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Let $G$ be a locally finite group having a subgroup $N$ of finite index $n$ that possesses a normal series

$$N = N_0 \geq N_1 \geq \cdots \geq N_k = 1$$

each of whose quotients $N_i/N_{i+1}$ is either locally nilpotent or satisfies an outer commutator law $w_i \equiv 1$. We show that $G$ contains a characteristic subgroup $H$ of finite index that has a characteristic series with the same properties. Moreover, the index of $H$ in $G$ is bounded by a function depending only on $n$, $k$ and the weight of the $w_i$.

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1. Introduction

Let $G$ be a group that has a subgroup $N$ of finite index with certain property $P$. Recently some attention was given to the question whether $G$ necessarily contains a characteristic subgroup of finite index with the property $P$. Since any finite-index subgroup $N < G$ contains a normal in $G$ subgroup of finite $(\text{dividing } |G : N|)$ index, in many instances we can assume that $N$ is normal in $G$. If $G$ is...
finitely generated, then $N$ contains a fully invariant (and even verbal) in $G$ subgroup of finite index (which depends on the number of generators). Therefore, for finitely generated groups the answer to the above question is positive whenever $P$ is inherited by subgroups.

For arbitrary groups it is known that the answer is positive for example for the following properties:

- $P$ is “abelian”; this fact is known as Passman’s lemma (see Chapter 12, Lemma 1.2 in [1] or Lemma 21.1.4 in [2]);
- $P$ is “nilpotent” or “locally nilpotent”, or “solvable”, or “locally finite” (consider the automorphic closure $\prod_{\alpha \in \text{Aut} G} N^\alpha$, which is a characteristic $P$-subgroup);
- $P$ is “nilpotent of given class $c$” (see Lemma 3 in [3]);
- $P$ is “satisfying a given outer (multilinear) commutator law $w$” (original proof can be found in [4], other shorter proofs are given in [5,6]).

It is also known that if $P$ is “periodic of exponent $p$”, then the answer is negative for large primes $p$ (see an example in [6]).

In the present paper we consider the property for $N$ to have a normal series

$$N = N_0 \geq N_1 \geq \cdots \geq N_k = 1$$

of finite length $k$ each of whose quotients $N_i/N_{i+1}$ is either locally nilpotent or satisfies an outer commutator law.

Our main result is the following theorem.

**Theorem 1.1.** Let $G$ be a locally finite group that contains a normal subgroup $N$ of finite index $n$ that possesses a normal series

$$N = N_0 \geq N_1 \geq \cdots \geq N_k = 1$$

each of whose quotients $N_i/N_{i+1}$ is either locally nilpotent or satisfies an outer commutator law $w_i \equiv 1$. Then $G$ contains a finite-index characteristic subgroup $H$ that has a characteristic series

$$H = H_0 \geq H_1 \geq \cdots \geq H_k = 1$$

whose quotients $H_i/H_{i+1}$ have the same properties as the quotients $N_i/N_{i+1}$, that is, if $N_i/N_{i+1}$ is locally nilpotent, then so is $H_i/H_{i+1}$, and if $N_i/N_{i+1}$ satisfies an outer commutator identity $\omega \equiv 1$, then so does $H_i/H_{i+1}$. Moreover, the index of $H$ in $G$ is bounded by a function depending only on $n$, $k$ and the weights of the $w_i$.

We mention that the particular case of the above theorem where all quotients $N_i/N_{i+1}$ satisfy outer commutator laws follows easily from the results obtained earlier in [4] (also in [5] or [6]). On the other hand, the case where all quotients $N_i/N_{i+1}$ are locally nilpotent follows from the Hirsch–Plotkin theorem that in any group a product of normal locally nilpotent subgroups is again locally nilpotent. So it is a combination of outer commutator laws and local nilpotency, where Theorem 1.1 gives a new result. We do not know whether the theorem remains valid if the assumption that $G$ is locally finite is removed from the hypothesis.

2. Star words and corresponding subgroups

If $X$, $Y$ are subsets of a periodic group $G$, we denote by $[X, Y]_s$ the subgroup generated by all the commutators $[x, y]$, where $x \in X$, $y \in Y$ and the elements $x$, $y$ have coprime orders, that is

$$[X, Y]_s = \langle [x, y] \mid x \in X, y \in Y \text{ and } (|x|, |y|) = 1 \rangle. \quad (1)$$
Obviously, if \(X\) and \(Y\) are normal (characteristic) subgroups then the subgroup \([X, Y]\)_s is normal (characteristic) as well. It is easy to see that a locally finite group \(G\) is locally nilpotent if and only if 
\([G, G]_s = 1\).

**Lemma 2.1.** If \(G\) is a locally finite group, then \([G, G]_s\) is the locally nilpotent residual of \(G\).

**Proof.** We need to prove that \(G/A\) is locally nilpotent for some normal subgroup \(A\) if and only if 
\([G, G]_s \leq A\). If \(G/A\) is locally nilpotent, it is obvious that \([G, G]_s \leq A\). Let now \([G, G]_s \leq A\). We take two elements \(x, y \in G\) such that their images \(\bar{x}, \bar{y}\) in the quotient \(G/A\) have prime-power orders for different primes: \(|\bar{x}| = p^r\) and \(|\bar{y}| = q^s\). Let \(|x| = p^rn\) with \((n, p) = 1\) and \(|y| = q^m\) with \((q, m) = 1\). By \((1)\) we have \([x^n, y^m] \in [G, G]_s\), so \([x^n, y^m] \in A\). We choose the positive integers \(k\) and \(l\) such that
\[nk \equiv 1(\text{mod} \ p)\] and 
\[ml \equiv 1(\text{mod} \ q)\]. Then
\[1 \equiv [x^n, y^m] = [(x^n)^k, (y^m)^l] \equiv [x, y](\text{mod} \ A)\].
(Here we use the property that if \([a, b] \in N\), then \([a^k, b^l] \in N\) for any normal subgroup \(N\) and any positive integers \(k, l\).) Thus, any two elements of prime-power orders for different primes commute in the quotient \(G/A\) and therefore \(G/A\) is locally nilpotent. \(\square\)

Recall that outer commutator words are group words obtained by nesting commutators using always different variables. Thus \([x_1, x_2], [x_3, x_4, x_5], x_6\) is an outer commutator word, but the Engel word \([x_1, x_2, x_2, x_2]\) is not.

The concept of a star word can be introduced as follows. Let \(x_1, x_2, \ldots\) be variables. A star word of weight 1 is just a variable \(x_i\). A star word of weight \(t > 1\) is either a formal expression of the form
\[W(x_1, \ldots, x_t) = [U(x_1, \ldots, x_r), V(x_{r+1}, \ldots, x_t)]\] or \(W(x_1, \ldots, x_t) = [U(x_1, \ldots, x_r), V(x_{r+1}, \ldots, x_t)]_s\),
where \(U\) and \(V\) are star words of weights \(r\) and \(t - r\) respectively.

If \(G\) is a group and \(w\) a group word, the verbal subgroup \(W(G)\) corresponding to the word \(w\) is the subgroup generated by all \(w\)-values in \(G\). If \(W\) is a star word, we can define the corresponding subgroup \(W(G)\) of \(G\) in the following way. If \(W\) is of weight 1, then \(W(G)\) is precisely \(G\). Suppose that \(W\) is of weight \(t \geq 2\) and assume by induction that subgroups corresponding to star words of weight at most \(t - 1\) are already defined. If
\[W(x_1, \ldots, x_t) = [U(x_1, \ldots, x_r), V(x_{r+1}, \ldots, x_t)],\]
then \(W(G) = [U(G), V(G)]\). If
\[W(x_1, \ldots, x_t) = [U(x_1, \ldots, x_r), V(x_{r+1}, \ldots, x_t)]_s,\]
then \(W(G) = [U(G), V(G)]_s\) is the subgroup defined in \((1)\). A little more generally, if \(W = W(x_1, \ldots, x_t)\) and \(X_1, \ldots, X_t\) are subsets of \(G\), the symbol \(W(X_1, \ldots, X_t)\) stands for the corresponding subgroup of \(G\). In the particular case where \(X_1 = X_2 = \cdots = X_t = G\) we obviously have
\[W(X_1, \ldots, X_t) = W(G)\].

Thus, every star word can be treated as a function defined on the set of all subgroups of any locally finite group \(G\). Note that if \(H\) is a normal (characteristic) subgroup of a group \(G\) then \(W(H)\) is also a normal (characteristic) subgroup.

If \(W_1 = W_1(x_1, \ldots, x_t)\) and \(W_2 = W_2(y_1, \ldots, y_s)\) are two star words, we denote by \(W_1 \circ W_2\) the star word
\[W_1(W_2(y_{11}, \ldots, y_{s1}), \ldots, W_2(y_{1t}, \ldots, y_{st})).\]
By induction we define \(W_1 \circ W_2 \circ \cdots \circ W_n = (W_1 \circ W_2 \circ \cdots \circ W_{n-1}) \circ W_n\).
It is easy to see that the subgroup \((W_1 \circ W_2)(G)\) corresponding to the star word \((W_1 \circ W_2)\) is equal to the subgroup \(W_1(W_2(G))\). We also mention that the operation \(\circ\) is associative on the set of star words, that is
\[
(W_1 \circ W_2) \circ W_3 = W_1 \circ (W_2 \circ W_3).
\]

**Lemma 2.2.** A locally finite group \(G\) has a normal series

\[
G = G_0 \geq G_1 \geq \cdots \geq G_k = 1
\]

each of whose quotients \(G_i/G_{i+1}\) is either locally nilpotent or satisfies an outer commutator law \(w_i \equiv 1\) if and only if \(W(G) = 1\), where \(W = W_{k-1} \circ \cdots \circ W_1 \circ W_0\) with \(W_i = w_i\) if the quotient \(G_i/G_{i+1}\) satisfies the outer commutator law \(w_i \equiv 1\), and \(W_i = [x, y]_s\) if the quotient \(G_i/G_{i+1}\) is locally nilpotent.

**Proof.** This is easy by induction on \(k\). Suppose that \(G\) possesses the above series. If \(k \leq 1\), the result follows from Lemma 2.1. Assume that \(k \geq 2\) and set \(U = W_{k-1} \circ \cdots \circ W_2 \circ W_1\). By induction \(U(G_1) = 1\). It is clear that \(W_0(G) \leq G_1\) so \(W(G) = (U \circ W_0)(G) = U(W_0(G)) = 1\), as required.

If \(W(G) = 1\), then the required normal series is
\[
G = G_0 \geq G_1 \geq \cdots \geq G_k = 1,
\]
where \(G_i = W_{i-1}(W_{i-2}(G)) = (W_{i-1} \circ \cdots \circ W_1 \circ W_0)(G)\).

3. Proof of the main result

The next lemma plays a crucial role for the proof of Theorem 1.1.

**Lemma 3.1.** Let \(G\) be a locally finite group. If \(A, B, Y\) are normal subgroups of \(G\), then \([AB, Y]_s = [A, Y]_s[B, Y]_s\).

**Proof.** The inclusion \([AB, Y]_s \geq [A, Y]_s[B, Y]_s\) is obvious, so we prove \([AB, Y]_s \subseteq [A, Y]_s[B, Y]_s\). Let \(A, B\) be normal subgroups of \(G\). Choose an element \(h \in [AB, Y]_s\) and write

\[
h = [x_1, y_1] \cdots [x_n, y_n],
\]

where \(x_1, \ldots, x_n \in AB, y_1, \ldots, y_n \in Y\) and \((|x_i|, |y_i|) = 1\) for all \(i\).

We represent every \(x_i\) as a product \(a_{i_1}b_{i_1}a_{i_2}b_{i_2} \cdots a_{i_k}b_{i_k}\) of some elements \(a_{ij} \in A, b_{ij} \in B\) such that \((|a_{i_1}|, |y_{i_1}|) = 1\) and \((|b_{i_1}|, |y_{i_1}|) = 1\). This is always possible. Indeed, let \(x_i = a_ib_i\), where \(a_i \in A, b_i \in B\), and \(|x_i| = p_1^{n_1} \cdots p_s^{n_s}\) for some primes \(p_1, p_2, \ldots, p_t\). For every \(1 \leq j \leq t\) we set \(n_j = p_1^{n_1} \cdots p_{j-1}^{n_{j-1}} p_{j+1}^{n_{j+1}} \cdots p_s^{n_s}\). The element \(x_i^{n_j}\) is a \(p_j\)-element and belongs to some \(p_j\)-Sylow subgroup of the finite group \(H = \langle a, b \rangle\). We have \(H = \langle a \rangle^H b^H\) and use the fact that a Sylow subgroup of a product of normal subgroups is equal to a product of Sylow subgroups of the factors. Therefore \(x_i^{n_j} = a_{p_j}b_{p_j}\) for some \(p_j\)-elements \(a_{p_j} \in \langle a \rangle^H \leq A, b_{p_j} \in \langle b \rangle^H \leq B\). There are integers \(f_1, \ldots, f_t\) such that \(f_1n_1 + f_2n_2 + \cdots + f_tn_t = 1\). So

\[
x_i = \prod_{j=1}^t (a_{p_j} b_{p_j})^{f_j},
\]

and we get the required representation of \(x_i\).
Using repeatedly the identity \([ab, c] = [a^b, c^b][b, c]\) we obtain that the commutator \([x_i, y_i] = [a_i, y_i] \cdots [b_i, y_i]\) is equal to a product of some conjugates of the commutators \([a_i, y_i], [b_i, y_i], \ldots, [b_i, y_i]\).

Taking into consideration that \(A, B\) are normal and \(\langle [a_i], [y_i]\rangle = 1, \langle [b_i], [y_i]\rangle = 1\) we obtain that \([a_i^g, y_i^g] \in [A, Y], [b_i^g, y_i^g] \in [B, Y]\) for all \(g \in G\) and for all \(j\). Thus, the element \(h = [x_1, y_1] \cdots [x_n, y_n]\) is the product of some commutators from \([A, Y]\) and \([B, Y]\). Since the subgroups \([A, Y]\) and \([X, Y]\) are normal, \(h \in [A, Y]_n[B, Y]_n\).

**Corollary 3.2.** Let \(W\) be a star word of weight \(t\), and \(A_1, \ldots, A_t\) normal subgroups of a group \(G\).

\[
W\left( A_1, \ldots, A_{i-1}, \prod_{N \in \mathcal{N}} N, A_{i+1}, \ldots, A_t \right) = \prod_{N \in \mathcal{N}} W\left( A_1, \ldots, A_{i-1}, N, A_{i+1}, \ldots, A_t \right)
\]

for any set \(\mathcal{N}\) of normal subgroups.

**Proof.** We expand \(W(A_1, \ldots, A_{i-1}, \prod_{N \in \mathcal{N}} N, A_{i+1}, \ldots, A_t)\) by applying repeatedly the equalities \([A, BC] = [A, B][A, C]\) and \([A, BC]_s = [A, B]_s[A, C]_s\). This is possible because all the “star-commutator subgroups” arising in this process are normal. In the end we obtain the product

\[
\prod_{N \in \mathcal{N}} W\left( A_1, \ldots, A_{i-1}, N, A_{i+1}, \ldots, A_t \right).
\]

In what follows \(W_\sigma(X_1, \ldots, X_t)\) denotes \(W(X_{\sigma(1)}, \ldots, X_{\sigma(t)})\), where \(\sigma\) is a permutation of degree \(t\).

**Theorem 3.3.** Let \(W(x_1, \ldots, x_t)\) be a star word. Then, in any group, the number of finite-index subgroups which are maximal (by inclusion) among all normal subgroups \(N\) such that \(W(N) = 1\) is finite. Moreover, the number of such subgroups of index \(\leq n\) does not exceed

\[
2^{F_{t-1}(n)}, \text{ where } F^k(x) \text{ is the } k\text{-th iteration of the function } F(x) = xn^{2^x}.
\]

**Proof.** The proof mimicks that of Theorem 1’ in [6]. We include it for completeness. Let \(\mathcal{N}\) be the set of finite-index subgroups \(N\) of \(G\) that are maximal by inclusion among all normal subgroups satisfying the property \(W(N) = 1\). We wish to prove that \(\mathcal{N}\) is finite. If the set \(\mathcal{N}\) is empty, then we have nothing to prove. Otherwise, consider a subgroup \(G_0 \in \mathcal{N}\). This subgroup satisfies the property that

\[
W_\sigma(G_0, \ldots, G_0) = W(G_0) = 1 \quad \text{for all } \sigma \in S_t.
\]

where \(S_t\) is the symmetric group of degree \(t\).

The subgroup \(G_0\) has finite index. Therefore, the set of subgroups \(\{NG_0 \mid N \in \mathcal{N}\}\) is finite and coincides with the set \(\{N_1G_0 \mid N_1 \in \mathcal{N}_1\}\), where \(\mathcal{N}_1\) is some finite subset of \(\mathcal{N}\). The subgroup

\[
G_1 = G_0 \cap \bigcap_{N \in \mathcal{N}_1} N
\]

has finite index and satisfies the equality

\[
W_\sigma(G_1, \ldots, G_1, NG_0) = 1 \quad \text{for all } \sigma \in S_t \text{ and for all } N \in \mathcal{N}.
\]

Indeed, by the choice of \(\mathcal{N}_1\), each product \(NG_0\), where \(N \in \mathcal{N}_1\), coincides with a product \(N_1G_0\) for some group \(N_1 \in \mathcal{N}_1\) and \(N_1 \supseteq G_1 \subseteq G_0\). Therefore, by Corollary 3.2
The subgroup $G_1$ has finite index. Therefore, the set of subgroups \{ $NG_1 \mid N \in \mathcal{N}$ \} is finite and coincides with the set \{ $N_2 G_1 \mid N_2 \in \mathcal{N}_2$ \} for some finite subset $\mathcal{N}_2$ of $\mathcal{N}$. The subgroup

$$G_2 = G_1 \cap \bigcap_{N \in \mathcal{N}_2} N$$

has finite index and satisfies the equality

$$W_\sigma(G_2, \ldots, G_2, NG_1, NG_1) = 1 \quad \text{for all } \sigma \in S_t \text{ and for all } N \in \mathcal{N}.$$ 

Indeed, by the choice of $\mathcal{N}_2$, each product $NG_1$, where $N \in \mathcal{N}$, coincides with a product $N_2 G_1$ for some group $N_2 \in \mathcal{N}_2$ and $N_2 \supseteq G_2 \subseteq G_1 \subseteq G_0$. Therefore, by Corollary 3.2

$$W_\sigma(G_2, \ldots, G_2, NG_1, NG_1) = W_\sigma(G_2, \ldots, G_2, N_2 G_1, N_2 G_1)$$

$$= W_\sigma(G_2, \ldots, G_2, N_2) W_\sigma(G_2, \ldots, G_2, N_2, G_1)$$

$$\times W_\sigma(G_2, \ldots, G_2, G_1, N_2) W_\sigma(G_2, \ldots, G_2, G_1, G_1)$$

$$\subseteq W_\sigma(N_2, \ldots, N_2, N_2) W_\sigma(G_1, \ldots, G_1, N_2, G_1)$$

$$\times W_\sigma(G_1, \ldots, G_1, G_1, N_2) W_\sigma(G_0, \ldots, G_0, G_0, G_0).$$

The first factor of the last product is trivial, because the group $N_2$ satisfies the property $W(N_2) = 1$. The second and the third factors are trivial by (3). The fourth factor is trivial by (2).

Continuing in the same manner, we finally obtain a finite-index subgroup $G_{t-1}$ such that

$$W_\sigma(NG_{t-1}, \ldots, NG_{t-1}) = 1 \quad \text{for all } \sigma \in S_t \text{ and for all } N \in \mathcal{N}.$$ 

In view of the maximality of all these subgroups $N$ this means that $G_{t-1} \subseteq N$ for all $N \in \mathcal{N}$, i.e. $G_{t-1} \subseteq \bigcap_{N \in \mathcal{N}} N$ and, therefore, the intersection $\bigcap_{N \in \mathcal{N}} N$ has finite index in $G$. The finiteness of the index implies the finiteness of the set $\mathcal{N}$, as required.

To obtain the bound it is sufficient to note that if all subgroups in $\mathcal{N}$ have index not larger than $n$, then

$$|G : G_k| \leq |G : G_{k-1}| n^{k|\mathcal{N}_k|} \quad \text{and} \quad |\mathcal{N}_k| \leq 2^{|G : G_{k-1}|} \quad \text{(this is a very rough estimate)}.$$ 

Therefore,

$$|G : G_k| \leq |G : G_{k-1}| n^{k|G : G_{k-1}|} = n^{k|G : G_{k-1}|}$$

and, i.e. $|G : G_{t-1}| \leq F^{t-1}(n)$ and $|\mathcal{N}| \leq 2^{F^{t-1}(n)}$,

where $F^k(x)$ is the $k$-th iteration of the function $F(x) = xn^2$.

Proof of Theorem 1.1. In view of Lemma 2.2, Theorem 1.1 is a straightforward consequence of Theorem 3.3. Indeed, let the subgroup $N$ be maximal by inclusion among all normal subgroups satisfying the hypothesis of Theorem 1.1. We set $W = W_{k-1} \circ \cdots \circ W_1 \circ W_0$, where $W_i = w_i$ if the quotient
\( N_i/N_{i+1} \) satisfies the outer commutator law \( w_i \equiv 1 \) and \( W_\ell = [x, y] \) if the quotient \( N_i/N_{i+1} \) is locally nilpotent. By Lemma 2.2 the subgroup \( W(N) \) is trivial and consequently \( W(N^\alpha) = 1 \) for any automorphism \( \alpha \) of \( G \). The subgroup

\[
H = \bigcap_{\alpha \in \text{Aut } G} N^\alpha
\]

is characteristic. By Theorem 3.3 the index \( [G : H] \) is bounded by a function depending only on \( [G : N] \) and the weight of the star word \( W \). The required series is

\[
H = H_0 \geq H_1 \geq \cdots \geq H_k = 1,
\]

where \( H_i = W_{i-1}(W_{i-2}(H)) = (W_{i-1} \circ \cdots \circ W_1 \circ W_0)(H) \). In fact, all the \( H_i \) are characteristic as subgroups corresponding to some star words; \( H_i/H_{i+1} \) is locally nilpotent by Lemma 2.1 if \( N_i/N_{i+1} \) is locally nilpotent, and \( H_i/H_{i+1} \) satisfies the outer commutator law \( w_i \equiv 1 \) if \( N_i/N_{i+1} \) satisfies the same commutator law \( w_i \equiv 1 \).

We can also adapt the proof of Klyachko and Melnikova [5] to obtain a better bound for the index.

**Theorem 3.4.** Let \( W(x_1, \ldots, x_t) \) be a star word. If a group \( G \) contains a normal finite-index subgroup \( N \) such that \( W(N) = 1 \), then \( G \) contains a characteristic and even invariant with respect to all surjective endomorphisms subgroup \( H \) such that \( W(H) = 1 \) and \( \log_2 [G : H] \leq f^{t-1}(\log_2 [G : N]) \), where \( f^k(x) \) is the \( k \)-th iteration of the function \( f(x) = x(x+1) \).

**Proof.** The proof is virtually the same as the proof in [5]; we should only replace the outer commutator \( \omega(x_1, \ldots, x_n) \), which defines the outer commutator law \( w \equiv 1 \) in the hypothesis of theorem in [5], by the star word \( W(x_1, \ldots, x_t) \) and use Corollary 3.2 instead of the similar property for the ordinary commutators subgroups and outer commutator words. \( \Box \)

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**References**


