J. Math. Anal. Appl. 358 (2009) 98–109

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Growth estimates in the Hardy–Sobolev space of an annular domain with applications

H. Meftahi, F. Wielonsky *

Laboratoire de Mathématiques P. Painlevé, UMR CNRS 8524 – Bat. M2, Université des Sciences et Technologies Lille, F-59655 Villeneuve d'Ascq Cedex, France

ARTICLE INFO

Article history:
Received 10 February 2009
Available online 23 April 2009
Submitted by P. Koskela

Keywords:
Hardy–Sobolev space
Annular domain
Interpolation scheme
Inverse problem

ABSTRACT

We give an explicit estimate on the growth of functions in the Hardy–Sobolev space $H^{k,2}(G_s)$ of an annulus. We apply this result, first, to find an upper bound on the rate of convergence of a recovery interpolation scheme in $H^{1,2}(G_s)$ with points located on the outer boundary of G_s. We also apply this result for the study of a geometric inverse problem, namely we derive an explicit upper bound on the area of an unknown cavity in a bounded planar domain from the difference of two electrostatic potentials measured on the boundary, when the cavity is present and when it is not.

1. Introduction

Let $H^2(\mathbb{D})$ denote the Hardy space of functions analytic in the unit disk $\mathbb{D}$ having L^2 boundary values on the unit circle $\mathbb{T}$, and let $H_0^2(\mathbb{C}\setminus s\overline{\mathbb{D}})$, $0<s<1$, be the Hardy space of functions analytic in the complement of $s\overline{\mathbb{D}}$, with L^2 boundary values on $s\mathbb{T}$ and vanishing at infinity. Moreover, let us denote by G_s the annulus $G_s=\mathbb{D}\setminus s\mathbb{D}$. We define the Hardy space $H^2(G_s)$ of the annulus G_s to be the orthogonal direct sum

$$H^2(G_s)=H^2(\mathbb{D})\oplus H_0^2(\mathbb{C}\setminus s\overline{\mathbb{D}}).$$

For equivalent definitions and more properties of the Hardy space of the annulus, we refer the reader to [9,21].

For $m\geqslant 1$, we also define the Hardy–Sobolev space $H^{m,2}(G_s)$ of order m as the subspace of functions f in $H^2(G_s)$ such that the derivatives $f^{(j)}$, $1\leqslant j\leqslant m$, belong to $H^2(G_s)$. The space $H^{m,2}(G_s)$ is a Hilbert space endowed with the scalar product

$$\langle f,g\rangle_{H^{m,2}(G_s)}=\sum_{l=0}^{m}\langle f^{(l)},g^{(l)}\rangle_{L^2(\partial G_s)},$$

where

$$\langle f,g\rangle_{L^2(\partial G)}=\langle f,g\rangle_{L^2(\mathbb{T})}+\langle f,g\rangle_{L^2(s\mathbb{T})}.$$

Consequently, the norm in $H^{m,2}(G_s)$ can be written as

$$\|g\|^2_{H^{m,2}(G_s)}=\|g\|^2_{W^{m,2}(\mathbb{T})}+\|g\|^2_{W^{m,2}(s\mathbb{T})},$$

* Corresponding author.
E-mail addresses: Houcine.Meftahi@math.univ-lille1.fr (H. Meftahi), Franck.Wielonsky@math.univ-lille1.fr (F. Wielonsky).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.04.040

where we use the norms of the Sobolev spaces $W^{m,2}(\mathbb{T})$ on $\mathbb{T}$ and $W^{m,2}(s\mathbb{T})$ on $s\mathbb{T}$, given by

$$\|g\|^2_{W^{m,2}(\mathbb{T})} = \sum_{l=0}^{m} \|g^{(l)}\|^2_{L^2(\mathbb{T})} = \frac{1}{2\pi}\sum_{l=0}^{m}\int \left|g^{(l)}\left(e^{i\theta}\right)\right|^2 d\theta = \sum_{n\in\mathbb{Z}} w_{m,n}|g_n|^2, \tag{1.1}$$

$$\|g\|^2_{W^{m,2}(s\mathbb{T})} = \sum_{l=0}^{m} \|g^{(l)}\|^2_{L^2(s\mathbb{T})} = \frac{1}{2\pi}\sum_{l=0}^{m}\int \left|g^{(l)}\left(se^{i\theta}\right)\right|^2 d\theta = \sum_{n\in\mathbb{Z}} \mu_{m,n}s^{2n}|g_n|^2, \tag{1.2}$$

with $g(z) = \sum_{n\in\mathbb{Z}} g_n z^n$, $z \in G_s$, and

$$w_{m,n} = 1 + n^2 + n^2(n-1)^2 + \cdots + n^2(n-1)^2\cdots(n-m+1)^2, \tag{1.3}$$

$$\mu_{m,n} = 1 + n^2 s^{-2} + \cdots + n^2(n-1)^2\cdots(n-m+1)^2 s^{-2m}. \tag{1.4}$$

Note that $H^{m,2}(G_s)$ admits the set

$$(e_n)_{n\in\mathbb{Z}} = \left(\frac{z^n}{\sqrt{l_{m,n}}}\right)_{n\in\mathbb{Z}}, \quad l_{m,n} = w_{m,n} + \mu_{m,n}s^{2n},$$

as a complete orthonormal set, and consequently, has the following reproducing kernel

$$K_m(x,y) = \sum_{k=-\infty}^{\infty} \frac{x^k \bar{y}^k}{l_{m,n}}. \tag{1.5}$$

Let $0 \leqslant k < m$ be two integers, and consider a function g with a fixed norm in the Hardy–Sobolev space $H^{m,2}(G_s)$. Then, it has been proved in [15] that the Sobolev norm of degree k of g on the inner boundary $s\mathbb{T}$ of the annulus is controlled by the corresponding norm taken on the outer boundary $\mathbb{T}$. Such norm estimates, in the disk or in the annulus, have been applied in [8,15], to obtain stability results for the inverse problem of recovering a Robin coefficient on the non-accessible boundary of a planar domain.

This result complements, in some sense, the Hadamard's three-circle theorem that describes the growth of an analytic function in an annulus from its values on the boundaries.

We give here a version of this property, where we make explicit the dependence with respect to the radius s of the inner boundary of the annulus. This will be important for one of the applications we have in mind.

Theorem 1.1. *Let $0 \leqslant k < m$ be two integers. Assume g is a function in $H^{m,2}(G_s)$ with $\|g\|_{W^{k,2}(\mathbb{T})} \leqslant 1$. Then, we have*

$$\|g\|_{W^{k,2}(s\mathbb{T})} \leqslant \left(\frac{2}{e\log(1/\|g\|_{W^{k,2}(\mathbb{T})})}\right)^{m-k} \left[\left(es|\log s|\right)^{m-k}\|g\|_{W^{m,2}(s\mathbb{T})} + (m-k)^{m-k}\right]. \tag{1.6}$$

Note that the authors of [15] derive their results in general weighted Hardy spaces, see [15, Proposition 7]. Note also that this kind of results extends to the doubly connected case the estimates established in [7] for subsets of the boundary of the disk $\mathbb{D}$.

In Section 2, we display the proof of Theorem 1.1, and in each of the two subsequent sections, we describe a different application. Namely, in the spirit of [7], we consider, in Section 3, a given function in $H^{1,2}(G_s)$ and a sequence $(f_n)_{n\geqslant 0}$ of functions in $H^{1,2}(G_s)$ of minimal norm, interpolating f on points of the outer boundary $\mathbb{T}$ of G_s. We show that the scheme is convergent with a rate which is inversely proportional to the logarithm of the maximal distance between the points of $\mathbb{T}$ and the points of the interpolation scheme, see Theorem 3.1.

As a second application, we study in Section 4 the geometric inverse problem of recovering a cavity in a bounded planar domain from the measurements of electrostatic potentials corresponding to a given flux on the outer boundary of the domain. More precisely, we get an upper estimate on the area of the unknown cavity, granted some a priori hypotheses on the regularity of the conformal map which sends the domain onto an annulus, see Theorem 4.1. The proof relies on the use of conformal maps and the previously established growth estimates in Hardy–Sobolev spaces of the annulus.

2. Growth estimates in the Hardy–Sobolev space of the annulus

In this section, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. We follow the scheme of proof of [15, Proposition 7]. We want to estimate

$$\|g\|^2_{W^{k,2}(s\mathbb{T})} = \sum_{n\leqslant -N} \mu_{k,n}s^{2n}|g_n|^2 + \sum_{n=-N+1}^{\infty} \mu_{k,n}s^{2n}|g_n|^2 = \sigma_1 + \sigma_2, \quad \text{say.}$$

On one hand, we have

$$\sigma_1 \leqslant \sup_{n\leqslant -N}\left(\frac{\mu_{k,n}}{\mu_{m,n}}\right)\|g\|^2_{W^{m,2}(s\mathbb{T})},$$

and, in view of the definition (1.4) of $\mu_{m,n}$,

$$\frac{\mu_{k,n}}{\mu_{m,n}} \leqslant \frac{s^{2(m-k)}}{(n-k)^2\cdots(n-m+1)^2} \leqslant \frac{s^{2(m-k)}}{(N+k)^{2(m-k)}},$$

where the last inequality holds true as soon as $0 < N+k$. Hence,

$$\sigma_1 \leqslant \frac{s^{2(m-k)}}{(N+k)^{2(m-k)}}\|g\|^2_{W^{m,2}(s\mathbb{T})}. \tag{2.1}$$

On the other hand, from the definitions (1.3)–(1.4), we see immediately that $\mu_{k,n} \leqslant s^{-2k}w_{k,n}$. Hence,

$$\sigma_2 \leqslant \sum_{n=-N+1}^{\infty} s^{-2(N+k-1)}w_{k,n}|g_n|^2 \leqslant s^{-2(N+k-1)}\epsilon^2. \tag{2.2}$$

Let us choose $N+k = 1 + \lfloor \log\epsilon/2\log s\rfloor > 0$. From (2.1), (2.2), and the inequalities

$$\frac{\log\epsilon}{2\log s} \leqslant N+k \leqslant 1 + \frac{\log\epsilon}{2\log s},$$

it follows that

$$\|g\|^2_{W^{k,2}(s\mathbb{T})} \leqslant \frac{1}{(\log\|g\|_{W^{k,2}(\mathbb{T})})^{2(m-k)}}\left[(2s\log s)^{2(m-k)}\|g\|^2_{W^{m,2}(s\mathbb{T})} + \|g\|_{W^{k,2}(\mathbb{T})}\left(\log\|g\|_{W^{k,2}(\mathbb{T})}\right)^{2(m-k)}\right]. \tag{2.3}$$

Using the fact that $x|\log x|^n \leqslant (n/e)^n$ for $x \in [0,1]$, we deduce that

$$\|g\|^2_{W^{k,2}(s\mathbb{T})} \leqslant \frac{1}{(\log\|g\|_{W^{k,2}(\mathbb{T})})^{2(m-k)}}\left[(2s\log s)^{2(m-k)}\|g\|^2_{W^{m,2}(s\mathbb{T})} + \left(\frac{2(m-k)}{e}\right)^{2(m-k)}\right],$$

and (1.6) follows from taking square roots. □

3. First application: a convergent interpolation scheme in $H^{1,2}(G_s)$

In this section, we study an interpolation scheme to recover a function in $H^{1,2}(G_s)$ from its values at some points on the outer boundary of the annulus G_s. A similar scheme has been studied previously in the case of a disk, see [7]. It consists in the following. Let $S_n = \{x_1,\dots,x_n\}$ be a set of n distinct points on $\mathbb{T}$. As usual, we will say that $f_n \in H^{1,2}(G_s)$ interpolates $f \in H^{1,2}(G_s)$ on S_n if

$$\forall i \in \{1,\dots,n\},\quad f_n(x_i) = f(x_i). \tag{3.1}$$

Now, we consider a nested sequence of sets

$$S_1 \subset S_2 \subset \cdots$$

and we set $S = \bigcup_n S_n$. Here, we assume that $\overline{S} = \mathbb{T}$. Of course, then, S will be a uniqueness set, meaning that two distinct functions in $H^{1,2}(G_s)$ cannot agree on S. Indeed, it is known that a function in the Hardy space $H^2(G_s)$ which vanishes on a subset of the boundary of G_s of positive Lebesgue measure is identically zero, see [11].

Condition (3.1) does not determine f_n uniquely. Among all the functions in $H^{1,2}(G_s)$ satisfying (3.1), we shall pick the only one with minimal norm. To perform this, we may decompose the space $H^{1,2}(G_s)$ as $H^{1,2}(G_s) = Z_n \oplus U_n$ where Z_n is the closed subspace of functions vanishing on S_n and U_n is its orthogonal complement. Then, we define $f_{S_n} = \Pi_n(f)$ where Π_n denotes the orthogonal projection on U_n, and get in this way the so-called minimum interpolation sequence (f_{S_n}) to f with respect to the scheme of points $(S_n)_{n\geqslant 1}$. From general results on Hilbert spaces, it is known that f_{S_n} converges to f in $H^{1,2}(G_s)$ hence converges uniformly in $\overline{G_s}$, see [7, Section 2] for details. From a practical point of view, it is also important to note that f_{S_n} can be computed from the values $f(x_i)$, $i = 1,\dots,k$, by simply solving a linear system of equations. Actually, f_{S_n} admits the following expressions

$$f_{S_n}(x) = \sum_{i=1}^{n}\lambda_{i,n}K_1(x,x_i),$$

where the vector $\lambda_n = (\lambda_{1,n}, \ldots, \lambda_{n,n})^T$ is any solution of the linear system $A_n\lambda_n = B_n$ with B_n the vector of values $(f(x_i))_{i=1,\ldots,n}$ and A_n the Gram matrix of the functions $K_1(x, x_i)$ where $K_1(x, y)$ denotes the reproducing kernel of $H^{1,2}(G_s)$, see (1.5).

We now state our result which estimates the rate of convergence of the interpolating sequence f_{S_n} to the limit function f.

Theorem 3.1. *Consider a function $f \in H^{1,2}(G_s)$ with $\|f\|_{H^{1,2}(G_s)} \leqslant 1$. Assume $(S_n)_{n\geqslant 1}$ is a sequence of interpolation sets with $\overline{S} = \mathbb{T}$ and set $h_n = \sup_{x\in\mathbb{T}} d(x, S_n)$. Then, for any $\epsilon > 0$, there exists $N > 0$ large enough such that for any $n \geqslant N$, we have that*

$$\|f - f_{S_n}\|_{H^2(G_s)} \leqslant \frac{4+\epsilon}{e\log(1/h_n)}.$$

Proof. Set $g_n = f - f_{S_n}$. We write the points x_k of S_n as $x_k = e^{i\theta_k}$ and we consider the covering of $\mathbb{T}$ with n intervals $I_K = [\theta_k^-, \theta_k^+]$ having at most one endpoint in common, each I_k containing θ_k. We may assume that $d(\theta_k^-, \theta_k)$ and $d(\theta_k^+, \theta_k)$ are less than or equal to h_n. We have

$$\int_{\mathbb{T}} |g_n|^2\, d\theta = \sum_{k=1}^{n} \int_{I_k} |g_n|^2\, d\theta.$$

Moreover, for $e^{i\gamma}$ in I_k,

$$\left|g_n\left(e^{i\gamma}\right)\right|^2 \leqslant \left(\int_{\theta_k}^{\gamma} \left|g_n'\left(e^{it}\right)\right| dt\right)^2 \leqslant h_n \int_{\theta_k}^{\gamma} \left|g_n'\left(e^{it}\right)\right|^2 dt,$$

hence

$$\int_{\theta_k}^{\theta_k^+} |g_n|^2\, d\theta \leqslant h_n \int_{\theta_k}^{\theta_k^+} \left|g_n'\right|^2 d\theta \int_{\theta_k}^{\theta_k^+} d\theta \leqslant h_n^2 \int_{\theta_k}^{\theta_k^+} \left|g_n'\right|^2 d\theta.$$

Since a similar inequality holds for the integral from θ_k^- to θ_k, we get that

$$\int_{I_k} |g_n|^2\, d\theta \leqslant h_n^2 \int_{I_k} \left|g_n'\right|^2 d\theta,$$

and consequently,

$$\|g_n\|_{L^2(\mathbb{T})}^2 \leqslant h_n^2 \|g_n\|_{H^{1,2}(G_s)}^2 \leqslant h_n^2, \tag{3.2}$$

where the last inequality follows from the fact that g_n is a projection of f which is assumed to be of norm less than 1 in $H^{1,2}(G_s)$. Using inequality (1.6) with $k = 0$ and $m = 1$, the previous inequality, and the fact that $\|g_n\|_{H^{1,2}(s\mathbb{T})} \leqslant 1$, we get that

$$\|g_n\|_{L^2(s\mathbb{T})} \leqslant \frac{2}{e\log 1/\|g_n\|_{L^2(\mathbb{T})}}\left(\|g_n\|_{H^{1,2}(s\mathbb{T})} + 1\right) \leqslant \frac{4}{e\log 1/h_n},$$

which implies together with (3.2) that

$$\|g_n\|_{H^2(G_s)}^2 \leqslant \left(\frac{4}{e\log 1/h_n}\right)^2 + h_n^2,$$

from which the assertion in the theorem follows. □

Note that the rate of convergence of some approximation scheme in the Hardy–Sobolev space of the unit disk with a constraint on a subarc of $\mathbb{T}$ has been obtained in [5], see in particular [5, Theorem 6.2]. Recovering of functions in the Hardy–Sobolev space of a horizontal strip was analysed in [16]. In this respect, results about recovery of functions in a more general setting can be found in [17], see also the comprehensive monograph [18].

4. Second application: estimating the area of a cavity from boundary measurements

We consider a planar bounded simply-connected domain Ω with boundary Γ, containing a cavity, that is a connected, simply-connected open set D with boundary $\gamma \subset \Omega$. From a physical point of view, we assume that Ω is an electrically conducting body, of constant conductivity 1, while the cavity is perfectly insulating, that is of conductivity 0. Our aim is to give an estimate of the size of the cavity from comparing measurements performed on the boundary Γ of Ω, first when the domain is safe, that is there is no cavity, and second when a cavity is present. The procedure consists in prescribing the same flux of current φ on the boundary Γ, and measuring the corresponding electrostatic potentials $\widetilde{u}$ and u, in both cases. When there is no cavity, the potential $\widetilde{u}$ satisfies the Neumann problem

$$\Delta\widetilde{u} = 0, \quad \text{in } \Omega, \qquad \frac{\partial\widetilde{u}}{\partial n} = \varphi, \quad \text{on } \Gamma, \tag{4.1}$$

and when the cavity is present, the situation is modeled by the following equations,

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \setminus \overline{D}, \\ \dfrac{\partial u}{\partial n} = \varphi, & \text{on } \Gamma, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \gamma, \end{cases} \tag{4.2}$$

where $\partial/\partial n$ denotes the partial derivative with respect to the outer normal unit vector. We assume that the boundaries Γ and γ of the domain are of class $C^{m,\alpha}$, with $m \geqslant 1$, $0 < \alpha < 1$, and that the flux φ is $C^{m-1,\alpha}$ on Γ. In order that a solution to (4.1) or (4.2) exists, the compatibility condition

$$\int_{\Gamma} \varphi(z)\, ds(z) = 0, \tag{4.3}$$

must hold. In this case, a solution to (4.1) or (4.2) indeed exists and is determined up to an additive constant. In the sequel, we impose the additional normalization conditions,

$$\int_{\Gamma} \widetilde{u}(z)\, ds(z) = 0, \qquad \int_{\Gamma} u(z)\, ds(z) = 0,$$

to ensure uniqueness of a solution $\widetilde{u}$ or u. With the smoothness assumptions made above, it is known from classical regularity results that the solutions $\widetilde{u}$ and u are $C^{m,\alpha}$ on $\overline{\Omega} \setminus D$, see [2].

Let us now turn to the geometric inverse problem of recovering the cavity. Concerning identifiability, one boundary measurement $u_{|\Gamma}$ determines the cavity D, granted some smoothness assumptions on γ, see [3]. Note that this is in contrast with the problem of recovering a 1-dimensional crack, since two boundary measurements are necessary in this case. As is the rule for this type of inverse problems, only weak (i.e. logarithmic) stability results hold true. This can be seen as a motivation for obtaining less precise, though still interesting, information on the unknown cavity.

Our result aims at giving such information, namely on the size of the cavity. For more information on this type of questions, one may consult [4,10,14].

Our study is based on using conformal maps and norm estimates in Hardy–Sobolev spaces of the annulus.

In order to use the norm estimates as established in Theorem 1.1, we will need to consider a conformal mapping ψ from an annulus $G_s = \mathbb{D} \setminus s\mathbb{D}$, $0 < s < 1$, where $\mathbb{D}$ denotes the unit disk, onto the domain $\Omega \setminus D$. We recall that s is uniquely determined by $\Omega \setminus D$. It is the inverse of a conformal invariant, the so-called conformal radius of $\Omega \setminus D$. A well-known result of Warschawski [19, Theorems 3.5 and 3.6] asserts that a conformal mapping from $\mathbb{D}$ onto the inner domain of a Jordan curve of class $C^{m,\alpha}$, $m \geqslant 1$, $0 < \alpha < 1$, has derivatives, up to order m, which admits continuous extensions to $\overline{\mathbb{D}}$. Moreover, the first derivative is non-vanishing on the unit circle $\mathbb{T}$. Reasoning as in the proof of [6, Proposition 4.2], one may extend the Warschawski result to the doubly connected annulus G_s. This implies in particular that the moduli of the derivatives of ψ, up to order m, are bounded above on the closure of G_s, and that $|\psi'|$ is also bounded below. For our analysis to go through, we actually need to restrict ourselves to domains such that these bounds are absolute constants. Consequently, given $m \geqslant 1$ and two real numbers $0 < \lambda < \Lambda$, we introduce the class of "admissible" domains $\Omega \setminus D$ such that the following property holds true:

***H*(m, λ, Λ).** *Any conformal map ψ from the annulus G_s onto $\Omega \setminus D$, where $1/s \geqslant 1$ is the conformal radius of $\Omega \setminus D$, mapping the outer boundary $\mathbb{T}$ of G_s to the outer boundary Γ of $\Omega \setminus D$, satisfies*

$$\lambda \leqslant |\psi'(z)| \leqslant \Lambda, \quad z \in \overline{G}_s. \tag{4.4}$$

Furthermore, if $m \geqslant 2$, the higher-order derivatives of the map ψ also satisfies

$$|\psi^{(l)}(z)| \leqslant \Lambda, \quad z \in \overline{G}_s,\ 2 \leqslant l \leqslant m. \tag{4.5}$$

Note that if (4.4)–(4.5) holds true for some conformal map, then it holds true for any conformal map from G_s to $\Omega \setminus D$ sending the outer boundary $\mathbb{T}$ of G_s onto the outer boundary Γ of Ω. This is a simple consequence of the fact that the only automorphisms of G_s, sending the outer boundary $\mathbb{T}$ onto itself, are the rotations.

Generally speaking, property $\boldsymbol{H(m,\lambda,\Lambda)}$ means that the distortion of the conformal map ψ is controlled by the two constants λ and Λ. As far as we know, there is no simple geometric characterisation of this property. The only reference, related to this problem, we know of, is [22], which shows, among other results, that the derivative of conformal maps from the unit disk to nearly circular regions cannot be too far from 1. Note that this assertion pertains to the simply connected case. It would be interesting to have an analog for the doubly connected case.

Throughout we assume that the flux is generated by two electrodes applied on parts of the boundary Γ. Hence, we consider two non-negative functions $\eta_1, \eta_2 \in C^{m-1,\alpha}(\Gamma)$ whose supports are disjoint subarcs Γ_1 and Γ_2 of Γ, with

$$\int_{\Gamma_j} \eta_j = 1, \quad j = 1, 2. \tag{4.6}$$

Then, the current density on Γ is taken as

$$\varphi = \eta_1 - \eta_2. \tag{4.7}$$

Note that the total flux $\int_\Gamma \varphi(s)\,ds$ of φ on the boundary Γ vanishes, in accordance with the assumption (4.3).

Next, consider the harmonic function $\widetilde{u}$ in Ω, which represents the potential corresponding to the flux φ on Γ when there is no cavity in Ω. Because of (4.3), it has a harmonic conjugate $\widetilde{v}$ in Ω, defined up to an additive constant, which is obtained by integrating the flux $\varphi(s)$ with respect to arc-length on the boundary Γ. Let

$$\widetilde{f}(z) = \widetilde{u}(z) + i\widetilde{v}(z), \quad z \in \Omega. \tag{4.8}$$

It will be shown that $\widetilde{f}$ is actually a conformal map from Ω to its image, see Lemma 4.2. Moreover, from the smoothness assumptions on the contour Γ and the flux φ, we know that $\widetilde{u} \in C^{1,\alpha}(\Gamma)$, and $\partial\widetilde{v}/\partial s = \partial\widetilde{u}/\partial n = \varphi \in C^{0,\alpha}(\Gamma)$. Hence, $\widetilde{v} \in C^{1,\alpha}(\Gamma)$ as well, which shows in particular that $\widetilde{f}(\Gamma)$ is of class $C^{1,\alpha}$. Applying the Warschawski theorem in the simply connected domain Ω, we obtain that $\widetilde{f}'$ admits a non-vanishing continuous extension to $\overline{\Omega}$. Consequently, setting

$$m_{\widetilde{f}'} = \inf_{z\in\overline{\Omega}} \left|\widetilde{f}'(z)\right|, \qquad M_{\widetilde{f}'} = \sup_{z\in\overline{\Omega}} \left|\widetilde{f}'(z)\right|, \tag{4.9}$$

we have that $0 < m_{\widetilde{f}'} < M_{\widetilde{f}'} < \infty$. Note that the two parameters $m_{\widetilde{f}'}$ and $M_{\widetilde{f}'}$ only depend on the safe domain Ω and are determined from the data on the boundary since the function $\widetilde{f}$ can be reconstructed from these data, and moreover, the above-mentioned Warschawski theorem ensures that

$$\widetilde{f}'(z) = \lim_{\zeta\to z} \frac{\widetilde{f}(z) - \widetilde{f}(\zeta)}{z-\zeta} \quad \text{as } \zeta \to z,\ z, \zeta \in \Gamma. \tag{4.10}$$

In particular, (4.10) could be used to estimate $m_{\widetilde{f}'}$ and $M_{\widetilde{f}'}$ numerically.

Finally, we denote by $|A|$ the planar Lebesgue measure of a measurable subset A of the plane. Since, from a practical point of view, it seems reasonable that the area of the unknown cavity D is not too large, relatively to that of the domain Ω, we will assume throughout that there exists an absolute constant $\rho < 1$ such that $|D|/|\Omega| \leqslant \rho$.

We are now in a position to state our main result.

Theorem 4.1. *Consider an admissible domain $\Omega \setminus D$ satisfying the property $\boldsymbol{H(m,\lambda,\Lambda)}$. Assume that the difference of potentials u and $\widetilde{u}$ is not too large, more precisely that*

$$\sigma := \|u - \widetilde{u}\|_{L^2(\Gamma)}/\sqrt{\lambda} < 1.$$

Then, the area of the cavity D is bounded above as follows,

$$|D| \leqslant \frac{C_1}{|\log\sigma|}, \quad \text{if } m = 1, \tag{4.11}$$

while,

$$|D| \leqslant \sup_{0<\delta<1} \min\left(\frac{C_m}{\delta^{m-1}|\log\sigma|^m}, \delta^2|\Omega|\right), \quad \text{if } m > 1. \tag{4.12}$$

In (4.11)–(4.12)*, the constant C_m, $m \geqslant 1$, is explicitly given by*

$$C_m = \sqrt{\frac{2\pi\Lambda}{\lambda}\frac{M_{\widetilde{f}'}}{m^2_{\widetilde{f}'}}}\left(\frac{2}{e}\right)^m \left(\frac{2m\pi\lambda^2(1+\Lambda)^{m-\frac{1}{2}}}{|\Omega|(1-\rho)}\left(\frac{\Lambda}{\lambda}\right)^{2m} C'_m\|\varphi\|_{W^{m-1,2}(\Gamma)} + m^m\right),$$

where C'_m is a computable constant that depends only on m (e.g. $C'_1 = 1$, $C'_2 = \sqrt{5}/2$).

4.1. Proof of Theorem 4.1

Step 1. Building a conformal map from the data and using the Green formula. We start with a lemma.

Lemma 4.2. *Assume the flux is given by* (4.6)–(4.7)*. Then, the analytic function* $\widetilde{f}$ *defined by* (4.8) *maps conformally the domain* Ω *onto its image* $\Omega_0 := \widetilde{f}(\Omega)$.

The assertion in the lemma has already appeared in the framework of inverse problems, and has also been extended to more general situations. For completeness, let us give a proof.

Proof. The Rado theorem [20] states that a harmonic function h in a disk, continuous on the boundary, with a critical point z_0 inside the disk, assumes the value $h(z_0)$ at least in four distinct points of the boundary. Since $\widetilde{v}$ is not constant, we have

$$\inf_{z\in\overline{\Omega}} \widetilde{v}(z) < \widetilde{v}(z) < \sup_{z\in\overline{\Omega}} \widetilde{v}(z), \quad z \in \Omega,$$

by the maximum principle. Since such values are taken exactly twice on Γ, we deduce from the Rado theorem applied with $\widetilde{v}$ on Ω (which is possible by considering a Riemann map from the disk to the simply-connected domain Ω) that $\widetilde{v}$ has no critical point inside Ω. Then, for a given point $z_0 \in \Omega$, there is only one level curve passing through z_0, and it is an analytic arc joining two distinct points of the boundary. First, from what precedes, we know that on this arc, the derivative $\partial\widetilde{u}/\partial s = \partial\widetilde{v}/\partial n$ never vanishes, hence is of constant sign. Consequently, each value of $\widetilde{u}$ is taken only once on this arc. Second, two level curves of $\widetilde{v}$ can only correspond to different values since otherwise, $\widetilde{v}$ would be constant inside the domain delimited by these two curves, hence also in Ω. The injectivity of $\widetilde{f}$, and the fact that it is a conformal map, follow from these two assertions. $\square$

Now, we consider a cavity D and its image D_0 by the map $\widetilde{f}$. The area of D satisfies

$$|D| = \int_D dx\,dy = \int_{D_0} \left|\left(\widetilde{f}^{-1}\right)'(z)\right|^2 dx\,dy \leqslant m_{\widetilde{f}'}^{-2} \int_{D_0} dx\,dy,$$

where $m_{\widetilde{f}'}$ has been defined in (4.9). Consider the harmonic function $\widetilde{v}_0$ in D_0 such that $\widetilde{v}_0 = \widetilde{v} \circ \widetilde{f}^{-1}$. From the definition of $\widetilde{f}$, we have that $\widetilde{v}_0(z) = y$, where $z = x + iy$. Hence, applying the Green formula on D_0, we get

$$\int_{D_0} dx\,dy = \int_{D_0} \left|\nabla\widetilde{v}_0(z)\right|^2 dx\,dy = \int_{\partial D_0} \widetilde{v}_0(z)\frac{\partial\widetilde{v}_0}{\partial n}(z)\,ds.$$

Let $v_0 = v \circ \widetilde{f}^{-1}$ with v the harmonic conjugate of u in $\Omega \setminus D$. Since D is insulating, v is constant on γ, and since we are free to choose this constant, we may assume that $v = 0$ on γ, or equivalently $v_0 = 0$ on ∂D_0. Hence, the last integral can be rewritten as

$$\int_{\partial D_0} (\widetilde{v}_0 - v_0)(z)\frac{\partial\widetilde{v}_0}{\partial n}(z)\,ds \leqslant \int_{\partial D_0} \left|(\widetilde{v}_0 - v_0)(z)\right| ds \leqslant M_{\widetilde{f}'} \int_{\gamma} \left|(\widetilde{v} - v)(z)\right| ds,$$

recall (4.9) for the definition of $M_{\widetilde{f}'}$. Applying the Schwarz inequality to the last integral, we get

$$\int_{\gamma} \left|(\widetilde{v} - v)(z)\right| ds \leqslant \text{length}(\gamma)^{1/2}\|\widetilde{v} - v\|_{L^2(\gamma)} \leqslant \text{length}(\gamma)^{1/2}\|\widetilde{f} - f\|_{L^2(\gamma)},$$

where $f = u + iv$ on $\Omega \setminus D$. Summing up, we obtain the following upper bound for the area of the cavity,

$$|D| \leqslant \text{length}(\gamma)^{1/2}\frac{M_{\widetilde{f}'}}{m_{\widetilde{f}'}^2}\|\widetilde{f} - f\|_{L^2(\gamma)} \leqslant \left(\frac{2\pi}{\lambda}\right)^{1/2}\frac{M_{\widetilde{f}'}}{m_{\widetilde{f}'}^2}\|\widetilde{f} - f\|_{L^2(\gamma)}, \tag{4.13}$$

where the last inequality comes from

$$\text{length}(\gamma) = \int_{\gamma} ds = \int_{s\mathbb{T}} \left|\left(\psi^{-1}\right)'(s)\right| ds \leqslant \frac{2\pi s}{\lambda} \leqslant \frac{2\pi}{\lambda}.$$

Step 2. Applying a norm inequality in the Hardy–Sobolev space $H^{m,2}(G_s)$ of the annulus. As explained in the introduction, to use the result in Section 2, we need to transport our original problem onto the annulus G_s, where s^{-1} is the conformal radius of the domain $\Omega \setminus D$. We thus define the following functions on $\overline{G}_s$,

$$\widetilde{u}_1 = \widetilde{u} \circ \psi, \qquad \widetilde{f}_1 = \widetilde{f} \circ \psi, \qquad u_1 = u \circ \psi, \qquad f_1 = f \circ \psi,$$

which are obtained from the corresponding functions $\widetilde{u}$, $\widetilde{f}$, u, f on $\overline{\Omega} \setminus D$ through the map ψ. Since $\widetilde{u}$ is a solution to (4.1), the function $\widetilde{u}_1$ satisfies

$$\Delta \widetilde{u}_1(\zeta) = 0, \quad \zeta \in \mathbb{D}, \qquad \frac{\partial \widetilde{u}_1}{\partial n}(\zeta) = \varphi_1(\zeta) := (\varphi \circ \psi)(\zeta)\left|\psi'(\zeta)\right|, \quad \zeta \in \mathbb{T}. \tag{4.14}$$

Similarly, since u is a solution to (4.2), the function u_1 is a solution to the Neumann problem in G_s,

$$\begin{cases} \Delta u_1(\zeta) = 0, & \zeta \in G_s, \\ \dfrac{\partial u_1}{\partial n}(\zeta) = \varphi_1(\zeta), & \zeta \in \mathbb{T}, \\ \dfrac{\partial u_1}{\partial n}(\zeta) = 0, & \zeta \in s\mathbb{T}. \end{cases} \tag{4.15}$$

Note that, since φ is $C^{m-1,\alpha}$ on Γ and ψ admits a $C^{m,\alpha}$ extension to $\overline{G}_s$, the flux φ_1 is $C^{m-1,\alpha}$ on $\mathbb{T}$, and in particular belongs to $W^{m-1,2}(\mathbb{T})$. Let

$$\varphi_1\left(e^{i\theta}\right) = \sum_{k \neq 0} \alpha_k e^{ik\theta},$$

be its Fourier expansion. The coefficient α_0 is zero because the total flux of φ_1 on $\mathbb{T}$ vanishes, recall the compatibility condition (4.3). Then, it is not difficult to check that the potentials $\widetilde{u}_1$ and u_1 admit the following expressions in polar coordinates,

$$\widetilde{u}_1(r, \theta) = \sum_{k \neq 0} \operatorname{sgn}(k) \frac{\alpha_k}{k} r^{|k|} e^{ik\theta},$$

$$u_1(r, \theta) = \sum_{k \neq 0} \frac{\alpha_k}{k(1 - s^{2k})}\left(r^k + \frac{s^{2k}}{r^k}\right) e^{ik\theta}.$$

From the previous expressions for $\widetilde{u}_1$ and u_1, one deduces the following expansion for $f_1 - \widetilde{f}_1$,

$$(f_1 - \widetilde{f}_1)(z) = 2 \sum_{k > 0} \frac{s^{2k}}{k(1 - s^{2k})}\left(\alpha_k z^k + \frac{\overline{\alpha}_k}{z^k}\right).$$

Hence, the norm of $f_1 - \widetilde{f}_1$ in the space $L^2(s\mathbb{T})$ satisfies

$$\|f_1 - \widetilde{f}_1\|^2_{L^2(s\mathbb{T})} \leqslant \frac{4s^2}{(1 - s^2)^2} \|\varphi_1\|^2_{L^2(\mathbb{T})}, \tag{4.16}$$

and for the derivatives, one checks, after some calculations, that

$$\left\|(f_1 - \widetilde{f}_1)^{(l+1)}\right\|^2_{L^2(s\mathbb{T})} \leqslant \frac{4}{s^{2l}(1 - s^2)^2}(l + 1)^2 \left\|\varphi_1^{(l)}\right\|^2_{L^2(\mathbb{T})}, \quad l \geqslant 0.$$

Summing up (4.16) and the previous inequalities for $l = 0, \dots, m - 1$, we get for the Sobolev norm of order m on $s\mathbb{T}$,

$$\|f_1 - \widetilde{f}_1\|^2_{W^{m,2}(s\mathbb{T})} \leqslant \frac{4m^2}{s^{2m-2}(1 - s^2)^2} \|\varphi_1\|^2_{W^{m-1,2}(\mathbb{T})}. \tag{4.17}$$

Note that this explicit inequality is an instance in the annulus of classical boundary regularity results for the solutions of linear elliptic equations in general domains, see e.g. [12, Chapter 8] or [23, Chapter 5].

Now, we estimate $\|f_1 - \widetilde{f}_1\|_{L^2(s\mathbb{T})}$ with respect to $\|f_1 - \widetilde{f}_1\|_{L^2(\mathbb{T})}$ by applying Proposition 1.1 in the Hardy–Sobolev space $H^{m,2}(G_s)$. Choosing $k = 0$, and assuming that

$$\epsilon_1 := \|f_1 - \widetilde{f}_1\|_{L^2(\mathbb{T})} \leqslant 1, \tag{4.18}$$

we get from (1.6), (4.17), and the fact that $es|\log s| \leqslant 1$, $s \in [0, 1]$, that

$$\|f_1 - \widetilde{f}_1\|_{L^2(s\mathbb{T})} \leqslant \left(\frac{2}{e|\log \epsilon_1|}\right)^m \left(\frac{2m}{s^{m-1}(1 - s^2)} \|\varphi_1\|_{W^{m-1,2}(\mathbb{T})} + m^m\right). \tag{4.19}$$

Since $\Omega \setminus D$ is an admissible domain satisfying the hypothesis $\boldsymbol{H(m,\lambda,\Lambda)}$, we have that

$$\epsilon_1^2 = \|f_1 - \widetilde{f}_1\|^2_{L^2(\mathbb{T})} \leqslant \Big(\sup_{s\in\Gamma}\big|\big(\psi^{-1}\big)'(s)\big|\Big)\|f - \widetilde{f}\|^2_{L^2(\Gamma)} \leqslant \lambda^{-1}\|u - \widetilde{u}\|^2_{L^2(\Gamma)}. \tag{4.20}$$

In the last inequality, we have assumed that $v - \widetilde{v} = 0$ on Γ. Indeed, since the same flux φ is applied on the outer boundary of Ω and $\Omega \setminus D$, $v - \widetilde{v}$ is constant there and the freedom we have on $\widetilde{v}$, which is determined up to a constant, allows one to make this difference equal to zero. From (4.20), we see that the assumption (4.18) on ϵ_1 is granted as soon as

$$\|u - \widetilde{u}\|^2_{L^2(\Gamma)} \leqslant \lambda. \tag{4.21}$$

Step 3. Estimate of the area of the cavity in the original domain. In this final step, we get our sought upper bound on the area of the unknown cavity by putting together (4.13) and (4.19). Before doing that, we need two preliminary results.

First, we show that the norm $\|\varphi_1\|_{W^{m-1,2}(\mathbb{T})}$ can be bounded by $\|\varphi\|_{W^{m-1,2}(\Gamma)}$ times a constant that depends on the conformal map ψ and its derivatives up to order m. When $m = 1$, we have

$$\|\varphi_1\|_{L^2(\mathbb{T})} \leqslant \Lambda^{1/2}\|\varphi\|_{L^2(\Gamma)}. \tag{4.22}$$

When $m \geqslant 2$, the Sobolev norm of order $m-1$ of φ_1 involves the derivatives $\partial^n\varphi_1/\partial s^n$, $0 \leqslant n \leqslant m-1$. Since

$$\varphi_1(\zeta) = \frac{\partial u_1}{\partial n}(\zeta) = \left(\frac{\partial u}{\partial n}\circ\psi\right)(\zeta)\big|\psi'(\zeta)\big|,$$

the Faa' Di Bruno formula for the nth derivative of a composite function tells us that

$$\frac{\partial^n\varphi_1}{\partial s^n}(\zeta) = \sum \frac{(n+1)!}{k_1!\cdots k_{n+1}!}\frac{\partial^k\varphi}{\partial s^k}\big(\psi(\zeta)\big)\big(\big|\psi'(\zeta)\big|\big)^{k_1}\cdots\left(\frac{1}{n!}\frac{\partial^n|\psi'|}{\partial s^n}(\zeta)\right)^{k_{n+1}},$$

where $k+1 = k_1 + \cdots + k_{n+1}$, and the sum ranges over the non-negative integers $k_1,\dots,k_{n+1}$ such that

$$k_1 + 2k_2 + \cdots + (n+1)k_{n+1} = n+1.$$

To estimate the lth derivative of $|\psi'|$, $0 \leqslant l \leqslant n$, we may write

$$\frac{\partial^l|\psi'|}{\partial s^l} = \frac{\partial^l}{\partial s^l}\big((\psi'\overline{\psi'})^{1/2}\big),$$

which can be also expanded by the Faa' Di Bruno formula. This will lead to an expression involving the derivatives of $\psi'\overline{\psi'}$ up to order l in the numerator and the $(2l-1)$th power of $|\psi'|$ in the denominator. Now, using the Leibniz formula, we get that

$$\frac{\partial^l}{\partial s^l}(\psi'\overline{\psi'}) = \sum_{j=0}^{l}\binom{l}{j}\frac{\partial^j\psi'}{\partial s^j}\frac{\partial^{l-j}\overline{\psi'}}{\partial s^{l-j}},$$

whose modulus can be bounded by the modulus of the derivatives $\partial^j\psi'/\partial s^j$, $0 \leqslant j \leqslant l \leqslant m-1$. Since ψ' is analytic in G_s and its derivative admits a continuous extension to $\mathbb{T}$ (when $m \geqslant 2$), we have that

$$\frac{\partial\psi'}{\partial s}\big(e^{i\theta}\big) = iz\psi''(z), \quad z = e^{i\theta},$$

see e.g. [11, Theorem 3.11] for a version in the Hardy space $H^1(\mathbb{D})$. Similarly for the derivatives of order n, $1 \leqslant n \leqslant m-1$, we obtain

$$\frac{\partial^n\psi'}{\partial s^n}\big(e^{i\theta}\big) = P^n(\psi')(z), \quad z = e^{i\theta},$$

where P denotes the differential operator $P(f)(z) = izf'(z)$. This shows that the modulus of the derivatives $\partial^n\psi'/\partial s^n$, $0 \leqslant n \leqslant m-1$, can, in turn, be bounded by the modulus of the derivatives $\psi^{(n)}$, $1 \leqslant n \leqslant m$, and consequently by an expression that depends only on the constant Λ in (4.4)–(4.5). From the above discussion, it can be proved that

$$\left|\frac{\partial^n\varphi_1}{\partial s^n}(\zeta)\right| \lesssim (1+\Lambda)^n\left(\frac{\Lambda}{\lambda}\right)^{2n}\max_{0\leqslant j\leqslant n}\left(\left|\frac{\partial^j\varphi}{\partial s^j}\right|\right)\big|\psi'(\zeta)\big|,$$

where the symbol $\lesssim$ means an inequality up to a constant in the right-hand side that depends only on the order of derivation. This implies for the Sobolev norms that

$$\|\varphi_1\|_{W^{m-1,2}(\mathbb{T})} \lesssim (1+\Lambda)^{m-\frac{1}{2}}\left(\frac{\Lambda}{\lambda}\right)^{2m-2}\|\varphi\|_{W^{m-1,2}(\Gamma)}. \tag{4.23}$$

We leave the details to the reader.

Second, we need to check that the radius s of the inner boundary of G_s does not come too close to 1. This is a simple consequence of the assumption made before the statement of Theorem 4.1 that the ratio $|D|/|\Omega|$ is less than a constant $\rho < 1$. Indeed,

$$|\Omega| - |D| = \int_{G_s} \left|\psi'(\zeta)\right|^2 dx\,dy \leqslant \Lambda^2 \pi \left(1 - s^2\right), \tag{4.24}$$

so that

$$|\Omega|(1-\rho)/\left(\pi \Lambda^2\right) \leqslant \left(1 - s^2\right). \tag{4.25}$$

From (4.13) and (4.19), together with (4.20), (4.22), (4.24) and the inequality

$$\| f - \widetilde{f} \|^2_{L^2(\gamma)} \leqslant \Lambda \| f_1 - \widetilde{f}_1 \|^2_{L^2(s\mathbb{T})},$$

it is straightforward to check that the inequalities (4.11) and (4.12) stated in Theorem 4.1 hold true. For the second inequality, note that, when $m > 1$, the upper bound on $|D|$ obtained from the previous analysis happens to tend to infinity if s tends to 0. Indeed, this comes from the occurrence of the factor s^{m-1} in the denominator of the right-hand side of (4.19). Anyway, if s is small, so is $|D|$, because of the classical inequality of Carleman, see [13, p. 503], which states that $|D|/|\Omega| \leqslant s^2$. This allows us to upper estimate the area of D by the minimum of these two bounds, which both depend on s, an unknown parameter in $(0, 1)$. Hence, we have to consider all possible value of s in $(0, 1)$, and this leads to (4.12).

4.2. An example: the class of eccentric annuli

In this section, we illustrate our previous results by considering an example of a specific class of admissible domains denoted by $\mathcal{D}_d$. The domains of the class $\mathcal{D}_d$ are the eccentric annuli, that is the annular domains $\mathcal{G}_{a,r}$ whose inner boundary is a circle $|z - a| = r$ and outer boundary the unit circle $\mathbb{T}$. Without loss of generality, we assume that the center a of the inner circle is a positive number. Moreover, we assume that $0 < a + r < 1 - d < 1$, with d some positive real number less than 1, e.g. $0 < d < 1/2$. Hence, there is a minimal separation between the circles $|z - a| = r$ and $|z| = 1$, in other words, the cavity D is not too close to the boundary of the domain Ω.

Proposition 4.3. *The class $\mathcal{D}_d$ is an admissible set of domains, whose elements satisfy, for any $m \geqslant 1$, the property $\boldsymbol{H(m, \lambda, \Lambda_m)}$ with*

$$\lambda = \frac{d}{2-d}, \qquad \Lambda_m = m!(2-d)(1-d)^{m-1}/d^m.$$

Proof. Let $\mathcal{G}_{a,r}$ be an element of $\mathcal{D}_d$. We describe the conformal map ψ, or rather its inverse ψ^{-1} from $\mathcal{G}_{a,r}$ to an annulus G_s, for some appropriate value of s. The map ψ^{-1} is a bilinear transform that can be determined from the following property: bilinear transforms map inverse points with respect to circles to inverse points. Following [1, Example 5.7.8], we then consider two positive real numbers α and β that are inverse with respect to both circles $|z| = 1$ and $|z - a| = r$. Consequently, they satisfy the relations,

$$\alpha\beta = 1, \qquad (\alpha - a)(\beta - a) = r^2.$$

We choose α and β so that α lies inside, and β outside both circles. Then, we choose ψ^{-1} as the bilinear transform that maps α and β onto 0 and ∞ respectively, namely

$$\psi^{-1}(z) = \kappa \frac{z - \alpha}{\alpha z - 1}.$$

With this choice, the circles $|z| = 1$ and $|z - a| = r$ are mapped onto circles centered at 0. The constant $\kappa = -1$ is chosen so that circle $|z| = 1$ is mapped onto itself. Hence,

$$\psi^{-1}(z) = \frac{z - \alpha}{1 - \alpha z} \quad \text{and} \quad \psi(z) = \frac{z + \alpha}{1 + \alpha z}. \tag{4.26}$$

The inner circle of G_s maps to $|z - a| = r$ by ψ. Let us determine its radius s. It is easily checked that the set $|z - \alpha|/|z - \beta| = k$ is a circle with respect to which α and β are inverse. Moreover, the center of this circle is $(\alpha - k^2\beta)/(1 - k^2)$. Hence, choosing k such that $(\alpha - k^2\beta)/(1 - k^2) = a$, or equivalently $k^2 = (a - \alpha)/(a - \beta)$, the above set coincides with $|z - a| = r$, and it is clear from (4.26) that its image by ψ^{-1} is the circle of radius

$$s = \frac{k}{\alpha} = \frac{1}{\alpha}\sqrt{\frac{\alpha - a}{\beta - a}} = \frac{\beta(\alpha - a)}{r} = \frac{r^2 + a(\alpha - a)}{r}.$$

Note that from the last expression for s, we deduce that

$$r \leqslant s \leqslant a + r \leqslant 1 - d < 1, \tag{4.27}$$

where $0 < d < 1$ has been defined at the beginning of this section.

The derivatives of ψ are given by

$$\psi^{(n)}(z) = (-1)^{n+1} \frac{n!(1-\alpha^2)\alpha^{n-1}}{(1+\alpha z)^{n+1}}, \quad n \geqslant 1.$$

Hence,

$$\inf_{z \in \bar{G}_s} \left|\psi'(z)\right| = \left|\psi'(1)\right| = \frac{1-\alpha}{1+\alpha} \geqslant \frac{d}{2-d}, \tag{4.28}$$

and, for $n \geqslant 1$,

$$\sup_{z \in \bar{G}_s} \left|\psi^{(n)}(z)\right| = \left|\psi^{(n)}(-1)\right| = n!\alpha^{n-1} \frac{1+\alpha}{(1-\alpha)^n} \leqslant n! \frac{(2-d)(1-d)^{n-1}}{d^n}.$$

Since, by assumption, $0 < d < 1/2$, we deduce that

$$\sup_{z \in \bar{G}_s} \left|\psi^{(n)}(z)\right| \leqslant m! \frac{(2-d)(1-d)^{m-1}}{d^m}, \quad 1 \leqslant n \leqslant m, \tag{4.29}$$

which shows the assertion in the proposition. □

As a final result, we give a version of Theorem 4.1 for domains in the class $\mathcal{D}_d$ when $m = 1$.

Theorem 4.4. *Consider a domain $\mathcal{G}$ in the class $\mathcal{D}_d$, and assume that*

$$\sigma := \sqrt{\frac{2-d}{d}} \|u - \widetilde{u}\|_{2,\mathbb{T}} < 1.$$

Then, the area of the inner disk D of $\mathcal{G}$, that is the area of the unknown cavity, is bounded above as follows,

$$|D| \leqslant \frac{C}{|\log \sigma|},$$

where the constant C is explicitly given by

$$C = 2M_{\widetilde{f}'} \sqrt{2\pi(1-d)} \big(2\|\varphi\|_{L^2(\Gamma)} + 1\big) / \big(ed^2 m^2_{\widetilde{f}'}\big).$$

Proof. The assertion is easily obtained from the general case and some minor modifications. □

References

[1] M.J. Ablowitz, A.S. Fokas, Complex Variables, Introduction and Applications, Cambridge Texts Appl. Math., 1997.
[2] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, Comm. Pure Appl. Math. 12 (1959) 623–727.
[3] G. Alessandrini, E. Beretta, E. Rosset, S. Vessella, Optimal stability for inverse elliptic boundary value problems with unknown boundaries, Ann. Sc. Norm. Super. Pisa 29 (2000) 755–806.
[4] G. Alessandrini, A. Morassi, E. Rosset, Size estimates, in: Inverse Problems: Theory and Applications, in: Contemp. Math., vol. 333, Amer. Math. Soc., Providence, RI, 2003, pp. 1–33.
[5] L. Baratchart, J. Grimm, J. Leblond, J. Partington, Asymptotic estimates for interpolation and constrained approximation in H^2 by diagonalization of Toeplitz operators, Integral Equations Operator Theory 45 (2003) 269–299.
[6] L. Baratchart, F. Mandréa, E.B. Saff, F. Wielonsky, 2-D inverse problem for the Laplacian: A meromorphic approximation approach, J. Math. Pures Appl. 86 (2006) 1–41.
[7] L. Baratchart, M. Zerner, On the recovery of functions from pointwise boundary values in a Hardy–Sobolev class of the disk, J. Comput. Appl. Math. 46 (1993) 255–269.
[8] S. Chaabane, I. Fellah, M. Jaoua, J. Leblond, Logarithmic stability estimates for a Robin coefficient in two-dimensional Laplace inverse problems, Inverse Problems 20 (2004) 47–59.
[9] I. Chalendar, J.R. Partington, Approximation problems and representations of Hardy spaces in circular domains, Studia Math. 136 (1999) 255–269.
[10] J. Cheng, Y.C. Hon, M. Yamamoto, Conditional stability estimation for an inverse boundary problem with non-smooth boundary in $\mathbb{R}^3$, Trans. Amer. Math. Soc. 353 (2001) 4123–4138.
[11] P. Duren, Theory of H_p Spaces, Academic Press, San Diego, CA, 1970.
[12] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, New York, 1983.
[13] P. Henrici, Applied and Computational Complex Analysis, vol. 3, Wiley–Interscience, 1993.

[14] M. Ikehata, Size estimation of inclusion, J. Inverse Ill-Posed Probl. 6 (1998) 127–140.
[15] J. Leblond, M. Mahjoub, J.R. Partington, Analytic extensions and Cauchy-type inverse problems on annular domains: Stability results, J. Inverse Ill-Posed Probl. 14 (2006) 189–204.
[16] K.Y. Osipenko, The Hardy–Littlewood–Polya inequality for analytic functions in Hardy–Sobolev spaces, Math. Sb. 197 (2006) 315–334.
[17] J.R. Partington, Recovery of functions by interpolation and sampling, J. Math. Anal. Appl. 198 (1996) 301–309.
[18] J.R. Partington, Interpolation, Identification and Sampling, Oxford University Press, 1997.
[19] C. Pommerenke, Boundary Behavior of Conformal Maps, Springer-Verlag, Berlin, New York, 2003.
[20] T. Radó, The problem of the least area and the problem of Plateau, Math. Z. 32 (1930) 763–796.
[21] D.E. Sarason, The H^p spaces of an annulus, Mem. Amer. Math. Soc. 56 (1965) 1–78.
[22] E.J. Specht, Estimates on the mapping function and its derivatives in conformal mapping of nearly circular regions, Trans. Amer. Math. Soc. 71 (1951) 183–196.
[23] M.E. Taylor, Partial Differential Equations I, Basic Theory, Springer-Verlag, New York, 1996.