From the classical to the generalized von Kármán and Marguerre–von Kármán equations

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Abstract

In this work, we describe and analyze two models that were recently proposed for modeling generalized von Kármán plates and generalized Marguerre–von Kármán shallow shells.

First, we briefly review the “classical” von Kármán and Marguerre–von Kármán equations, their physical meaning, and their mathematical justification. We then consider the more general situation where only a portion of the lateral face of a nonlinearly elastic plate or shallow shell is subjected to boundary conditions of von Kármán type, while the remaining portion is free. Using techniques from formal asymptotic analysis, we obtain in each case a two-dimensional boundary value problem that is analogous to, but is more general than, the classical equations.

In particular, it is remarkable that the boundary conditions for the Airy function can still be determined on the entire boundary of the nonlinearly elastic plate or shallow shell solely from the data.

Following recent joint works, we then reduce these more general equations to a single “cubic” operator equation, which generalizes an equation introduced by Berger and Fife, and whose sole unknown is the vertical displacement of the shell. We next adapt an elegant compactness method due to Lions for establishing the existence of a solution to this operator equation.

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1. Introduction

The history of the justification and generalization of the classical von Kármán’s theory of plates and Marguerre-von Kármán’s theory of shallow shells has almost one century. The two-dimensional von Kármán equations for nonlinearly elastic plates were originally proposed in [17] by the American engineer of Hungarian origin Theodore von Kármán (1881–1963).

Using a formal asymptotic analysis, Ciarlet [3] showed that the von Kármán equations can be fully justified by means of the leading term of a formal asymptotic expansion (in which the “small” parameter is the thickness of the plate) of the exact three-dimensional equations of nonlinear elasticity associated with a specific class of boundary conditions that characterizes such plates.

This work clearly delineated the validity of these “limit” two-dimensional equations and it clarified the nature of admissible boundary conditions for the three-dimensional model from which these equations are derived.

In all the classical and generalized models presented in this paper we restrict ourselves to St Venant-Kirchhoff materials, but our conclusions apply as well to more general elastic materials. As shown by Davet [12] in the case of a clamped plate, the constitutive equation may be replaced by that of the most general nonlinearly elastic, homogeneous, and isotropic, material whose reference configuration is a natural state. Since this replacement does not modify the form of the “limit” two-dimensional equations, we can conclude that they have a generic character.

2. Classical von Kármán equations

Greek indices, corresponding to the “horizontal” coordinates, vary in the set \{1, 2\}, while Latin indices vary in the set \{1, 2, 3\} (except when they are used for indexing sequences), the index 3 being that of the “vertical” coordinate. The summation convention with respect to repeated indices is systematically used.

Let \( \omega \) be a domain in the horizontal plane \( \mathbb{R}^2 \) spanned by vectors \( \mathbf{e}_2 \), i.e., a bounded and connected open subset of \( \mathbb{R}^2 \) with a Lipschitz-continuous boundary \( \gamma \), the set \( \omega \) being locally on a single side of \( \gamma \). The unit outer normal vector \( (\mathbf{n}_2) \) and the unit tangent vector \( (\mathbf{t}_2) \) along the boundary \( \gamma \) are related by \( \mathbf{t}_1 = -\mathbf{n}_2 \) and \( \mathbf{t}_2 = \mathbf{n}_1 \). The outer normal and tangential derivative operators \( v_2 \partial_2 \) and \( \mathbf{t}_2 \partial_2 \) along \( \gamma \) are denoted \( \partial_n \) and \( \partial_t \).

For any \( \varepsilon > 0 \), define the set \( \Omega^\varepsilon = \omega \times [\varepsilon - \varepsilon] \), with \( \Omega^\varepsilon_+ = \omega \times \{\varepsilon\} \) as its “upper face” and \( \Omega^\varepsilon_- = \omega \times \{-\varepsilon\} \) as its “lower face”. The unit outer normal vector along the boundary \( \partial \Omega^\varepsilon \) of the set \( \Omega^\varepsilon \) is denoted by \( (\mathbf{n}_\varepsilon) : \partial \Omega^\varepsilon \rightarrow \mathbb{R}^3 \); in particular then, \( (\mathbf{n}_1^\varepsilon) = (v_1, v_2, 0) \) along the lateral face \( \gamma \times [-\varepsilon, \varepsilon] \).

Assume that the set \( \Omega^\varepsilon = \overline{\omega} \times [-\varepsilon, \varepsilon] \) is the reference configuration of a nonlinearly elastic plate, subjected to applied body forces acting in its interior \( \Omega^\varepsilon \) of density \( (f_1^\varepsilon) : \Omega^\varepsilon \rightarrow \mathbb{R}^3 \), to applied surface forces acting on its upper and lower faces of density \( (g_1^\varepsilon) : \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon \rightarrow \mathbb{R}^3 \), and to applied surface forces parallel to the horizontal plane spanned by the vectors \( \mathbf{e}_z \) acting on the lateral face \( \gamma \times [-\varepsilon, \varepsilon] \), whose only the resultant density \( (h_1^\varepsilon, h_2^\varepsilon, 0) : \gamma \rightarrow \mathbb{R}^3 \) per unit length, obtained by integration across the thickness, is known along \( \gamma \).

Let \( \lambda^\varepsilon \) and \( \mu^\varepsilon \) denote the Lamé constants of the elastic material constituting the plate and let \( (\sigma_{ij}^\varepsilon) : \overline{\Omega}^\varepsilon \rightarrow \mathbb{S}^3 \) denote the second Piola–Kirchhoff stress tensor field, where \( \mathbb{S}^3 \) denotes the set of all symmetric matrices of order 3.
The three-dimensional problem then consists in finding the displacement vector field $\mathbf{u}^e = (u_i^e) : \overline{\Omega}^{\varepsilon} \to \mathbb{R}^3$ that satisfies the equilibrium equations (see, e.g., Ciarlet [4]):

$$-\partial_j (\sigma_{ij}^e + \sigma_{kj}^e \partial_k \hat{u}_i^e) = f_i^e \quad \text{in } \Omega^e,$$

$$\sigma_{ij}^e + \sigma_{kj}^e \partial_k u_i^e = g_i^e \quad \text{on } \Gamma_+ \cup \Gamma_-,$$

$$\frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} (\sigma_{2\beta}^e + \sigma_{k\beta}^e \partial_k u_2^e) v_\beta \, dx_3^e = h_2^e \quad \text{on } \gamma,$$

the constitutive equation:

$$\sigma_{ij}^e = E_{pp}^e (\mathbf{u}^e) \delta_{ij} + 2\mu E_{ij}^e (\mathbf{u}^e),$$

where

$$E_{ij}^e (\mathbf{u}^e) = \frac{1}{2} (\sigma_{ij}^e u_j^e + \sigma_{ji}^e u_i^e + \sigma_{km}^e \partial_j^e u_m^e u_m^e),$$

and the boundary conditions:

$$u_x^e \text{ independent of } x_3^e \quad \text{and } \quad u_3^e = 0 \text{ on } \gamma \times [-\varepsilon, \varepsilon].$$

These specific boundary conditions of “von Kármán type” involving the stress tensor field and the displacement field $(u^e) : \Omega^e \to \mathbb{R}^3$ along the lateral face $\gamma \times [-\varepsilon, \varepsilon]$ of the plate were introduced by Ciarlet [3] in order to justify the classical von Kármán equations. They mean in particular that any vertical segment along the lateral face can only undergo horizontal translations.

We assume that all applied forces are vertical, i.e., $f_2^e = 0$ and $g_2^e = 0$, and we impose the following compatibility conditions for the applied surface forces acting along the lateral face of the plate:

$$\int_{\gamma} h_1^e \, d\gamma = \int_{\gamma} h_2^e \, d\gamma = \int_{\gamma} (x_1 h_2^e - x_2 h_1^e) \, d\gamma = 0.$$

Their mathematical justification is to give rise to a well-defined problem in an appropriate quotient space, the displacements being then defined only up to horizontal infinitesimal rigid displacement. For details, see Ciarlet [5, Section 5.1]. These compatibility conditions satisfied by the functions $h_2$ have another mathematical justification, viz., ensuring that the functions $\Phi_0$ and $\Phi_1$ introduced below are unambiguously defined. In addition, they also have a mechanical interpretation, simply expressing that the horizontal forces acting on the shell are in static equilibrium.

As a first step toward the asymptotic analysis for this problem, we re-write it in a variational form and then “scale” it over a domain that is independent of $\varepsilon$. More specifically, we let $\Omega = \omega \times ]-1, 1[ = \Gamma_+ = \omega \times \{1\}$, $\Gamma_-=\omega \times \{-1\}$ and with each point $x \in \overline{\Omega}$, we associate the point $x^\varepsilon \in \overline{\Omega}^{\varepsilon}$ through the bijection

$$\pi^\varepsilon : x = (y, x_3) \in \overline{\Omega} \to x^\varepsilon = (x_i^\varepsilon) = (y, \varepsilon x_3) \in \overline{\Omega}^{\varepsilon}.$$

With the displacement field $\mathbf{u}^e = (u_i^e) : \overline{\Omega}^{\varepsilon} \to \mathbb{R}^3$, we next associate the scaled displacement field $\mathbf{u}(\varepsilon) = (u_i(\varepsilon)) : \overline{\Omega} \to \mathbb{R}^3$ defined by

$$u_x^e(x^\varepsilon) = \varepsilon^2 u_x(\varepsilon)(x), \quad u_3^e(x^\varepsilon) = \varepsilon u_3(\varepsilon)(x) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon.$$
We then make the following fundamental assumptions on the data $\lambda^e, \mu^e, f_3^e, g_3^e$, and $h_z^e$ regarding their asymptotic behavior as functions of $\varepsilon$:

$$\lambda^e = \lambda \text{ and } \mu^e = \mu,$$

$$f_3^e(x^e) = \varepsilon^3 f_3(x), \text{ for all } x^e = \pi^e x \in \Omega^e,$$

$$g_3^e(x^e) = \varepsilon^4 g_3(x), \text{ for all } x^e = \pi^e x \in I^e_+ \cup I^e_-,$$

$$h_z^e(y) = \varepsilon^2 h_z(y), \text{ for all } y \in \gamma,$$

where the constants $\lambda > 0$, $\mu > 0$ and the functions $f_3 \in L^2(\Omega)$, $g_3 \in L^2(I^e_+ \cup I^e_-)$, and $h_z \in L^2(\gamma)$, are all independent of $\varepsilon$. These scalings and assumptions on the data produce an equivalent “scaled” variational problem, posed over the set $\Omega = \overline{\omega} \times [-1, 1]$, which is now independent of $\varepsilon$. The specific form of this problem then suggests that we use the method of formal asymptotic expansions, i.e., we let $u(x) = u_0 + \varepsilon u^1 + \varepsilon^2 u^2 + \varepsilon^3 u^3 + \cdots$ and we equate to zero the factors of the successive powers $\varepsilon^p$, $p \geq 0$, until the leading term $u_0$ can be fully identified. For details, see again ibid., Chapters 4 and 5.

Without loss of generality, we assume that the origin 0 belongs to the boundary $\gamma$ of the set $\omega$ and we denote by $\gamma(y)$ the arc, oriented in the usual manner, joining the origin 0 to the point $y$ along $\gamma$.

If the data are sufficiently regular and the above compatibility conditions hold, then the vector field $u^0 = (u_{x}^0)$ is a scaled Kirchhoff–Love displacement field, i.e., $u_3^0$ is independent of $x_3$ and there exist $\zeta_3 \in H^3(\omega)$ and $\zeta_3 \in H^2(\omega) \cap H^4(\omega)$ such that

$$u_3^0 = \zeta_3 \text{ and } u_{x}^0 = \zeta_x - x_3 \partial_x \zeta_3.$$

Moreover, if the set $\omega$ is simply-connected, the two-dimensional vector field $\zeta = (\zeta_i)$ can be computed as follows: one first solves the scaled von Kármán equations: Find $(\zeta_3, \Phi) : \overline{\omega} \to \mathbb{R}^2$ such that

$$\frac{8\mu(\lambda + \mu)}{3(\lambda + 2\mu)} \Delta^2 \zeta_3 = [\Phi, \zeta_3] + p_3 \text{ in } \omega,$$

$$\Delta^2 \Phi = -\frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} [\zeta_3, \zeta_3] \text{ in } \omega,$$

$$\zeta_3 = \partial_y \zeta_3 = 0 \text{ on } \gamma,$$

$$\Phi = \Phi_0 \text{ and } \partial_y \Phi = \Phi_1 \text{ on } \gamma,$$

where $\Phi$ is the scaled Airy stress function,

$$\Phi_0(y) = -y_1 \int_{\gamma(y)} h_2 \, d\gamma + y_2 \int_{\gamma(y)} h_1 \, d\gamma + \int_{\gamma(y)} (x_1 h_2 - x_2 h_1) \, d\gamma, \quad y \in \gamma,$$

$$\Phi_1(y) = -v_1(y) \int_{\gamma(y)} h_2 \, d\gamma + v_2(y) \int_{\gamma(y)} h_1 \, d\gamma, \quad y \in \gamma,$$

$$p_3 = g_3(., 1) + g_3(., -1) + \int_{-1}^1 f_3 \, dx_3.$$
and

\[ \{ \Phi, \zeta \} = \hat{c}_{11} \Phi \hat{c}_{22} \zeta + \hat{c}_{22} \Phi \hat{c}_{11} \zeta - 2 \hat{c}_{12} \Phi \hat{c}_{12} \zeta \]

denotes the Monge-Ampère form.

In order that this boundary value problem be expressed in terms of “physical” quantities, it remains to “de-scale” the unknowns and the data. To this end, we define the transverse displacement \( \zeta^e_3 : \overline{\omega} \to \mathbb{R} \) of the middle surface of the plate and the Airy stress function \( \Phi^e : \overline{\omega} \to \mathbb{R} \) through the following de-scalings:

\[ \zeta^e_3 = \varepsilon \zeta_3 \quad \text{and} \quad \Phi^e = \varepsilon^2 \Phi \quad \text{in} \quad \overline{\omega}. \]

The de-scaled functions \( \zeta^e_3 \) and \( \Phi^e \) then satisfy the “classical” von Kármán equations:

\[ \frac{8\mu^e(\lambda^e + \mu^e)}{3(\lambda^e + 2\mu^e)} \varepsilon^3 \Delta^e_3 \zeta^e_3 = \varepsilon [\Phi^e, \zeta^e_3] + p^e_3 \quad \text{in} \quad \omega, \]

\[ \Delta^2 \Phi^e = -\frac{\mu^e(3\lambda^e + 2\mu^e)}{\lambda^e + \mu^e} \frac{[\zeta^e_3, \zeta^e_3]}{3} \quad \text{in} \quad \omega, \]

\[ \zeta^e_3 = 0 \quad \text{on} \quad \gamma, \]

\[ \Phi^e = \Phi^e_0 \quad \text{and} \quad \partial_y \Phi^e = \Phi^e_1 \quad \text{on} \quad \gamma, \]

where

\[ \Phi^e_0(y) = -y_1 \int_{\gamma(y)} h_2^e \, d\gamma + y_2 \int_{\gamma(y)} h_1^e \, d\gamma + \int_{\gamma(y)} (x_1 h_2^e - x_2 h_1^e) \, d\gamma, \quad y \in \gamma, \]

\[ \Phi^e_1(y) = -v_1(y) \int_{\gamma(y)} h_2^e \, d\gamma + v_2(y) \int_{\gamma(y)} h_1^e \, d\gamma, \quad y \in \gamma, \]

\[ p^e_3 = g^e_3(., \varepsilon) + g^e_3(., -\varepsilon) + \int_{-\varepsilon}^\varepsilon f^e_3 \, dx^e_3. \]

Remarks. (a) The coefficient \( (8\mu^e(\lambda^e + \mu^e)/3(\lambda^e + 2\mu^e)) \varepsilon^3 \) is the flexural rigidity of the plate and the coefficient \( \mu^e(3\lambda^e + 2\mu^e)/(\lambda^e + \mu^e) \) denotes the Young modulus of the elastic material constituting the plate.

(b) As in the case of a clamped plate, it is remarkable that a quasilinear second-order problem has been replaced by a semilinear fourth-order problem, whose mathematical analysis may be therefore expected to be simpler, as exemplified by the available bifurcation theory for the von Kármán equations, which has no counterpart for the original three-dimensional problem.

(c) Which boundary conditions are appropriate for the three-dimensional problem is a question of particular importance, in as much as the von Kármán equations are sometimes used when they should not be. One of the noteworthy virtues of the above justification is that it clearly identifies those boundary conditions that are admissible for the corresponding three-dimensional problem, and from which the boundary conditions for the Airy stress function must be in turn derived in a specific fashion.
Once the equations are justified, one can carry out their mathematical analysis, regarding questions of existence, regularity, multiplicity, and bifurcation of their solutions; for details and references, see Ciarlet [5, Chapter 5].

Note that we can re-write the scaled von Kármán equations in a simpler form, by introducing the new unknowns
\[ \xi = D^{-1} E^{\frac{1}{2}} \xi_3 \quad \text{and} \quad \psi = D^{-1} \phi, \]
where \( D = \frac{8\mu(\lambda + \mu)}{3(\lambda + 2\mu)} \) and \( E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \).

In this manner, we find that the pair \((\xi, \psi)\) satisfies the canonical von Kármán equations:
\[ \begin{aligned}
\Delta^2 \xi &= [\psi, \xi] + f \quad \text{in} \ \omega, \\
\Delta^2 \psi &= -[\xi, \xi] \quad \text{in} \ \omega, \\
\xi &= \partial_N \xi = 0 \quad \text{on} \ \gamma, \\
\psi &= \psi_0 \quad \text{and} \quad \partial_N \psi = \psi_1 \quad \text{on} \ \gamma,
\end{aligned} \]
where \( f, \psi_0, \psi_1 \) are given functions. We can further transform these canonical equations into an even more condensed form, by reducing their solutions to that of a single nonlinear operator equation in the unknown \( \xi \).

More precisely, one can show (see [11] or [5, Chapter 5]) that the pair \((\xi, \psi) \in H^2_\lambda(\omega) \times H^2(\omega)\) satisfies the canonical von Kármán equations if and only if \( \xi \in H^2_\lambda(\omega) \) satisfies the operator equation
\[ C(\xi) + (I - A)\xi - F = 0, \]
and \( \psi \) is given by
\[ \psi = \theta_0 - B(\xi, \xi), \]
where \( B : H^2(\omega) \times H^2(\omega) \to H^2_0(\omega) \) is a bilinear operator, \( C : H^2_0(\omega) \to H^2_0(\omega) \) is a nonlinear operator, \( A : H^2_0(\omega) \to H^2_0(\omega) \) is a linear operator, and \( F \in H^2_0(\omega), \theta_0 \in H^2(\omega) \) are known functions (the complete definitions of these various operators and functions are found in Ciarlet [5, Chapter 5]).

We emphasize that the nonlinearity of the von Kármán equations lies in the operator \( C \), which is “cubic”, in the sense that
\[ C(\alpha \xi) = \alpha^3 C(\xi), \]
for any \( \alpha \in \mathbb{R} \) and \( \xi \in H^2_0(\omega) \).

The reduced von Kármán equation generalizes a cubic operator equation introduced by Berger [1]; see also Berger and Fife [2].

### 3. Generalized von Kármán equations

In this section, we present a first generalization of the classical von Kármán equations due to Ciarlet and Gratie [6], where only a portion of the lateral face is subjected to boundary conditions of von Kármán’s type, the remaining portion being subjected to a boundary condition of free edge. We showed that a formal asymptotic analysis of the scaled three-dimensional solution still leads in this case to a two-dimensional
boundary value problem that is analogous to, but is more general than, the classical equations presented in Section 2. In particular, it is remarkable that the boundary conditions for the Airy function can still be determined solely from the data.

More specifically, consider a nonlinearly elastic plate again with reference configuration \( \Omega^0 = \overline{\omega} \times [-\varepsilon, \varepsilon] \), subjected to the same body forces in its interior and surface forces on its upper and lower faces as before, and to von Kármán surface forces only on a portion \( \gamma_1 \times [-\varepsilon, \varepsilon] \) of its lateral face, where \( \gamma_1 \) is a relatively open subset of \( \gamma \) satisfying \( 0 < \text{length}(\gamma_1) < \text{length}(\gamma) \). The remaining portion \( \gamma_2 \times [-\varepsilon, \varepsilon] \) of the lateral face, where \( \gamma_2 = \gamma - \gamma_1 \), is free. The unknown displacement field \( u^e = (u_i^e) \) thus satisfies the following three-dimensional problem:

\[
-\partial_j (\sigma_{ij}^e + \sigma_{kj}^e \partial_k u_i^e) = f_i^e \quad \text{in} \: \Omega^e,
\]

\( u_x^e \) independent of \( \chi_3^e \) and \( u_3^e = 0 \) on \( \gamma_1 \times [-\varepsilon, \varepsilon] \),

\[
\frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} (\sigma_{x\beta}^e + \sigma_{k\beta}^e \partial_k u_x^e) \nu_\beta \, \text{d}x_3^e = h_2^e \quad \text{on} \: \gamma_1,
\]

\[
(\sigma_{ij}^e + \sigma_{kj}^e \partial_k u_i^e) n_j^e = 0 \quad \text{on} \: \gamma_2 \times [-\varepsilon, \varepsilon],
\]

\[
(\sigma_{ij}^e + \sigma_{kj}^e \partial_k u_i^e) n_j^e = g_i^e \quad \text{on} \: \omega \times [-\varepsilon, \varepsilon],
\]

where

\[
\sigma_{ij}^e = \partial_j^e E_{pp}^e (u^e) \delta_{ij} + 2 \mu^e E_{ij}^e (u^e) \quad \text{and} \quad E_{ij}^e (u^e) = \frac{1}{2} (\partial_i^e u_j^e + \partial_j^e u_i^e + \partial_i^e u_m^e \delta_j^e \partial_m^e) .
\]

Applying again the method of formal asymptotic expansions to the solution of this more general problem with the same scalings as before, we first put it in a variational form and scale it over the fixed set \( \overline{\Omega} = \overline{\omega} \times [-1, 1] \). We then show that the leading term \( u^0 \) of the asymptotic expansion of the three-dimensional solution \( u^e : \overline{\Omega} \to \mathbb{R}^3 \) is again a Kirchhoff–Love displacement field, viz., \( u_x^0 = \zeta_x - x_3 \partial_x \zeta_3 \) and \( u_3^0 = \zeta_3 \), where \( \zeta = (\zeta_i) \) satisfies a two-dimensional problem, which can be expressed either as a variational problem or as a boundary value problem.

Assume that \( \omega \) is simply-connected with a smooth boundary \( \gamma \) and that \( \zeta_3 \in H^3(\omega) \) and \( \zeta_3 \in H^4(\omega) \). Then the functions \( \tilde{h}_x \in L^2(\gamma) \) defined by \( \tilde{h}_x = h_x \) on \( \gamma_1 \) and \( \tilde{h}_x = 0 \) on \( \gamma_2 \) necessarily satisfy the compatibility relations

\[
\int_\gamma \tilde{h}_1 \, \text{d}\gamma = \int_\gamma \tilde{h}_2 \, \text{d}\gamma = \int_\gamma (x_1 \tilde{h}_2 - x_2 \tilde{h}_1) \, \text{d}\gamma = 0 ,
\]

and there exists an Airy function \( \Phi \in H^4(\omega) \) such that

\[
\partial_{11} \Phi = N_{22} , \quad -\partial_{12} \Phi = N_{12} = N_{21} , \quad \partial_{22} \Phi = N_{11} \quad \text{in} \: \omega
\]

where

\[
N_{\alpha \beta} = \frac{4 \lambda \mu}{\lambda + 2 \mu} E^0_{\sigma \alpha} (\zeta) \delta_{\alpha \beta} + 4 \mu E^0_{\alpha \beta} (\zeta) \quad \text{with} \quad E^0_{\alpha \beta} (\zeta) = \frac{1}{2} \{ \partial_x \zeta_\alpha + \partial_\beta \zeta_x + \partial_\alpha \zeta_\beta + \partial_\beta \zeta_\alpha \} .
\]
Finally, the pair \((\zeta_3, \Phi)\) ∈ \(H^4(\omega) \times H^4(\omega)\) satisfies the following scaled generalized von Kármán equations:

\[
\frac{8\mu(\lambda + \mu)}{3(\lambda + 2\mu)} \Delta^2 \zeta_3 = [\Phi, \zeta_3] + p_3 \quad \text{in } \omega, \\
\Delta^2 \Phi = -\frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \zeta_3 \quad \text{in } \omega, \\
\zeta_3 = \partial_y \zeta_3 = 0 \quad \text{on } \gamma_1, \\
m_{x\beta} n_\gamma = 0 \quad \text{on } \gamma_2, \\
\partial_x m_{x\beta} n_\gamma + \partial_\tau (m_{x\beta} n_\gamma) = 0 \quad \text{on } \gamma_2, \\
\Phi = \Phi_0 \quad \text{and} \quad \partial_y \Phi = \Phi_1 \quad \text{on } \gamma,
\]

where \(\Phi_0, \Phi_1\) are known functions of \((\widetilde{h}_2)\), given by (with \(y(\gamma)\) being as before the oriented arc joining 0 to \(y\) along \(\gamma\)):

\[
\Phi_0(y) = -y_1 \int_{\gamma(y)} \widetilde{h}_2 \, d\gamma + y_2 \int_{\gamma(y)} \widetilde{h}_1 \, d\gamma + \int_{\gamma(y)} (x_1 \widetilde{h}_2 - x_2 \widetilde{h}_1) \, d\gamma, \quad y \in \gamma, \\
\Phi_1(y) = -v_1(y) \int_{\gamma(y)} \widetilde{h}_2 \, d\gamma + v_2(y) \int_{\gamma(y)} \widetilde{h}_1 \, d\gamma, \quad y \in \gamma,
\]

and

\[
m_{x\beta} = -\frac{1}{3} \left\{ \frac{4\lambda \mu}{\lambda + 2\mu} \Delta \zeta_3 n_\gamma + 4\mu \partial_\tau \zeta_3 \right\}, \\
p_3 = \int_{-1}^{1} f_3 \, dx_3 + g_3(., +1) + g_3(., -1).
\]

An important conclusion is thus that the boundary conditions on the Airy function \(\Phi : \overline{\omega} \to \mathbb{R}\) can still be determined from the sole knowledge of the data, even if \(\text{length}(\gamma_2) > 0\).

**Remark.** (a) The pair \((\zeta_3, \Phi)\) satisfies a boundary value problem that generalizes the classical von Kármán equations found in Section 2, which correspond to the case where \(\gamma_2 = \emptyset\).

(b) It would be interesting to see if other three-dimensional boundary conditions, such as boundary conditions corresponding to live von Kármán surface forces, or boundary conditions of simple support on \(\gamma_2 \times [-\varepsilon, \varepsilon]\), similarly lead to other “generalized” von Kármán equations.

(c) The expressions \(N_{x\beta}\) and \(m_{x\beta}\) respectively denote the scaled stress resultants, and the scaled bending moments.

Once the generalized von Kármán equations are derived under the assumption that their solutions \((\zeta_3, \Phi)\) are smooth, they can be studied on their own sake, in particular as regards the existence of less smooth solutions.

More specifically, define the function space

\[
V(\omega) = \{ \eta \in H^2(\omega); \; \eta = \partial_{y} \eta = 0 \; \text{on} \; \gamma_1 \}.
\]
Then, as shown by Ciarlet, Gratie, and Sabu [9], solving the generalized von Kármán equations in the sense of distributions amounts to finding \( \xi = D^{-\frac{1}{2}} E^\frac{1}{2} \zeta_3 \in V(\omega) \) that satisfies the operator equation
\[
\widetilde{C}(\xi) + (I - \widetilde{A}) \xi - \widetilde{F} = 0,
\]
where the “cubic” operator \( \widetilde{C} : V(\omega) \rightarrow V(\omega) \) (“cubic” in the sense that \( \widetilde{C}(\alpha \eta) = \alpha^3 \widetilde{C}(\eta) \) for all \( \alpha \in \mathbb{R} \) and \( \eta \in V(\omega) \)), the linear operator \( \widetilde{L} : V(\omega) \rightarrow V(\omega) \), and the element \( \widetilde{F} \in V(\omega) \) generalize those found in the operator equation corresponding to the classical von Kármán equations; cf. Section 2.

Moreover, if the norms \( ||h_2||_{L^2(\gamma_1)} \) are small enough, the generalized von Kármán equations have at least one solution \( (\zeta_3, \Phi) \in H^2(\omega) \times H^2(\omega) \) in the sense of distributions. The proof of this existence result, where a compactness method due to Lions [21] plays a key role, is also found in Ciarlet et al. [9].

4. Classical Marguerre–von Kármán equations

The classical Marguerre–von Kármán equations are two-dimensional equations for a nonlinearly elastic shallow shell subjected to boundary conditions analogous to those of a von Kármán plate. They were initially proposed by Marguerre [22] and von Kármán and Tsien [18], and later justified by Ciarlet and Paumier [10] by means of a formal asymptotic analysis.

Consider a three-dimensional nonlinearly elastic shell with the reference configuration \( \{\hat{\Omega}^\varepsilon\}^- \), where \( \hat{\Omega}^\varepsilon = \Theta^\varepsilon(\Omega^\varepsilon), \Omega^\varepsilon = \omega \times [-\varepsilon, \varepsilon] \), \( \omega \) is a horizontal domain in \( \mathbb{R}^2 \), and the mapping \( \Theta^\varepsilon : \{\Omega^\varepsilon\}^- \rightarrow \mathbb{R}^3 \) is given by
\[
\Theta^\varepsilon(x^\varepsilon) = (y, \theta^\varepsilon(y)) + x_3^\varepsilon a_3^\varepsilon(y) \quad \text{for all} \quad x^\varepsilon = (y, x_3^\varepsilon) \in \overline{\Omega}^\varepsilon,
\]
where \( a_3^\varepsilon \) is a continuously varying unit vector normal to the middle surface \( \Theta^\varepsilon(\omega) \) of the shell and \( \theta^\varepsilon : \overline{\omega} \rightarrow \mathbb{R} \) is a function of class \( C^3 \) that satisfies \( \partial_\gamma \theta^\varepsilon = 0 \) along \( \gamma \), i.e., \( \theta^\varepsilon \) is constant along \( \gamma \) and the lateral face \( \Theta^\varepsilon(\gamma \times [-\varepsilon, \varepsilon]) \) of the shell is vertical. We denote by \( (n^\varepsilon) : \partial \hat{\Omega}^\varepsilon \rightarrow \mathbb{R}^3 \) the unit outer normal vector along the boundary of the set \( \hat{\Omega}^\varepsilon \).

Following the definition proposed by Ciarlet and Paumier [10], we say that a shell is shallow if the mapping \( \theta^\varepsilon : \overline{\omega} \rightarrow \mathbb{R} \) that measures the deviation of the middle surface of the reference configuration of the shell from a plane is (up to an additive constant) of the order of the thickness of the shell, i.e., of the order \( \varepsilon \). Consequently, all the derivatives \( \partial_\gamma \theta^\varepsilon, \partial_\gamma^2 \theta^\varepsilon, \) etc., are also of the order of \( \varepsilon \), and the radii of the curvature of the shell is of the order of \( \varepsilon^{-1} \). In other words, there exists a function \( \theta \in C^3(\overline{\omega}) \) independent of \( \varepsilon \) such that
\[
\theta^\varepsilon(y) = \varepsilon \theta(y) \quad \text{for all} \quad y \in \overline{\omega}.
\]

The shell is subjected to applied body forces in its interior \( \Theta^\varepsilon(\Omega^\varepsilon) \) of density \( \hat{f}^\varepsilon = (0, 0, \hat{f}_3^\varepsilon) : \hat{\Omega}^\varepsilon \rightarrow \mathbb{R}^3 \); to surface forces of density \( \hat{g}_3^\varepsilon = (0, 0, \hat{g}_3^\varepsilon) : \hat{f}_+^\varepsilon \cup \hat{f}_-^\varepsilon \rightarrow \mathbb{R}^3 \) on its upper and lower faces \( \hat{f}_+^\varepsilon = \Theta^\varepsilon(\Gamma^\varepsilon_+) \) and \( \hat{f}_-^\varepsilon = \Theta^\varepsilon(\Gamma^\varepsilon_-) \), where \( \Gamma^\varepsilon_+ = \omega \times [+\varepsilon] \) and \( \Gamma^\varepsilon_- = \omega \times [-\varepsilon] \); and to applied surface forces of von Kármán’s type analogous to those considered in Section 2 on the entire lateral face \( \Theta^\varepsilon(\gamma \times [-\varepsilon, \varepsilon]) \), whose only the resultant \( (\hat{h}_1^\varepsilon, \hat{h}_2^\varepsilon, 0) \) after integration across the thickness is given along the curve \( \gamma^\varepsilon = \Theta^\varepsilon(\gamma) \).
As before, the functions \( \hat{h}_x^e : \gamma^e \rightarrow \mathbb{R} \) must be such that the functions \( h_x^e = \hat{h}_x^e \circ \Theta^e \) satisfy the compatibility conditions

\[
\int_\gamma h_x^e \, d\gamma = \int_\gamma h_x^e \, d\gamma = \int_\gamma (x_1 h_x^e - x_2 h_y^e) \, d\gamma = 0.
\]

The unknown in the three-dimensional formulation is the displacement field \( \hat{u}^e = (\hat{u}^e) : \{ \hat{\Omega}^e \}^\gamma \rightarrow \mathbb{R}^3 \), where the functions \( \hat{u}^e_i : \{ \hat{\Omega}^e \}^\gamma \rightarrow \mathbb{R} \) represent the Cartesian components of the displacement, i.e., \( \hat{u}^e_i(\hat{x}^e) e_i \) is the displacement of the point \( \hat{x}^e = \Theta^e(x^e) \in \{ \hat{\Omega}^e \}^\gamma \), where \( e_i \) denote the basis in \( \mathbb{R}^3 \). The components \( \hat{\sigma}^e_{ij} \) of the second Piola–Kirchhoff tensor field are now expressed as functions of the coordinates \( \hat{x}^e \). Assume as before that the nonlinearly elastic material constituting the shell is a St Venant–Kirchhoff material, with Lamé constants \( \lambda^e \) and \( \mu^e \). Then the displacement vector field \( \hat{u}^e \) solves the following three-dimensional problem, consisting of the equilibrium equations:

\[
-\hat{\sigma}^e_{ij}(\hat{\sigma}^e_{ij} + \hat{\sigma}^e_{kj} \hat{\sigma}^e_{ik}) = \delta^e_i \quad \text{in} \, \hat{\Omega}^e,
\]

\[
(\hat{\sigma}^e_{ij} + \hat{\sigma}^e_{kj} \hat{\sigma}^e_{ik}) \hat{u}^e_j \circ \Theta^e = \hat{g}^e_i \circ \Theta^e \quad \text{on} \, \Gamma_+^e \cup \Gamma_-^e,
\]

the constitutive equations:

\[
\hat{\sigma}^e_{ij} = \lambda^e \hat{E}_{pp}(\hat{u}^e) \delta_{ij} + 2\mu^e \hat{E}_{ij}(\hat{u}^e) \quad \text{where} \quad \hat{E}_{ij}(\hat{u}^e) = \frac{1}{2}(\hat{\sigma}^e_{ij} \hat{u}^e_{ij} + \hat{\sigma}^e_{ij} \hat{u}^e_{ij} + \hat{\sigma}^e_{ij} \hat{u}^e_{ij} \hat{u}^e_{ij}),
\]

and the boundary conditions along the lateral face of the shell:

\[
\hat{u}^e_x \text{ independent of } \hat{x}^e_3 \quad \text{and} \quad \hat{u}^e_3 = 0 \quad \text{on} \, \Theta^e(\gamma \times [-\varepsilon, \varepsilon]),
\]

\[
\frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \{ (\hat{\sigma}^e_{2x} + \hat{\sigma}^e_{kx} \hat{\sigma}^e_{kx}) \circ \Theta^e \} \nu_{\beta} \, dx^e_3 = \hat{h}^e_x \circ \Theta^e \quad \text{on} \, \gamma,
\]

which are highly reminiscent of those corresponding to a von Kármán plate.

As shown by Ciarlet and Paumier [10], the outcome of the asymptotic analysis is then the following: In the scaled two-dimensional formulation, the unknowns are the scaled vertical displacement \( \zeta_3 : \overline{\omega} \rightarrow \mathbb{R} \) of the middle surface of the shell and the scaled Airy stress function \( \Phi : \overline{\omega} \rightarrow \mathbb{R} \), and the pair \((\zeta_3, \Phi) \in \{ H_0^2(\omega) \cap H^4(\omega) \} \times H^4(\omega) \) satisfies the following scaled classical Marguerre–von Kármán equations:

\[
\frac{8\mu(\lambda + \mu)}{3(\lambda + 2\mu)} \Delta^2 \zeta_3 = [\Phi, \zeta_3 + \theta] + p_3 \quad \text{in} \, \omega,
\]

\[
\Delta^2 \Phi = -\frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} [\zeta_3, \zeta_3 + 2\theta] \quad \text{in} \, \omega,
\]

\[
\zeta_3 = \partial_n \zeta_3 = 0 \quad \text{on} \, \gamma,
\]

\[
\Phi = \Phi_0 \quad \text{and} \quad \partial_n \Phi = \Phi_1 \quad \text{on} \, \gamma,
\]
5. Generalized Marguerre–von Kármán equations

Let there be given a domain \( \omega \) in \( \mathbb{R}^2 \). Let \( \gamma_1 \) be a relatively open subset of \( \gamma \) such that \( \text{length}(\gamma_1) > 0 \) and \( \text{length}(\gamma_2) > 0 \), where \( \gamma_2 = \gamma - \gamma_1 \). Consider as above a \textit{nonlinearly elastic shell} occupying in its reference configuration the set \( \{ \hat{\Omega}^\varepsilon \}^- \), where \( \hat{\Omega}^\varepsilon = \Theta^\varepsilon(\Omega^\varepsilon) \), \( \Omega^\varepsilon = \omega \times \varepsilon - \varepsilon, \varepsilon \) [2], and the mapping \( \Theta^\varepsilon : \{ \Omega^\varepsilon \}^- \rightarrow \mathbb{R}^3 \) is defined by

\[
\Theta^\varepsilon(y, x_3) = (y, \Theta^\varepsilon(y)) + x_3 \mathbf{a}_3^\varepsilon(y),
\]

for all \( (y, x_3^\varepsilon) \in (\overline{\Omega})^\varepsilon \), where \( \mathbf{a}_3^\varepsilon \) is a continuously varying unit vector normal to the surface \( \Theta^\varepsilon(\overline{\omega}) \) and \( \Phi^\varepsilon : \overline{\omega} \rightarrow \mathbb{R} \) is a function of class \( C^3 \) that satisfies \( \Phi^\varepsilon = \partial_y \Phi^\varepsilon = 0 \) along \( \gamma_1 \). We again assume that there exists a function \( \theta \in C^3(\overline{\omega}) \) \textit{independent of} \( \varepsilon \) such that

\[
\theta^\varepsilon(y) = \varepsilon \theta(y) \quad \text{for all} \quad y \in \overline{\omega}.
\]

The shell is subjected to body forces of density \( (f_1^\varepsilon, f_2^\varepsilon) = (0, 0, f_3^\varepsilon) : \hat{\Omega}^\varepsilon \rightarrow \mathbb{R}^3 \) in its interior \( \Theta^\varepsilon(\Omega^\varepsilon) \); to surface forces of density \( (g_1^\varepsilon, g_2^\varepsilon) = (0, 0, g_3^\varepsilon) : \hat{\Gamma}_+^\varepsilon \cup \hat{\Gamma}_-^\varepsilon \rightarrow \mathbb{R}^3 \) on its upper and lower faces \( \hat{\Gamma}_+^\varepsilon := \Theta^\varepsilon(\Gamma_+^\varepsilon) \) and \( \hat{\Gamma}_-^\varepsilon := \Theta^\varepsilon(\Gamma_-^\varepsilon) \), where \( \Gamma_+^\varepsilon = \omega \times \{ +\varepsilon \} \) and \( \Gamma_-^\varepsilon = \omega \times \{ -\varepsilon \} \); and to applied surface forces of von Kármán’s type on the portion \( \Theta^\varepsilon(\gamma_1 \times [-\varepsilon, \varepsilon]) \) of its lateral face, the remaining portion \( \Theta^\varepsilon(\gamma_2 \times [-\varepsilon, \varepsilon]) \) being \textit{free}. The surface forces of von Kármán type are horizontal and only their resultant \( (\hat{h}_1^\varepsilon, \hat{h}_2^\varepsilon, 0) : \gamma_1^\varepsilon \rightarrow \mathbb{R}^3 \) after integration across the thickness is given along \( \gamma_1^\varepsilon = \Theta^\varepsilon(\gamma_1) \).

The unknown in the three-dimensional formulation is the displacement field \( \hat{u}^\varepsilon = (\hat{u}_1^\varepsilon) : \{ \hat{\Omega}^\varepsilon \}^- \rightarrow \mathbb{R}^3 \), where the functions \( \hat{u}_1^\varepsilon : \{ \hat{\Omega}^\varepsilon \}^- \rightarrow \mathbb{R} \) are its Cartesian components. The unknown displacement field

\[
\hat{u}_1^\varepsilon(y, x_3) = \Theta^\varepsilon(y) + x_3 \mathbf{a}_3^\varepsilon(y).
\]
\( \mathbf{u}^e = (u^e_i) : \Omega^e \rightarrow \mathbb{R}^3 \) then satisfies the following three-dimensional boundary value problem:

\[
- \hat{\sigma}_{ij}^e (\hat{\sigma}_{ikj}^e + \hat{\sigma}_{kj}^e \hat{u}_i^e) = \hat{f}_i^e \quad \text{in} \quad \hat{\Omega}^e,
\]

\( \hat{u}_3^e \) independent of \( \hat{\lambda}_3^e \) and \( \hat{u}_3^e = 0 \) on \( \Theta^e(\gamma_1 \times [-\varepsilon, \varepsilon]) \),

\[
\frac{1}{\varepsilon} \int_{\varepsilon}^1 \left( (\hat{\sigma}_{2\beta}^e + \hat{\sigma}_{\kappa\lambda}^e \hat{\lambda}_k \hat{u}_2^e) \circ \Theta^e \right) \nu_{\beta} \, ds^e = \hat{h}_x^e \circ \Theta^e \quad \text{on} \quad \gamma_1,
\]

\[
(\hat{\sigma}_{ij}^e + \hat{\sigma}_{kij}^e \hat{u}_i^e) n_j^e \circ \Theta^e = 0 \quad \text{on} \quad \gamma_2 \times [-\varepsilon, \varepsilon],
\]

\[
(\hat{\sigma}_{ij}^e + \hat{\sigma}_{kij}^e \hat{u}_i^e) n_j^e \circ \Theta^e = \hat{g}_i^e \circ \Theta^e \quad \text{on} \quad \Gamma^e_+ \cup \Gamma^e_-,
\]

where

\[
\hat{\sigma}_{ij}^e = \hat{\lambda}_i^e \hat{E}_{pp}^e (\hat{\mathbf{u}}^e) \delta_{ij} + 2 \mu^e \hat{E}_{ij}^e (\hat{\mathbf{u}}^e),
\]

\[
\hat{E}_{ij}^e (\hat{\mathbf{u}}^e) = \frac{1}{2} \left( \frac{\hat{\lambda}_i^e}{\hat{\lambda}_j^e} \hat{u}_j^e + \frac{\hat{\lambda}_j^e}{\hat{\lambda}_i^e} \hat{u}_i^e + \hat{\lambda}_j^e \hat{u}_i^e \hat{\lambda}_m^e \hat{u}_m^e \right),
\]

and \( \hat{n}_3^e \) is the unit outer normal vector along the boundary of the set \( \hat{\Omega}^e \).

The boundary conditions along the portion of the lateral face with \( \gamma_1^e \) as its middle line, viz.,

\( \hat{u}_3^e \) independent of \( \hat{\lambda}_3^e \) and \( \hat{u}_3^e = 0 \) on \( \Theta^e(\gamma_1 \times [-\varepsilon, \varepsilon]) \),

mean that only horizontal displacements of equal direction and magnitude are allowed along each vertical segment of the subset \( \Theta^e(\gamma_1 \times [-\varepsilon, \varepsilon]) \) of the lateral face of the shell.

The boundary conditions on \( \Theta^e(\gamma_1 \times [-\varepsilon, \varepsilon]) \) are thus the same as those used by Ciarlet and Paumier [10] for justifying the “classical” Marguerre–von Kármán equations, which correspond to the special case where \( \gamma_1 = \gamma \) (see Section 4). The main novelty is thus that boundary conditions can change their type along the lateral face \( \Theta^e(\gamma_1 \cup \gamma_2 \times [-\varepsilon, \varepsilon]) \).

Gratie [16] has then applied the method of formal asymptotic expansions to this problem, writing the scaled variational problem on the fixed domain \( \Omega = \omega \times [0, 1] \). Then the two-dimensional displacement problem \( P(\omega) \) obtained as the outcome of the asymptotic analysis is the following. Define the function space

\[
\mathbf{V}(\omega) = \{ \eta = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega) ; \eta_3 = \hat{\partial}_x \eta_3 = 0 \text{ on} \ \gamma_1 \}.
\]

Then there exists \( \zeta = (\zeta_i) \in \mathbf{V}(\omega) \) such that the components of the leading term \( \mathbf{u}^0 = (u^0_i) \) of the formal asymptotic expansion of the scaled three-dimensional solution \( \mathbf{u}(\varepsilon) = \mathbf{u}^0 + \varepsilon \mathbf{u}^1 + \varepsilon^2 \mathbf{u}^2 + \varepsilon^3 \mathbf{u}^3 + \cdots \) are again of the form

\[
u^0_3 = \zeta_3 = \zeta_3.
\]
Assume that the set \( \omega \) is simply-connected, that its boundary \( \gamma \) is smooth enough, and that \( \zeta = (\zeta_i) \) is such that \( \zeta \in H^3(\omega) \) and \( \zeta_3 \in H^4(\omega) \). Then the functions \( \tilde{h}_z \in L^2(\gamma) \) defined by \( \tilde{h}_z = h_z \) on \( \gamma_1 \) and \( \tilde{h}_z = 0 \) on \( \gamma_2 \) necessarily satisfy the compatibility relations
\[
\int_{\gamma} \tilde{h}_1 \, d\gamma = \int_{\gamma} \tilde{h}_2 \, d\gamma = \int_{\gamma} (x_1 \tilde{h}_2 - x_2 \tilde{h}_1) \, d\gamma = 0.
\]

Moreover, there exists a scaled Airy function \( \Phi \in H^4(\omega) \), uniquely determined by the conditions \( \Phi(0) = \partial_1 \Phi(0) = \partial_2 \Phi(0) = 0 \), such that
\[
\overline{N}_{11} = \partial_2 \Phi, \quad \overline{N}_{12} = \overline{N}_{21} = -\partial_{12} \Phi, \quad \overline{N}_{22} = \partial_1 \Phi \quad \text{in} \ \omega,
\]
where
\[
\overline{N}_{z\beta} = \frac{4\lambda\mu}{\lambda + 2\mu} \frac{E_0}{2\pi} (\zeta) \delta_{z\beta} + 4\mu E_0 (\zeta) \in L^2(\omega),
\]
with
\[
E_0 (\zeta) = \frac{1}{2} \{ \partial_{z \beta} \zeta + \partial_\beta \zeta_\beta + \partial_{z_3} \zeta_3 \partial_\beta \zeta_3 + \partial_\beta \partial_\beta \partial_{z_3} \zeta_3 + \partial_\beta \partial_\beta \partial_{z_3} \zeta_3 \} \in L^2(\omega).
\]

Finally, the pair \((\zeta_3, \Phi) \in H^4(\omega) \times H^4(\omega)\) satisfies the following scaled generalized Marguerre–von Kármán equations:
\[
\frac{8\mu(\lambda + \mu)}{3(\lambda + 2\mu)} \Delta^2 \zeta_3 = [\Phi, \zeta_3 + \theta] + p_3 \quad \text{in} \ \omega,
\]
\[
\Delta^2 \Phi = -\frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} [\zeta_3, \zeta_3 + 2\theta] \quad \text{in} \ \omega,
\]
\[
\zeta_3 = \partial_\gamma \zeta_3 = 0 \quad \text{on} \ \gamma_1,
\]
\[
m_{z\beta} \nu_{z \beta} = 0 \quad \text{on} \ \gamma_2,
\]
\[
\partial_\gamma m_{z\beta} \nu_{z \beta} + \partial_\gamma (m_{z\beta} \nu_{z \beta} \gamma_{\beta}) = 0 \quad \text{on} \ \gamma_2,
\]
\[
\Phi = \Phi_0 \quad \text{and} \quad \partial_\gamma \Phi = \Phi_1 \quad \text{on} \ \gamma,
\]
where
\[
m_{z\beta} = -\frac{1}{3} \left\{ \frac{4\lambda\mu}{\lambda + 2\mu} \Delta \zeta_3 \delta_{z\beta} + 4\mu \partial_{z\beta} \zeta_3 \right\},
\]
\[
\Phi_0(y) = -y_1 \int_{\gamma(y)} \tilde{h}_2 \, d\gamma + y_2 \int_{\gamma(y)} \tilde{h}_1 \, d\gamma + \int_{\gamma(y)} (x_1 \tilde{h}_2 - x_2 \tilde{h}_1) \, d\gamma, \quad y \in \gamma,
\]
\[
\Phi_1(y) = -v_1 \int_{\gamma(y)} \tilde{h}_2 \, d\gamma + v_2 \int_{\gamma(y)} \tilde{h}_1 \, d\gamma, \quad y \in \gamma.
\]

Remark. If \( \theta \equiv 0 \) in \( \overline{\omega} \), the shallow shell becomes a plate and the generalized Marguerre–von Kármán equations reduce to the generalized von Kármán equations.
Once the generalized Marguerre–von Kármán equations are derived under the assumption that their solutions are smooth, they can, like the generalized von Kármán equations, be studied for their own sake, in particular regarding the existence of less smooth solutions.

The first result in this direction consists in showing that finding a solution \((\zeta, \Phi)\) in the sense of distributions to these equations amounts to solving a single “cubic” operator equation, whose single unknown is (proportional to) the function \((\zeta + \theta)\), the scaled Airy function \(\Phi\) being then obtained by solving a linear boundary value problem.

**Theorem 1.** Assume that the domain \(\omega\) is simply-connected and that the functions \(\tilde{h}_x \in L^2(\gamma)\) defined by \(h_x = h_x\) on \(\gamma_1\) and \(h_x = 0\) on \(\gamma_2\) satisfy the compatibility relations

\[
\int_\gamma \tilde{h}_1 \, d\gamma = \int_\gamma \tilde{h}_2 \, d\gamma = \int_\gamma (x_1 \tilde{h}_2 - x_2 \tilde{h}_1) \, d\gamma = 0.
\]

Let \(E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}, \zeta = \sqrt{E} \zeta, \tilde{\theta} = \sqrt{E} \theta, \) and \(f = \sqrt{E} p_3, \) and define the function space

\[V(\omega) = \{\eta \in H^2(\omega): \eta = \partial_\nu \eta = 0 \text{ on } \gamma_1\}.
\]

Then there exist a “cubic” mapping \(\tilde{C} : V(\omega) \rightarrow V(\omega), \) a linear mapping \(\tilde{L} : V(\omega) \rightarrow V(\omega), \) and an element \(\tilde{F} \in V(\omega)\) (their precise definitions are found in Ciarlet and Gratie [7]) such that a pair \((\zeta, \Phi) \in V(\omega) \times H^2(\omega)\) satisfies the scaled generalized Marguerre–von Kármán equations in the sense of distributions if and only if the function \(\tilde{\zeta} = (\zeta + \tilde{\theta}) \in V(\omega)\) satisfies the operator equation

\[
\tilde{C}(\tilde{\zeta}) + (I - \tilde{L}) \tilde{\zeta} - \tilde{F} = 0.
\]

The scaled Airy function \(\Phi \in H^2(\omega)\) is then given as the unique solution in the sense of distributions of

\[
\Delta^2 \Phi = -[\zeta - \tilde{\theta}, \zeta + \tilde{\theta}] \text{ in } \omega,
\]

\[
\Phi = \Phi_0 \quad \text{and} \quad \partial_\nu \Phi = \Phi_1 \quad \text{on } \gamma.
\]

Our second task consists in establishing the existence of solutions to the generalized Marguerre–von Kármán equations, by making use of the operator equation found in Theorem 1.

**Theorem 2.** Let the assumptions be as in Theorem 1. If the norms \(\|h_x\|_{L^2(\gamma_1)}\) are small enough, the generalized Marguerre–von Kármán equations have at least one solution \((\zeta, \Phi) \in V(\omega) \times H^2(\omega)\) in the sense of distributions.

**Proof.** By Theorem 1, it suffices to establish that the cubic operator equation has at least one solution \(\tilde{\zeta} \in V(\omega)\). For any \((\zeta, \eta) \in V(\omega) \times V(\omega), \) we introduce the inner product:

\[
((\zeta, \eta)) = -\int_\omega m_{x \beta} \partial_{x \beta} \eta \, d\omega = \frac{1}{3} \int_\omega \left\{ \frac{4\lambda \mu}{\lambda+2\mu} \Delta \zeta \Delta \eta + 4\mu \partial_{x \beta} \zeta \partial_{x \beta} \eta \right\} \, d\omega.
\]

The space \(V(\omega)\) will be henceforth considered as equipped with this inner product, which makes it a Hilbert space since its associated norm, denoted \(\|\cdot\|\), is equivalent to the norm \(\|\cdot\|_{H^2(\omega)}\) over \(V(\omega)\).
Assume henceforth that therefore there exist a subsequence of the various operators defined in Theorem 1 imply that there is a constant applied to each continuous mapping for all \( X \in R^m \).

Let \( X \cdot Y \) and \(|X|\) denote the Euclidean inner product of \( X, Y \in R^m \) and norm of \( X \in R^m \). Then we define for each \( m \geq 1 \) a mapping \( \Phi^m = (\phi^m_i)_{i=1}^m : R^m \to R^m \) by letting

\[
\Phi^m(X) \cdot X = (\tilde{C}(J^m(X)) + (I - \tilde{L})(J^m(X)) - \tilde{F}, J^m(X)) \\
\geq \left( 1 - c_0 \sum_x \|h_x\|_{L^2(\gamma_1)} \right) |X|^2 - \|\tilde{F}\||X|.
\]

Assume henceforth that

\[
\sum_x \|h_x\|_{L^2(\gamma_1)} < c_0^{-1}
\]

and let

\[
M = \left( 1 - c_0 \sum_x \|h_x\|_{L^2(\gamma_1)} \right)^{-1} \|\tilde{F}\|
\]

so that

\[
\Phi^m(X) \cdot X \geq 0 \quad \text{for all } X \in R^m \text{ that satisfy } |X| = M.
\]

Then a corollary of the Brouwer fixed point theorem (see, e.g., Lions [21, Chapter 1, Lemma 4.3]) applied to each continuous mapping \( \Phi^m : R^m \to R^m \) shows that, for each \( m \geq 1 \), there exists \( X^m \in R^m \) such that \( |X^m| \leq M \) and \( \Phi^m(X^m) = 0 \).

Equivalently, there exists for each \( m \geq 1 \) a function \( \xi^m = J^m(X^m) \in V^m \) such that

\[
\|\xi^m\| \leq M \quad \text{and} \quad (\tilde{C}(\xi^m) + (I - \tilde{L})\xi^m - \tilde{F}, \eta) = 0, \quad \text{for all } \eta \in V^m.
\]

Therefore there exist a subsequence \( (\xi^n)_{n=1}^\infty \) of the sequence \( (\xi^m)_{m=1}^\infty \) and an element \( \tilde{\xi} \in V(\omega) \) such that \( \xi^n \rightharpoonup \tilde{\xi} \) in \( V(\omega) \). Given any \( \eta \in V(\omega) \), there exist functions \( \eta^n \in V^n \) such that \( \eta^n \to \eta \in V(\omega) \), so that

\[
((\tilde{C}(\xi^n) + (I - \tilde{L})\xi^n - \tilde{F}, \eta^n)) = 0, \quad \text{for all } n \geq 1.
\]
The cubic mapping $\tilde{C}$ and the linear mapping $\tilde{L}$ are such that

$$((\tilde{C}(\xi^n), \eta^n)) \to ((\tilde{C}(\tilde{\xi}), \eta)) \quad \text{and} \quad ((\tilde{L}(\xi^n), \eta^n)) \to ((\tilde{L}(\tilde{\xi}), \eta)).$$

Passing to the limit as $n \to \infty$, we thus obtain

$$((\tilde{C}(\tilde{\xi}) + (I - \tilde{L})\tilde{\xi} - \tilde{F}, \eta)) = 0, \quad \text{for any } \eta \in V(\omega).$$

Hence $\tilde{\xi}$ is a solution to the cubic operator equation and the proof is complete. □

6. Concluding remarks

(a) It is most likely, although this claim remains to be substantiated by a proof, that all the equations reviewed in this paper, from the classical von Kármán equations to the generalized Marguerre–von Kármán equations, and so far only obtained by means of a formal asymptotic procedure, can be rigorously justified from the equations of three-dimensional nonlinear elasticity by means of a convergence theorem as $\varepsilon \to 0$ based on gamma-convergence theory, in the spirit of the landmark contributions of Le Dret and Raoult [20] and Friesecke et al. [13].

(b) A characteristic of the operator equation found in Theorem 1 is the “loss of strict positivity” incurred by its cubic part, since the relation

$$((\tilde{C}(\xi^n), \eta^n)) = \int_\omega B(\xi, \eta)[\xi, \eta] \, d\omega = \|\Delta B(\xi, \eta)\|_{L^2(\omega)}^2 \geqslant 0 \quad \text{for all } \eta \in V(\omega),$$

needed in the proof of Theorem 2, shows that there exist (easily constructed) nonzero functions $\eta \in V(\omega)$ that satisfy $[\eta, \eta] = 0$ in $\omega$ when $\text{length}(\gamma_2) > 0$. By contrast, $((\tilde{C}(\eta), \eta)) > 0$ for all $\eta \in V(\omega) = H^2_0(\omega)$ when $\gamma_2 = \emptyset$ (see Ciarlet [5, Theorem 5.8-2]). This observation thus precludes the recourse to the topological degree as in Goeleven et al. [14] or to pseudo-monotone operators as in Gratie [15], for solving the operator equation.

(c) How to get rid of the assumption that the norms $\|h_2\|_{L^2(\gamma_1)}$ are “small enough” in Theorem 2 seems to be a challenging question, and especially so since the recourse to techniques from the calculus of variations, as in Ciarlet and Rabier [11], is precluded. Indeed, there is no longer an associated functional to minimize, because, by contrast with the case when $\gamma_1 = \gamma$, the linear mapping $\tilde{L}$ is no longer symmetric with respect to the inner product $((.,.))$.

(d) The lack of positivity of the nonlinear term similarly precludes the use of standard numerical analysis techniques for establishing the convergence of the finite element approximation, such as those used by Kesavan [19]. There are good hopes, however, that the compactness method of Lions used in the proof of Theorem 2 can be also used for establishing the weak convergence in the space $V(\omega)$ of a finite element approximation; see [8].

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References