Basis of the Identities of the Matrix Algebra of Order Two over a Field of Characteristic $p \neq 2$

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DEDICATED TO THE MEMORY OF MY TEACHERS, COLLEAGUES, AND FRIENDS, MIKHAIL GAVRILOV AND LJUBOMIR DAVIDOV

In this paper we prove that the polynomial identities of the matrix algebra of order 2 over an infinite field of characteristic $p \neq 2$ admit a finite basis. We exhibit a finite basis consisting of four identities, and in “almost” all cases for $p$ we describe a minimal basis consisting of two identities. The only possibilities for $p$ where we do not exhibit minimal bases of these identities are $p = 3$ and $p = 5$. We show that when $p = 3$ one needs at least three identities, and we conjecture a minimal basis in this case. In the course of the proof we construct an explicit basis of the vector space of the central commutator polynomials modulo the ideal of the identities of the matrix algebra of order two.

INTRODUCTION

Let $M_2$ be the matrix algebra of order two over a field $K$, and let $sl_2$ be the Lie algebra of all traceless matrices in $M_2$. Polynomial identities in the algebras $M_2$ and $sl_2$ have been attracting the attention of a large number of researchers for the past 30 years. Let us recall some of the most important results describing the identities in these algebras.

When the characteristic of the field $K$ equals zero, Razmyslov [15] found finite bases of the identities satisfied by $M_2$ and by $sl_2$ as well; see for example [16]. He also proved that the variety of Lie algebras generated by $sl_2$ is Spechtian. (This means that its identities admit a finite basis and

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the same holds for every subvariety of this variety.) In fact, in [15] it was
proved that whenever $K$ is an infinite field of characteristic $\neq 2$ the
multilinear polynomial identities of $M_2$ are consequences of a finite
collection of such identities. Since over a field of characteristic 0 the
identities in an algebra are determined by the multilinear ones, the last
fact resolved the case of characteristic 0. Obviously such a reason cannot
be applied to algebras over fields of positive characteristic.

Consequently, in [19], Tki proved that all identities of $M_2$ admit a basis
consisting of four identities; further, Drensky [4] obtained a minimal basis
consisting of two identities for the algebra $M_2$, and Filippov [7] proved that
the identities of the Lie algebra $sl_2$ follow from a single identity. In this
case, when the base field $K$ is of characteristic 0, the identities satisfied by
$M_2$ have been described in detail. These descriptions are very tight; they
include various numerical characterizations of the ideal of identities in
$M_2$. See for example [5, 8, 16, 17] for the “state of art” as well for further
bibliographic references concerning the quantitative study of the identities
in $M_2$ when char $K = 0$.

Considering the algebra $M_2$ over a finite field $K$, it was proved in [13]
that its identities follow from two of them.

Now let us turn to the remaining possibility for the field $K$. Suppose that
$K$ is infinite and of positive characteristic. Vasilovsky in [20] proved that
when char $K \neq 2$ the polynomial identities of the Lie algebra $sl_2$ admit a
basis consisting of a single identity (it is the same identity as in the case of
char $K = 0$). On the other hand, Vaughan-Lee (see [22, 23]) proved that
when char $K = 2$ then the identities of $M_2$ considered as a Lie algebra
with respect to the commutator operation do not admit finite bases. Some
information about the identities in $M_2$ over an infinite field, and particu-
larly in the case char $K \neq 2$, can be found in [1, 9, 10, 14], but this
information is far from being as complete as that available in the case of
characteristic 0.

We note that some further partial results in the case char $K = 2$ were
obtained in [6, 12].

The main goal of the present paper is to describe a finite basis of the
polynomial identities for the associative algebra $M_2$ over an infinite field
$K$ with char $K \neq 2$. Note that the finite set of identities we describe is
quite large. But using standard facts about the representations of the
symmetric and the general linear groups, we prove that when char $K > 5$
the minimal basis of identities for $M_2$ consists of two polynomials. They
are the same as in characteristic 0. The problem of reducing the set of our
identities in the cases char $K = 3$ and 5 still remains open. Some com-
ments and conjectures on this topic are given, too.

Our approach is combinatorial. Since our algebras are over fields of
positive characteristic, their identities cannot be determined (generally
speaking) by multilinear ones. So we make use of the characteristic-free description of the invariants of the classical groups as obtained in [2, 3] and of methods and results obtained on its basis in [10, 11, 14, 20, 21]. Due to the infinity of the base field we consider multihomogeneous polynomials only. Furthermore, we restrict our attention to the commutator polynomials since they determine the T-ideals whenever the field is infinite. We construct a model for the relatively free algebra in the variety of algebras determined by the identities of $M_2$. This construction is done in several steps. It depends on a linear transformation defined in the respective relatively free algebras, which we denote as $L$. It should be noted that the proof of the correctness of $L$ is far from obvious although quite technical.

1. PRELIMINARIES

Throughout the paper we consider associative and Lie algebras over a fixed infinite field $K$ of characteristic $p \neq 2$. Denote as $M_2$ the associative matrix algebra of order two over $K$, and let $sl_2$ be the Lie algebra of the traceless matrices in $M_2$. Since char $K \neq 2$, then $sl_2$ is a simple three-dimensional Lie algebra.

1.1. Polynomial Identities

Let $X = \{x_1, x_2, \ldots\}$ be a countable set of symbols, and denote by $A(X)$ the free associative algebra freely generated by the set $X$ over $K$. Sometimes for the sake of convenience we shall use the letters $x, y, z$, etc., with or without indices, for the free generators of $A(X)$. The elements of $A(X)$ are called polynomials. We denote by $[a, b] = ab - ba$ and $a \circ b = (1/2)(ab + ba)$ the commutator and the Jordan product, respectively, of $a, b \in A(X)$. All commutators will be left normed; in other words, $[a, b, c] = [[[a, b], c]$. If $L(X)$ is the free Lie algebra freely generated by the set $X$ then $L(X)$ is canonically isomorphic to the subalgebra of the adjoint Lie algebra of $A(X)$ that is generated by the set $X$. We shall identify $L(X)$ with this subset of $A(X)$.

The polynomial $f(x_1, \ldots, x_n) \in A(X)$ is a polynomial identity for the associative algebra $R$ if $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in R$. We denote by $T(R)$ the T-ideal of $R$. Thus $T(R)$ is the ideal of all identities for $R$. The variety $\text{var} R$ is the class of all algebras satisfying the identities in $T(R)$. The quotient algebra $A(X)/T(R)$ is the relatively free algebra of countable rank in $\text{var} R$; every algebra $S \in \text{var} R$ is a homomorphic image of $A(X)/T(R)$, provided that $S$ can be generated by some countable set of elements. Sometimes we shall write $\text{var} T(R)$ instead of $\text{var} R$.

Analogously, one defines the above concepts for Lie algebras.
Now suppose that $R$ is an associative algebra and let $S$ be a vector subspace of $R$. Suppose further that $S$ generates the algebra $R$. Then the polynomial $f(x_1, \ldots, x_n) \in A(X)$ is a weak polynomial identity for the pair $(R, S)$ if $f(s_1, \ldots, s_n) = 0$ for every $s_1, \ldots, s_n \in S$. The weak identities for the pair $(R, S)$ form the ideal $T(R, S)$ in $A(X)$. Note that while the $T$-ideals are closed with respect to endomorphisms of $A(X)$, the ideals of weak identities need not be. But they are closed with respect to linear transformations of variables. For further details concerning weak identities, see for example [16] and [10, 11].

We will be interested in the situation when the vector space $S$ is actually a Lie subalgebra of the adjoint Lie algebra of $R$. Thus we shall consider pairs $(R, S)$ consisting of an associative algebra $R$ and a Lie algebra $S$, $S \subseteq R$. In this case the ideal $T(R, S)$ is closed with respect to Lie substitutions. In other words, if $f(x_1, \ldots, x_n) \in T(R, S)$ then $f(y_1, \ldots, y_n) \in T(R, S)$ for every $y_1, \ldots, y_n \in L(X)$. We shall consider ideals of weak identities of this type only. Note that these ideals are called ideals of weak Lie identities, or sometimes ideals of identities for representations of Lie algebras. For convenience we shall omit the word “Lie” and call them simply “weak identities.”

If $f$ and $g$ are two polynomials in $A(X)$ then the identity $g$ is a consequence of $f$ if $g$ belongs to the $T$-ideal $(f)^T$ in $A(X)$ generated by $f$. The identity $f \in A(X)$ follows from the set of identities $\{f_i \mid i \in I\} \subseteq A(X)$ if $f$ belongs to the $T$-ideal in $A(X)$ generated by the set $\{f_i \mid i \in I\}$. Analogously, one defines a consequence of a weak identity. Note that in the last case we shall consider weak Lie identities together with the corresponding rule for the consequences. Hence the ideal of weak identities generated by the set $\{f_i \mid i \in I\} \subseteq A(X)$ is the ordinary ideal in $A(X)$ generated by the set $\{f_i(g_1, \ldots, g_n) \mid i \in I; g_j \in L(X)\}$. The identities $\{f_i \mid i \in I\}$ form a basis of the $T$-ideal $T$ if $T$ coincides with the $T$-ideal generated by these polynomials.

One defines the variety of pairs, the relatively free pair, and so on, in the same manner as in the case of algebras; see for example [10, 11, 16] for the exact definition.

Our field $K$ is infinite. Hence every identity is equivalent to the collection of identities obtained by its multihomogeneous components; see for example [24, Corollary 1.3.2]. Thus we may and shall consider multihomogeneous identities only. The same holds without any modification for weak identities, too.

Let $x_1, x_2, \ldots, u_1, u_2, \ldots$, be a basis of the vector space $L(X)$ where $u_i$ are commutators of length $\geq 2$. Then the Poincaré–Birkhoff–Witt Theorem gives us that the elements

$$\left\{x_{i_1}^{n_1} \cdots x_{i_m}^{n_m} u_j^{i_1} \cdots u_j^{i_n} \mid i_1 < \cdots < i_m, j_1 < \cdots < j_n, m, n \geq 0 \right\}$$
form a basis of the algebra $A(X)$. (When $m = n = 0$ we define the corresponding element as 1.) The commutators $u_1, u_2, \ldots$ generate the subalgebra $B(X)$ of $A(X)$. The polynomials in $B(X)$ are linear combinations of products of (possibly long) commutators. These polynomials are called proper (or commutator) polynomials. It is known that if $T$ is $T$-ideal in $K(X)$ then $T$ is generated as a $T$-ideal by the intersection $T \cap B(X)$; see [18]. In fact, the statement in [18] was proved for a field of characteristic 0 but its proof holds without any modification for every infinite field. Note that a similar assertion holds for ideals of weak identities; see for example [6].

Denote by $s_n(x_1, \ldots, x_n) = \sum_{\sigma} (-1)^{\sigma} x_{\sigma(1)} \cdots x_{\sigma(n)}$ the standard polynomial of degree $n$. Here $\sigma$ runs over the permutations of the symmetric group $S_n$ acting on $\{1, 2, \ldots, n\}$, and $(-1)^{\sigma}$ stands for the sign of $\sigma \in S_n$.

We shall need the following notation. Let $1 \leq k \leq n$, then $S_n^k$ stands for the set of all permutations $\sigma \in S_n$ such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k + 1) < \cdots < \sigma(n)$. Obviously, $S_n^k$ is a set of representatives of the lateral classes of $S_n$ modulo its subgroup $S_k \times S_{n-k}$. Here $S_k$ acts on $\{1, \ldots, k\}$ and $S_{n-k}$ acts on $\{k + 1, \ldots, n\}$.

1.2. The Algebras $M_2$ and $sl_2$

Assume that $E$ is an extension of the field $K$, and denote $R_E = R \otimes_K E$. It is well known that $T(R_E) = T(R) \otimes_K E$. In other words, the algebra $R_E$ satisfies the same identities with coefficients from $K$ as the algebra $R$ does. We shall see later that the coefficients of our defining identities lie in the minimal subfield of $K$. Hence without loss of generality we can assume that the field $K$ is algebraically closed; let $i \in K$, $i^2 = -1$. Then the matrices

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ e_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

form a basis of the vector space $M_2$, and $e_1, e_2, e_3$ are a linear basis of $sl_2$. It is obvious that the pair $(M_2, sl_2)$ satisfies the weak identity $[x_1, x_2, x_3] = 0$, and since char $K \neq 2$ it is equivalent to $[x_1^2, x_2] = 0$. Therefore one defines a nondegenerated symmetric bilinear form on $sl_2$ by setting $(a, b) = a \cdot b$, $a, b \in sl_2$. (Observe that $(a_1 e_1 + a_2 e_2 + a_3 e_3)^2 = -(a_{11}^2 + a_{12}^2 + a_{13}^2)e_0$, and identify the vector space $Ke_0$ with the field $K$.) The content of [10] is that all weak identities for the pair $(M_2, sl_2)$ are consequences of the weak identity $[x_1^2, x_2] = 0$.

For a matrix $a \in M_2$ we denote by $\text{tr} \ a$ its trace, therefore $a_0 = a - (1/2)\text{tr} \ ae_0 \in sl_2$ for every $a \in M_2$. Sometimes we shall write $a' = a - (1/2)\text{tr} \ a$, identifying again $Ke_0$ with $K$. The center of the algebra $M_2$ consists of all scalar matrices, $C(M_2) = \{ \lambda e_0 \mid \lambda \in K \}$. If one substitutes a
central element in a commutator then obviously one obtains 0. Thus if 
f(x_1, \ldots, x_n) \in B(X) is a proper polynomial then 
f(a_1, \ldots, a_n) = f(a_1', \ldots, a_n') \text{ for every } a_1, \ldots, a_n \in M_2. \text{ Therefore the proper polynomial } f
is an identity for \ M_2 \text{ if and only if it vanishes on } \mathfrak{s l}_2.

Let \ f \in B(X) \text{ be a proper polynomial. The above observation yields that } f
is a polynomial identity for \ M_2 \text{ if and only if it vanishes on } \mathfrak{s l}_2. \text{ Throughout the paper we shall use the term “weak identity” in the sense of “weak identity for the pair } (M_2, \mathfrak{s l}_2).”

1.3. Invariants and Double Tableaux

Given the commuting variables \ \{x_{ij} \mid i \geq 1, 1 \leq j \leq n\}, one considers the vectors \ x_i = (x_{i1}, \ldots, x_{in}) \text{ and denotes by } \ V \text{ their } K-\text{span. One defines on } \ V \text{ a symmetric bilinear form by setting } x_i \odot x_j = x_{i1}x_{j1} + \cdots + x_{in}x_{jn}; \text{ this form is nondegenerated. We recall the descriptions of the algebras of the invariants of the orthogonal group } O_n \text{ and of the special orthogonal group } SO_n \text{ obtained in [2, Section 5]. Note that the descriptions in [2] do not depend on the characteristic of } K. \text{ First, the algebra of the } O_n-\text{invariants coincides with the (commutative) polynomial algebra } K[(x_i \odot x_j)]. \text{ The latter algebra admits a basis indexed by the doubly standard tableaux } T \text{ of the following form; see [2, Theorem 5.1]}:

\[
T = \begin{pmatrix}
p_{11} & p_{12} & \cdots & p_{1m_1} & q_{11} & q_{12} & \cdots & q_{1m_1} 
p_{21} & p_{22} & \cdots & p_{2m_2} & q_{21} & q_{22} & \cdots & q_{2m_2} 
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots 
p_{k1} & p_{k2} & \cdots & p_{km_k} & q_{k1} & q_{k2} & \cdots & q_{km_k}
\end{pmatrix}.
\]

Here \ n \geq m_1 \geq m_2 \geq \cdots \geq m_k \geq 0 \text{ and } p_{ij}, q_{ij} \text{ are positive integers. The double tableau } T \text{ is doubly standard if the ordinary tableau } T' \text{ obtained from } T \text{ by inserting every row with } q \text{'s immediately after its corresponding row with } p \text{'s yields the standard tableau}

\[
T' = \begin{pmatrix}
p_{11} & p_{12} & \cdots & p_{1m_1} 
q_{11} & q_{12} & \cdots & q_{1m_1} 
p_{21} & p_{22} & \cdots & p_{2m_2} 
q_{21} & q_{22} & \cdots & q_{2m_2} 
\vdots & \vdots & \ddots & \vdots 
p_{k1} & p_{k2} & \cdots & p_{km_k} 
q_{k1} & q_{k2} & \cdots & q_{km_k}
\end{pmatrix}.
\]

This means that the entries of } T' \text{ are strictly increasing along its rows and are increasing (with possible repetitions) along its columns. One associates to } T \text{ the following polynomial in } K[(x_i \odot x_j)].
If \( T = (p_1 p_2 \cdots p_m \mid q_1 q_2 \cdots q_m) \) is a row tableau then one defines
\[
\tilde{\varphi}(T) = \sum (-1)^\sigma (x_{p_1} \circ x_{q_{\sigma(1)}})(x_{p_2} \circ x_{q_{\sigma(2)}}) \cdots (x_{p_m} \circ x_{q_{\sigma(m)}}) = \det(x_{p_1} \circ x_{q_i}).
\]
Here \( \sigma \) runs over the symmetric group \( S_m \). If \( T^{(1)}, \ldots, T^{(k)} \) are the rows of \( T \) then \( \tilde{\varphi}(T) = \tilde{\varphi}(T^{(1)} \cdots \tilde{\varphi}(T^{(k)})) \). Sometimes we identify the double tableau \( T \) with the corresponding polynomial \( \tilde{\varphi}(T) \).

In order to describe the \( SO_n \)-invariants one considers the \( n \times n \) determinants of type \( \det(x_{i_1}, \ldots, x_{i_n}) \) whose rows are formed by the coordinates of the vectors \( x_{i_1}, \ldots, x_{i_n} \). Denote the last determinant as \( \langle i_1, i_2, \ldots, i_n \rangle \). Then according to Theorem 5.6 of [2] the invariants of \( SO_n \) coincide with \( K[(x_{i_1} \circ x_i)] \) and \( \det(x_{i_1}, \ldots, x_{i_n}) \). All determinants of the latter form. It has a basis consisting of all doubly standard tableaux as described above plus all of the products \( \langle i_1, \ldots, i_n \rangle \tilde{\varphi}(T) \) that give rise to standard tableaux. This means that by forming the ordinary tableau \( T' \) from \( T \) and putting the row \( (i_1 i_2 \cdots i_n) \) on top of it; the obtained tableau is standard.

Using the terminology of [10, 11], one says that the polynomial
\[
\varphi(T) = \langle i_1, i_2, \ldots, i_n \rangle \tilde{\varphi}(T) = \det(x_{i_1}, \ldots, x_{i_n}) \cdot \tilde{\varphi}(T^{(1)} \cdots \tilde{\varphi}(T^{(k)}))
\]
represents some \( n \)-tableau \( T \). This means that one associates to the above polynomial a double tableau with the first \( n \) entries of the first row void. One conveniently assigns them the values \( -n, -(n - 1), \ldots, -1 \). If all entries of the double tableau \( T \) are positive integers then one sets \( \varphi(T) = \tilde{\varphi}(T) \). In this case \( T \) is called the 0-tableau (or simply tableau). One defines the \( k \)-tableau, \( 1 \leq k \leq n \), as a double tableau whose first \( k \) entries of its first row are void. Assign them the values \( -k, -(k - 1), \ldots, -1 \). See [10, 11] for details.

Now we define a partial order on the set of the double \( (k-) \)tableaux. If \( T \) is a double \( (k-) \)tableau, we put \( m(T) = (m_1, m_1, m_2, m_2, \ldots, m_k, m_k) \), where \( m_i \) stand for the lengths of the respective (half) lines of \( T \), and
\[
p(T) = (p_1, p_2, \ldots, p_{1m_1}, q_1, q_2, \ldots, q_{1m_1}, p_{21}, \ldots, q_{km_k}),
\]
and call them the shape and form of \( T \), respectively. Let
\[
d(T) = (d_{-n}, \ldots, d_{-1}, d_1, d_2, \ldots)
\]
be the contents of \( T \). This means that \( d_i \) is the number of entries of \( T \) that are equal to \( i, i = \pm 1, \pm 2, \ldots \).

Now suppose that \( T \) and \( Q \) are double tableaux with the same contents \( d(T) = d(Q) \). We say that \( T \succ Q \) if \( m(T) > m(Q) \) in the usual lexico-
graphical order or if \( m(T) = m(Q) \) but \( p(T) < p(Q) \) in the same lexicographical order; see [2, 10, 11, 20].

2. IDENTITIES IN \( M_2 \)

In this section we introduce some basic identities in the matrix algebra \( M_2 \) and describe the \( T \)-ideal generated by them in terms of the transformation defined below.

**Lemma 2.1.** The following polynomials are identities in \( M_2 \):

1. the standard identity \( s_4 = s_4(x_1, x_2, x_3, x_4); \)
2. the Hall’s identity \( h_5 = [[x_1, x_2], x_3, x_4], x_5]; \) and
3. the Lie identity \( v_5 = [y, z, [t, x], x] + [y, x, [z, x], t]. \)

**Remarks.**

1. The standard identity \( s_4 \) is the identity of minimal degree satisfied by \( M_2 \), and the only identities of degree 4 in \( M_2 \) are the scalar multiples of \( s_4 \). The standard polynomial \( s_4 \) can be written in the form
   \[
   s_4 = 2([x_1, x_2] \circ [x_3, x_4] - [x_1, x_3] \circ [x_2, x_4] + [x_1, x_4] \circ [x_2, x_3]).
   \]
2. It is known that when \( \text{char } K = 0 \) the identities \( s_4 \) and \( h_5 \) form a basis of the identities of \( M_2 \); see [4].
3. One writes the identity \( h_5 \) in its equivalent form
   \[
   [w_1, x] \circ w_2 = -w_1 \circ [w_2, x],
   \]
   where \( w_1 = [x_1, x_2], w_2 = [x_3, x_4]; \) see for example [16, pp. 204–207]. We shall refer to the last identity as \( h_5 \), as well.
4. The identity \( v_5 \) is a basis of the identities for the Lie algebra \( sl_2 \). This was proved in [7] in the case \( \text{char } K = 0 \) and by Vasilovsky in [20] for the arbitrary infinite field \( K, \text{char } K \neq 2 \).

Recall that the vector space \( L(X) \) has a basis consisting of the elements \( x_i \) and of the commutators \( u_j \). Following [20], we define a linear transformation \( L(a, b), a, b \in L(X), \) on the Lie subalgebra of \( L(X) \) generated by all \( u_j \), in the following way. If \([w_1, w_2] \) is a commutator in \( L(X) \) then

\[
[w_1, w_2]L(a, b) = (1/16)([w_1, a, b, w_2] + [w_1, b, a, w_2] + [w_1, a, w_2, b] + [w_1, b, w_2, a] - [w_2, a, b, w_1] - [w_2, b, a, w_1] - [w_2, a, w_1, b] - [w_2, b, w_1, a]).
\]
It is easy to verify that modulo the Jacobi identity, the following equality holds in every Lie algebra.

\[
[w_1, w_2]L(a, b) = (1/8)\left([w_1, a, b, w_2] + [w_1, b, a, w_2]\right)
\]

Since the equality \( [c, a, b] = 4((a \circ b) \circ c - a \circ (b \circ c)) \) holds in every associative algebra, one immediately obtains that the weak identity

\[
[x, y]L(a, b) = [x, y] \circ (a \circ b)
\]

holds. Thus we have the identity for \( M_2 \),

\[
[x_1, x_2] \circ (a \circ b) = (1/8)\left([x_1, a, b, x_2] + [x_1, b, a, x_2]\right)
\]

where \( a = [x_3, x_4], b = [x_5, x_6] \) are commutators of length 2. We refer to this identity as \( r_6 \). Note that the following three identities hold for the Lie algebra \( sl_2 \).

\[
[x_1, x_2]L(a, b) = [[x_1, x_2]L(a, b), x_3] \\
x_1L(x_2, b) - [x_2, a]L(x_1, b) = (1/4)[x_1, x_2, b, a] \\
x_1L(x_2, b) - [x_2, a]L(x_1, b) = [x_1, x_2]L(c, d) - [x_1, x_2]L(c, d) L(a, b)
\]

See [20], identities (3), (4), and (5), for their deduction. (These identities are consequences of \( v_3 \), as proved in [20].)

**Lemma 2.2** [16, Lemma 41.2, p. 205]. *Let \( f \in B(X) \) be a proper polynomial. Then modulo the identities \( h_5 \) and \( r_6 \), the polynomial \( f \) can be represented in the form \( f = l + g \) where \( l \in L(X) \) is a Lie polynomial and \( g \) is a linear combination of products of the type \( g_i \circ [x_i, x_n] \). Here \( g_i \in L(X) \) are Lie polynomials and \( x_n \) is a fixed (but arbitrary) variable such that \( f \) depends on \( x_n \).*

Suppose that \( f \in B(X) \) is a proper polynomial, and let \( f = l + g \) as in the lemma above. One easily verifies that \( f \) is an identity for \( M_2 \) if and only if \( l \) and \( g \) are identities for \( M_2 \), too (see for example [16, p. 206]). Therefore, if \( T_1 \) is the \( T \)-ideal generated by the polynomials \( s_3, h_5, v_5 \), and \( r_6 \), then \( T_1 \subset T(M_2) \), and one may choose a basis of the identities of \( M_2 \), modulo \( T_1 \), among the identities of the form \( g \).

Now we extend the transformation \( L(a, b) \) to the polynomials of type \( g \). Let \( g = g_1 \circ g_2 \) where \( g_1 \) and \( g_2 \) are commutators of length \( \geq 2 \), then we put \( gL(a, b) = g_1 \circ (g_2 L(a, b)) \). Due to the weak identities

\[
g_1 \circ (g_2 L(a, b)) = g_1 \circ (g_2 \circ (a' \circ b')) = g_1 \circ (g_2 \circ (a' \circ b'))
\]

\[
= g_1 \circ (a' \circ b') \circ g_2 = (g_1 L(a, b)) \circ g_2,
\]
one can apply $L(a,b)$ to either $g_1$ or $g_2$. (Note that since we deal with commutator polynomials it is sufficient to consider traceless matrices only.)

This implies the following identities for $M_2$.

**Lemma 2.3.** The identities

\[
([x_1, a, b, x_2] + [x_1, b, a, x_2] - [x_2, a, x_1, b] - [x_2, b, x_1, a]) \circ [y_1, y_2] \\
= ([y_1, a, b, y_2] + [y_1, b, a, y_2] - [y_2, a, y_1, b] - [y_2, b, y_1, a]) \\
\times \circ [x_1, x_2],
\]

and

\[
\begin{align*}
([x_1, a, b, x_2, x_3] + [x_1, b, a, x_2, x_3] - [x_2, a, x_1, b, x_3] \\
- [x_2, b, x_1, a, x_3]) \circ [y_1, y_2] \\
= ([y_1, a, b, y_2] + [y_1, b, a, y_2] - [y_2, a, y_1, b] \\
- [y_2, b, y_1, a]) \circ [x_1, x_2, x_3]
\end{align*}
\]

hold in the algebra $M_2$.

**Proof.** Use the weak identity $[x, y]L(a,b) = [x, y]\circ(a \circ b)$ and the fact that $a \circ b$ is central in $M_2$ when $a, b \in \mathfrak{sl}_2$. One may suppose that the respective elements do belong to $\mathfrak{sl}_2$ since the polynomials in the lemma are proper. The second identity follows from Identity (3) of [20], using the above reasoning. ■

**Remarks.**

1. The above lemma justifies the definition of the transformation $L$. Later we shall prove that the definition of $L$ is correct and that $L$ is linear.

2. The linearization of the identity $\psi$ yields the polynomial

\[
[x_1, x_2, [x_3, x_4], x_5] + [x_1, x_2, [x_3, x_5], x_4] \\
+ [x_1, x_4, [x_2, x_3], x_5] + [x_1, x_5, [x_2, x_4], x_3],
\]

that is, an identity for $\mathfrak{sl}_2$ and for $M_2$. Since char $K \neq 2$, the linearization is harmless, and the latter polynomial is equivalent as an identity to $\psi$. We refer to this identity as $\psi$, as well.

3. Write the linearized $\psi$ as $\psi = \sum \alpha_i [a_i, b_i]$, where $\alpha_i \in K$ and $a_i, b_i$ are commutators, deg $a_i \geq 2$. Then the polynomial $\psi' = \sum \alpha_i (a_i \circ [b_i, x_i])$ is an identity for $M_2$ since it is a weak identity. Indeed, it can be written as a weak identity in the form $\sum \alpha_i [a_i, b_i] \circ x_i$, and the latter is obviously a weak identity (see for example [16, Remark 41.4, p. 204]).
**Proposition 2.1.** All multilinear identities for $M_2$ follow from the identities $s_4, h_5, v_5, r_6,$ and $v'_5$.

**Proof.** Combine the main result of [20] with [16, Remark 41.4, p. 204].

**Remark.** The above proposition yields immediately that when char $K = 0$ all identities for $M_2$ follow from the ones listed in it. Of course, when char $K > 0$ the multilinear elements in a T-ideal might not determine the T-ideal.

Now we extend the T-ideal $T_1$. Denote by $T_2$ the T-ideal generated by $T_1$, the identities of Lemma 2.3, and $v'_5$. Since all these are identities for $M_2$ we have that $T_2 \subset T(M_2)$.

**Lemma 2.4.** Let $f \in B(X)$ be a proper polynomial that is multilinear and skew-symmetric in four of its variables. Then $f$ is an identity for $M_2$ and $f$ belongs to the T-ideal $T_2$.

**Proof.** If a proper and multilinear polynomial is skew-symmetric in four of its variables it vanishes on $M_2$. This is easy to verify since one has to check whether the polynomial vanishes on the basic elements of $sl_2$ and since $\dim sl_2 = 3$. Then we apply Proposition 2.1.

**Lemma 2.5.** Every proper polynomial $f \in B(X)$ of even degree $\geq 4$ can be written, modulo the T-ideal $T_2$, in the form

$$f = \sum_i \alpha_i [x_{i_1}, x_{i_2}]^o [x_{i_3}, x_{i_4}] \prod_j L(a_{ij}, b_{ij})$$

for suitable $a_{ij}, b_{ij} \in X$ and $\alpha_i \in K$. Further, one may suppose that $i_1 \leq i_2$, $i_3 \leq i_4$, and $i_1 \leq i_3$. When $\deg f = 4$ one may require in addition that $i_2 \leq i_4$.

**Proof.** It is sufficient to consider the case $f = u \circ v$ where $u$ and $v$ are commutators, $\deg u = 2$, and $\deg v \geq 2$. But then, if $\deg v \geq 4$, according to equality (4) from [20], one writes $v$ as a linear combination of elements of the form $wL(x, y)$ where $\deg w = \deg v - 2$, and so on. Obviously the transformations $L(a, b)$ commute and obey the associative rule. Thus, the first part of the lemma is done.

Further, one may write, up to a sign, that $i_1 \leq i_2$ and $i_3 \leq i_4$ and that $i_1 \leq i_3$. Note that for the last inequality one may need to use the identity $h_5$.

Let $i_2 > i_4$. Then due to the identity $s_4$ one obtains

$$[x_{i_1}, x_{i_2}]^o [x_{i_3}, x_{i_4}] = -[x_{i_1}, x_{i_2}]^o [x_{i_3}, x_{i_4}] + [x_{i_1}, x_{i_2}]^o [x_{i_3}, x_{i_4}],$$

and the righthand terms satisfy the condition of the lemma.
LEMMA 2.6. Every proper polynomial \( f \in B(X) \) of odd degree \( \geq 5 \) can be written, modulo the \( T \)-ideal \( T_2 \), in the form

\[
f = \sum_i \alpha_i \sum_{\sigma} (-1)^{\sigma} [x_{i_{\sigma(1)}}, x_{i_{\sigma(2)}}] \circ [x_{i_{\sigma(3)}}, x_{i_{\sigma(4)}}, x_{i_{\sigma(5)}}] \prod_j L(a_{ij}, b_{ij})
\]

for suitable \( a_{ij}, b_{ij} \in X \) and \( \alpha_i \in K \). (In what follows sums of the type \( \sum_{\sigma} \) will be taken over the respective symmetric group.)

Further, when \( \deg f = 5 \) one may suppose that \( i_1 \leq i_2 \leq i_3 \), \( i_4 \leq i_5 \) and \( i_1 \leq i_4 \).

Proof. Using the identities (3) and (4) from [20], one represents \( f \) as a linear combination of elements \( u \cdot w \prod_j L(a_{ij}, b_{ij}) \) where \( \deg u = 2 \) and \( \deg w = 3 \). It is easy to observe that

\[
4[x_1, x_2] \circ [x_3, x_4, x_5] = \sum_{\sigma} (-1)^{\sigma} [x_{\sigma(1)}, x_{\sigma(2)}] \circ [x_{\sigma(3)}, x_4, x_5] \\
- \sum_{\sigma} (-1)^{\sigma} [x_{\sigma(1)}, x_{\sigma(2)}] \circ [x_{\sigma(4)}, x_3, x_5].
\]

The identity

\[
\sum_{\sigma} (-1)^{\sigma} [x_{\sigma(1)}, x_{\sigma(2)}] \circ [x_{\sigma(3)}, x_4, x_5] = 0
\]

holds for \( M_2 \), and it is a consequence of \( h_5 \) and of the trivial identity (valid in every associative algebra) \( \sum_{\sigma} (-1)^{\sigma} [x_{\sigma(1)}, x_{\sigma(2)}] = 0 \). (Note that the identity under consideration also follows from \( s_4 \).) Hence

\[
\sum_{\sigma} (-1)^{\sigma} [x_{\sigma(1)}, x_{\sigma(2)}] \circ [x_{\sigma(3)}, x_4, x_5] \\
= \sum_{\sigma} (-1)^{\sigma} [x_{\sigma(1)}, x_{\sigma(2)}] \circ [x_{\sigma(3)}, x_5, x_4]
\]

modulo the \( T \)-ideal \( T_2 \), and we may suppose that \( i_4 \leq i_5 \) in the statement of the lemma.

In order to prove the last statement, we proceed as follows. Let \( i_1 > i_4 \), then modulo \( T_2 \) we have

\[
0 = \sum_{\sigma \in S_4} (-1)^{\sigma} [x_{\sigma(1)}, x_{\sigma(2)}] \circ [x_{\sigma(3)}, x_{\sigma(4)}, x_5] \\
= \sum_{\sigma \in S_4} (-1)^{\sigma} f_{\sigma(1), \sigma(2), \sigma(3), \sigma(4), 5},
\]

where \( f_{1, 2, 3, 4, 5} = \sum_{\sigma} (-1)^{\sigma} [x_{\sigma(1)}, x_{\sigma(2)}] \circ [x_{\sigma(3)}, x_4, x_5] \). Therefore

\[
f_{i_1, i_2, i_3, i_4, i_5} = \pm f_{i_4, i_1, i_2, i_3, i_5} \pm f_{i_4, i_1, i_3, i_2, i_5} \pm f_{i_4, i_2, i_3, i_1, i_5},
\]
Consider the lexicographical order on the permutations of the indices \(\{i_1, i_2, i_3, i_4, i_5\}\), supposing that \(i_5\) is the last and that the first three elements form an ascending sequence. The indices of the terms on the right-hand side are larger than those on the left-hand side. We repeat (if necessary) this process until we obtain the desired property. Thus the proof of the lemma is complete.

**Remarks.** 1. The identity
\[
\sum_{\sigma \in S_4} (-1)\sigma [x_{\sigma(1)}, x_{\sigma(2)}] \circ [x_{\sigma(3)}, x_{\sigma(4)}, x_5]
\]

belongs to the T-ideal generated by \(s_4\). This can be verified in the following way. Write
\[
s_4 = s_4(x_1, x_2, x_3, x_4) = (1/2) \sum (\sigma) [x_{\sigma(1)}, x_{\sigma(2)}] [x_{\sigma(3)}, x_{\sigma(4)}],
\]

where \(\sigma\) runs over \(S_4\). The equality \([ab, c] = a[b, c] + [a, c]b\) holds for every \(a, b, c\), in every associative algebra. Then
\[
2[s_4, x_3] = \sum (-1)\sigma [x_{\sigma(1)}, x_{\sigma(2)}] [x_{\sigma(3)}, x_{\sigma(4)}, x_5]
\]

\[
= \sum (-1)\sigma [x_{\sigma(1)}, x_{\sigma(2)}] [x_{\sigma(3)}, x_{\sigma(4)}, x_5]
\]

\[
+ \sum (-1)\sigma [x_{\sigma(1)}, x_{\sigma(2)}] [x_{\sigma(3)}, x_{\sigma(4)}]
\]

\[
= 2 \sum (-1)\sigma [x_{\sigma(1)}, x_{\sigma(2)}] \circ [x_{\sigma(3)}, x_{\sigma(4)}, x_5],
\]

since obviously the signs of the permutations \((\sigma(1), \sigma(2), \sigma(3), \sigma(4))\) and \((\sigma(3), \sigma(4), \sigma(1), \sigma(2))\) coincide. Thus our identity follows from the standard identity \(s_4\). Note that our identity is multilinear and it is skew-symmetric in four variables, i.e., it must follow from the identities \(s_4\), \(h_5\), \(v_5\), \(r_5\), and \(v_5\) for the T-ideal generated by \(s_4\). Since it is of degree 5 it would follow from \(s_4\), \(h_5\), and \(v_5\). We showed that it follows from \(s_4\) only.

2. Note that the polynomials
\[
[x_{i_1}, x_{i_2}] \circ [x_{i_3}, x_{i_4}], \quad i_1 < i_2, i_3 < i_4; i_1 \leq i_3, i_2 \leq i_4,
\]

and
\[
\sum_{\sigma} (-1)\sigma [x_{i_{\sigma(1)}}, x_{i_{\sigma(2)}}] \circ [x_{i_{\sigma(3)}}, x_{i_{\sigma(4)}}, x_{i_5}], \quad i_1 < i_2 < i_3 < i_4; i_1 \leq i_5; i_1 < i_4,
\]

are linearly independent modulo the T-ideal of the identities for \(M_2\). This will be proved in the next section. Then they form the basis of the vector space of all proper polynomials of the form \(g = g_1 \circ g_2\) where \(g_1\) and \(g_2\).
are commutators, \( \deg g \leq 5 \) in \( A(X)/T_2 \) as shown in the previous two lemmas.

In the next two sections we shall prove the correctness of the definition of the transformation \( L \) in the following way. First we define \( L(a, b) \) on the proper polynomials \( g \) of degrees 4 and 5 using the basis given above. This obviously is compatible with the definition of \( L(a, b) \) given in [20]. Then we shall use induction on the degree of the respective polynomials in order to define \( L(a, b) \) on the basic elements only, and then we extend it by linearly.

3. GENERIC MATRICES AND DOUBLE TABLEAUX

Let \( K[\xi] = K[\xi_{ij}^{(r)} | i, j = 1, 2; r = 1, 2, \ldots] \) be the commutative polynomial algebra in variables \( \xi_{ij}^{(r)} \). The \( K \)-algebra generated by the \( 2 \times 2 \) matrices

\[
g_r = \begin{pmatrix}
\xi_{11}^{(r)} & \xi_{12}^{(r)} \\
\xi_{21}^{(r)} & \xi_{22}^{(r)}
\end{pmatrix}, \quad r = 1, 2, \ldots,
\]

is isomorphic to the relatively free algebra of countable rank in the variety \( \text{var} M_2 \). Make the \( K \)-linear transformation of the variables \( \xi_{ij}^{(r)} \) in order to obtain \( g_r = \alpha_r e_1 + \beta_r e_2 + \gamma_r e_3 + \delta_r e_0 \). The elements \( \alpha_r, \beta_r, \gamma_r, \delta_r \in K[\xi] \) are algebraically independent and we consider the polynomial algebra \( K[\alpha_r, \beta_r, \gamma_r | r = 1, 2, \ldots] \).

Recall that the multiplication among \( e_1, e_2, e_3 \) is given by

\[
e_1 e_2 = -e_2 e_1 = e_3, \quad e_2 e_3 = -e_3 e_2 = e_1, \quad e_3 e_1 = -e_1 e_3 = e_2, \quad e_i^2 = -e_0.
\]

Therefore \( [e_1, e_2] = 2 e_3, \quad [e_2, e_3] = 2 e_1, \quad [e_3, e_1] = 2 e_2, \quad e_i \circ e_j = 0 \) if \( i \neq j \).

As in [1], we define a lexicographical order on \( K[\alpha_r, \beta_r, \gamma_r] \), setting

\[
\alpha_1 > \beta_1 > \gamma_1 > \alpha_2 > \beta_2 > \gamma_2 > \alpha_3 > \cdots > \alpha_m > \beta_m > \gamma_m > \cdots.
\]

Let \( B(M_2) \) be the image of \( B(X) \) under the canonical projection

\[A(X) \to A(X)/T(M_2).\]

Then \( B(M_2) = B_2 \oplus L_2 \), where \( B_2 \) is the subspace of \( B(M_2) \) consisting of all proper central polynomials and \( L_2 \) stands for the subspace of all Lie polynomials; see for example [16, pp. 205, 206]. We may consider only the subspace \( B_2 \) due to the main theorem of [20]. First we show that the definition of the transformation \( L \) is correct on the vector space \( B_2 \), and then we exhibit a basis of this vector space.
Lemma 3.1. The transformation \( L(a, b) \) is a well-defined linear operator on the vector space \( B_2 \).

Proof. First we prove that if \( f \in B_2 \) is an identity for \( M_2 \) then the polynomial \( fL(a, b) \in B_2 \) is also. But \( fL(a, b) \in B_2 \) according to the definition of \( L(a, b) \). The second statement follows easily from the weak identity \( fL(a, b) = f \circ (a \circ b) \). If \( f \in B_2 \) is an identity for \( M_2 \) then obviously \( f \circ (a \circ b) \) is a weak identity. Therefore \( fL(a, b) \) is a weak identity. But \( fL(a, b) \in B_2 \) is a commutator polynomial and we obtain that \( fL(a, b) \) is an identity for \( M_2 \) as well. The linearity of the transformation \( L(a, b) \) is obvious. Thus the lemma is proved. Note that when \( a \) and/or \( b \) belong to \( M_2 \) but not to \( sl_2 \) then one must write \( fL(a, b) = f \circ (a' \circ b') \) in the expression for the corresponding weak identity.)

The above lemma shows that the operators \( L(a, b) \) generate a commutative subalgebra of the algebra of the linear transformations of \( B_2 \). We shall need some properties of this subalgebra.

Denote by \( Adm \) the set of all doubly standard tableaux \( T \) such that \( m = 2 \) or \( 3 \), \( T \) is either 0-tableau or 3-tableau, and if \( T \) is 3-tableau then \( m = 0 \). To every such tableau we associate a polynomial \( f \) in \( B_2 \) in the following manner. Let

\[
T = \begin{pmatrix}
  p_{11} & p_{12} & \cdots & p_{1m_1} \\
  p_{21} & p_{22} & \cdots & p_{2m_2} \\
  \vdots & \vdots & \ddots & \vdots \\
  p_{k1} & p_{k2} & \cdots & p_{km_k}
\end{pmatrix}
\begin{pmatrix}
  q_{11} & q_{12} & \cdots & q_{1m_1} \\
  q_{21} & q_{22} & \cdots & q_{2m_2} \\
  \vdots & \vdots & \ddots & \vdots \\
  q_{k1} & q_{k2} & \cdots & q_{km_k}
\end{pmatrix}
\]

be a tableau in \( Adm \). If \( m_1 = 2 \) we define \( f_T \) as

\[
f_T = [x_{p_{11}}, x_{p_{12}}] \circ [x_{q_{11}}, x_{q_{12}}] l_2 \cdots l_k,
\]

where \( l_j \) stands for the transformation

\[
l_j = \sum_{\sigma} (-1)^{\sigma} L(x_{p_{1j}}, x_{q_{1(\sigma)}}) \cdots L(x_{p_{mj}}, x_{q_{m(\sigma)}}), \quad \sigma \in S_m, m = m_j.
\]

When \( m_1 = 3 \) and \( T \) is 0-tableau we set

\[
f_T = \frac{1}{2} \sum_{\sigma} (-1)^{\sigma} [x_{p_{11}}, x_{p_{12}}] \circ [x_{q_{1(\sigma)}}, x_{q_{1(\sigma)}}, x_{q_{1(\sigma)}}] L(x_{p_{13}}, x_{q_{1(\sigma)}}) l_2 \cdots l_k,
\]

where the sum runs over \( \sigma \in S_3 \).

Finally, when \( m_1 = 3 \) and \( T \) is 3-tableau we define

\[
f_T = \frac{1}{2} \sum_{\sigma} (-1)^{\sigma} \left( \sum_{\tau} (-1)^{\tau} [x_{q_{1(\tau)}}, x_{q_{1(\tau)}}, x_{q_{1(\tau)}}] \circ [x_{q_{1(\tau)}}, x_{p_{21}}, x_{p_{22}}] l_2 \right) l_3 \cdots l_k
\]
for $\sigma \in S_3$, $\tau \in S_{m_2}$. Here $I'_2 = L(x_{p_{21}}, x_{q_{21}})\cdots L(x_{p_{2m_2}}, x_{q_{2m_2}})$. Let us recall that in this case one necessarily has $m_2 \geq 1$.

**Lemma 3.2.** The polynomials $f_T$ just defined belong to $B$, and they are central polynomials for $M_2$.

**Proof.** The proof of the lemma is obvious since $u \circ v$ is central for $M_2$ provided that $u$ and $v$ are commutators of degrees $\geq 2$.

**Proposition 3.1.** The polynomials $f_T$, $T \in \text{Adm}$, are linearly independent modulo the $T$-ideal $T(M_2)$.

**Proof.** We exploit an idea from [1, Section 3]. Substitute $x_r$ for the generic traceless matrix $y_r = \alpha_r e_1 + \beta_r e_2 + \gamma_r e_3$, and write $f_T(y_1, \ldots, y_n)$ as

$$f_T(y_1, \ldots, y_n) = \sum \left( \prod_{i=1}^n \alpha_i^{r_i} \beta_i^{s_i} \gamma_i^{t_i} \right) \lambda_{s,t} e_0.$$ 

Note that we must obtain a multiple of the matrix $e_0$ since the polynomial $f_T(y_1, \ldots, y_n)$ is central.

We shall prove that the largest summand in $f_T$ in the lexicographical order on $K[\alpha_1, \beta_1, \gamma_1]$ equals

$$\lambda \prod_{i=1}^n \alpha_{p_{i1}} \alpha_{q_{i1}} \prod_{i=1}^n \beta_{p_{i2}} \beta_{q_{i2}} \prod_{i=1}^n \gamma_{p_{i3}} \gamma_{q_{i3}} e_0, \quad 0 \neq \lambda \in K.$$ 

Here $u, v, w$ stand for the number of (half) rows in $T$ of lengths 3, 2, and 1, respectively. If $T$ is a 3-tableau then $\alpha_{p_{i1}}, \beta_{p_{i2}}, \gamma_{p_{i3}}$ do not appear in the above products.

But the statement for the largest summand follows from the weak identity $[x, y]L(a, b) = [x, y] \circ (a \circ b)$ and from the following obvious fact (see [1, proof of Proposition 3.1]): If $f$ and $g \in B_2$ have the largest summands $f_1$ and $g_1$, respectively, and if $f_1 g_1 \neq 0$ then the largest summand of $fg$ equals $f_1 g_1$.

The multiplication table for $e_1, e_2, e_3$ yields that the largest summand of $[y_1, y_2]$ equals $2\alpha_1 \beta_2 e_3$, and that of $y_3 \circ y_4$ equals $\alpha_3 \alpha_4 e_0$. Thus after a simple calculation one obtains the largest summand of $f_T$. We give as an example the largest summand of the polynomial

$$f_T = \left( \frac{1}{2} \right) \sum_{\sigma} (-1)^{\sigma} [x_{\alpha(1)}, x_{\alpha(2)}] \circ [x_{\alpha(3)}, x_4, x_3].$$ 

It equals $2 \alpha_1 \beta_2 \gamma_3 \alpha_4 \alpha_5 e_0$. The largest summands of the polynomials $f_T$ for $T = (12 \mid 34)$ and for $T = (123 \mid 456)$ are respectively of the forms $\pm 2 \alpha_1 \beta_2 \alpha_3 \beta_4 e_0$, and $\pm 2 \alpha_1 \beta_2 \gamma_3 \alpha_4 \beta_5 \gamma_6 e_0$. 

Note that the coefficients \( \lambda \) obtained will be of the form \( \pm 2^n \) and hence \( \lambda \neq 0 \) in \( K \). Now by applying several times the weak identity \([x, y]L(a, b) = [x, y] \circ (a \circ b)\) one obtains the largest summand of \( f_T \).

Therefore the largest summands in a linear combination of \( f_T, T \) standard, cannot cancel each other, and the polynomials \( f_T, T \in \text{Adm} \), are linearly independent in \( B_2 \).

Remarks. 1. In fact, following [1, Proposition 3.1], one can easily prove that the above polynomials form a basis for \( B_3 \); cf. [14, Section 5]. We shall prove this later as a consequence of another result. Note that this would give a detailed description of the basis of \( B_2 \); cf. [14, Section 5], as well as [10, Theorem 2.9].

2. Another proof of the proposition can be obtained following verbatim the proofs of Lemmas 2.6 and 2.7 and using Corollary 3.3 of [10].

3. We use the “French” notation for the tableaux but not the “English” one. Thus compared to [1] we transpose the rows and columns of the tableaux. Of course this is a technical detail and it cannot cause confusion.

4. The polynomials of Lemmas 2.5 and 2.6, of degrees respectively 4 and 5, correspond to the double tableaux

\[
(i_1 \ i_2 \mid i_3 \ i_4), \quad \begin{pmatrix} -3 & -2 & -1 \\ i_4 \\ i_5 \end{pmatrix},
\]

that give rise to the ordinary tableaux

\[
\begin{pmatrix} i_4 & i_2 \\ i_3 & i_4 \end{pmatrix}, \quad \begin{pmatrix} i_1 & i_2 & i_3 \\ i_4 \\ i_5 \end{pmatrix},
\]

where we conventionally erased the first (void) line of the second tableau.

4. STRAIGHTENING DOUBLE TABLEAUX

Here we show that modulo a certain T-ideal contained in \( T(M_2) \), every proper polynomial is expressible as a linear combination of polynomials \( f_T, T \in \text{Adm} \). Since our T-ideal will be finitely based this will prove our main result, namely that \( T(M_2) \) is finitely based as T-ideal. In the course of the proof we show that the transformation \( L \) is a well-defined linear operator on the respective vector space.

We start with the T-ideal \( T_2 \). Denote by \( A_i = K(X)/T_i \) the relatively free algebra defined by the T-ideal \( T_i \).
LEMMA 4.1. The following identity holds in $M_2$:

$$
\sum_{\sigma} (-1)^{\sigma} [x_1, x_2] \circ [y_{\sigma(1)}, y_{\sigma(2)}] L(x_3, y_{\sigma(3)})
$$

$$
= \sum_{\sigma} (-1)^{\sigma} [x_{\sigma(1)}, x_{\sigma(2)}] L(x_{\sigma(3)}, y_3) \circ [y_1, y_2].
$$

Proof. It is sufficient to check whether this is a weak identity. Using [10, identity (10)], one obtains that the left-hand-side polynomial equals, as a weak identity,

$$
\sum_{\sigma} (-1)^{\sigma} [x_1, x_2] \circ [y_{\sigma(1)}, y_{\sigma(2)}] L(x_3, y_{\sigma(3)})
$$

$$
= 4 \sum_{\sigma} (-1)^{\sigma} (x_{\sigma(1)} \circ y_1)(x_{\sigma(2)} \circ y_2)(x_{\sigma(3)} \circ y_3).
$$

Analogously, the right-hand side equals

$$
\sum_{\sigma} (-1)^{\sigma} [x_{\sigma(1)}, x_{\sigma(2)}] L(x_{\sigma(3)}, y_3) \circ [y_1, y_2]
$$

$$
= 4 \sum_{\sigma} (-1)^{\sigma} (x_{\sigma(1)} \circ y_1)(x_{\sigma(2)} \circ y_2)(x_{\sigma(3)} \circ y_3),
$$

and we are done since both expressions equal $4 \det(x_i \circ y_j)$ in $sl_2$ where the determinant is of order $3$, and $1 \leq i \leq 3, 1 \leq j \leq 3$.

Let $w_6$ be the identity of the lemma and let $T_3$ be the T-ideal generated by $T_2$ and by $w_6$. Then it is immediate that $T_2 \subseteq T_3 \subseteq T(M_2)$.

LEMMA 4.2. Denote by

$$
\varphi(x_1, x_2, x_3, y_1, y_2, y_3) = \varphi(1, 2, 3, 4, 5, 6)
$$

$$
= \sum_{\sigma} (-1)^{\sigma} [x_1, x_2] \circ [y_{\sigma(1)}, y_{\sigma(2)}] L(x_3, y_{\sigma(3)})
$$

the polynomial on the left-hand side of the previous lemma. Then in the algebra $A_3 = K(X)/T_3$ one has the equalities

$$
\sum_{\sigma \in S_4} (-1)^{\sigma} \varphi(\sigma(1), \sigma(2), \sigma(3), \sigma(4), 5, 6) = 0,
$$

$$
\sum_{\sigma \in S_4} (-1)^{\sigma} \varphi(1, \sigma(2), \sigma(3), \sigma(4), \sigma(5), 6) = 0,
$$

$$
\sum_{\sigma \in S_4} (-1)^{\sigma} \varphi(1, 2, \sigma(3), \sigma(4), \sigma(5), \sigma(6)) = 0,
$$

where $\sigma$ runs over the group $S_4^i$ acting on the respective indices.

Proof. The polynomial $\varphi(1, 2, 3, 4, 5, 6)$ is skew-symmetric in $x_1, x_2, x_3$. The first of the equalities we want to prove is obtained by this polynomial having made skew-symmetrisation in $x_1, x_2, x_3, y_1$. But the resulting polynomial is still proper and is skew-symmetric in four variables. Therefore it
vanishes on $A_3$. The second and third equalities are deduced by repeating the same argument.

In order to simplify and unify our further considerations we make use of some weak identities. Let

$$T = (p_1 \cdots p_r r_1 \cdots r_{k+1} \cdots r_{n+1} q_1 \cdots q_{k-1})$$

be a row tableau, $n = s + k \leq 3$, and denote

$$\tilde{T} = \sum (-1)^{\sigma} \tilde{\phi}(p_1 \cdots p_r r_{\sigma(1)} \cdots r_{\sigma(k+1)} \cdots r_{\sigma(n+1)} q_1 \cdots q_{k-1}).$$

The summation is over all $\sigma \in S_{n+1}^k$.

**Lemma 4.3** [2, Lemma 5.2]. The polynomial $\tilde{T}$ is a weak identity for the pair $(M_2, s_1)$.

**Lemma 4.4** [20, Lemma 1.5]. Let $1 \leq k \leq m \leq n \leq 3$, and let

$$T = \left( \begin{array}{ccccccc} s_1 & \cdots & s_k & s_{k+1} & \cdots & s_n & q_1 & \cdots & q_{k-1} & r_{k+1} & \cdots & r_{n+1} \\ r_1 & \cdots & r_k & q_{k+1} & \cdots & q_m & t_1 & \cdots & t_{k-1} & t_k & \cdots & t_m \end{array} \right)$$

be a double tableau. Then the sum

$$\sum (-1)^{\sigma} \tilde{\phi}$$

$$\times \left( \begin{array}{ccccccc} s_1 & \cdots & s_k & s_{k+1} & \cdots & s_n & q_1 & \cdots & q_{k-1} & r_{\sigma(k+1)} & \cdots & r_{\sigma(n+1)} \\ r_{\sigma(1)} & \cdots & r_{\sigma(k)} & q_{k+1} & \cdots & q_m & t_1 & \cdots & t_{k-1} & t_k & \cdots & t_m \end{array} \right)$$

is equal as a weak identity to a sum $\sum \alpha_i \tilde{\phi}(T_i)$, $\alpha_i \in K$, where $T_i$ are double tableaux having their first rows of the form $(\cdots \cdots | r_1 \cdots r_{n+1})$ if $n \leq 2$. When $n = 3$ the above sum equals 0. Here $\sigma$ runs over $S_{n+1}^k$.

**Lemma 4.5.** The identity

$$4 \sum_{\sigma \in S_n} (-1)^{\sigma} [x_1, x_2] \cdot [y_1, y_2] L(z_1, t_{\sigma(1)}) L(z_2, t_{\sigma(2)}) L(z_3, t_{\sigma(3)})$$

$$= \sum_{k < l} \left( \sum_{\sigma \in S_k} (-1)^{\sigma} [x_1, x_2] \cdot [t_{\sigma(k)}, t_{\sigma(l)}] L(t_{\sigma(j)}, z_j) \right)$$

$$\times L(z_k, y_1) L(z_l, y_2)$$

holds for the algebra $M_2$ where $(j, k, l) = (1, 2, 3)$.

**Proof.** The proof is the same as the proof of Identity (14) from [20].

Let $g = g_1 \circ g_2$ be a proper polynomial where $g_1$ and $g_2$ are commutators of degrees $\geq 2$. We denote by $gL(p_1 \cdots p_n \mid q_1 \cdots q_m)$ the polynomial

$$\sum_{\sigma \in S_n} (-1)^{\sigma} g_1 \circ g_2 L(x_{p_1}, x_{q_{\sigma(i)}}) \cdots L(x_{p_n}, x_{q_{\sigma(n)}}).$$
If $T$ is a double tableau with rows $T^{(1)}, \ldots, T^{(k)}$ we denote by $L(T)$ the transformation $L(T) = L(T^{(1)}) \cdots L(T^{(k)})$. This transformation is still defined only on the polynomial $g_2$. Due to [20] we know that it is linear if restricted only on one of the multiples (in our case, $g_2$).

**Proposition 4.1.** Let $V$ be the vector space spanned in $A_3$ by all products $g = g_1 \circ g_2$ where $g_1$ and $g_2$ are commutators in the variables $X$. Then $L(a, b)$ is a well-defined linear transformation on $V$.

**Proof.** When $\deg g = 4$ and 5 the statement of the proposition was proved in the remark at the end of the last section. (Note that using the linear independence of the standard tableaux one defines $L(a, b)$ only for them, and then one extends it by linearity.)

When $\deg g_1 = 2$ and $\deg g_2 = 4$ one uses Lemmas 4.1 and 4.2 and the results of the previous section in order to prove that the polynomials that correspond to standard tableaux are linearly independent. Then they form the basis of the respective vector subspace of $V$. Define $L(a, b)$ on this vector space, and then again using Lemmas 4.1 and 4.2; extend the definition of $L(a, b)$ to the case $\deg g = 8$; and so on.

Analogously, when $\deg g_2 = 5$ first we apply Identity (4) from [20] to $g_2$, and thus we can induct on the degree in analogy with the above considerations.

Hence $L(a, b)$ is well defined; its linearity is evident.

We have already shown that in $A_3$ every proper polynomial is a linear combination of polynomials of the form $f_T$, where $T$ are double 0- or 3-tableaux with $m_1 \geq 2$ and if $T$ is a 3-tableau then $m_2 > 0$. Thus it is sufficient to prove that every double tableau of the above type equals a linear combination of the standard ones.

**Lemma 4.6.** Let $T$ be double tableau and suppose that $f_T \neq 0$ in $A_3$. Then $m_1 = 2$ or 3.

**Proof.** If some of the (half) rows of $T$ were of length $\geq 4$ then the polynomial $f_T$ would be skew-symmetric in four variables. But since $f_T$ is a proper polynomial then necessarily $f_T = 0$ in $A_3$ as shown above.

Hence one obtains the following identities for $M_2$.

**Corollary 4.1.** Let $g_1$ and $g_2$ be commutators of degrees $\geq 2$, and let $T = (p_1 \cdots p_s r_1 \cdots r_k | r_{k+1} \cdots r_{s+1} q_1 \cdots q_{k-1})$ be a row tableau, $n = s + k \leq 3$. Then

$$\sum (-1)^s g_1 \circ g_2 L(p_1 \cdots p_s r_{\sigma(1)} \cdots r_{\sigma(k)} | r_{\sigma(k+1)} \cdots r_{\sigma(n+1)} q_1 \cdots q_{k+1}) = 0$$
is an identity for $M_2$ where the sum is over all permutations $\sigma \in S_{n+1}^k$.

**Corollary 4.2.** Let $T$ be the double tableau from Lemma 4.4, and let $g_1, g_2$ be commutators of degrees $\geq 2$. Then the identity

$$\sum_{\sigma \in S_{n+1}^k} (-1)^\sigma g_1 \circ g_2 L(\sigma(T)) = \sum \alpha_i g_1 \circ g_2 L(T_i)$$

holds in $M_2$. Here $T_i$ are the same as in Lemma 4.4, and $\sigma(T)$ are the double tableaux on the right-hand side of the equality of Lemma 4.4.

**Proof.** Both corollaries follow easily from Lemmas 4.3 and 4.4.

Denote by $T_4$ the T-ideal generated by $T_3$ and by the identities from Lemma 4.5 and Corollaries 4.1 and 4.2. Since all these are identities for $M_2$ we have again $T_4 \subseteq T(M_2)$.

**Proposition 4.2.** Let $T$ be a 0-tableau with $m_1 = 2$ or $3$, and suppose that $T$ is not standard. Then in $A_4$ the polynomial $f_T$ equals a linear combination of polynomials $f_Q$ for some 0-tableaux $Q$. These tableaux $Q$ have the same contents as $T$, their first half rows are of lengths 2 or 3, and all of them are larger than $T$ in the order defined on the set of the double tableaux.

**Proof.** The proof of the proposition is similar to the proof of Proposition 2.1 of [20]. Suppose that the first row of $T$ is of length 3. If the standardness of $T$ is violated somewhere below the first row one applies Corollaries 4.1 and 4.2. If the violation appears in the first row then one applies Lemmas 4.1 and 4.2. Hence the only case that remains to be considered is when the violation of the standardness occurs between elements of the first and of the second row. Then one applies Lemma 4.2. Thus the case when the first row of $T$ is of length 3 is done.

If the length of the first row of $T$ equals 2 then one starts straightening, as in [20, Proposition 2.1], one of the commutators. If there appears a row of length 3 then we apply Lemma 4.5, and thus we reduce this case to the previous. Hence we may suppose that all rows are of length $\leq 2$. Then Lemma 2.5 and the rules for “operating” with the transformations $L(a, b)$ yield our statement.

**Proposition 4.3.** Let $T$ be a 3-tableau that is not standard. Then in $A_4$ the polynomial $f_T$ equals a linear combination of polynomials $f_Q$ for some 3-tableaux $Q$. These tableaux $Q$ have the same contents as $T$ and all of them are larger than $T$ in the order defined on the set of the double tableaux.

**Proof.** The proof is fairly similar to that of the last but one proposition. One uses Lemma 2.6 and repeats the reasoning above.

**Theorem 4.1.** Let $g \in A_4$ be a nonzero proper polynomial. Then $g$ can be represented in $A_4$ as a linear combination: $g = \sum \alpha_w f_{T_w}$ where $T_w \in \text{Adm}$ are standard, $\alpha_w \in K$, and at least one $\alpha_w \neq 0$. 
Proof. The proof of the theorem is a straightforward union of the last two propositions.

Note that the last theorem implies the finite basis property for the T-ideal \( T(M_2) \).

**Theorem 4.2.** The T-ideal \( T(M_2) \) is finitely based.

Proof. We have proved already that the polynomials \( f_T \), \( T \in \text{Adm} \), form a basis of the span of all commutator central polynomials for \( M_2 \). On the other hand, we just proved that these polynomials form a basis of the span of all polynomials of the form \( g = g_1 \circ g_2 \) for \( g_1 \) and \( g_2 \) commutators, modulo the T-ideal \( T_4 \). This means that

\[ T_4 = T(M_2). \]

But the T-ideal \( T_4 \) is finitely based, and this proves our theorem.

5. MINIMISATION OF THE BASIS

In this section we shall prove that some of the identities that define the T-ideal \( T_4 \) are consequences of other identities in \( T_4 \), and thus we shall obtain some smaller basis of identities for \( M_2 \). Let us first make a list of the identities that define \( T_4 \). These are the following.

\[ s_4 = 2([[x_1, x_2] \circ [x_3, x_4]] - [x_1, x_3] \circ [x_2, x_4] + [x_1, x_4] \circ [x_2, x_3]) \quad (1) \]

\[ h_5 = [[x_1, x_2] \circ [x_3, x_4], x_2] \]

\[ h_5 = [x_1, x_2, x_3] \circ [x_4, x_5] + [x_1, x_2] \circ [x_4, x_5, x_3]; \quad (2) \]

\[ v_5 = [y, z, [t, x], x] + [y, x, [z, x], t]; \quad (3) \]

\[ v'_5 = [x_1, x_2, [x_3, x_4]] \circ [x_5, x_6] + [x_1, x_2, [x_3, x_5]] \circ [x_4, x_6] \]

\[ + [x_1, x_4, [x_2, x_5]] \circ [x_3, x_6] + [x_1, x_5, [x_2, x_4]] \circ [x_3, x_6]; \quad (4) \]

\[ r_6 = [x_1, x_2] \circ ([[[x_3, x_4] \circ [x_5, x_6]]] - (1/8)([[x_1, x_3, x_4], [x_5, x_6], x_2] \]

\[ - [x_1, [x_3, x_5], [x_3, x_4], x_2] - [x_2, [x_3, x_4], x_1, [x_5, x_6]] \]

\[ - [x_2, [x_3, x_5], x_1, [x_3, x_4]]); \quad (5) \]

\[ [x_1, x_2] \circ [y_1, y_2] L(a, b) = [x_1, x_2] L(a, b) \circ [y_1, y_2]; \quad (6) \]

\[ [x_1, x_2, x_3] \circ [y_1, y_2] L(a, b) = [x_1, x_2, x_3] L(a, b) \circ [y_1, y_2]; \quad (7) \]

\[ \sum_{\sigma \in S_3} (-1)^{\sigma} [x_1, x_2] \circ [y_{\sigma(1)}, y_{\sigma(2)}] L(x_3, y_{\sigma(3)}) \]

\[ - \sum_{\sigma \in S_3} [x_{\sigma(1)}, x_{\sigma(2)}] L(x_{\sigma(3)}, y_3) \circ [y_1, y_2]; \quad (8) \]
\[
4 \sum_{\sigma \in S_3} (-1)^\sigma [x_1, x_2] \circ [y_1, y_2] L(z_1, t_{\sigma(1)}) L(z_2, t_{\sigma(2)}) L(z_3, t_{\sigma(3)}) = \sum_{k \leq l} \left( \sum_{\sigma \in S_3} (-1)^\sigma [x_1, x_2] \circ [t_{\sigma(k)}, t_{\sigma(l)}] L(t_{\sigma(j)}, z_j) \right) \times L(z_k, y_1) L(z_l, y_2), \text{ where } \{j, k, l\} = \{1, 2, 3\} \quad (9)
\]

\[
\sum_{\sigma \in S_{n+1}^k} (-1)^\sigma g_1 \circ g_2 \times L(p_1 \cdots p_k r_{\sigma(1)} \cdots r_{\sigma(k)} \cdots r_{\sigma(n+1)} q_1 \cdots q_{k+1}) \quad (10)
\]

\[
\sum_{\sigma \in S_{n+1}^k} (-1)^\sigma g_1 \circ g_2 L(\sigma(T)) = \sum \alpha_1 g_1 \circ g_2 L(T_i), \quad (11)
\]

where \( g_1 \) and \( g_2 \) are commutators, \( \deg g_j \geq 2 \), and \( \sigma(T) \) and \( T_i \) are defined as in Lemma 4.4 and Corollary 4.2.

**Lemma 5.1.** The equality

\[
\psi_5 = [y, z, [t, x], x] + [y, x, [z, x], t]
\]

\[
= s_4(z, y, x, t) + x s_4(z, y, x, t) - s_4(x, y, x, t) = s_4(z, xy, x, t)
\]

holds for every \( x, y, z, t \) in the free associative algebra \( A(X) \).

**Proof.** The proof consists of direct and rather tedious verification: one expands both sides and cancels out all terms, which takes at least half an hour. See for example [9, Lemma 11].

**Corollary 5.1.** Every Lie polynomial that is an identity for \( M_2 \) is a consequence of \( s_4 \).

**Proof.** We have mentioned already that the identity \( \psi_5 \) is a basis of the identities for the Lie algebra \( sl_2 \) (see [20, Theorem 1]). As an associative polynomial it follows from \( s_4 \) due to the previous lemma.

Thus the identity \( \psi_5 \) is redundant in our basis. As mentioned at the beginning, the multilinear proper identities of \( M_2 \) follow from \( s_4 \), \( h_5 \), \( v_5 \), \( \psi_5 \), and \( \psi_6 \) (see [16, Remark 41.1, p. 204]). Hence the identities \( s_4 \), \( h_5 \), \( v_5 \), and \( \psi_6 \) form a basis of the polynomial identities for \( M_2 \).

**Lemma 5.2.** Let \( \text{char } K = 3 \). Then the identity \( \psi_6 \) does not belong to the \( T \)-ideal of \( A(X) \) generated by \( s_4 \) and \( h_5 \).

**Proof.** This was proved in [9, Proof of Theorem 10]. The proof consists in showing that the infinite-dimensional Grassmann (or exterior) algebra \( G \) over an infinite field \( K \) of characteristic 3 satisfies the identities \( s_4 \) and \( h_5 \), but it does not satisfy \( \psi_6 \).
Thus when char $K = 3$ a minimal basis of the identities for $M_2$ must consist of at least three identities since $s_4$, $h_5$, and $r_6$ are independent in this case. On the other hand, when char $K > 5$ one uses the decomposition of the $S_n$-module of the multilinear proper polynomials as in the case of characteristic 0; see [4]. Since one considers identities of degrees $\leq 6$ in this case the complete reducibility is preserved. Thus when char $K \geq 7$ one has a minimal basis of the identities for $M_2$ consisting of $s_4$ and $h_5$. We state all this as our last theorem.

**THEOREM 5.1.** Let $K$ be an infinite field of characteristic $\neq 2$. Then the polynomial identities for the algebra $M_2$ admit a basis consisting of the identities $s_4$, $h_5$, $t'_5$, and $r_6$. Further,

(i) when char $K \geq 7$ the identities $s_4$ and $h_5$ form a minimal basis;

(ii) when char $K = 3$ the identities $s_4$, $h_5$, and $r_6$ are independent.

**OPEN QUESTIONS.**

1. Describe minimal bases of the identities for $M_2$ in the cases of char $K = 3$ and 5.

2. Describe the basis of the identities for $M_2$ in the case char $K = 2$.

We conjecture that when char $K = 3$ the minimal basis of identities for $M_2$ would consist of the identities $s_4$, $h_5$, and $r_6$. The case char $K = 5$ should yield the same basis as that when char $K > 5$ (and char $K = 0$).

Concerning the second question we make no guess there.

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