Note

The Ramsey numbers for disjoint unions of graphs

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Received 23 June 2006; received in revised form 2 April 2007; accepted 11 April 2007

Available online 25 April 2007

Abstract

For given graphs $G$ and $H$, the Ramsey number $R(G, H)$ is the smallest natural number $n$ such that for every graph $F$ of order $n$: either $F$ contains $G$ or the complement of $F$ contains $H$. In this paper we investigate the Ramsey number of a disjoint union of graphs $R(\bigcup_{i=1}^{k} G_i, H)$. For any natural integer $k$, we contain a general upper bound, $R(kG, H) \leq R(G, H) + (k - 1)|V(G)|$. We also show that if $m = 2n - 4$, $2n - 8$ or $2n - 6$, then $R(kS_n, W_m) = R(S_n, W_m) + (k - 1)n$. Furthermore, if $|G_i| > (|G_i| - |G_{i+1}|)(\chi(H) - 1)$ and $R(G_i, H) = (\chi(H) - 1)(|G_i| - 1) + 1$, for each $i$, then $R(\bigcup_{i=1}^{k} G_i, H) = R(G_k, H) + \sum_{i=1}^{k-1} |G_i|$.

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Keywords: Graph; Ramsey number; Disjoint union of graphs

1. Introduction

Throughout this paper, all graphs are finite and simple. Let $G$ be any graph with the vertex set $V(G)$ and the edge set $E(G)$. The order of $G$, written as $|G|$ denotes the number of vertices of $G$. The graph $\overline{G}$, the complement of $G$, is obtained from the complete graph on $|G|$ vertices by deleting the edges of $G$. The number of vertices in a maximum independent set of $G$ is denoted by $\chi(G)$. A graph $H = (V', E')$ is a subgraph of $G$ if $V' \subseteq V(G)$ and $E' \subseteq E(G)$. For $S \subseteq V(G)$, $G[S]$ represents the subgraph induced by $S$ in $G$. If $G$ is a graph and $H$ is a subgraph of $G$, then denote $G[V(G) \setminus V(H)]$ by $G \setminus H$.

Given two graphs $G$ and $H$, the Ramsey number $R(G, H)$ is defined as the smallest natural number $n$ such that for any graph $F$ on $n$ vertices, either $F$ contains $G$ or $\overline{F}$ contains $H$. Chvátal and Harary [6] established a useful and general lower bound on the exact Ramsey numbers $R(G, H)$ as follows.

Theorem A (Chvátal and Harary [6]). Let $G$ and $H$ be two graphs (not necessarily different) with no isolated vertices. Then the following lower bound holds:

$$R(G, H) \geq (\chi(G) - 1)(n(H) - 1) + 1,$$

where $\chi(G)$ is the chromatic number of $G$ and $n(H)$ is the number of vertices in the largest component of $H$.

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0012-365X/S - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2007.04.026
This result of Chvátal and Harary has motivated various authors to determine the Ramsey numbers $R(G, H)$ for many combinations of graphs $G$ and $H$, see the nice survey paper [9].

Let $T_n$ be a tree on $n$ vertices and let $W_m$ be a wheel on $m + 1$ vertices that consists of a cycle $C_m$ with one additional vertex being adjacent to all vertices of $C_m$. A star $S_n$ is the graph on $n$ vertices with one vertex of degree $n - 1$, called the center, and $n - 1$ vertices of degree 1.

There are several known results on Ramsey numbers for combination of stars and wheels were established. For instance, Surahmat et al. showed in [10] that for $n \geq 3$,

$$R(S_n, W_4) = \begin{cases} 2n + 1 & \text{if } n \text{ is even}, \\ 2n - 1 & \text{if } n \text{ is odd}. \end{cases}$$

They also showed that $R(S_n, W_5) = 3n - 2$ for $n \geq 3$. This result was strengthened by Chen et al. [4] who showed that this Ramsey number remains the same, even if $m$ is odd and $n \geq m - 1 \geq 2$. For even $m$, Zhang and Zhang [11] established $R(S_n, W_6) = 2n + 1$ and

$$R(S_n, W_8) = \begin{cases} 2n + 1 & \text{if } n \text{ is odd}, \\ 2n + 2 & \text{if } n \text{ is even}. \end{cases}$$

In [7], Hasmawati et al. established the following theorem.

**Theorem B (Hasmawati et al. [7]).** If $n$ is odd and $n \geq 5$, then

$$R(S_n, W_m) = \begin{cases} 3n - 4 & \text{if } m = 2n - 4, \\ 3n - 6 & \text{if } m = 2n - 6 \text{ or } 2n - 8. \end{cases}$$

In this paper, we study the Ramsey numbers for a disjoint union of graphs. Let $G_i$ be any graph with vertex set $V_i$ and edge set $E_i$, $i = 1, 2, \ldots, k$. The union $G = \bigcup_{i=1}^{k} G_i$ has the vertex set $V = \bigcup_{i=1}^{k} V_i$ and the edge set $E = \bigcup_{i=1}^{k} E_i$. The union of s vertex-disjoint copies of $G$ is denoted by $sG$. The joint $G = G_1 + G_2$ has the vertex set $V = V_1 \cup V_2$ and the edge set $E = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$.

In 1975, Burr et al. [3] determined the upper bound and the lower bounds on the Ramsey numbers of disjoint unions of graphs.

**Theorem C (Burr et al. [3]).** Let $G$ and $H$ be graphs of order $n_1$ and $n_2$, respectively. Then

$$n_1s + n_2t - D \leq R(sG, tH) \leq n_1s + n_2t - D + k,$$

where $D = \min\{s\chi(G), t\chi(H)\}$ and $k$ is a constant depending only on $G$ and $H$.

We observe here that if $s = t = 1$, and the chromatic number $\chi(G)$ or $\chi(H)$ is at least 3, then the lower bound of Chvátal and Harary is better than the lower bound of Burr et al.

Recently, Baskoro et al. [2] determined the Ramsey numbers for multiple copies of a star versus a wheel and for a forest versus a complete graph. Their results are given in the following three theorems.

**Theorem D (Baskoro et al. [2]).** If $m$ is odd and $3 \leq m \leq 2n - 1$, then $R(kS_n, W_m) = 3n - 2 + (k - 1)n$.

**Theorem E (Baskoro et al. [2]).** For $n \geq 3$,

$$R(kS_n, W_4) = \begin{cases} (k + 1)n & \text{if } n \text{ is even and } k \geq 2, \\ (k + 1)n - 1 & \text{if } n \text{ is odd and } k \geq 1. \end{cases}$$

**Theorem F (Baskoro et al. [2]).** Let $n_i \geq n_{i+1}$ for $i = 1, 2, \ldots, k - 1$. If $m$ is such that $n_i > (n_i - n_{i+1})(m - 1)$ for every $i$, then

$$R(\bigcup_{i=1}^{k} T_{n_i}, K_m) = R(T_{n_k}, K_m) + \sum_{i=1}^{k-1} n_i.$$
Motivated by these result, in this paper we gave the general upper bound and we consider the Ramsey numbers for multiple copies of stars $kS_n$ versus a wheel $W_m$. We also consider a general form of Theorem F, in which the graphs are not restricted to be trees or complete graphs. The main results are presented in the following theorems.

**Theorem 1.** For connected graphs $G$, and $H$, and $k \geq 1$ $R(kG, H) \leq R(G, H) + (k - 1)|V(G)|$.

**Theorem 2.** If $n$ is odd and $n \geq 5$, then $R(kS_n, W_m) = R(S_n, W_m) + (k - 1)n$ for $m = 2n - 4$, $2n - 6$ or $2n - 8$.

**Theorem 3.** Let $H$ and $G_i$ be connected graphs with $|G_i| > |G_{i+1}|, i = 1, 2, \ldots, k - 1$. If $|G_i| > |(H) - 1)|(|G_i| - 1) + 1$ for each $i$, then $R(\bigcup_{i=1}^{k} G_i, H) = R(G_k, H) + \sum_{i=1}^{k-1} |G_i|$.

2. The proofs of theorems

**Proof of Theorem 1.** Let $G$ and $H$ be connected, we show that $R(kG, H) \leq R(G, H) + (k - 1)|V(G)|$ applying an induction on $k$. It is trivial to see that the assertion holds for $k = 1$. Assume the theorem holds for any $r < k$. Let $F$ be a graph with order $R(G, H) + (k - 1)|V(G)|$. Suppose $\overline{F}$ contains no $H$. By induction hypothesis $F$ contains $(k - 1)G$. Now, write $T = F \setminus (k - 1)G$. Thus, $|T| = R(G, H)$. Since $\overline{T}$ contains no $H$, then $T$ must contain $G$. Hence, $F$ contains $(k - 1)G \bigcup G$. Therefore, we have $R(kG, H) \leq R(G, H) + (k - 1)|V(G)|$. □

**Proof of Theorem 2.** By Theorem 1, we have $R(kS_n, W_m) \leq R(S_n, W_m) + (k - 1)n$.

For $m = 2n - 4$ we consider $F \simeq K_{kn-1} \bigcup K_{n^2-n-2}$. The graph $F$ has $(3n - 5) + (k - 1)n$ vertices and contains no $kS_n$. Observe that $\overline{F}$ contains no $W_m$. Hence, $R(kS_n, W_m) \geq (3n - 4) + (k - 1)n = R(S_n, W_m) + (k - 1)n$ for $m = 2n - 4$.

In showing the lower bound for $m = 2n - 6$ or $2n - 8$, we use $F_1 \simeq K_{kn-1} \bigcup (n-3/2)K_2 + ((n-3)/2)K_2$. The graph $F_1$ has $(3n - 7) + (k - 1)n$ vertices and contains no $kS_n$. We observe that $\overline{F}$ contains no $W_m$. Therefore, we have $R(kS_n, W_m) \geq (3n - 6) + (k - 1)n = R(S_n, W_m) + (k - 1)n$ for $m = 2n - 6$ or $2n - 8$. □

**Proof of Theorem 3.** Let $|G_i| = n_i$ for $i = 1, 2, \ldots, k$. We will show that $R(\bigcup_{i=1}^{k} G_i, H) = (\chi(H) - 1)(nk - 1) + \sum_{i=1}^{k-1} n_i + 1$ if $R(G_i, H) = (\chi(H) - 1)(n_i - 1) + 1$. Consider $F \simeq (\chi(H) - 2)K_{nk-1} \bigcup K_s$ where $s = 1 - \sum_{i=1}^{k} n_i$. This graph $F$ has $(\chi(H) - 1)(nk - 1) + \sum_{i=1}^{k-1} n_i$ vertices and contains no $\bigcup_{i=1}^{k} G_i$. Furthermore, its complement contains no $H$. Then, we have $R(\bigcup_{i=1}^{k} G_i, H) \geq (\chi(H) - 1)(nk - 1) + \sum_{i=1}^{k-1} n_i$.

Next, we will show that $R(\bigcup_{i=1}^{k} G_i, H) \leq (\chi(H) - 1)(nk - 1) + \sum_{i=1}^{k-1} n_i$. First, we show that $R(G_1 \bigcup G_2, H) \leq (\chi(H) - 1)(n_1 - 1) + 1 + \sum_{i=1}^{k-1} n_i$. Let $F_1$ be a graph with $|F_1| = (\chi(H) - 1)(n_1 - 1) + 1 + \sum_{i=1}^{k-1} n_i$. Suppose $\overline{F_1}$ contains no $H$ and let $n_1 - n_2 = q$. Then $n_2 = n_1 - q$ and $|F_1| = (\chi(H) - 1)(n_1 - q + 1) + 1 + \sum_{i=1}^{k-1} n_i$. We can write $|F_1| = (\chi(H) - 1)(n_1 - 1) + 1 + 1 + n_1 - n_2 = q$. Since $n_1 - n_2 < 1$, it follows that $T$ contains $G_1 \bigcup G_2$. Therefore, $F$ contains $G_1 \bigcup G_2$. Hence, we have $R(G_1 \bigcup G_2, H) \leq (\chi(H) - 1)(n_2 - 1) + 1 + \sum_{i=1}^{k-1} n_i$.

Write $T = F_1 \setminus G_1$. Thus, $|T| = (\chi(H) - 1)(n_2 - 1) + 1$. Since $\overline{T}$ contains no $H$ and $|T| = (\chi(H) - 1)(n_2 - 1) + 1$, it follows that $T$ contains $G_2$. Therefore, $F$ contains $G_1 \bigcup G_2$. Hence, we have $R(G_1 \bigcup G_2, H) \leq (\chi(H) - 1)(n_2 - 1) + 1 + \sum_{i=1}^{k-1} n_i$.

Now, by induction, assume the theorem holds for any $r < k$, namely $R(\bigcup_{i=1}^{r} G_i, H) \leq (\chi(H) - 1)(n_r - 1) + 1 + \sum_{i=1}^{r-1} n_i$. We shall show that $R(\bigcup_{i=1}^{k} G_i, H) \leq (\chi(H) - 1)(nk - 1) + 1 + \sum_{i=1}^{k-1} n_i$.

Let $F_2$ be a graph with $|F_2| = (\chi(H) - 1)(nk - 1) + 1 + \sum_{i=1}^{k-1} n_i$. Suppose $\overline{F_2}$ contains no $H$. By induction, $F_2$ contains $\bigcup_{i=1}^{k-1} G_i$. Let $B = F_2 \setminus \bigcup_{i=1}^{k-1} G_i$ and $Q = F_2[B]$. Then $|Q| = (\chi(H) - 1)(n_k - 1) + 1 = R(G_k, H)$. Consequently, $Q$ contains $G_k$. So, $F_2$ contains $\bigcup_{i=1}^{k-1} G_i$. Therefore, $R(\bigcup_{i=1}^{k} G_i, H) \leq (\chi(H) - 1)(n_k - 1) + 1 + \sum_{i=1}^{k-1} n_i$.

Hence, we have $R(\bigcup_{i=1}^{k} G_i, H) = (\chi(H) - 1)(nk - 1) + 1 + \sum_{i=1}^{k-1} n_i$ or $R(\bigcup_{i=1}^{k} G_i, H) = R(G_k, H) + \sum_{i=1}^{k-1} n_i$. □

3. Remarks

In Table 1, we present the Ramsey numbers for some combinations of graphs which equal the lower bound of Chvátal and Harary [6]. We can use Theorem 3 to determine many other Ramsey numbers for disjoint unions.
Table 1
The Ramsey number $R(G, H)$

<table>
<thead>
<tr>
<th>$G, H$</th>
<th>$(\chi(H) - 1)(n(G) - 1) + 1$</th>
<th>interval</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_n, W_5$</td>
<td>$3n - 2$</td>
<td>$n \geq 3$</td>
<td>[1]</td>
</tr>
<tr>
<td>$T_n, K_m$</td>
<td>$(n - 1)(m - 1) + 1$</td>
<td>for any $n, m$</td>
<td>[5]</td>
</tr>
<tr>
<td>$C_n, W_m$</td>
<td>$3n - 2$</td>
<td>odd $m, m \geq 5, n \geq \frac{5m - 9}{2}$</td>
<td>[10]</td>
</tr>
<tr>
<td>$S_n, W_m$</td>
<td>$3n - 2$</td>
<td>odd $m, n \geq 3, m \leq 2n - 1$</td>
<td>[7]</td>
</tr>
<tr>
<td>$C_n, C_m$</td>
<td>$2n - 1$</td>
<td>odd $m, 3 \leq m \leq n$</td>
<td>[9]</td>
</tr>
<tr>
<td>$S_{1+n}, C_m$</td>
<td>$m$</td>
<td>even $m, m \geq 2n$</td>
<td>[8]</td>
</tr>
</tbody>
</table>

of graphs $R(\bigcup_{i=1}^{n}G_i, H)$ from the ‘known’ Ramsey numbers $R(G_i, H)$’s, particularly as given in the following corollary.

**Corollary 1.**

1. For odd $m$, $m \leq 5$ and $n \leq 5m - 9/2$, $R(kC_n, W_m) = R(C_n, W_m) + (k - 1)n$.
2. For odd $m$ and $3 \leq m \leq n$, $R(kC_n, C_m) = R(C_n, C_m) + (k - 1)n$.
3. For even $m$ and $m \geq 2n$, $R(kS_{1+n}, C_m) = R(S_{1+n}, C_m) + (k - 1)n$.

Note that these disjoint unions of graphs may consists of different graphs. Thus, Theorem 3 can used to determine $R(k_1C_n \bigcup k_2S_n, W_m)$ for odd $m, m \geq 5$ and $n \geq (5m - 9)/2$, $R(k_1C_n \bigcup k_2T_n \bigcup k_3S_n, W_5)$, and if $n \geq 8$, then $R(k_1C_n \bigcup k_2T_{n-1} \bigcup k_3S_{n-3}, W_5)$, for some integers $k, k_1, k_2$ and $k_3$.

**Acknowledgment**

The authors are thankful to the referees for a number of comments that helped to improve the presentation of the manuscript.

**References**