Netlike partial cubes, V: Completion and netlike classes

Norbert Polat
L.A.E., Université Jean Moulin (Lyon 3), 6 cours Albert Thomas, 69355 Lyon Cedex 08, France

A R T I C L E   I N F O

Article history:
Received 18 May 2008
Received in revised form 20 January 2009
Accepted 22 January 2009
Available online 23 February 2009

Keywords:
Hypercube
Partial cube
Netlike partial cube
Median cycle property
Median graph
Cellular bipartite graph
Geodesic convexity
Gated set
Completion
Netlike class

A B S T R A C T

We define a completion of a netlike partial cube G by replacing each convex 2n-cycle C of G with n ≥ 3 by an n-cube admitting C as an isometric cycle. We prove that a completion of G is a median graph if and only if G has the Median Cycle Property (MCP) (see N. Polat, Netlike partial cubes III. The Median Cycle Property, Discrete Math.). In fact any completion of a netlike partial cube having the MCP is defined by a universal property and turns out to be a minimal median graph containing G as an isometric subgraph. We show that the completions of the netlike partial cubes having the MCP preserves the principal constructions of these graphs, such as: netlike subgraphs, gated amalgams and expansions. Conversely any netlike partial cube having the MCP can be obtained from a median graph by deleting some particular maximal finite hypercubes. We also show that, given a netlike partial cube G having the MCP, the class of all netlike partial cubes having the MCP whose completions are isomorphic to those of G share different properties, such as: depth, lattice dimension, semicube graph and crossing graph.

© 2009 Elsevier B.V. All rights reserved.

1. Introduction

The class of netlike partial cubes was introduced in Part I [20] of this series of papers as a class of partial cubes containing median graphs, even cycles, benzenoid graphs and cellular bipartite graphs as particular elements. Among netlike partial cubes are those which have the Median Cycle Property (MCP), i.e. the property that any triple of vertices admits a unique median or a unique median cycle (see [21]). These particular partial cubes will be simply called MC-netlike partial cubes in this paper. Special MC-netlike partial cubes are median graphs and cellular bipartite graphs.

The concept of completion of a mathematical structure is a very common one: completion of an ordered set, of a metric space and more generally of a uniform space, of a valued field, the Stone–Čech compactification of a topological space, etc. The completion is generally defined or characterized up to isomorphism by a uniform property. This is such a process of construction of a richer structure that we will apply in this paper, in a very natural way, to all netlike partial cubes and especially to those which have the MCP. More precisely, the aim of this completion is to isometrically embed, if possible, any netlike partial cube into a minimal netlike partial cube all of whose triples of vertices have a unique median, hence in other words into a minimal median graph.

Median graphs are the netlike partial cubes all of whose convex cycles are 4-cycles, while a netlike partial cube may have convex cycles of any even length. Because the convex hull of any isometric cycle of a median graph is a hypercube, we may expect to obtain a median graph by “filling” the “holes” formed by the convex cycles of a netlike partial cube G of length greater than 4 by adequate hypercubes. As we saw at the end of [22], what we usually get is not even a netlike partial cube. Actually, the results of this construction, which we call the completions of G, are isomorphic almost median graphs (Corollary 3.3). A completion of a netlike partial cube G turns out to be a median graph if and only if G has the MCP.

E-mail address: norbert.polat@univ-lyon3.fr.

0012-365X/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2009.01.019
As can be expected of a good definition of such a concept, the completions of MC-netlike partial cubes preserve the main constructions of these graphs, that is: netlike subgraphs, gated amalgams and C-peripheral expansions (Section 4).

Conversely, from any median graph, by deleting some special maximal finite hypercubes minus one of their isometric cycles of maximal length, we obtain an MC-netlike partial cube, and moreover all MC-netlike partial cubes can be obtained in this way. This gives a very simple process of construction of all MC-netlike partial cubes from median graphs (Section 5).

This leads us to consider, for each MC-netlike partial cube G, the class of all MC-netlike partial cubes whose completions are isomorphic to those of G. Such a class is called the netlike class of G. Special elements of this class are evidently the maximal ones, that is the median graphs which are isomorphic to the completions of G, and the minimal ones which are the MC-netlike partial cubes which are isomorphic to what is called the skeleton of G. The elements of a netlike class, in addition of having isomorphic completions and isomorphic skeletons, turn out to share other interesting properties. For example, they have the same depth, the same lattice dimension, isomorphic semicube graphs and isomorphic crossing graphs (Section 6).

Finally we point out different examples in which completion can be used to infer some properties of MC-netlike partial cubes from analogous properties of median graphs.

2. Preliminaries

2.1. Graphs

The graphs we consider are undirected, without loops or multiple edges, and may be finite or infinite. Let G be a graph. If x ∈ V(G), the set Nc(x) := {y ∈ V(G) : xy ∈ E(G)} is the neighborhood of x in G, Nc(x) := {x} ∪ Nc(x) is the closed neighborhood of x in G and dc(x) := |Nc(x)| is the degree of x in G. For a set X of vertices of G we put Nc[X] := ∪x∈X Nc[x] and Nc(x) := Nc[X] \ X, we denote by G[X] the subgraph of G induced by X, and we set G − X := G[V(G) \ X]. Moreover we denote by βc(G) the boundary of a subgraph H of G, that is the set of all vertices of H that have a neighbor in G − H.

A path P = (x0, ..., xn) is a graph with V(P) = {x0, ..., xn}, xi ≠ xj if i ≠ j, and E(P) = {xi, xi+1} : 0 ≤ i < n}. A path P = (x0, ..., xn) is called an (x0, xn)-path, x0 and xn are its endpoints, while the other vertices are called its inner vertices, n = |E(P)| is the length of P. If x and y are two vertices of a path P, then we denote by P[x, y] the subpath of P whose endpoints are x and y.

A cycle C with V(C) = {x1, ..., xn}, xi ≠ xj if i ≠ j, and E(C) = {xi, xi+1} : 1 ≤ i < n} ∪ {xn, x0}, will be denoted by (x1, ..., xn). The non-negative integer n = |E(C)| is the length of C, and a cycle of length n is called an n-cycle and is often denoted by Cn.

Let G be a connected graph. The usual distance between two vertices x and y, that is, the length of an (x, y)-geodesic (=shortest (x, y)-path) in G, is denoted by dc(x, y). A connected subgraph H of G is isometric in G if dG(x, y) = dc(x, y) for all vertices x and y of H. The (geodesic) interval Ic(x, y) between two vertices x and y of G is the set of vertices of all (x, y)-geodesics in G.

2.2. Convexities

A convexity on a set X is an algebraic closure system C on X. The elements of C are the convex sets and the pair (X, C) is called a convex structure. See van de Vel [25] for a detailed study of abstract convex structures. Several kinds of graph convexities, that is convexities on the vertex set of a graph G, have already been investigated. We will principally work with the geodesic convexity, that is the convexity on V(G) which is induced by the geodesic interval operator Ic. In this convexity, a subset C of V(G) is convex provided it contains the geodesic interval Ic(x, y) for all x, y ∈ C. The convex hull coC(A) of a subset A of V(G) is the smallest convex set which contains A. The convex hull of a finite set is called a polytope. A subset H of V(G) is a half-space if H and V(G) − H are convex. We denote by Ic the pre-hull operator of the geodesic convex structure of G, i.e. the self-map of P(V(G)) such that Ic(A) := ∪x,y∈A Ic(x, y) for each A ⊆ V(G). The convex hull of a set A ⊆ V(G) is then coC(A) = ∪A∈A Ic(A). Furthermore we say that a subgraph of a graph G is convex if its vertex set is convex, and by the convex hull coC(H) of a subgraph H of G we mean the smallest convex subgraph of G containing H as a subgraph, that is coC(H) := G[coC(V(H))].

2.3. Netlike partial cubes

First we recall some properties of partial cubes, that is of isometric subgraphs of hypercubes. Partial cubes are particular connected bipartite graphs.

For an edge ab of a graph G, let

\[ W^G_{ab} := \{x ∈ V(G) : dc(a, x) < dc(b, x)\}, \]

\[ U^G_{ab} := Nc(W^G_{ba}). \]

Where no confusion is likely, we will simply denote W_{ab} and U_{ab} by W_{ab} and U_{ab}, respectively. Note that the sets W_{ab} and W_{ba} are disjoint and that V(G) = W_{ab} ∪ U_{ab} if G is bipartite and connected.
Two edges $xy$ and $uv$ are in the Djoković–Winkler relation $\Theta$ if
\[ d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u). \]

If $G$ is bipartite, the edges $xy$ and $uv$ are in relation $\Theta$ if and only if $d_G(x, u) = d_G(y, v)$ and $d_G(x, v) = d_G(y, u)$. The relation $\Theta$ is clearly reflexive and symmetric.

**Theorem 2.1** (Djoković [8, Theorem 1] and Winkler [26]). A connected bipartite graph $G$ is a partial cube if and only if it has one of the following properties:

(i) For every edge $ab$ of $G$, the sets $W_{ab}$ and $W_{ba}$ are convex (and thus are half-spaces).

(ii) The relation $\Theta$ is transitive.

Note that every interval and every polytope of a partial cube are finite. We now recall two characterizations of median graphs that we will often use.

**Proposition 2.2** (Bandelt [1] (see also [14, Theorem 5])). Let $G$ be a connected bipartite graph. The following assertions are equivalent:

(i) $G$ is a median graph.

(ii) The sets $U_{ab}$ and $U_{ba}$ are convex for each edge $ab$ of $G$.

(iii) The convex closure of any isometric cycle of $G$ is a hypercube.

We denote by $CV(G)$ (resp. $3V(G)$) the set of vertices of a graph $G$ which belong to a cycle of $G$ (resp. whose degree is at least 3). We say that a set $A \subseteq V(G)$ is $C$-convex (resp. $(3)$-convex) if $CV(G[I_C(A)]) \subseteq A$ (resp. $3V(G[I_C(A)]) \subseteq A$). The set of $C$-convex subsets of $V(G)$ and the one of $(3)$-convex subsets of $V(G)$ are convexities on $V(G)$ which are finer than the geodesic convexity.

**Lemma 2.3** (Polat [21, Corollary 2.7]). If $A$ is a $C$-convex set of a connected graph $G$, then $I_C(A)$ is convex.

By relaxing the type of convexity in Bandelt’s characterization of a median graph (Proposition 2.2(ii)) we obtain what we call a netlike partial cube.

**Definition 2.4.** We say that a partial cube $G$ is netlike if, for each edge $ab$, $U_{ab}$ and $U_{ba}$ are $C$-convex.

Thus median graphs are netlike partial cubes. Clearly even cycles are also netlike partial cubes, and moreover any convex subgraph of a netlike partial cube is a netlike partial cube. We have the following characterization of netlike partial cubes:

**Proposition 2.5.** A partial cube $G$ is netlike if and only if it has the following two properties:

(i) For each edge $ab$ of $G$, the sets $U_{ab}$ and $U_{ba}$ are $(3)$-convex.

(ii) The convex hull of each non-convex isometric cycle of $G$ is a hypercube.

If $H$ is a subgraph of a partial cube $G$, we denote by $\Theta_C(H)$ (or simply $\Theta(H)$ if no confusion is likely) the set of the $\Theta$-classes of all edges of $H$.

**Lemma 2.6** (Polat [20, Proposition 6.3]). Let $C_0$, $C_1$ be two distinct convex cycles of a netlike partial cube $G$. If $\Theta(C_0) \cap \Theta(C_1) \neq \emptyset$, then either $C_0$ and $C_1$ are disjoint or their intersection is a $K_2$, and moreover $|\Theta(C_0) \cap \Theta(C_1)| = 1$ whenever the length of at least one of these cycles is greater than 4.

A netlike partial cube $G$ such that, for each edge $ab$, $I_C(U_{ab})$ and $I_C(U_{ba})$ induce trees, is called a linear partial cube.

**Lemma 2.7.** [Polat [20, Theorem 7.4]] Let $G$ be a partial cube. The following assertions are equivalent:

(i) $G$ is linear.

(ii) $G$ is a netlike partial cube which contains no hypercube of dimension greater than 2.

(iii) $G$ is a netlike partial cube whose isometric cycles are convex.

**Lemma 2.8** (Polat [20, Lemma 6.1]). Let $ab$ be an edge of a netlike partial cube $G$. Then any convex cycle of $G[U_{ab}]$ is a 4-cycle.

**Lemma 2.9** (Polat [20, Corollary 7.2]). A netlike partial cube is a median graph if and only if any of its convex cycles is a 4-cycle.

Let $H$ be an induced subgraph (or its vertex set) of a graph $G$ and $x \in V(G)$. The gate of $x$ in $H$ is a vertex $y$ of $H$ such that $y \in I_C(x, z)$ for every $z \in V(H)$. The subgraph $H$ is said to be gated if each $x \in V(G)$ has a gate in $H$ (see [9]). Obviously, every gated subgraph is convex. Conversely any convex subgraph of a median graph is gated. However, this is clearly not true for netlike partial cubes. The following result characterizes the convex subgraphs of a netlike partial cube which are gated.

**Proposition 2.10** (Polat [20, Theorem 6.2]). A convex subgraph $H$ of a netlike partial cube is gated if and only if every convex cycle which has at least three vertices in common with $H$ is a cycle of $H$.

Note that a partial cube $G$ being an isometric subgraph of some hypercube $Q$, any hypercube in $G$ is then a convex subgraph of $Q$, and thus is gated in $G$. By [20, Corollary 6.4], any convex cycle of a netlike partial cube is gated.
2.4. The median cycle property

We recall that, if \(u_0, u_1, u_2\) are three vertices of a graph \(G\), then a median of the triple \((u_0, u_1, u_2)\) is any element of the intersection \(I_C(u_0, u_1) \cap I_C(u_1, u_2) \cap I_C(u_2, u_0)\), and that a median cycle (see [2]) of \((u_0, u_1, u_2)\) is a gated cycle \(C\) of \(G\) such that for all \(i, j, k \in \{0, 1, 2\}\), if \(x_i\) is the gate of \(u_i\) in \(C\), then: \([x_i, x_j] \subseteq I_C(u_i, u_j)\) if \(i \neq j\), and \(d_C(x_i, x_j) = d_C(x_i, x_k) + d_C(x_k, x_j)\).

**Definition 2.11.** A graph \(G\) has the *Median Cycle Property* (*MCP*) for short, if every triple of vertices of \(G\) admits a unique median or a unique median cycle.

Medians graphs have the *MCP* by definition. Cellular bipartite graphs are other examples of graphs which have the *MCP*. The cellular bipartite graphs are the graphs which can be obtained from single edges and even cycles by successive gated amalgamations. These graphs were defined and studied by Bandelt and Chepoi [2]. They showed in particular [2, Proposition 3] that the cellular bipartite graphs have the *MCP* and that they are partial cubes.

**Lemma 2.12** (Polat [21, Proposition 3.4]). Let \(G\) be the gated amalgam of two partial cubes \(G_1\) and \(G_2\). Then \(G\) has the *MCP* if and only if \(G_1\) and \(G_2\) have the *MCP*.

We call a triple of convex cycles such that at least one of them has length greater than 4, any pair of them has exactly an edge in common and they all intersect in one vertex, a *tricycle*. Note that a netlike partial cube cannot contain \(Q_3^-\) (i.e. \(Q_3\) minus a vertex) as a convex subgraph.

**Lemma 2.13** (Polat [21, Lemma 2.15]). A netlike partial cube contains no tricycle such that at least one of the cycles is a 4-cycle.

**Proposition 2.14** (Polat [21, Theorem 2.5]). Let \(G\) be a netlike partial cube. The following assertions are equivalent:

(i) \(G\) has the *MCP*.

(ii) \(G\) contains no tricycle.

From now on, a netlike partial cube that has the *MCP* will be called an *MC-netlike partial cube*.

3. Completion of a netlike partial cube

3.1. Definition and general properties

Let \(G\) be a bipartite graph. We denote by \(\Gamma(G)\) the set of all convex cycles of \(G\) of length greater than 4. We say that a family \(H = (H_C)_{C \in \Gamma(G)}\) of hypercubes is *suitable* for \(G\) if, for all \(C, C' \in \Gamma(G)\):

- \(H_C\) is an \(n\)-cube if \(C\) is a \(2n\)-cycle;
- \(C\) is an isometric cycle of \(H\);
- \(H_C \cap G = C\);
- \(H_C \cap H_{C'} = C \cap C'\) if \(C \neq C'\).

For any \(C \subseteq \Gamma(G)\), \(H_C := (H_C)_{C \subseteq \Gamma(G)}\) is called a *suitable subfamily of hypercubes* for \(G\), and we denote by \(G^H\) the graph \(G^H := \bigcup_{C \in \Gamma(G)} H_C\).

In particular the graph \(G^H\) is called a completion of \(G\), and more precisely the *completion of \(G\) with respect to the suitable family \(H)*.

If \(G^H\) and \(G^{H'}\) are two completions of a graph \(G\), then there is clearly an isomorphism of \(G^H\) onto \(G^{H'}\) whose restriction on \(V(G)\) is the identity function. For this reason we will sometimes refer to the *completion of \(G\)*, keeping in mind that this graph is defined up to isomorphism. We will often denote by \(G^+\) a completion of \(G\), even a non-specific one. In other words we will consider \(G^+\) as a generic notation for the different completions of \(G\). See Fig. 1 for an example of a graph and its completion.

From now on, if \(G\) is partial cube and if \(G^* := G^{H_C}\) for some suitable subfamily \(H_C\) of hypercubes for \(G\), then, for any edges \(ab\) and \(uv\) of \(G\) and \(G^*\) respectively, we will denote \(W_{ab}^{G^*}, U_{uv}^{G^*}\), and \(U_{uv}^{G^*}\) by \(W_{ab}, U_{uv}, W_{uv}\) and \(U_{uv}\) respectively.

**Lemma 3.1.** Let \(G\) be a bipartite graph, \(H = (H_C)_{C \subseteq \Gamma(G)}\) a suitable family of hypercubes for \(G\), \(C \subseteq \Gamma(G)\), and \(G^* := G^{H_C}\). Then \(G^*\) is an isometric subgraph of \(G^+\). More precisely, for all \(x, y \in V(G)\) and each \((x, y)\)-geodesic \(P\) in \(G^+\), there exists an \((x, y)\)-geodesic \(R\) in \(G\) such that \(P \cap G \subseteq R\).

**Proof.** Let \(x, y \in V(G)\) and \(P\) an \((x, y)\)-geodesic in \(G^+\). We are done if \(P\) is a path of \(G\). Suppose that \(P - G \neq \emptyset\) and let \(z \in V(P - G)\). Then, because \(x, y \in V(G)\), there exist two vertices \(u, v \in V(P \cap G)\) such that \(z \in P, u, v\) of \(P\) passes through \(z\) and \(z\) is an isometric cycle of \(H_C\) for some \(C \in H\). Hence, because \(C\) is an isometric cycle of \(H_C\), it follows that \(C\) contains a \((u, v)\)-geodesic \(Q\). Then \(P' := (P - P[u, v]) \cup Q\) is an \((x, y)\)-geodesic such that \(P \cap G \subseteq P' \cap G\). Therefore, by repeating this argument, we can obtain an \((x, y)\)-geodesic \(R\) such that \(P \cap G \subseteq R \subseteq G\). \(\square\)

**Lemma 3.2.** Let \(G\) be a netlike partial cube, \(C \subseteq \Gamma(G)\), \(H_C := (H_C)_{C \subseteq \Gamma(G)}\) a suitable subfamily of hypercubes for \(G\), and \(G^* := G^{H_C}\). Then \(G^*\) is a partial cube such that, for any edge \(ab\) of \(G^*\) and all non-adjacent vertices \(x, y \in U_{ab}^*\) if \(I_C(x, y) \cap U_{ab}^* = \{x, y\}\), then \(I_C(x, y) \subseteq V(C)\) for some cycle \(C \in \Gamma(G) - C\) that contains an edge which is \(\Theta\)-equivalent to \(ab\).
Let $11$ Any completion of an netlike partial cube is an almost-median graph.

(a) We will show that $G^*$ is a partial cube by proving that the sets $W_{ab}^*$ and $W_{ba}^*$ are convex for any edge $ab$ of $G^*$, or equivalently for any edge $ab$ of $G$.

First note that for any $C \in \mathcal{C}$, because $C$ is gated in $G$, if $u$ and $v$ are the gates of $a$ and $b$ in $C$, respectively, and if $u \neq v$, then $u$ and $v$ are adjacent and the edges $uv$ and $ab$ are $\Theta$-equivalent. Moreover the gate of a vertex $x$ in $G$ in $C$ is also the gate of $x$ in $H_C$, because $G$ is isometric in $G^*$ and $C = \beta_{G^*}(H_C)$. It follows that any edge $xy$ of $H_C$ is in relation $\Theta$ with $ab$ if and only if it is in relation $\Theta$ with $uv$.

Let $ab \in E(G)$. Suppose that $W_{ab}^*$ is not convex. Then there is a geodesic $P = \langle x_0, \ldots, x_n \rangle$ with $x_0, x_n \in U_{ab}^*$ and $x_i \in W_{ab}^*$ for $1 \leq i \leq n - 1$, and whose length $n$ is as small as possible. $P$ is not a path of $H_C$ for any $C \in \mathcal{C}$, since otherwise we should have $I_C(x_0, x_n) \subseteq U_{ab}^*$, contrary to the properties of $P$. Let $i$ be the smallest integer such that $x_i \notin V(G)$. Assume that $i > 1$. Then, by the definition of $G^*$, there exists $C_0 \in \mathcal{C}$ such that $x_i \in U_{ab}^* \cap V(C_0)$ for $0 \leq i \leq 1$. Let $x'_i$ be the projection of $x_i$ in $U_{ab}^*$. Then $x'_i \in I_C(x_0, x_i)$ because $H_0^C$ is a hypercube. It follows that $\langle x'_i, x_i, \ldots, x_n \rangle$ is a geodesic of $G^*$ joining two vertices of $U_{ab}^*$, whose inner vertices lie in $W_{ab}^*$ and whose length is less than $n$, contrary to the choice of $P$. Therefore $i = 1, x'_1 = x_0$ and $x_1 \in U_{ab}^* \cap V(C_0)$. Analogously we have $x_{n-1} \in U_{ab}^* \cap V(C_1)$ for some $C_1 \in \mathcal{C}$.

Note that at least $x_0$ or $x_n$ does not belong to $V(G)$, since otherwise $x_1$ and $x_{n-1}$ could not belong to $I_C(x_0, x_n)$ because $W_{ab}^*$ is convex, and thus not to $I_C(x_0, x_n)$ because $G$ is isometric in $G^*$. For $i \in \{0, n\}$ let $u_i, v_i$ be an edge of $C_i$, which is $\Theta$-equivalent to $ab$, and that we choose so that $u_0$ and $v_0$ are the gates of $v_0$ and $v_0$ in $C_0$, respectively, and also so that $u_0$ and $v_0$ are the gates of $u_0$ and $v_0$ in $C_0$, respectively. Obviously $u_i \neq x_i$ if $x_i \notin V(G)$ for $i \in \{0, n\}$.

Due to the fact that $U_{ab}^* = C$-convex and since $x_1, x_{n-1} \in I_C(U_{ab}^*)$, it follows that $v_0$ and $v_n$ belong to any $(x_1, x_{n-1})$-geodesic in $G$. Hence, because $G$ is isometric in $G^*$, we have

$$d_C(x_0, x_n) \leq d_C(x_0, u_0) + d_C(u_0, v_0) + d_C(v_0, x_n),$$

$$= d_C(x_1, v_0) + d_C(v_0, v_n) + d_C(v_n, x_{n-1}),$$

$$= d_C(x_1, x_{n-1}) = n - 2,$$

contrary to the hypothesis.

Consequently $W_{ab}^*$ is convex, and analogously so is $W_{ba}^*$. Therefore $G^*$ is a partial cube.

(b) Let $ab \in E(G^*)$. Suppose that there are two non-adjacent vertices $x, y \in U_{ab}^*$ such that $I_C(x, y) \cap U_{ab}^* = \{x, y\}$. Note that, for any vertex $u \in U_{ab}^* - V(G)$, $N_C(u) \subseteq U_{ab}^* \cup U_{ab}^*$. Hence $x, y \in V(G)$. Let $x'$ and $y'$ be the projections of $x$ and $y$ in $U_{ab}^*$, respectively, $P$ an $(x, y')$-geodesic and $P'$ an $(x', y')$-geodesic in $G$. Then $C := \langle x', x \rangle \cup P \cup \langle y', y' \rangle \cup P'$ is a convex cycle of $G$, since $G$ is netlike, and thus $C \in \Gamma(G)$. Moreover the edges $xx'$ and $yy'$ of $C$ are $\Theta$-equivalent to $ab$. Assume that $C \in \mathcal{C}$. Then $I_C(x, x') \subseteq U_{ab}^*$, contrary to the hypothesis. Therefore $C \notin \Gamma(G) - \mathcal{C}$, and $I_C(x, y) \subseteq V(C)$. □

We recall that a partial cube $G$ is an almost-median graph (see [11]) if, for any edge $ab$ of $G$, the subgraphs $G[U_{ab}]$ and $G[U_{ba}]$ are isometric in $G$.

Corollary 3.3. Any completion of a netlike partial cube is an almost-median graph.

Proof. Let $H = (H_C)_{C \in \Gamma(G)}$ be a suitable family of hypercubes for a netlike partial cube $G$, and $G^+ := G^H$ a completion of $G$. By Lemma 3.2, $G^+$ is a partial cube. Suppose that the subgraph $G^+\[U_{ab}^*\]$ is not isometric. Then there are two non-adjacent vertices $x, y \in U_{ab}^*$ such that $I_C(x, y) \cap U_{ab}^* = \{x, y\}$. By Lemma 3.2, $I_C(x, y) \subseteq V(C)$ for some cycle $C \in \Gamma(G) - \mathcal{C}$, contrary to the hypothesis that $C \notin \Gamma(G)$. Hence $G^+$ is an almost-median graph. □

An almost-median graph is not necessarily the completion of a netlike partial cube. This is for example the case of the graph $Q_3$. Furthermore, as we showed in the last part of [22], $G^+$ is generally not a median graph, and not even a netlike partial cube. For example the graph $G^+$ in Fig. 1 contains $Q_3^-$ as an isometric subgraph.

Let $G$ be a netlike partial cube. We denote by $\Gamma^c(G)$ the set of all elements of $\Gamma(G)$ that are not members of a tricycle of $G$.

Fig. 1. A benzenoid graph and its completion.
Lemma 3.4. Let $G$ be a netlike partial cube, $C \subseteq \Gamma^\ast(G)$, $H_C = (H_C)_{C \in C}$ a suitable subfamily of hypercubes for $G$, and $G^\ast := G^{H_C}$. For any edge $ab$ of $G^\ast$ and all non-adjacent vertices $x, y \in U^a_{ab}$, if there is an $(x, y)$-geodesic $P$ of $G^\ast$ whose inner vertices do not belong to $U^y_{ab}$, then $P$ is a path of some cycle $C \subseteq \Gamma^\ast(G) - C$ that contains an edge which is $\Theta$-equivalent to $ab$.

**Proof.** By Lemma 3.2, $G^\ast$ is a partial cube. Let $ab \in E(G^\ast)$. Suppose that there are two non-adjacent vertices $x, y \in U^a_{ab}$ and an $(x, y)$-geodesic $P$ whose inner vertices do not belong to $U^y_{ab}$, and such that $P$ is not a path of any cycle $C \subseteq \Gamma^\ast(G) - C$ that contains an edge which is $\Theta$-equivalent to $ab$. Then the inner vertices of $P$ do not belong to $V(C)$. Hence, for the same reason as in (b) of the proof of Lemma 3.2, $x, y \in V(G)$.

By Lemma 3.1, there exists an $(x, y)$-geodesic $Q$ such that $V(Q) \subseteq V(G)$. Because $U_{ab}$ is $C$-convex, $P \cup \Delta$ cannot be a cycle of $G$. Hence $P - G$ is not a cycle. Suppose that there are two distinct cycles, $C_1, C_2 \subseteq \Gamma^\ast(G)$ such that $(P - G) \cap H_C$ and $(P - G) \cap H_C$ are not empty. By the proof of Lemma 3.1, we can replace $P \cap H_C$ by a path $W$ of $C^\ast$ such that $P' = (P - H_C) \cup W$ is an $(x, y)$-geodesic. Because $d_{C^\ast}(x, y)$ is as small as possible, the inner vertices of $P'$ do not belong to $U^y_{ab}$. By repeating this argument, and since the vertices of any $(x, y)$-geodesic in $G$ distinct from $Q$ must be contained in $U_{ab}$ because this set is $C$-convex, it follows that we can assume that $P - \{x, y\}$ is a path of $H_C - C$ for some $C \subseteq \Gamma^\ast(G)$, and hence $Q$ is a path of $C$. Because $H_C$ is a hypercube, by the minimality of $d_{C^\ast}(x, y)$, it follows that $d_{C^\ast}(x, y) = 2$.

Let $u, v$ be the two common neighbors of $x$ and $y$ in $H_C$, with $u \in V(Q)$. Then $u \in U^a_{ab}$ and $v \notin U^a_{ab}$. Let $x', y'$ and $u'$ be the projections of $x, y$ and $u$ in $U^a_{ab}$, respectively. Because, by Lemma 2.6, two convex cycles of a netlike partial cube have at most one edge in common, it follows that $u' \notin V(C)$ and that $x', y' \notin V(G)$. Hence there exist two distinct $C_x, C_y \subseteq \Gamma^\ast(G)$ such that $x, x', u', u' \in V(C_x)$ and $y, y', u' \in V(C_y)$. Therefore $(C_x, C_y)$ is a tricycle of $G$, contrary to the assumption that $C \subseteq \Gamma^\ast(G)$.

It follows that $d_{C^\ast}(x, y) \subseteq V(C)$ for some cycle $C \subseteq \Gamma^\ast(G) - C$ which contains an edge $\Theta$-equivalent to $ab$. □

Proposition 3.5. Let $G$ be a netlike partial cube, $C \subseteq \Gamma^\ast(G)$, $H_C = (H_C)_{C \in C}$ a suitable subfamily of hypercubes for $G$, and $G^\ast := G^{H_C}$. Then $G^\ast$ is a netlike partial cube if and only if $C \subseteq \Gamma^\ast(G)$.

**Proof.** By Lemma 3.2, $G^\ast$ is a partial cube.

(a) Assume that $C \subseteq \Gamma^\ast(G)$ and let $ab \in E(G^\ast)$. We will denote by $I_{ab}(G)$ and $C_{ab}$ the set of all cycles containing an edge $\Theta$-equivalent to $ab$ and which belongs to $\Gamma^\ast(G)$ and $C$, respectively.

We show that $U^a_{ab}$ is $C$-convex. Suppose that there is a cycle $G^\ast[I_{ab}(G)]$ which is not contained in $G^\ast[U^a_{ab}]$. Then there clearly exists such a cycle $C^\ast$ which is isometric in $G^\ast$. Because $V(C^\ast) \nsubseteq U^a_{ab}$, there are two non-adjacent vertices $x, y \in V(C^\ast) \cap U^a_{ab}$ and an $(x, y)$-geodesic $P$ of $C^\ast$ whose inner vertices do not belong to $U^a_{ab}$. By Lemma 3.4, $P$ is a path of some cycle in $I_{ab}(G) - C$. Note that there may be several such paths in $C^\ast$ with the same properties as $P$.

On the other hand, there may also be some geodesic of $C^\ast$ of length at least 2, whose endvertices belong to $V(G)$, and whose inner vertices belong to $U^a_{ab} - U^a_{ab}$. Denote by $W(C^\ast)$ the set of all such geodesics of $C^\ast$. Any $W \in W(C^\ast)$ is then a geodesic of a hypercube $H_{C_W}$ for some cycle $C_W \subseteq C_{ab}$, whose endvertices lie in $C_W$. Denote by $W_C$ the geodesic of $C_W$ joining the endvertices of $W$. If $W$ and $W'$ are two distinct elements of $W(C^\ast)$, then $W_C \cap W'_C = W \cap W'$ since, by Lemma 2.6, $C_W \cap C_W'$ is empty or is an edge $\Theta$-equivalent to $ab$. Besides, if $R$ is a geodesic in $C^\ast \cap C$ joining two distinct $C_0, C_1 \subseteq I_{ab}(G)$, then no inner vertex $x$ of $R$ may lie on $W_C$ for any $W \in W(C^\ast)$ such that $C_W$ is distinct from $C_0$ and $C_1$, since otherwise we should have $I_{ab}(x, y) \subseteq V(H_{C_W})$ for any $y \in V(W)$ by the convexity of $H_{C_W}$, contrary to the hypothesis that $C^\ast$ is isometric in $G^\ast$. Moreover $V(R) \subseteq I^2_{ab}(U^a_{ab})$ because $V(C) \cap W_C \subseteq I_x(U^a_{ab})$ for each cycle $C \subseteq I_{ab}(G)$.

Therefore, by replacing each $W \in W(C^\ast)$ by the corresponding geodesic $W_C$, we obtain a cycle $C_G$ of $G^\ast$—which is generally not isomorphic in $G$—whose vertex set is contained in $I^2_{ab}(U^a_{ab})$. Because $I^2_{ab}(U^a_{ab})$ is convex by Lemma 2.3 since $U^a_{ab}$ is $C$-convex, it follows that $V(C_G) \subseteq I_{ab}(U^a_{ab})$. On the other hand $V(C_G) \nsubseteq U^a_{ab}$ because of the existence of the path $P$ above. This gives rise to a contradiction with the fact that $U^a_{ab}$ is $C$-convex. Therefore $U^a_{ab}$ is $C$-convex, and analogously so is $U^a_{ab}$.

Consequently $G^\ast$ is netlike.

(b) Conversely, assume that $C \subseteq \Gamma^\ast(G)$, and let $(C_1, C_2, C_3)$ be a tricycle in $G$ such that $C_3 \subseteq C$. Then $C_1, C_2, C_3 \subseteq \Gamma^\ast(G)$ by Lemma 2.13. Let $c_0, c_1, c_2, c_3$ be the vertices such that $G_{C_0}$ is the common edge of $C_1$ and $C_4$ for all triple $(i, j, k)$ of distinct elements of the set $\{1, 2, 3\}$. Let $x$ be the common neighbor of $c_1$ and $c_2$ in $H_{C_3}$ distinct from $c_0$. Then $(c_1, c_0, c_2, x, c_1)$ is a cycle of $G^\ast[I_{ab}(U^a_{ab})]$ with $x \notin U^a_{ab}$. Therefore $U^a_{ab}$ is $C$-convex, and thus $G^\ast$ is not a netlike partial cube. □

3.2. Netlike completion

If $G$ is a netlike partial cube and if $H_C := (H_C)_{C \subseteq \Gamma^\ast(G)}$ is a suitable subfamily of hypercubes for $G$, then the netlike partial cube $G^{H_C}$ is called a netlike completion of $G$, and more precisely the netlike completion of $G$ with respect to the suitable family $H_C$.

Let $G$ be a netlike partial cube, $G^\ast$ the completion of $G$ with respect to a suitable family $(H_C)_{C \subseteq \Gamma^\ast(G)}$, and $G^\ast$ the netlike completion of $G$ with respect to the suitable subfamily $(H_C)_{C \subseteq \Gamma^\ast(G)}$. Then, by Proposition 3.5, $G$ has the MCP if and only if $G^\ast = G^\ast$. Actually we have the following result:

**Theorem 3.6.** Let $G$ be a netlike partial cube and $G^\ast$ any of its completions. The following assertions are equivalent:

(i) $G$ has the MCP.

(ii) $G^\ast$ is a median graph.
G is a netlike partial cube.

(iv) The convex hull of each non-convex isometric cycle of G is a hypercube.

Proof. The implication (ii)⇒(iii) is obvious, (iii)⇒(iv) is a consequence of Proposition 2.5, whereas (ii)⇒(i) follows from the definition of a median graph. It remains to prove the implications (i)⇒(ii) and (iv)⇒(ii).

(i)⇒(ii). If G has the MCP, then \( I^0(G) = I^r(G) \) and \( G^+ \) is a netlike completion of G. Hence, by Lemma 3.4, \( U^+_G \) is convex for any edge \( ab \) of \( G^+ \). Therefore, by Proposition 2.2, \( G^+ \) is a median graph.

(iv)⇒(ii). By Corollary 3.3, \( G^+ \) is an almost-medium graph. Hence, by (iv) and [20, Proposition 7.1], G is a median graph.

From now on we will only deal with MC-netlike partial cubes. Hence any completion will be a netlike completion. So, for simplicity, we will only use the term of completion.

Remark 3.7. Let \( G^+ \) be a completion of an MC-netlike partial cube G, and let \( (u, v, w) \) be a triple of vertices of G. Then, because G is an isometric subgraph of \( G^+ \) by Lemma 3.1, and because a triple of vertices has a unique median in a median graph, it follows that, if \( (u, v, w) \) has a median \( m \) in G, then \( m \) is also its median in \( G^+ \).

Now suppose that \( (u, v, w) \) has a median cycle \( C \) in G, and let \( x, y, z \) be the gates in \( C \) of \( u, v, w \), respectively. Then the median of \( (u, v, w) \) in \( G^+ \) is equal to the median of \( (x, y, z) \) in \( G^+ \). Indeed, let \( m \) be the median of \( (x, y, z) \) in \( G^+ \). Then

\[
m \in I_{G^+}(x, y) \cap I_{G^+}(y, z) \cap I_{G^+}(z, x) \subseteq I_{G^+}(u, v) \cap I_{G^+}(v, w) \cap I_{G^+}(w, u)
\]

by the definition of a median cycle. Hence \( m \) is the median of \( (u, v, w) \) in \( G^+ \).

As several constructions in mathematics, the concept of completion of an MC-netlike partial cube can be defined or characterized by a universal property. To state such a property, we must choose which morphisms we have to use. Those which seem to be the most adequate in this study are the interval preserving functions.

Let \( G \) and \( F \) be two graphs. A function \( f : V(G) \rightarrow V(F) \) is said to be interval preserving if

\[
f(I_{G^+}(x, y)) \subseteq I_{F^+}(f(x), f(y)) \quad \text{for all } x, y \in V(G).
\]

Any interval preserving function \( f : G \rightarrow F \) is a contraction of \( G \) into \( F \), that is \( d_F(f(x), f(y)) \leq d_G(x, y) \) for all \( x, y \in V(G) \), but obviously any contraction is not necessarily interval preserving.

In the following proposition we list some simple properties whose proofs are routine and are left to the reader.

**Proposition 3.8.** Let \( G \) and \( F \) be two MC-netlike partial cubes, and \( f : G \rightarrow F \) an interval preserving function. We have the following properties:

(i) If \( P \) is an \( (x, y) \)-geodesic in \( G \) for some \( x, y \in V(G) \), then \( f(P) \) is an \( (f(x), f(y)) \)-geodesic in \( F \).

(ii) If \( X \) is an isometric subgraph of \( G \), then \( f(X) \) is isometric in \( F \).

(iii) If \( C \) is an isometric cycle of \( G \), then \( f(C) \) is either an isometric cycle of \( F \) or a path of length at most 1.

(iv) If a triple \( (u, v, w) \) of vertices of \( G \) has a median \( m \) in \( G \), then \( f(m) \) is the median of \( f(u), f(v), f(w) \) in \( F \).

(v) If a triple \( (u, v, w) \) of vertices of \( G \) has a median cycle \( C \) in \( G \), and if \( f(u), f(v), f(w) \) also has a median cycle \( C' \) in \( F \), then \( C' = f(C) \).

If \( G \) and \( F \) are median graphs, then, by Proposition 3.8(i), any interval preserving function \( f : G \rightarrow F \) is median preserving.

**Theorem 3.9.** Any completion \( G^+ \) of an MC-netlike partial cube \( G \) has the following universal property: for any interval preserving function \( g \) of \( G \) into a median graph \( M \), there exists a unique interval preserving function \( g^+ : G^+ \rightarrow M \) which extends \( g \).

**Proof.** Let \( H = (H_C)_{C \in I^0(G)} \) be a suitable family of hypercubes for \( G \), and let \( G^+ = G^+ \). We define the extension \( g^+ \) of \( g \) as follows. Let \( x \in V(H_C - C) \). Then \( x \) is the median of some triple \( (u, v, w) \) of vertices of \( C \). We define \( g^+(x) \) as the median of \( (g(u), g(v), g(w)) \) in \( M \). If \( g(C) \) is a cycle, then, by Proposition 3.8, \( g(C) \) is an isometric cycle of \( M \), and hence its convex hull in \( M \) is a hypercube \( H_g(C) \), since \( M \) is a median graph (see Proposition 2.2(iii)). Note that \( H_g(C) \) coincides with \( g(C) \) if the length of this cycle is 4.

We now show that \( I_{G^+}(x, y) \subseteq I_{M^+}(g^+(x), g^+(y)) \) for any \( x, y \in V(G^+) \). We distinguish four cases.

Case 1. \( x, y \in V(H_C) \) for some \( C \in I^0(G) \).

By the definition of \( g^+ \), we clearly have \( g^+(H_C) = H_g(C) \), and moreover, for all \( x, y \in V(H_C) \),

\[
I_{M^+}(g^+(x), g^+(y)) = g^+(I_{C^+}(x, y)).
\]

Case 2. \( x, y \in V(G) \).

Let \( P \) be an \( (x, y) \)-geodesic. We will show that \( g^+(V(P)) \subseteq I_{M^+}(g^+(x), g^+(y)) \) by induction on the number \( \gamma(P) \) of elements of \( I^r(G) \) which meet \( P \). This is clear if \( \gamma(P) = 0 \) because \( g^+(V(P)) = g(V(P)) \) and \( g \) is interval preserving. Suppose that this is true if \( \gamma(P) \leq p \) for some non-negative integer \( p \). Let \( P = \langle x_0, \ldots, x_n \rangle \) with \( x_0 = x \) and \( x_n = y \) be such that \( \gamma(P) = p + 1 \).

Let \( i \) be the smallest integer such that \( x_i \in V(C) \) for some \( C \in I^0(G) \). Then \( x_i \) is the gate of \( x \) in \( C \) (and thus in \( H_C \)). Then \( \langle x_0, \ldots, x_i \rangle \subseteq I_C(x_0, x_i) \), and thus \( g^+(\langle x_0, \ldots, x_i \rangle) = g(\langle x_0, \ldots, x_i \rangle) \subseteq I_{M^+}(g^+(x_0), g^+(x_i)) \). On the other hand, if \( j \) is the greatest integer such that \( x_j \in V(C) \), then \( x_j \) is the gate of \( y \) in \( C \), and \( \langle x_i, \ldots, x_j \rangle \) is a geodesic of \( H_C \). Hence
$g^+([x_0, \ldots, x_j]) \subseteq I_H(x_j)$. \(g^+(x_j) = I_M(g^+(x_j), g^+(x_j))\). Therefore

$$g^+([x_0, \ldots, x_j]) \subseteq I_M(g^+(x_j), g^+(x_j)). \tag{1}$$

Now \(Q := (x_j, x_n)\) is an \((x_j, x_n)\)-geodesic in \(G^+\) such that \(y(Q) = p\). Then, by the induction hypothesis,

$$g^+(V(Q)) \subseteq I_M(g^+(x_j), g^+(x_j)) \tag{2}$$

Hence, by (1) and (2)

$$g^+(V(P)) \subseteq I_M(g^+(x_0), g^+(x_1)) \cup I_M(g^+(x_1), g^+(x_0)) = I_M(g^+(x), g^+(y)).$$

Consequently \(g^+(I_{C^+}(x_1, y)) \subseteq I_M(g^+(x), g^+(y)). \)

Case 3. \(x \in V(H_C)\) for some \(C \in \Gamma(G)\) and \(y \in V(G)\).

Let \(y'\) be the gate of \(y\) in \(C\). Then \(I_{C^+}(x, y') = I_{C^+}(x, y') \cup I_{C^+}(x', y')\). By Cases 1 and 2, \(g^+(I_{C^+}(x, y')) = I_M(g^+(x), g^+(y'))\) and \(g^+(I_{C^+}(x', y')) \subseteq I_M(g^+(x'), g^+(y')).\) It follows that \(g^+(I_{C^+}(x, y)) \subseteq I_M(g^+(x), g^+(y)).\)

Case 4. \(x \in V(H_{C'})\) for some \(C, C' \in \Gamma(G)\) with \(C \neq C'\).

Let \(y'\) be the gate of \(y\) in \(C\), and \(x'\) the gate of \(x\) in \(C'\). Note that \(y'\) is the gate of \(x'\) in \(C\), and \(x'\) is the gate of \(y'\) in \(C'\). Then \(I_{C^+}(x, y') = I_{C^+}(x, y') \cup I_{C^+}(y', x') \cup I_{C^+}(x', y')\). Hence, as in Case 3, by Cases 1 and 2 we obtain that \(g^+(I_{C^+}(x, y)) \subseteq I_M(g^+(x), g^+(y)).\)

Consequently \(g^+\) is an interval preserving function. The uniqueness of this extension is clear. \(\square\)

**Theorem 3.10.** Let \(G\) be an MC-netlike partial cube. Then the completions of \(G\) are the median graphs which contain \(G\) as an isometric subgraph and which are minimal with respect to the subgraph relation.

**Proof.** Let \(G^+\) be the completion of \(G\) with respect to a suitable family \((H_C)_{C \in \Gamma(G)}\) of hypercubes. Then, by Theorem 3.6 and Lemma 3.1, \(G^+\) is a median graph which contains \(G\) as an isometric subgraph. Let \(M\) be a median graph which contains \(G\) as an isometric subgraph. Let \(g : G \to M\) be such that \(g(x) = x\) for all \(x \in V(G)\). \(g\) is obviously an interval preserving function. By Theorem 3.9, there exists a unique interval preserving function \(g^+ : G^+ \to M\) which extends \(g\).

Because \(M\) is a median graph, and \(G\) an isometric subgraph of \(M\), it follows by Proposition 2.2(iii) that \(c_{M}(C) = : H_c\) is a hypercube for every \(C \in \Gamma(G)\). Then, because \(g^+\) is median preserving, for any \(C \in \Gamma(G)\) the restriction of \(g^+\) to \(V(H_C)\) is an isomorphism of \(H_c\) onto \(H_c\).

We show that \(g^+\) is injective. Suppose that \(g^+(x) = g^+(y) =: u\) for some distinct \(x, y \in V(G^+)\). Then, by the properties of \(G^+\), there are two cycles \(C, C' \in \Gamma(G)\) such that \(x \in V(H_C - C)\) and \(y \in V(H_{C'} - C')\). Then \(I_{C^+}(x, y)\) meets \(V(C)\) and \(V(C')\). On the other hand

$$g^+(I_{C^+}(x, y)) = I_M(g^+(x), g^+(y)) = \{u\}$$

because \(g^+\) is interval preserving. Hence, for any \(v \in I_{C^+}(x, y) \cap V(C)\), we have

$$v = g(v) = g^+(v) = u,$$

which is impossible because \(v \in V(C)\) and \(u \in V(H_C - C)\).

Therefore \(g^+\) is injective. It follows, on the one hand that \(G^+\) is minimal with respect to the subgraph relation; and on the other hand that if \(M\) is minimal with respect to the subgraph relation, then \(g^+\) is an isomorphism of \(G^+\) onto \(M\), and thus \(M\) is also a completion of \(G\). \(\square\)

## 4. Completion with respect to netlike subgraphs, gated amalgams and expansions

In this section we will study the behavior of completion with respect to netlike subgraphs, gated amalgams and expansions of MC-netlike partial cubes.

### 4.1. Completion and netlike subgraphs

In [22] we define the concept of netlike subgraph as follows:

**Definition 4.1.** A subgraph \(F\) of a netlike partial cube \(G\) is called a netlike subgraph of \(F\) if \(F\) is isometric in \(G\) and if, for every triple \((u, v, w)\) of vertices of \(F\) which has a median \(m\) in \(G\), then \(m \in V(F)\) (and thus is the median of \((u, v, w)\) in \(F\) by the uniqueness of the median and the fact that \(F\) is isometric in \(G\)).

We can easily notice that the netlike subgraphs of a median graph are the median subgraphs of this graph. By [22, Proposition 4.4], a netlike subgraph of a netlike partial cube is also a netlike partial cube. Note that, contrary to median subgraphs, an isometric subgraph \(F\) of a netlike partial cube \(G\) which is netlike in its own right is not necessarily a netlike subgraph of \(G\), as is shown by the example of a 6-cycle in a 3-cube. On the other hand, a median subgraph \(F\) of a median graph \(G\) can be defined as a connected induced subgraph of \(G\) such that, for every triple \((u, v, w)\) of vertices of \(F\), the median of \((u, v, w)\) in \(G\) belongs to \(F\).
Proposition 4.2 (Polat [21, Theorem 5.2]). Let $G$ be an MC-netlike partial cube, and $F$ a subgraph of $G$. Then $F$ is a netlike subgraph of $G$ if and only if $F$ is a connected induced subgraph of $G$ such that, for every triple $(u, v, w)$ of vertices of $F$, the median or the median cycle of $(u, v, w)$ in $G$ is a vertex or a cycle of $F$, respectively.

Note that by Proposition 3.8 we obtain immediately:

Proposition 4.3. Let $G$ and $F$ be two MC-netlike partial cubes, and $f : G \to H$ an interval preserving function. Then $f(G)$ is a netlike subgraph of $F$.

Lemma 4.4 (Polat [21, Theorem 5.3]). Let $F$ be a netlike subgraph of an MC-netlike partial cube $G$. Then any convex cycle of $F$ is convex in $G$, and thus $\Gamma(F) \subseteq \Gamma(G)$.

The following result shows that completion preserves netlike subgraphs of MC-netlike partial cubes.

Theorem 4.5. Let $G$ be an MC-netlike partial cube, $G^+$ the completion of $G$ with respect to a suitable family $(H_C)_{C \in \Gamma(G)}$ of hypercubes, and $F$ a netlike subgraph of $G$. Then the completion $F^+$ of $F$ with respect to $(H_C)_{C \in \Gamma(F)}$ is a median subgraph of $G^+$.

Proof. $F^+$ is a subgraph of $G^+$ such that $\beta_{G^+}(F^+) = \beta_{G^+}(F)$. Furthermore $F$ is isometric in $G$, and $G$ is isometric in $G^+$ by Lemma 3.1. Hence $F$ is isometric in $G^+$. It follows that $F^+$ is an isometric subgraph of $G^+$ because $\beta_{G^+}(F^+) = \beta_{G^+}(F)$. Therefore $F^+$ is a median subgraph of $G^+$ since it is a median graph in its own right.

4.2. Completion and gated amalgams

Following Mulder [17], a graph $G$ is the gated amalgam of two graphs $G_0$ and $G_1$ if $G_0$ and $G_1$ are two intersecting gated subgraphs of $G$ whose union is $G$. We also say that $G$ is the gated amalgam along the intersection of these gated subgraphs.

Proposition 4.6 (Polat [20, Theorem 6.5]). The gated amalgam of two MC-netlike partial cubes is an MC-netlike partial cube.

Note that, by Proposition 2.10, if $G$ is the gated amalgam of two MC-netlike partial cubes $G_0$ and $G_1$, then $\Gamma(G) = \Gamma(G_0) \cup \Gamma(G_1)$. The following result shows that completion preserves gated amalgams of MC-netlike partial cubes.

Theorem 4.7. Let $G$ be the gated amalgam of two MC-netlike partial cubes $G_0$ and $G_1$. Let $G^+$ be the completion of $G$ with respect to a suitable family $(H_C)_{C \in \Gamma(G)}$ of hypercubes, and, for $i = 0, 1$, let $G^+_i$ be the completion of $G_i$ with respect to $(H_C)_{C \in \Gamma(G_i)}$. Then $G^+$ is the gated amalgam of $G_0^+$ and $G_1^+$.

Proof. By Lemma 2.12, $G$ is an MC-netlike partial cube. Hence, by Theorem 3.6, $G^+_0$, $G^+_1$, and $G^+$ are median graphs.

By Proposition 2.10, $\beta_{G^+}(G_i^+) = \beta_G(G_i)$ for $i = 0, 1$. Hence $G^+ = G_0^+ \cup G_1^+$. Let $i \in \{0, 1\}$, $x, y \in \beta_{G^+}(G_i^+)$, and $P$ an $(x, y)$-geodesic in $G^+$. By Lemma 3.1 and because $x, y \in V(G_i)$, there exists an $(x, y)$-geodesic $R$ in $G$ such that $P \cap G_i \subseteq R$. Because $x, y \in V(G_i)$ and since $G_i$ is convex in $G$, it follows that $R$ is a path of $G_i$. Moreover, also because $G_i$ is convex in $G$, any element of $\Gamma(G_i) - \Gamma(G)$ has at most one vertex or two adjacent vertices in common with $G_i$. It follows that any $C \in \Gamma(G)$ such that $P \cap (H_C - C) \neq \emptyset$ must belong to $\Gamma(G_i)$. Hence $P$ is a path of $G_i^+$, which implies that $G_i^+$ is convex in $G^+$. Therefore, because $G^+$ is a median graph and since any convex subgraph of a median graph is gated, it follows that $G_i^+$ is also gated in $G^+$. Consequently $G^+$ is the gated amalgam of $G_0^+$ and $G_1^+$.  

4.3. Completion and expansions

We recall the definition of an expansion of a graph, a concept which was introduced by Mulder [18] to characterize median graphs and which was later generalized by Chepoi [6].

Definition 4.8. A pair $(V_0, V_1)$ of sets of vertices of a graph $G$ is called a proper cover of $G$ if it satisfies the following conditions:

- $V_0 \cap V_1 \neq \emptyset$ and $V_0 \cup V_1 = V(G)$;
- there is no edge between a vertex in $V_0 - V_1$ and a vertex in $V_1 - V_0$;
- $G[V_0]$ and $G[V_1]$ are isometric subgraphs of $G$.

Definition 4.9. An expansion of a graph $G$ with respect to a proper cover $(V_0, V_1)$ of $G$ is the subgraph of $G \boxtimes K_2$ induced by the vertex set $(V_0 \times \{0\}) \cup (V_1 \times \{1\})$ (where $\{0, 1\}$ is the vertex set of $K_2$).

We recall a result of Chepoi:

Proposition 4.10 (Chepoi [6, 7]). A finite graph is a partial cube if and only if it can be obtained from $K_1$ by a sequence of expansions.
Several theorems of this kind have been stated for different subclasses of partial cubes, see [11]. The first one is the following theorem of Mulder for median graphs. An expansion of a partial cube with respect to a proper cover \((V_0, V_1)\) is said to be convex if \(V_0 \cap V_1\) is convex.

**Proposition 4.11** ([Mulder [18,19]]). A finite graph is a median graph if and only if it can be obtained from \(K_1\) by a sequence of convex expansions.

For netlike partial cubes such a result is impossible. There exist netlike partial cubes which are not the expansion of any netlike partial cubes (see [21]).

**Definition 4.12.** A proper cover \((V_0, V_1)\) of a partial cube \(G\) is said to be \(c\)-peripheral if it has the following properties:

(P1) \(V_0 \cap V_1\) is \(c\)-convex in \(G[V_i]\) for \(i = 0, 1\);

(P2) \(I_{c}(V_0 \cap V_1)\) is gated;

(P3) \(V_i = J_{G[V_i]}(V_0 \cap V_1)\) for some \(i \in \{0, 1\}\);

(P4) any convex cycle of \(G[V_0 \cap V_1]\) is a 4-cycle.

**Definition 4.13.** An expansion of a partial cube \(G\) with respect to a \(c\)-peripheral proper cover of \(G\) is called a \(c\)-peripheral expansion of \(G\).

For finite MC-netlike partial cubes we proved in [21] a result which is of the same kind as Chepoi’s [6] and Mulder’s [18] (see Propositions 4.10 and 4.11).

**Proposition 4.14** ([Polat [21, Theorem 6.15]]). A finite graph is an MC-netlike partial cube if and only if it can be obtained from \(K_1\) by a sequence of \(c\)-peripheral expansions.

We recall that a proper cover \((V_0, V_1)\) of a graph \(G\) is peripheral if \(V_0 \cap V_1 = V_i\) for some \(i \in \{0, 1\}\), or equivalently if \(V_{1-i} = V(G)\). An expansion of a partial cube with respect to a peripheral proper cover is said to be peripheral. By [19, Lemma 9], a finite graph is a median graph if and only if it can be obtained from \(K_1\) by a sequence of peripheral convex expansions.

We will now show that completion preserves \(c\)-peripheral expansions of MC-netlike partial cubes.

**Theorem 4.15.** Let \(G\) be an MC-netlike partial cube, and \(G_1\) the expansion of \(G\) with respect to a \(c\)-peripheral proper cover \((V_0, V_1)\) of \(G\). Then, for any completion \(G^+\) of \(G\), the pair \((c_{G^+}(V_0), c_{G^+}(V_1))\) is a peripheral convex proper cover of \(G^+\), and the expansion \((G^+)_1\) of \(G^+\) with respect to \((c_{G^+}(V_0), c_{G^+}(V_1))\) is a completion of \(G_1\), i.e.

\[ (G^+_1) = (G_1^+). \]

**Proof.** Let \(G^+\) be the completion of \(G\) with respect to a suitable family \((H_C)_{C \in \Gamma(G)}\) of hypercubes. We will use the following notation:

- For \(i = 0, 1\), let \(\psi_i : V_i \rightarrow V(G_1)\) be such that \(\psi_i(x) := (x, i)\) for each \(x \in V_i\); and for any \(A \subseteq V(G)\) let \(\psi(A) := V_0(A \cap V_0) \cup \psi_1(A \cap V_1)\).

- Note that \(\psi_0(V_0)\) and \(\psi_1(V_1)\) are complementary half-spaces of \(G_1\).

- We will simply write \(G^+_1\) for \((G^+_1)\).

- For \(i = 0, 1\), we put \(\tilde{V}_i := c_{G^+}(V_i)\), and we denote by \(\tilde{\psi}_i\) the map of \(\tilde{V}_i\) in \(V(G^+_1)\) which corresponds to the map \(\psi_i\) defined above.

\((\tilde{V}_0, \tilde{V}_1)\) is clearly a convex proper cover of \(G^+\). We show that it is peripheral. Because \((V_0, V_1)\) is \(c\)-peripheral, by (P3), \(V_i = J_{G[V_i]}(V_0 \cap V_1)\) for some \(i \in \{0, 1\}\).

\[ \tilde{V}_i := c_{G^+}(V_i) = c_{G^+}(J_{G[V_i]}(V_0 \cap V_1)) = c_{G^+}(V_0 \cap V_1) \subseteq \tilde{V}_0 \cap \tilde{V}_1 \subseteq \tilde{V}_i. \]

It follows that \(\tilde{V}_0 \cap \tilde{V}_1 \subseteq \tilde{V}_i\).

Therefore \((\tilde{V}_0, \tilde{V}_1)\) is a peripheral convex proper cover of \(G^+\), which is a median graph by Theorem 3.6. Hence \(G^+_1\) is also a median graph by Proposition 4.11. By (P2), \(I_{c}(V_0 \cap V_1)\) is gated. Then \(V_0 \cap \tilde{V}_1 \cap V(G) = I_{c}(V_0 \cap V_1)\) and \(G^+\) is a median graph which is the completion of \(G^+_1\).

Denote by \(I_{G}(G)\) the set of all 4-cycles in \(I_{G}(V_0 \cap V_1)\) whose vertex sets are not contained in \(V_0 \cap V_1\). Then any \(C \in I_{G}(G)\) meets \(V_0 \cap V_1\) in two antipodal vertices and \(V_0 \cap V_1\), and in \(V_1 \cap V_0\) in exactly one vertex each. Hence \(\psi(C) = H_{\psi(C)}\). Let \(C \in I_{G}(G)\). Then \(C\) is 2n-cycle with \(n \geq 3\). If \(V(C) \leq I_{c}(V_0 \cap V_1)\), then, by (P3), \(C\), meets \(V_0 \cap V_1\) and \(V_1 \cap V_0\), and thus \(\psi(C)\) is a \((2n + 2)\)-cycle of \(G^+\), and \(\tilde{\psi}(H_{\psi(C)})\) is an \((n + 1)\)-cycle since \(H_{\psi(C)}\) is an n-cycle. Then \(\psi(C)\) and \(\psi(C)\) are linked by the equality \(\psi(C) = H_{\psi(C)}\), equality that is defined up to isomorphism.

If \(V(C) \leq I_{c}(V_0 \cap V_1)\), then \(C\) has at most two vertices in \(I_{c}(V_0 \cap V_1)\), and thus \(\psi(C) = C\) and \(\tilde{\psi}(H_{\psi(C)}) = H_{\psi(C)}\).

Therefore, because \(G^+\) is a median graph, \(\tilde{V}_0 \cap \tilde{V}_1 \cap V(G) = I_{c}(V_0 \cap V_1)\), the map \(\psi\) induces a bijection from \(I_{G}(G)\) onto \(G^+\) such that \(H_{\psi(C)} = \tilde{\psi}(H_{\psi(C)})\) with \(H_{\psi(C)} = C\). If \(C \in I_{G}(G)\) for every \(C \in I_{G}(G)\), then \(G^+_1\) is the completion of \(G_1\). □
5. Netlike classes

Definition 5.1. Let $G$ be an MC-netlike partial cube. We call netlike class of $G$, and we denote by $\mathbf{NC}(G)$, the class of all MC-netlike partial cubes whose completions are isomorphic to those of $G$.

Let us say that a finite hypercube $H$ of dimension $n_H$ of a partial cube $G$ is:
- maximal if it is not a proper subgraph of another hypercube;
- well-surrounded if there is an isometric $2n_H$-cycle $C_H$ of $H$ that contains $\beta_C(H)$. Such a cycle will be called a surrounding cycle of $H$.

Clearly any 2-cube is well-surrounded, and any well-surrounded hypercube of dimension greater than 2 is maximal. We denote by $\Sigma(G)$ the set of all well-surrounded hypercubes of $G$ of dimension greater than 2, and for each $H \in \Sigma(G)$ we choose a surrounding cycle $C_H$ of $H$. By the definition, $H \cap H' = C_H \cap C_{H'}$ for any two distinct $H, H' \in \Sigma(G)$.

For a subset $K$ of $\Sigma(G)$, we denote by $G_K$ the subgraph $G - \bigcup_{H \in K} (H - C_H)$ of $G$.

An element of $\Sigma(G)$ may have several surrounding cycles, but it is clear that if, for some $H \in K$, $C_H$ is a surrounding cycle of $H$ which is different from $C_H$, then $G_K := G - \bigcup_{H \in K} (H - C_H)$ is isomorphic to $G_K$.

Theorem 5.2. Let $G$ be an MC-netlike partial cube, and $K$ a subset of $\Sigma(G)$. Then $G_K \in \mathbf{NC}(G)$ and is isometric in $G$. Moreover, any MC-netlike partial cube $G' \in \mathbf{NC}(G)$ is isomorphic to the graph $(G^+)_K$ for some $K \subseteq \Sigma(G)$ and some completion $G^+$ of $G$.

Proof. (a) We distinguish two cases.

Case 1. $G$ is a median graph.

$G_K$ is a bipartite graph. Moreover $G$ is a completion of $G_K$ since a median graph contains no convex cycles of length greater than 4. Then, by Lemma 3.1, $G_K$ is isometric in $G$, and thus is a partial cube. We will use Proposition 2.5 to prove that $G_K$ is netlike. By the definition of $G_K$ and the fact that $G$ is a median graph, the convex-hull of each non-convex isometric cycle of $G_K$ is clearly a hypercube in $G$.

Let $ab$ be an edge of $G_K$, and let $x \in I_{ab}(U^G_{ab}) - U^G_{ab}$. Because $G$ is a median graph, the set $U^G_{ab}$ is convex, and thus $x$ has a projection $y$ in $U^G_{ab}$. Hence $x$ is a vertex of a surrounding cycle $C_H$ of some $H \in K$ that contains edges that are in relation $\Theta$ with $ab$. by the construction of $G_K$, $V(C_H) \subseteq I_{ab}(U^G_{ab}) \cup I_{ab}(U^G_{ba})$. Suppose that $x$ has a neighbor $u$ in $I_{ab}(U^G_{ab}) - V(C_H)$. Then $u \in I_{ab}(U^G_{ab}) - V(H)$, and thus $u \in U^G_{ab} - V(H)$ because $U^G_{ab}$ is convex. It follows that $u$ has a projection in $U^G_{ab}$, and that this projection is adjacent to $y$, contrary to the fact that $C_H$ is a surrounding cycle of $H$ and that $y \notin V(C_H)$. Therefore the degree of $x$ in $G_K[I_{ab}(U^G_{ab})] = 2$, which proves that $U^G_{ab}$ is $(3,3)$-convex.

Consequently $G_K$ is an MC-netlike partial cube whose completion is equal to the median graph $G$, and which thus belongs to the netlike class of $G$.

Case 2. $G$ is any MC-netlike partial cube.

Clearly $K$ is a suitable subfamily of hypercubes for $G_K$. Whence $(G_K)^K = G$. Then, by Lemma 3.1, $G_K$ is isometric in $G$. It follows that, if $G^+$ is any completion of $G$, then there is a completion of $G_K$ which is equal to $G^+$. Hence $G^+$ is a median graph because $G$ has the MGP, and is such that $K \subseteq \Sigma(G^+)$. Let $K^+ \subseteq \Sigma(G^+)$ be such that $G = (G^+)^K$. Then $G_K = (G^+)_{K \cup \Sigma(G)}$. By Case 1, this implies that $G_K \in \mathbf{NC}(G^+) = \mathbf{NC}(G)$.

(b) Let $G' \in \mathbf{NC}(G)$. Then, without loss of generality, we can suppose that $G'$ and $G$ have a common completion $G^+$. Let $H = (H_C)_{C \in \mathcal{P}(G^+)}$ be a suitable family of hypercubes for $G$ such that $G^+ = (G^+)_H$. Then each $C \in \mathcal{P}(G^+)$ is a surrounding cycle of $H_C$. Therefore $G' = (G^+)_H$.

□

Remark 5.3. (1) Let $G$ be a median graph, $K$ a subset of $\Sigma(G)$, and $(u, v, w)$ a triple of vertices of $G_K$. If $(u, v, w)$ has a median cycle in $G_K$, then, by Remark 3.7, this cycle is the surrounding cycle $C_H$ of the hypercube $H \in K$ such that $H - C_H$ contains the median of $(u, v, w)$ in $G$.

(2) If $G$ is a finite median graph such that each element of $\Sigma(G)$ is fixed by every automorphism of $G$, then the number of elements of $\mathbf{NC}(G)$, up to isomorphism, is $2^{\Sigma(G)}$. For example, $\mathbf{NC}(G)$ has exactly one (resp. two) element(s), up to isomorphism, if $G$ is a cube-free median graph, that is a linear median graph (resp. an $n$-cube $Q_n$ for some $n \geq 3$, and in this case these two elements are precisely $Q_n$ and $C_{2n}$).

Let $G$ be an MC-netlike partial cube. The graph $G_{\Sigma(G)}$ is called the skeleton of $G$, and is denoted by $Sk(G)$. The skeleton of $G$ is the smallest subgraph of $G$ that belongs to $\mathbf{NC}(G)$. Clearly any linear partial cube is the skeleton of some median graph. However, a skeleton of a median graph is not necessarily a linear partial cube. Take for example an $n$-cube $Q_n$ with $n \geq 3$, and let $Q'$ be the graph obtained from $Q$ by joining each vertex $x$ of $Q$ to a new vertex $y_x$, such that $y_x \neq y_{x'}$ if $x \neq x'$. Then $Q'$ is a median graph which is its own skeleton.

By Theorems 3.6 and 5.2 we immediately have the following characterization of MC-netlike partial cubes:

Theorem 5.4. Any MC-netlike partial cube is equal to $G_K$ for some median graph $G$ and subset $K$ of $\Sigma(G)$. In particular MC-linear partial cubes (and more specifically cellular bipartite graphs) are the skeletons of the (finite) median graphs all of whose maximal finite hypercubes are well-surrounded.
Remark 5.5. Given a netlike class \( \mathcal{A} \), a skeleton in \( \mathcal{A} \), considered up to isomorphism, can be seen as a “pattern” which is common to all elements of \( \mathcal{A} \), and thus which characterizes \( \mathcal{A} \). That is two MC-netlike partial cubes have the same pattern if and only if they belong to the same netlike class.

6. Netlike classes and common properties

The elements of any netlike class share different properties. In this section we will produce some of these common properties.

6.1. Depth

In [4], Bandelt and van de Vel introduced an invariant of convex structures—the depth—to study the structure of finite median graphs.

Definition 6.1. The depth of an \( \text{S4} \) convex structure is the supremum length of a chain of non-trivial half-spaces.

By [21, Section 4], we know that the geodesic convexity of an MC-netlike partial cube \( G \) has the separation property \( \text{S4} \) (Kakutani Separation Property): if \( C, D \subseteq V(G) \) are disjoint convex sets, then there is a half-space \( H \) that separates \( C \) from \( D \), that is, \( C \subseteq H \) and \( D \subseteq V(G) - H \).

In this subsection we will show that all elements of any netlike pattern have the same depth, and we will extend to any finite MC-netlike partial cube a result of Bandelt and van de Vel [4, Theorem 2.4] on median graphs. We need the following lemma.

Lemma 6.2. Let \( A \) be a convex set of an MC-netlike partial cube \( G \), and \( G^+ \) a completion of \( G \). Then \( \text{co}_{G^+}(A) \cap V(G) = A \), and moreover \( A \) and \( \text{co}_{G^+}(A) \) have the same depth.

Proof. (a) Clearly \( A \subseteq \text{co}_{G^+}(A) \cap V(G) \). Suppose that \( \text{co}_{G^+}(A) \cap V(G) \neq A \). Then there exists a vertex \( x \in \text{co}_{G^+}(A) \cap V(G) - A \). By the separation property \( \text{S4} \), there exists an edge \( ab \) of \( G \) such that \( A \subseteq W_{ab}^G \) and \( x \in W_{ab}^G \). Then \( \text{co}_{G^+}(A) \subseteq W_{ab}^G \) and \( x \in W_{ab}^G \), contrary to the fact that \( x \in \text{co}_{G^+}(A) \) by hypothesis.

Consequently \( \text{co}_{G^+}(A) \cap V(G) = A \).

(b) We know that the non-trivial half-spaces of a partial cube \( H \) are the sets \( W^H_{uv}, uv \in E(H) \). Hence, for any convex set \( C \) of \( H \), because \( H[A] \) is a convex subgraph of \( H \) and thus a partial cube, the half-spaces of \( C \) are the sets \( W^H_{uv} \cap C, uv \in E[H[A]] \).

(c) Let \( ab \in E(G^+[\text{co}_{G^+}(A)]) \). Then there exists an edge \( uv \) of \( G \) that is \( \Theta \)-equivalent to \( ab \). Suppose that no edge of \( G[A] \) is \( \Theta \)-equivalent to \( uv \). Then, because \( A \) is convex, it follows that \( A \) is contained in \( W^G_{ab} \) or in \( W^G_{co} \). Hence \( \text{co}_{G^+}(A) \) is contained in \( W^G_{ab} \) or in \( W^G_{co} \), contrary to the hypothesis. Therefore there is an edge of \( G[A] \) that is \( \Theta \)-equivalent to \( ab \).

Consequently, by (b), any chain of half-spaces of \( \text{co}_{G^+}(A) \) corresponds to a chain of half-spaces of \( A \) of the same length, and conversely. This implies that \( A \) and \( \text{co}_{G^+}(A) \) have the same depth. □

Theorem 6.3. Any two elements of a netlike class have the same depth.

Proof. Let \( G \) and \( G' \) be two MC-netlike partial cubes such that \( \text{NC}(G) = \text{NC}(G') \), and let \( G^+ \) and \( G'^+ \) be any of their completions, respectively. By Lemma 6.2, with \( A = V(G) \) we have that \( G \) and \( G^+ \) have the same depth, and with \( A = V(G') \) that \( G' \) and \( G'^+ \) have the same depth. Whence the result because \( G^+ \) and \( G'^+ \) are isomorphic. □

Next theorem extends [4, Theorem 2.4] to all finite MC-netlike partial cubes. In this result, by a strongly maximal hypercube of a netlike partial cube \( G \), we mean a hypercube of \( G \) which is not a proper subgraph of a hypercube or of a convex cycle of \( G \).

Theorem 6.4. A finite MC-netlike partial cube \( G \) has depth \( k \geq 2 \) if and only if there is a convex set \( A \subseteq V(G) \) of depth \( k - 2 \) meeting each strongly maximal hypercube and convex cycle of \( G \) of length greater than \( 4 \).

Proof. Let \( G^+ \) be any completion of \( G \). We are done if \( G = G^+ \) by [4, Theorem 2.4] since \( G^+ \) is a median graph. Assume that \( G \neq G^+ \). Suppose that there is a convex set \( A \subseteq V(G) \) of depth \( k - 2 \), for some \( k \geq 2 \), that meets each strongly maximal hypercube and convex cycle of \( G \) of length greater than \( 4 \). Then, by Lemma 6.2, \( \text{co}_{G^+}(A) \) has depth \( k - 2 \) and clearly meets each maximal hypercube of \( G^+ \). Therefore the median graph \( G^+ \) has depth \( k \) by [4, Theorem 2.4], and thus \( G \) also has depth \( k \) by Theorem 6.3.

Conversely suppose that the depth of \( G \) is \( k \geq 2 \). Then, by Theorem 6.3, the depth of \( G^+ \) is also \( k \). Hence, by [4, Theorem 2.4], there is a convex set \( A \subseteq V(G^+) \) of depth \( k - 2 \) that meets each strongly maximal hypercube. Let \( C \in I^+(G) \). Then \( A \cap V(H_C) \neq \emptyset \). Moreover, because the depth \( k \) of \( G^+ \) is at least \( 2 \), it follows that \( A \not\subseteq V(H_C) \). Hence \( A \cap V(C) \neq \emptyset \) since \( A \) is convex. Therefore \( A' := A \cap V(G) \), which is clearly a convex set of \( G \), meets each strongly maximal hypercube and convex cycle of \( G \) of length greater than \( 4 \). Furthermore \( A = \text{co}_{G^+}(A') \) and then, by Lemma 6.2, \( A' \) and \( A \) have the same depth \( k - 2 \). □
Corollary 6.5. The depth of a finite MC-netlike partial cube $G$ is $n \leq 2$ if and only if there exists a vertex which is common to all strongly maximal hypercubes and convex cycles of $G$ of length greater than 4.

Proof. This is obvious if $n = 0$ since $G$ has then exactly one vertex, and if $n = 1$ since $G$ is then either a hypercube or a convex cycle. For $n = 2$, the result is a consequence of Theorem 6.4. □

6.2. Lattice dimension and semicubic graphs

We recall that the dimension of a finite partial cube $G$ is the least non-negative integer $n$ such that $G$ is an isometric subgraph of an $n$-cube. The dimension of $G$ is denoted by $\dim(G)$ and is clearly equal to the number of $\Theta$-classes of $G$. It follows immediately that, if $G$ is an MC-netlike partial cube, then $\dim(G) = \dim(G')$ for all $G' \in \mathcal{NC}(G)$.

Similarly the lattice dimension $d(G)$ of a partial cube $G$ is defined as the least non-negative integer $n$ such that there exists an isometric embedding of $G$ into the $n$-dimensional grid $\mathbb{Z}^n$. We will show that, as the dimension, the lattice dimension of an MC-netlike partial cube is equal to that of its completion. For that we will use a result of Eppstein [10].

Following Eppstein, we call semicubes of a partial cube $G$ the sets $W_{ab}$, $ab \in E(G)$. The semicube graph of $G$ is the graph, denoted by $Sc(G)$, whose vertices are the semicubes of $G$, and where $W_{ab}W_{uv} \in E(Sc(G))$ if and only if $W_{ab} \cap W_{uv} \neq \emptyset$ and $W_{ab} \cup W_{uv} = V(G)$.

Note that the following conditions are clearly equivalent:

(i) $W_{ab}W_{uv} \in E(Sc(G))$.
(ii) $W_{vu} \subset W_{ab}$ (i.e. $W_{vu} \subseteq W_{ab}$ and $W_{vu} \neq W_{ab}$).
(iii) $W_{ba} \subset W_{uv}$.

We denote by $v(G)$ the matching number of a finite graph $G$, that is the greatest non-negative integer such that $G$ has a matching of cardinality $n$.

Proposition 6.6 (Eppstein [10, Theorem 1]). If $G$ is a finite partial cube, then $d(G) = \dim(G) - v(Sc(G))$.

Theorem 6.7. Let $G$ and $G'$ be two elements of a netlike class. Then $Sc(G)$ and $Sc(G')$ are isomorphic. If moreover $G$ is finite, then $d(G) = d(G')$.

Proof. (a) Let $G^+$ and $G'^+$ be any completions of $G$ and $G'$, respectively. Because $G^+$ and $G'^+$ are isomorphic, it suffices to show that $Sc(G)$ and $Sc(G'^+)$ are isomorphic.

We will use the simplified notation already introduced, that is, for an edge $ab$ of $G$: $W_{ab}$, $U_{ab}$, $W_{ab}^+$ and $U_{ab}^+$ will denote $W_{ab}^G$, $U_{ab}^G$, $W_{ab}^{G'}$ and $U_{ab}^{G'}$, respectively.

(a.1) Let $ab$ be an edge of $G$. Because $G$ has the MCP, $G^+$ is a median graph by Theorem 3.6, and thus $U_{ab}^+$ and $U_{ba}^+$ are convex. Let $H \in \Sigma(G^+)$ and $C \in \Gamma^+(G)$ be one of its surrounding cycles. If $V(H) \cap U_{ab}^+ \neq \emptyset$, then $V(H) \cap U_{ba}^+ \neq \emptyset$, and thus $C$ has an edge which is $\Theta$-equivalent to $ab$. Hence $V(C) \subseteq I_G(U_{ab}) \cap I_G(U_{ba})$.

Moreover, for any $C \in \Gamma^+(G)$, either $V(C) \subseteq W_{ab}$ or $V(C) \subseteq W_{ba}$ or $V(C) \subseteq I_G(U_{ab}) \cup I_G(U_{ba})$ (i.e. $ab$ is in relation $\Theta$ with some edge of $C$). Hence, by what precedes, either $V(H_C) \subseteq W_{ab}^+$ or $V(H_C) \subseteq W_{ba}^+$ or $V(H_C) \subseteq U_{ab}^+ \cup U_{ba}^+$, respectively.

Because the sets $W_{ab}$, $W_{ba}$, $I_G(U_{ab}) \cup I_G(U_{ba})$ are convex and thus induce MC-netlike partial cubes in $G$, it follows that

\begin{align*}
G^+[U_{ab}^+ \cup U_{ba}^+] &= (G[I_G(U_{ab}) \cap I_G(U_{ba})])^+ \\
G^+[W_{ab}^+] &= (G[W_{ab}])^+ \quad \text{and} \quad G^+[W_{ba}^+] &= (G[W_{ba}])^+.
\end{align*}

(3) (4)

(a.2) Let $ab$ and $uv$ be two edges of $G$. Suppose that $W_{vu} \subset W_{ab}$. Then, by (4), $W_{vu}^+ \subset W_{ab}^+$, and

$\emptyset \neq W_{uv} \cap W_{ab} \subseteq W_{uv} \cap W_{ab}^+ \subseteq W_{uv}^+ \cap W_{ab}^+$.

Hence $W_{vu}^+ \subset W_{ab}^+$.

Conversely, suppose that $W_{vu}^+ \subset W_{ab}^+$. Then, by (4),

$W_{vu} = W_{vu}^+ \cap V(G) \subseteq W_{ab}^+ \cap V(G) = W_{ab},$

and moreover $W_{vu} \neq W_{ab}$, since otherwise $W_{vu}^+ = W_{ab}^+$, contrary to the hypothesis. Therefore $W_{vu} \subset W_{ab}$.

Consequently

$W_{vu} \subset W_{ab} \iff W_{vu}^+ \subset W_{ab}^+.$

It follows that the map $W_{ab}^G \mapsto W_{ab}^{G'}$, $ab \in E(G)$, is an isomorphism of $Sc(G)$ onto $Sc(G'^+)$.

(b) Suppose that $G$ is finite. Then all elements of $\mathcal{NC}(G)$, and $G'$ in particular, are finite. We already know that $\dim(G) = \dim(G')$. Moreover, by (a), the graphs $Sc(G)$ and $Sc(G')$ are isomorphic, and thus have the same matching number. Therefore, by Eppstein’s result (Proposition 6.6), $d(G) = d(G')$. □

Remark 6.8. If we know the skeleton of some finite MC-netlike partial cube $G$, then we can faster compute the lattice dimension of $G$ in particular, and more generally of all elements of $\mathcal{NC}(G)$. Indeed, by Theorem 6.7 and [10, Theorem 2], $d(G)$ can be computed in time $O(mn + nr^2)$, where $n$, $m$ and $r$ are the number of vertices, the number of edges and the dimension of $Sk(G)$, respectively.
6.3. Crossing graphs

Let \( G \) be a partial cube. We say that two \( \Theta \)-classes \( A, B \) of edges of \( G \) cross if, for \( a_i a_j \in A \) and \( b_i b_j \in B \),

\[
W_{a_i a_{j-1}} \cap W_{b_i b_{j-1}} \neq \emptyset \quad \text{for all } i, j \in \{0, 1\}.
\]

Note that this definition is independent of the choice of the edges in \( A \) and \( B \).

From a result [15, Lemma 3.3] of Klavžar and Mulder, that was only stated for finite graphs but holds for infinite ones, we have:

**Lemma 6.9.** Let \( G \) be a partial cube, and \( A, B \) two \( \Theta \)-classes of edges of \( G \). Then \( A \) and \( B \) cross if and only if each of these \( \Theta \)-classes has a representative occurring in an isometric cycle \( C \) of \( G \), i.e. if \( E(C) \cap A \neq \emptyset \) and \( E(C) \cap B \neq \emptyset \).

The crossing graph of a partial cube \( G \) is the graph \( G^* \) whose vertices are the \( \Theta \)-classes of \( G \), and where two vertices are adjacent if they cross. The concept of crossing graph was introduced by Bandelt and Dress [3] under the name of incomparability graph, and extensively studied by Klavžar and Mulder [15].

**Theorem 6.10.** Let \( G \) and \( G' \) be two elements of a netlike class. Then \( G^* \) and \( G'^* \) are isomorphic.

**Proof.** Let \( G^+ \) and \( G'^+ \) be any completions of \( G \) and \( G' \), respectively. Because \( G^+ \) and \( G'^+ \) are isomorphic, it suffices to show that \( G^* \) and \( G'^* \) are isomorphic.

By the definition of \( G^+ \), it is clear that the map \( f : \Theta_C(xy) \mapsto \Theta_{C'}(xy), xy \in E(G) \), is a bijection of \( V(G^*) \) onto \( V(G'^*) \).

It follows, by Lemma 6.9, that \( f \) is a homomorphism of \( G^* \) onto \( G'^* \). It remains to prove that \( f^{-1} \) is also a homomorphism, that is, that for all \( xy, uv \in E(G), \Theta_C(xy) \cap \Theta_C(uv) \) cross whenever \( \Theta_{C'}(xy) \cap \Theta_{C'}(uv) \) cross. By Lemma 6.9, it suffices to show that, for each isometric cycle \( D \) of \( G^* \), there exists an isometric cycle \( C \) of \( G \) such that \( \Theta_{C'}(D) \subseteq \Theta_C(C) \).

This is obvious if \( D \) is a cycle of \( G \). Suppose that \( D \cap (H_C - C) \neq \emptyset \) for some \( C \in G \) and that \( \Theta_{C'}(D) \not\subseteq \Theta_{G'}(C) = \Theta_{C'}(H_C) \). Because \( G^+ \) is a median graph, \( H := co_{C'}(D) \) is a hypercube, and thus \( \Theta_{C'}(H) = \Theta_{C'}(D) \). Let \( x \in V(D \cap (H_C - C)) \).

Then \( x \) has a neighbor \( y \) in \( H \) such that \( \Theta_{C'}(xy) \not\cap \Theta_{C'}(H) \). It follows that \( y \not\in H(C) \), contrary to the definition of \( G^* \). Therefore \( \Theta_{C'}(D) \subseteq \Theta_C(C) \).

Consequently \( f \) is an isomorphism of \( G^* \) onto \( G'^* \). \( \square \)

From [15, Theorem 5.1] and [16, Theorem 7] restated in [15, Proposition 4.1], respectively, and by Remark 5.3(2), we obtain that if \( G \) is a finite MC-netlike partial cube, then:

- \( G^* \) is triangle-free if and only if \( G \) is a cube-free median graph, and thus \( NC(G) \) has exactly one element up to isomorphism.
- \( G^* \) is a complete graph if and only if \( G \) is a hypercube or an even cycle (more precisely, \( G^* = K_n \) for some \( n \geq 1 \) if and only if \( G = Q_n \) or \( G = C_{2n} \) if \( n \geq 2 \)), and thus \( NC(G) \) has exactly two elements up to isomorphism.

**Remark 6.11.** Two concepts similar to the crossing graph of a partial cube \( G \) have been studied: the \( \Theta \)-graph [5,12] and the \( \tau \)-graph [13] of \( G \). Both have the \( \Theta \)-classes of \( G \) as vertices. Two \( \Theta \)-classes \( A \) and \( B \) are adjacent whenever there exist edges \( a \in A \) and \( b \in B \) which are adjacent for the \( \Theta \)-graph, and which induce a convex path of length 2 for the \( \tau \)-graph.

Unlike the crossing graph, the \( \Theta \)-graph and the \( \tau \)-graph of two elements of a netlike class are generally not isomorphic. For example both the \( \Theta \)-graph and the \( \tau \)-graph of \( C_8 \) are a \( C_4 \), while the \( \Theta \)-graph and the \( \tau \)-graph of the completion \( Q_4 \) of \( C_8 \) are a \( K_4 \).

7. Concluding remarks

7.1

The concept of completion can be useful to derive some properties of MC-netlike partial cubes from analogous properties of median graphs. This is the process that we used to prove Theorem 6.4, and that we could have used to prove that the geodesic interval space of any MC-netlike partial cube is a Pash-Peano space [21, Theorem 4.1]. This is also what we will now do in order to show that in this way we easily obtain the restrictions to MC-netlike partial cubes of some of the main theorems of [23] that were proved for any netlike partial cubes, but with the requirement of a lot of preliminary results.

Let \( G \) be an MC-netlike partial cube, and \( G^+ \) any of its completions. Clearly \( G \) contains no isometric rays (i.e. one-way infinite paths) if and only if \( G^+ \) contains no isometric rays. Moreover if \( G^+ \) contains no isometric rays, then \( G^+ \) contains no infinite hypercubes, and hence any of its hypercubes is either a hypercube of \( G \) or a subhypercube of some well-surrounded hypercube of \( G^+ \).

On the other hand, any self-contraction (resp. isomorphism) \( f \) of \( G \) naturally extends to a self-contraction (resp. isomorphism) \( f^+ \) of \( G^+ \). Then, if all hypercubes of \( G^+ \) are finite, and if \( f^+ \) fixes some hypercube \( H \) of \( G^+ \) (i.e. if \( f \) is a hypercube of \( G \) and \( f^+ \) fixes \( H \) when \( H \) is a hypercube of \( G \), then either \( f \) fixes \( H \) or any surrounding cycle of \( H \) according to whether \( H \) is or is not a hypercube of \( G \).

It follows that the restrictions of [23, Theorems 6.5, 6.6 and 6.8] to MC-netlike partial cubes are immediate consequences of Tardif's analogous results [24, Theorem 1.2] on median graphs. For example, from [24, Theorem 1.2(2)], stating that "Every self-contraction of a median graph \( G \) fixes a finite hypercube of \( G \) if and only if \( G \) contains no isometric rays" we infer that "Every self-contraction of an MC-netlike partial cube \( G \) fixes a finite hypercube or a convex cycle of \( G \) if and only if \( G \) contains no isometric rays" [23, Theorems 6.6] restricted to MC-netlike partial cubes.
7.2

As is usually done for other notions of completion in mathematics, we could have generalized the completion of an MC-netlike partial cube $G$ by defining it entirely up to isomorphism. Hence the completions of $G$ would have been the median graphs in which $G$ is isometrically embeddable and which are minimal with respect to the subgraph relation, and moreover the netlike class of $G$ would have been the class of all MC-netlike partial cubes having the same completions as $G$.

However, the statement of most of the results and their proofs would have been less simple by the constant use of the concepts of embedding and isomorphism. This is why we have preferred to consider an MC-netlike partial cube as an isometric subgraph of any of its completions.

References