}
\text{ and } n \neq k + 1
\]
also leads to a contradiction.”

Page 108, line 15: replace “\( 0 \leq y < j - 1 \)” by “\( 0 \leq y \leq j \)”.

Page 108, after line 16, insert: “If \( y = j \), then \( S \subseteq P_{n-j-1} \), and we are done
by induction. So we assume that \( y \leq j - 1 \).”

Page 108, line 17, after “maximal” insert: “If \( y = 0 \), then by the comment
above we must have \( n = k + 1 \). But whenever \( n - 1 = k + y \), \( S \supseteq P_{n-j-1} \).
By \( P(n, q, k, 0, j) \) (actually by \( P'(n, q, k, 0, j) \)) any \( P_{n-j-1} \) in \( S \) and any other
point of $S$ generate a $P_{n-j}$ contained in $S$. This implies that, for any two points in $S$, all the points on the line joining them are also in $S$. Thus $S$ is a subspace. It follows that we can assume $n - 1 = k + y$. Since $n - k + 1$ we have in particular that $y > 0$.

Page 109, line 1. Replace “It” by “Since $n - 1 \neq k + y$, it...”

Page 109, line 3, replace the second inequality by a strict inequality.

Page 109, after line 6, insert: “$T \neq \emptyset$, then $S \subseteq P_{n-j-2}$ and we are done by induction. If $T \neq \emptyset$ let...”

Page 109, line 7, replace the comma at the end of line by a period.

Page 109, line 8, replace “then” by “Then”.

Delete from p. 109, line 14 to p. 110, line 3, and replace by: “which is not an integer. This completes the proof.”

This completes the correction and strengthening of the results [4]. It was mentioned there [4, p. 99] that for the higher dimensions things seem to be more difficult. Indeed, for any $r > 0$ there exists $j$ so that $P(n, q; k, r,j)$ does not characterize subspaces. For example, in $PG(4, 2)$, any set $S$ of 7 skew lines satisfies $P(4, 2; 2, 1, 2)$ and yet $S$ is not the set of lines of a plane.

Of course $S$ is also a set satisfying $P'(4, 2; 2, 1, 2)$. Apart from $P'$, $A'$ various other weakenings of hypotheses can be considered. For example, the condition that $|S| = \{k\}$ might be weakened. This was done for the analogous case of sets in [1]. (There, the counterexamples arise only when the condition on $|S|$ is relaxed and only under certain special conditions on $r$ and $j$.) In the case of spaces, the condition $|S| = \{k\}$ cannot be relaxed even for $r = 0$. An interesting counterexample is provided by the points of an ovoid in $PG(3, q)$. In fact, for $r = 0$ this is the only “counterexample” to Theorem 1 when $|S| \neq \{r\}$. (The details will appear elsewhere.) Thus the results in [4] are sharp in that the theorems there will not hold in general without some restrictions on $r, j$ or $|S|$.

**Adjustments to Proofs in [4]**

We list these by page and line numbers.

**Theorem 1.** (100, 11): replace “$\{0, [k^{-1}], [k]\}$” by “$\{0, 1, \frac{1}{0}, \ldots, \frac{1}{0}\}$”.

(100, 13): replace “$= 0$ or $q^{k-1}$” by “$\leq q^{k-1}$” and “$= 0$” by “$< q^{k-1}$”.

(100, 16): replace “$q^k = |S|$” by “$q^k - q^{k-1} < |S| - |S \cap A_{n-1}|$.”

**Theorem 2.** (101, 11): replace “$\leq [k^{-1}]$” by “$\leq [k^{-1}]$”.

(101, 15): replace “$= [k^{-1}]$” by “$> 0$”.

(101, 22): replace “a $[k^{-1}]$” by “at most a $[k^{-1}]$”.
(102, 1): replace “Equating” by “Comparing.”
(102, 2): replace “=” by “$$\geq$$”.
(102, 4): replace equation by “$$\frac{(q^r - 1)}{(q - 1) \cdot (q^k - 1)/q} \leq a \leq \frac{(q^n - 1)}{(q - 1)q}$$”.
(102, 10): replace “$$P(n, q; k, r, 1)$$” by “$$P'(n, q; k, r, 1)$$”.

Theorem 3. (102, 14): should read “$$|S \cap \{p_{r-1}\}| = \{\frac{k}{r}\} \ldots$$”.
(102, 16): “=” should be “$$\neq$$”.
(These two were simply misprints.)
(102, 18): replace “$$\{k_{-1}r\}$$” by “at most $$\{k_{-1}r\}$$.”
(102, 19): replace “=” by “$$\geq$$”.
(102, 20): replace “=” by “$$\leq$$”.

Theorem 4. (103, 2), (104, 17), (104, 31), replace “$$A(n, q; k, 0, j)$$” by “$$A'(n, q; k, 0, j)$$”.
(104, 31): replace “$$-0$$ or $$q^{k-1}$$” by “$$\leq q^{k-1}$$”.

Theorem 5. (106, 12), (106, 23), (106, 27), (108, 9): replace “$$P(n, q; k, 0, j)$$” by “$$P'(n, q; k, 0, j)$$”.
(108, 10): replace “=” by “$$\leq$$”.
(108, 13): replace “=” by “$$\geq$$”.
(108, 14): replace “=” by “$$\leq$$”.

REFERENCES