Liouville-type theorems for polyharmonic systems in $\mathbb{R}^N$

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Abstract

In this note, we consider the polyharmonic system $(-\Delta)^m u = v^\alpha$, $(-\Delta)^m v = u^\beta$ in $\mathbb{R}^N$ with $N > 2m$ and $\alpha \geq 1$, $\beta \geq 1$, where $(-\Delta)^m$ is the polyharmonic operator. For $1/(\alpha + 1) + 1/(\beta + 1) > (N - 2m)/N$, we prove the non-existence of non-negative, radial, smooth solutions. For $1 < \alpha, \beta < (N + 2m)/(N - 2m)$, we show the non-existence of non-negative smooth solutions. In addition, for either $(N - 2m)\beta < N\alpha + 2m$ or $(N - 2m)\alpha < N\beta + 2m$ with $\alpha, \beta > 1$, we show the non-existence of non-negative smooth solutions for polyharmonic system of inequalities $(-\Delta)^m u \geq v^\alpha$, $(-\Delta)^m v \geq u^\beta$. More general, we can prove that all the above results hold for the system $(-\Delta)^m u = v^\alpha$, $(-\Delta)^n v = u^\beta$ in $\mathbb{R}^N$ with $N > \max\{2m, 2n\}$ and $\alpha \geq 1$, $\beta \geq 1$.

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1. Introduction

The classical Liouville theorem from function theory says that every bounded entire function is constant. In terms of differential equation one has, for instance, a superharmonic function defined in the whole plane \( \mathbb{R}^2 \), which is bounded below, is constant. In this work we consider the following polyharmonic system:

\[
(-\Delta)^m u = v^\alpha, \\
(-\Delta)^m v = u^\beta \quad \text{in} \quad \mathbb{R}^N.
\]  

(1.1)

We are interested in Liouville-type results, i.e., we want to determine for which positive real values of the exponents \( \alpha \) and \( \beta \) \((u, v) = (0, 0)\) is the only non-negative entire solution of the system. Here, we say a solution \((u, v)\) of system (1.1) is one in the classical sense, i.e., \(u, v \in C^2(\mathbb{R}^N)\).

For \(m = 1\), this type of problems were well studied, see, for example, Mitidieri [7] proved that if \(\alpha, \beta > 1\) and

\[
\frac{1}{\alpha + 1} + \frac{1}{\beta + 1} > \frac{N - 2}{N},
\]

then the system

\[
\Delta u + u^\alpha = 0, \\
\Delta v + v^\beta = 0 \quad \text{in} \quad \mathbb{R}^N
\]

(1.2)

has no non-negative, radial \(C^2\) solutions. Souto in [8] showed that if

\[
\frac{1}{\alpha + 1} + \frac{1}{\beta + 1} > \frac{N - 2}{N - 1}, \quad \alpha, \beta > 0,
\]

then (1.2) has no positive solutions. A further result was given in a paper of de Figueiredo and Felmer [4], the authors proved that if

\[
0 < \alpha, \beta \leq \frac{N + 2}{N - 2} \quad \text{but not both equal to} \quad \frac{N + 2}{N - 2},
\]

then system (1.2) has no positive \(C^2\) solutions.

We point out that it is still an open problem whether (1.2) has no non-negative solutions under assumption

\[
\frac{N - 2}{N - 1} > \frac{1}{\alpha + 1} + \frac{1}{\beta + 1} > \frac{N - 2}{N} \quad \text{and} \quad \left(\alpha > \frac{N + 2}{N - 2} \quad \text{or} \quad \beta > \frac{N + 2}{N - 2}\right).
\]

As concerned with the generalization of the above mentioned results to the system (1.1) for \(m > 1\), the method of moving planes naturally comes to our mind. However, since the Maximum principle cannot be directly applied if we do not know enough information
about \((-\Delta)^i u\) and \((-\Delta)^i v\), \(i = 1, 2, \ldots, m - 1\). In this work, we first study the properties of \((-\Delta)^i u\), \((-\Delta)^i v\) \((i = 1, 2, \ldots, m - 1)\) for the positive solution \((u, v)\) of system (1.1), and then by using the method of moving planes, we show that the quoted results in [4,7] for system (1.2) can be extended to system (1.1). At the same time, as an immediately consequence of the positive properties of \((-\Delta)^i u\), \((-\Delta)^i v\), \(i = 1, 2, \ldots, m - 1\), we get the non-existence results for system of inequalities:

\[
(-\Delta)^m u \geq v^\alpha, \quad (-\Delta)^m v \geq u^\beta \quad \text{in } \mathbb{R}^N.
\]

Here are our main results.

**Theorem 1.1.** If \(N > 2m\), \(\alpha, \beta \geq 1\), but not both equal to 1 are such that

\[
\frac{1}{\alpha + 1} + \frac{1}{\beta + 1} > \frac{N - 2m}{N},
\]

then system (1.1) has no radial non-negative solutions in \(C^{2m}(\mathbb{R}^N)\).

**Theorem 1.2.** (I) If \(1 \leq \alpha, \beta \leq \frac{N + 2m}{N - 2m}\) but not both equal to 1 neither to \(\frac{N + 2m}{N - 2m}\), then the only non-negative \(C^{2m}\) solution of system (1.1) in the whole of \(\mathbb{R}^N\) is the trivial one: \((u, v) = (0, 0)\).

(II) If \(\alpha = \beta = \frac{N + 2m}{N - 2m}\), then \(u\) and \(v\) are radially symmetric with respect to some point of \(\mathbb{R}^N\).

**Theorem 1.3.** Suppose that \((u, v) \in C^{2m}(\mathbb{R}^N) \times C^{2m}(\mathbb{R}^N)\) is a non-negative solution of system (1.3). If \(\alpha, \beta \geq 1\) satisfying either

\[
(N - 2m)\beta < \frac{N}{\alpha} + 2m \quad \text{or} \quad (N - 2m)\alpha < \frac{N}{\beta} + 2m,
\]

then \((u, v) \equiv 0\).

Our proofs are strongly motivated by the works of de Figueiredo and Felmer [4], Lin [6], Xu [9], Mitidieri [7], Busca and Manásevich [1] and Felmer [3]. Indeed, the proof of Theorem 1.1 presented in Section 4 uses an idea of Mitidieri, which relies on the application of a Rellich-type identity, while the proof of Theorem 1.3 is new somehow. In order to prove Theorem 1.2, we will show that the entire positive solution of the system (1.1) must satisfy \((-\Delta)^i u \geq 0\), \((-\Delta)^i v \geq 0\) \((i = 1, 2, \ldots, m - 1)\). This is the key part of the whole paper and will be done in Section 2. Section 3 is devoted to the proof of the decay properties of solution after the Kelvin’s transform. In Section 5, we apply the method of the moving planes to prove Theorem 1.2. In our case, the main difficulty in applying the method comes from the fact that the Maximum principle cannot be applied directly to \((u, v)\). To overcome this difficulty, we follow an idea of Xu [9] for the single equation and apply the moving
planes method for both \((-\Delta)^i u, (-\Delta)^j v\) \((i = 1, 2, \ldots, m - 1)\) and \((u, v)\). Contrarily to [4] we use no additional change of variables except for Kelvin’s transforms.

As far as we know this is the first work concerning Liouville-type results for a system involving the polyharmonic operator. However, some related questions remain unsolved. For example, it is not known whether Theorem 1.2 is true for \(0 < \alpha, \beta < 1\), even for the case of \(m = 1\).

By using the same argument as we did for system (1.1), we can prove the corresponding results for system

\[
(-\Delta)^m u = v^\alpha, \quad (-\Delta)^n v = u^\beta \quad \text{in} \quad \mathbb{R}^N.
\]

**Theorem 1.1’.** If \(N > \max\{2m, 2n\}, \alpha, \beta \geq 1\), but not both equal to 1 are such that

\[
\frac{1}{\alpha + 1} + \frac{1}{\beta + 1} > \min\left\{\frac{N - 2m}{N}, \frac{N - 2n}{N}\right\},
\]

then system (1.1’) has no radial non-negative solutions in \(C^{2m}(\mathbb{R}^N)\).

**Theorem 1.2’.** If \(1 < \alpha, \beta < \min\{\frac{N + 2m}{N - 2n}, \frac{N + 2n}{N - 2m}\}\) but not both equal to 1 neither to \(\frac{N + 2m}{N - 2m}\) (in case \(m = n\)), then the only non-negative \(C^{2m}\) solution of system (1.1’) in the whole of \(\mathbb{R}^N\) is the trivial one: \((u, v) = (0, 0)\).

**Theorem 1.3’.** Suppose \((u, v) \in C^{2m}(\mathbb{R}^N) \times C^{2n}(\mathbb{R}^N)\) is a non-negative solution of system (1.3). If \(\alpha, \beta \geq 1\) satisfying

\[
(N - 2m)\beta < \frac{N}{\alpha} + 2n \quad \text{or} \quad (N - 2n)\alpha < \frac{N}{\beta} + 2m,
\]

then \((u, v) \equiv (0, 0)\).

Before the conclusion of this introduction, we would like to mention that Liouville theorems play a very important role in the study of the existence of the solutions for a non-variational system, which leads naturally to the questions answered by theorems of Liouville-type. For more details, we refer the readers to [2].

### 2. General superharmonic property of the solutions

Let \((u, v) \in C^{2m}(\mathbb{R}^N) \times C^{2m}(\mathbb{R}^N)\) be a positive solution of the system (1.1). We say that \((u, v)\) satisfies the general superharmonic property if

\[
(-\Delta)^i u(x) \geq 0, \quad (-\Delta)^j v(x) \geq 0, \quad \text{for} \quad i = 1, 2, \ldots, m - 1.
\]
In what follows, we always use \( \bar{u} \) to denote the spherical average of \( u \), i.e.,

\[
\bar{u}(r) := \frac{1}{\omega_N r^{N-1}} \int_{\partial B_r(0)} u \, d\sigma,
\]

where \( \omega_N \) denotes the measure of unit sphere in \( \mathbb{R}^N \).

**Lemma 2.1.** Assume \( w \in C^2(\mathbb{R}^N) \) satisfying

\[
\Delta \bar{w} \geq c(r - r_0)^p, \quad \text{for all } r \geq r_0, \quad \bar{w}'(r_0) \geq 0, \quad \bar{w}(r_0) \geq 0, \tag{2.2}
\]

where \( \Delta = \frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr} \), and \( p \geq 0, \ c > 0, \ r_0 \geq 0 \). Then

\[
\bar{w}(r) \geq \frac{c(r - r_0)^{p+1}}{2^{N-1}(p + 2)(N + p)}, \quad \text{for all } r \geq r_0. \tag{2.3}
\]

**Proof.** Multiplying (2.2) by \( r^{N-1} \) and integrating the inequality, we get, if \( r_0 \leq r \leq 2r_0 \),

\[
\bar{w}'(r) \geq c \int_{r_0}^{r} \left( \frac{s}{r} \right)^{N-1} (s - r_0)^p \, ds \geq c \int_{r_0}^{r} (s - r_0)^p \, ds = \frac{c(r - r_0)^{p+1}}{2^{N-1}(p + 1)}. \tag{2.4}
\]

If \( r > 2r_0 \), a similar computation leads to

\[
\bar{w}'(r) \geq \frac{c}{r^{N-1}} \int_{r_0}^{r} s^{N-1}(s - r_0)^p \, ds \geq \frac{c}{r^{N-1}} \int_{r_0}^{r} (s - r_0)^{p+N-1} \, ds
\]

\[= \frac{c(r - r_0)^{p+1}}{N + p} \left( \frac{r - r_0}{r} \right)^{N-1} \geq \frac{c(r - r_0)^{p+1}}{2^{N-1}(p + N)}. \tag{2.5}
\]

Combining (2.4) with (2.5), we obtain

\[
\bar{w}'(r) \geq \frac{c(r - r_0)^{p+1}}{2^{N-1}(p + N)}.
\]

Noting that \( \bar{w}(r_0) > 0 \), again, we integrate the above inequality and obtain the desired inequality of (2.3). \( \square \)

**Theorem 2.2.** If \( (u, v) \in C^{2m}(\mathbb{R}^N) \times C^{2m}(\mathbb{R}^N) \) is a positive solution of system (1.1) with \( \alpha\beta > 1 \) and \( N > 2m \). Then we have

\[
(-\Delta)^{i} u(x) \geq 0, \quad (-\Delta)^{i} v(x) \geq 0, \quad i = 1, 2, \ldots, m - 1.
\]
Proof. Step 1. We show that \((-\Delta)^{m-1}u(x) \geq 0\), and \((-\Delta)^{m-1}v(x) \geq 0\).
We prove this by contradiction. Without loss of generality, we assume \((-\Delta)^{m-1}u(0) < 0\).
Set \(w_{m-1} = (-\Delta)^{m-1}u, z_{m-1} = (-\Delta)^{m-1}v\). Denote by \(\bar{w}_{m-1}, \bar{z}_{m-1}, \bar{u}\) and \(\bar{v}\) the spherical average of \(w_{m-1}, z_{m-1}, u\) and \(v\), respectively. Then it is easy to see that
\[
\begin{align*}
-(-\Delta)^{m-1}\tilde{u} + \bar{w}_{m-1} &= 0, & \Delta\bar{w}_{m-1} + \tilde{v}\alpha \leq 0, \\
-(-\Delta)^{m-1}\tilde{v} + \bar{z}_{m-1} &= 0, & \Delta\bar{z}_{m-1} + \tilde{u}\beta \leq 0.
\end{align*}
\] (2.6)
Multiply the first inequality in (2.6) by \(r^{N-1}\) and integrate the resulting inequality, we have
\[
\frac{d}{dr} \bar{w}_{m-1} \leq 0,
\]
hence
\[
\bar{w}_{m-1}(r) \leq \bar{w}_{m-1}(0) = w_{m-1}(0) < 0, \quad \text{for all } r > 0. \tag{2.7}
\]
In the following, we discuss two cases:

Case 1. \(m\) is an odd number. It follows from (2.6),
\[
\Delta^{m-1}\bar{u} = \tilde{w}_{m-1} \leq \bar{w}_{m-1}(0) = w_{m-1}(0),
\]
multiplying the above inequality by \(r^{N-1}\) and integrating the resulting equality twice, we get
\[
\Delta^{m-2}\bar{u}(r) \leq \Delta^{m-2}\bar{u}(0) + \frac{w_{m-1}(0)}{2N}r^{2}.
\] (2.8)
Repeating the above process step by step, we obtain
\[
\Delta^{m-i}\bar{u}(r) \leq \Delta^{m-i}\bar{u}(0) + \sum_{l=1}^{i-1} \left[ \Delta^{m-i+l}\bar{u}(0) \frac{r^{2l}}{\prod_{k=1}^{l}(2k) \prod_{k=1}^{l}[N + 2(k - 1)]} \right] \tag{2.9}
\]
for \(i = 3, 4, 5, \ldots, m\). Consequently,
\[
\bar{u}(r) \leq \bar{u}(0) + \sum_{l=1}^{m-1} \left[ \Delta^{l}\bar{u}(0) \frac{r^{2l}}{\prod_{k=1}^{l}(2k) \prod_{k=1}^{l}[N + 2(k - 1)]} \right]. \tag{2.10}
\]
Since \(\Delta^{m-1}\tilde{u}(0) = \bar{w}_{m-1} = w_{m-1}(0) < 0\), we can deduce from (2.10) that \(\bar{u}(r) \to -\infty\) as \(r \to +\infty\), this is a contradiction since \(u(r)\) is positive.
Case 2. \( m \) is an even number. To proceed, we first prove some auxiliary facts. Let \( C_0 = \max\{N + 2(m - 1), 2^{(N-1)m}\} \), \( p_0 = 0 \), \( q_0 = 0 \), and \( \{p_k\}, \{q_k\} \) be the sequences defined inductively by

\[
p_{k+1} = \alpha q_k + 2m, \quad q_{k+1} = \beta p_k + 2m,
\]

then

\[
q_k \to \infty, \quad p_k \to \infty,
\]

thus

\[
\lim_{k \to \infty} \frac{\alpha q_{k-2} + C_0}{\alpha q_k + C_0} = \frac{1}{\alpha \beta}, \quad \lim_{k \to \infty} \frac{\beta p_{k-2} + C_0}{\beta p_k + C_0} = \frac{1}{\alpha \beta}.
\]

Therefore we can choose \( M > 1, \epsilon_1 > 0, C_1 > 0 \) such that

\[
\epsilon_1^{1+\alpha} M^{\alpha \beta - 1} > 1, \quad \frac{C_1^{\alpha(\alpha \beta - 1)}}{C_0^{2m(\alpha \beta - 1)+\alpha \beta+\alpha} (2m \beta + C_0)^{2m \alpha} (2m \alpha \beta + 2m \alpha + C_0)^{2m} > M,
\]

and

\[
\left( \frac{\alpha q_{k-2} + C_0}{\alpha q_k + C_0} \right)^{2m} \geq \epsilon_1, \quad \left( \frac{\beta p_{k-2} + C_0}{\beta p_k + C_0} \right)^{2m} \geq \epsilon_1, \quad k \geq 2.
\]

Let \( A_0 = C_1, B_0 = C_1 \), we define the sequences \( \{A_k\} \) and \( \{B_k\} \) inductively by

\[
A_{k+1} = \frac{B_k^\alpha}{C_0(\alpha q_k + C_0)^{2m}}, \quad B_{k+1} = \frac{A_k^\beta}{C_0(\beta p_k + C_0)^{2m}}.
\]

We claim that \( \frac{A_{2k+1}}{A_{2k-1}} \geq M, k \geq 1 \). We prove it by mathematical induction. Indeed, it follows from (2.12), that \( A_3/A_1 > M \). Suppose that \( A_{2k+1}/A_{2k-1} \geq M \) for some \( k \). Then from (2.13) we have

\[
\frac{A_{k+1}}{A_{k-1}} = \left( \frac{B_k}{B_{k-2}} \right)^\alpha \left( \frac{\alpha q_{k-2} + C_0}{\alpha q_k + C_0} \right)^{2m} \geq \epsilon_1 \left( \frac{B_k}{B_{k-2}} \right)^\alpha,
\]

\[
\frac{B_{k+1}}{B_{k-1}} = \left( \frac{A_k}{A_{k-2}} \right)^\beta \left( \frac{\beta p_{k-2} + C_0}{\beta p_k + C_0} \right)^{2m} \geq \epsilon_1 \left( \frac{A_k}{A_{k-2}} \right)^\beta.
\]

Thus

\[
\frac{A_{k+1}}{A_{k-1}} \geq \epsilon_1^{\alpha+1} \left( \frac{A_{k-1}}{A_{k-3}} \right)^{\alpha \beta}.
\]
Note that $\epsilon_1^{1+\alpha} M^{\alpha \beta - 1} > 1$, it follows from (2.14) that

$$\frac{A_{2(k+1)+1}}{A_{2(k+1)+1}^{-1}} \geq \epsilon_1^{\alpha+1} \left( \frac{A_{2k+1}}{A_{2k-1}} \right)^{\alpha \beta} \geq \epsilon_1^{\alpha+1} M^{\alpha \beta} = \epsilon_1^{\alpha+1} M^{\alpha \beta - 1} \cdot M \geq M.$$ 

This proves the claim. Thus, $A_{2k+1} \to +\infty$ as $k \to \infty$.

Now we return to the proof of $(-\Delta)^{m-1} u \geq 0$. Since $m$ is even, it follows from (2.6)

$$
\Delta^{m-1} \tilde{u}(r) = -\tilde{w}_{m-1}(r) \geq -\tilde{w}_{m-1}(0) = -w_{m-1}(0) > 0. 
$$

(2.15)

By the similar arguments as we did in the proof of Case 1, we have for $i = 2, \ldots, m$

$$
\Delta^{m-i} \tilde{u}(r) \geq \Delta^{m-i} \tilde{u}(0) + \sum_{l=1}^{i-2} \Delta^{m-i+l} \tilde{u}(0) r^{2l} \prod_{k=1}^l (2k) \prod_{k=1}^{l-1} [N + 2(k - 1)] \\
- \frac{w_{m-1}(0) r^{2(i-1)}}{\prod_{k=1}^{i-1} (2k) \prod_{k=1}^{l-1} [N + 2(k - 1)]},
$$

(2.16)

Since $w_{m-1} < 0$, there is an $\bar{r} > 0$ such that

$$
\tilde{u}(r) \geq A_0, \quad \text{for all } r \geq \bar{r},
$$

(2.17)

$$
\frac{d}{dr} \Delta^{m-i} \tilde{u}(r) \geq 0, \quad \Delta^{m-i} \tilde{u}(r) \geq 0, \quad i = 1, 2, \ldots, m, \quad r \geq \bar{r}.
$$

Therefore, from (2.6) we have

$$
\Delta^m \tilde{v} \geq \tilde{u}^{\beta} \geq A_0^\beta \geq 0, \quad \text{for all } r \geq \bar{r}.
$$

From here, one can proceed the similar arguments as we did for $\tilde{u}$ and show that there exists $r_0 \geq \bar{r}$ such that

$$
\tilde{v}(r) \geq B_0, \quad \text{for all } r \geq r_0,
$$

(2.18)

$$
\frac{d}{dr} \Delta^{m-i} \tilde{v}(r_0) \geq 0, \quad \Delta^{m-i} \tilde{v}(r_0) \geq 0, \quad i = 1, 2, \ldots, m.
$$

We show that

$$
\tilde{u}(r) \geq A_k (r - r_0)^{p_k}, \quad \tilde{v}(r) \geq B_k (r - r_0)^{q_k}, \quad \text{for } k \geq 0, \quad r \geq r_0. 
$$

(2.19)

In fact, for $k = 0$, (2.19) follows from (2.17), (2.18). Then (2.19) can be proved by induction. For example, for $k = 1$, by Lemma 2.1

$$
\tilde{u}(r) \geq B_0^\alpha (r - r_0)^{2m} \prod_{l=1}^m (2l) \prod_{l=1}^{m} [N + 2(i - 1)] \geq \frac{B_0^\alpha (r - r_0)^{2m}}{C_0^{2m+1}} = A_1 (r - r_0)^{p_1}
$$

(2.19)
and
\[ v(r) \geq \frac{A_0^\beta (r - r_0)^{2m}}{2(N-1)^m \prod_{i=1}^m (2i) \prod_{i=1}^m [N + 2(i - 1)]} \geq \frac{A_0^\beta (r - r_0)^{2m}}{C_0^{2m+1}} = B_1(r - r_0)^{q_1}. \]

Taking \( r = r_0 + 1 \) in (2.19) deduces that
\[ \bar{u}(r_0 + 1) \geq A_k \rightarrow \infty, \quad \text{as} \quad k \rightarrow \infty. \]

A contradiction. We complete the step 1.

**Step 2.** We show that \((-\Delta)^{m-i}u(x) \geq 0, (-\Delta)^{m-i}v(x) \geq 0, i = 2, 3, \ldots, m - 1.\)

We prove this by induction \( i \). In fact, by step 1, this is true for \( i = 1 \). Assume that \((-\Delta)^{m-i+1}u \geq 0, (-\Delta)^{m-i+1}v \geq 0.\) Let \((-\Delta)^{m-i}u = w_{m-i}, (-\Delta)^{m-i}v = z_{m-i}\) and suppose, by contrary, that there is \( x_0 \in \mathbb{R}^n \) such that \( w_{m-i}(x_0) < 0 \) (without loss of generalization, we assume that \( x_0 = 0 \)). Then system (1.1) can be written as
\[ (-\Delta)^{m-i}u = w_{m-i}, \quad (-\Delta)^i w_{m-i} = v^\alpha, \]
\[ (-\Delta)^{m-i}v = z_{m-i}, \quad (-\Delta)^i z_{m-i} = u^\beta. \]

From this, by using the similar arguments as we did in step 1, we have
\[ \tilde{w}_{m-i}(r) \leq \tilde{w}_{m-i}(0) = w_{m-i}(0) < 0, \quad \text{for all} \quad r > 0. \]

Again, we discuss two cases: \( m - i \) is an odd number and \( m - i \) is an even number. In both of the cases, we follow the same arguments in step 1 and lead to a contraction. Thus we complete the proof of step 2, and therefore Theorem 2.2. \( \square \)

The next lemma will be used in the next proposition.

**Lemma 2.3.** Suppose that \((u, v)\) is the positive solution of system (1.1), \( w_{m-1}, z_{m-1} \) are defined as before. Let \( 1 < \alpha, \beta \) be such that \( \alpha \beta \neq 1 \). Then there exists a positive constant \( C = C(n) \) such that
\[ \tilde{w}_{m-1}(r) \leq C r^{-(2(m-1)+(2m(\alpha+1))/((\alpha\beta-1))}, \quad \tilde{z}_{m-1}(r) \leq C r^{-(2(m-1)+(2m(\beta+1))/((\alpha\beta-1))). \]

**Proof.** Let
\[ u_{m-i} = (-\Delta)^{m-i}u, \quad i = 1, 2, \ldots, m - 1. \]

Then we have
\[ \Delta \tilde{u}_{m-1} + \tilde{v}^\alpha \leq 0, \]
\[ \Delta \tilde{u}_{m-i} + \tilde{u}_{m-i+1} = 0, \quad i = 2, 3, \ldots, m - 1, \]
\[ \Delta \tilde{u} + \tilde{u}_1 = 0. \]
Since \( \bar{u}, \bar{u}_i \geq 0 \) \( (i = 1, 2, \ldots, m-1) \), by [9, Lemma 3.1]
\[
\bar{u}_{m-1}(r) \geq cr^2 \bar{u}, \quad \bar{u}_{i-1}(r) \geq cr^2 \bar{u}_i, \quad i = m-1, \ldots, 2.
\]
Hence
\[
\bar{u} \geq cr^2 \bar{u}_1 \geq c^2 r^4 \bar{u}_2 \geq \cdots \geq c^{m-1} r^{2m-2} \bar{u}_{m-1} \geq c^m r^{2m} \bar{u} \alpha.
\] (2.20)
Similarly
\[
\bar{v} \geq cr^2 \bar{v}_1 \geq c^2 r^4 \bar{v}_2 \geq \cdots \geq c^{m-1} r^{2m-2} \bar{v}_{m-1} \geq c^m r^{2m} \bar{v} \beta.
\] (2.21)
Combine (2.20) with (2.21), we get
\[
\bar{u} \geq Cr^{2m(\alpha+1)} \bar{u} \alpha \beta.
\] (2.22)
It follows from (2.20)
\[
\bar{w}_{m-1} = \bar{u}_{m-1} \leq Cr^{-(2m-2)} \bar{u}(r) \leq Cr^{-(2m-2)} \frac{2m(\alpha+1)}{\alpha^2-1}.
\]
In a same way, one has
\[
\bar{z}_{m-1} = \bar{v}_{m-1} \leq Cr^{-(2m-2)} \bar{v}(r) \leq Cr^{-(2m-2)} \frac{2m(\beta+1)}{\beta^2-1}.
\]
Therefore we finish the proof of Lemma 2.3. \( \Box \)

**Proposition 2.4.** Let \((u, v) \in C^{2m}(\mathbb{R}^N) \times C^{2m}(\mathbb{R}^N)\) be a positive solution of system (1.1) with \(1 < \alpha, \beta \leq \frac{N+2m}{N-2m}\). Then
\[
\int_{\mathbb{R}^N \setminus B_1(0)} |x|^{2m-N} u^\beta \, dx \leq C, \quad \int_{\mathbb{R}^N \setminus B_1(0)} |x|^{2m-N} v^\alpha \, dx \leq C,
\]
for some constant \(C\).

**Proof.** First of all, we claim that there exists a sequence \(\{R_i\}\) such that
\[
-R_i^{2m-1} \bar{w}_{m-1}'(R_i) \to 0, \quad \text{as } R_i \to \infty, \quad \text{ (2.23)}
\]
\[
-R_i^{2m-1} \bar{z}_{m-1}'(R_i) \to 0, \quad \text{as } R_i \to \infty. \quad \text{ (2.24)}
\]
We only prove (2.23). The proof of (2.24) is similar.
Suppose by contrary that there exist constants \(\delta_0 > 0\) and \(r_0 > 0\) such that for all \(R > r_0, -R^{2m-1} \bar{w}_{m-1}'(R) \geq \delta_0 > 0\). By Lemma 2.3, we have
\[
\bar{w}_{m-1}(R) \leq cR^{-(2m-2+\frac{2m(\alpha+1)}{\alpha^2-1})},
\]
hence
\[-\bar{w}'_{m-1}(R_0)R = \int_{R}^{2R} (-\bar{w}'_{m-1}(r)) \, dr = \bar{w}_{m-1}(R) - \bar{w}_{m-1}(2R) \leq \bar{w}_{m-1}(R) \]
\[\leq c R^{-(2m-2 + \frac{2m(\alpha+1)}{\alpha(\alpha-1)})},\]
where \( R < R_0 < 2R \). So
\[-R^{2m-1} \bar{w}'_{m-1}(R) \leq CR^{\frac{2m(\alpha+1)}{\alpha(\alpha-1)}} \rightarrow 0, \quad \text{as} \quad R \rightarrow \infty, \]
this contradicts with the beginning assumptions. Our claim is proved. Since \(-\Delta w_{m-1} = v^\alpha\), by taking the spherical average on the both side of the equation and multiplying the resulting equation by \( r^{N-1} \), we obtain
\[(r^{N-1} \bar{w}'_{m-1})' + r^{N-1} \frac{1}{r^{N-1} \omega_N} \int_{\partial B_r} v^p \, d\sigma = 0. \tag{2.25} \]
Therefore, we have for any \( R > 1 \)
\[\int_{B_R \setminus B_1} |x|^{2m-N} v^\alpha \, dx = \int_1^R r^{N-1} \frac{1}{\omega_N r^{N-1}} \int_{\partial B_r} v^\alpha r^{2m-N} \, d\sigma \, dr \]
\[= - \int_1^R r^{2m-N} (r^{N-1} \bar{w}'_{m-1}(r))' \, dr \]
\[= -R^{2m-1} \bar{w}'_{m-1}(R) + \bar{w}'_{m-1}(1) + (2m - N)R^{2m-2} \bar{w}_{m-1}(R) \]
\[- (2m - N)(2m - 2) \int_1^R r^{2m-3} \bar{w}_{m-1}(r) \, dr - (2m - N)\bar{w}_{m-1}(1). \tag{2.26} \]
By our claim, there exists a sequence \( \{R_i\} \) such that the first term on the right-hand side of (2.26) goes to zero. On the other hand, it is easy to see that all the other terms on the right-hand side of (2.26) are finite for all \( R \). Thus for this sequence of \( \{R_i\} \), the right-hand side of (2.26) is finite. Note that \( \int_{B_R \setminus B_1} |x|^{2m-N} v^\alpha \, dx \) is monotone with respect to \( R \). Thus it is finite for all \( R \). Let \( R \) go to infinity, we get
\[\int_{\mathbb{R}^N \setminus B_1} |x|^{2m-N} v^\alpha \, dx \leq C.\]
Similarly we can prove that
\[ \int_{\mathbb{R}^N \setminus B_1} |x|^{2m-N} u^\beta \, dx \leq C. \]

3. Kelvin’s transformation and asymptotic behavior

Let \((u, v)\) be a \(C^{2m}(\mathbb{R}^N)\) positive solution of system (1.1). We introduce their Kelvin’s transformation
\[
    u^*(x) = |x|^{2m-N} u\left(\frac{x}{|x|^2}\right), \quad v^*(x) = |x|^{2m-N} v\left(\frac{x}{|x|^2}\right), \quad x \in \mathbb{R}^N \setminus \{0\}.
\]

**Lemma 3.1.** Let \(u^*\) be the Kelvin’s transform of \(u\), then it holds that
\[
    (-\Delta)^m u^*(x) = |x|^{-2m-N}((-\Delta)^m u)\left(\frac{x}{|x|^2}\right). \tag{3.1}
\]

**Proof.** We consider the polar coordinates, then
\[
    \Delta = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta \omega,
\]
where \(r \in \mathbb{R}, \omega \in S^{N-1}\) and \(S^{N-1}\) is the \((N-1)\)-dimensional sphere surface, then a carefully computation shows that
\[
    \Delta^m u^*(r, \omega) = r^{-2m-N} \left(\Delta^m u\right)\left(\frac{1}{r}, \omega\right),
\]
this is what we want after the coordinate translation (see also [7]).

**Lemma 3.2.** Let \((u, v) \in C^{2m}(\mathbb{R}^N) \times C^{2m}(\mathbb{R}^N)\) be a positive solution of system (1.1). Then \((u^*, v^*)\) satisfies
\[
    (-\Delta)^m u^* = |x|^{(N-2m)\alpha-(N+2m)(v^*)^\alpha}, \quad (-\Delta)^m v^* = |x|^{(N-2m)\beta-(N+2m)(u^*)^\beta} \quad \text{in} \ \mathbb{R}^N. \tag{3.2}
\]

Moreover, \((-\Delta)^i u^*(x), (-\Delta)^i v^*(x) (i = 1, 2, \ldots, m - 1)\) have the following asymptotic expansion at infinity:
\[
    (-\Delta)^i u^*(x) = \frac{e_i^f}{|x|^{N-2m+2i}} f_i\left(\frac{x}{|x|^2}\right), \tag{3.3}
\]
\[
    (-\Delta)^i v^*(x) = \frac{e_i^g}{|x|^{N-2m+2i}} g_i\left(\frac{x}{|x|^2}\right). \tag{3.4}
\]
where \( f_i(x), \ g_i(x) \) are smooth functions with \( f_i(0) = u(0), \ g_i(0) = v(0), \ c_i^f, c_i^g > 0 \) \((i = 1, 2, \ldots, m - 1)\) being constant. In particular, we have

\[
(-\Delta)^i u^*(x) > 0, \quad (-\Delta)^i v^*(x) > 0, \quad i = 1, 2, \ldots, m - 1,
\]

for \(|x|\) large enough.

**Proof.** By Lemma 3.1, a directly calculating yields (3.2).

Now we prove (3.3) by mathematical induction. The proof of (3.4) is similar.

Indeed, it is easy to see (3.3) is true for \( C_{f_0}^f = 1, \ f_0 = u \). Assume for some \( i \),

\[
(-\Delta)^i u^*(x) = \frac{c_i^f}{|x|^{N-2m+2i}} f_i \left( \frac{x}{|x|^2} \right).
\]

Then

\[
(-\Delta)^{i+1} u^*(x) = (-\Delta) \left( \frac{c_i^f}{|x|^{N-2m+2i}} f_i \left( \frac{x}{|x|^2} \right) \right)
\]

\[
= \frac{c_i^f}{|x|^{N-2m+2i+2}} \left\{ (2m - 2i - 2)(N - 2m + 2i) f_i \left( \frac{x}{|x|^2} \right)
\]

\[
+ 4(m - i - 1) \sum x_j |x|^2 \left( \frac{\partial f_i}{\partial x_j} \right) \left( \frac{x}{|x|^2} \right) - |x|^{-2} (\Delta f_i) \left( \frac{x}{|x|^2} \right) \right\}.
\]

Set

\[
(2m - 2i - 2)(N - 2m + 2i) f_{i+1}(x)
\]

\[
= (2m - 2i - 2)(N - 2m + 2i) f_i(x) + 4(m - i - 1) \sum x_i \frac{\partial f_i}{\partial x_i}(x) - |x|^2 \Delta f_i(x),
\]

\[
C_{i+1}^f = (2m - 2i - 2)(N - 2m + 2i) C_i^f,
\]

then

\[
(-\Delta)^{i+1} u^*(x) = \frac{C_{i+1}^f}{|x|^{N-2m+2i+2}} f_{i+1} \left( \frac{x}{|x|^2} \right)
\]

and

\[
f_{i+1}(0) = f_i(0) = u(0).
\]

Since \( C_i^f > 0 \) for \( i = 0, 1, \ldots, m - 1 \),

\[
(-\Delta)^i u^*(x) = \frac{C_i^f u(0)}{|x|^{N-2m+2i}} \left( 1 + O \left( \frac{1}{|x|} \right) \right) > 0,
\]

for \(|x|\) large enough. The proof is completed. \( \square \)
Proposition 3.3. Let \((u^*, v^*)\) be defined as above, then
\[
(-\Delta)^iy^*(x) > 0, \quad (-\Delta)^zv^*(x) > 0,
\]
for \(i = 1, 2, 3, \ldots, m - 1\) and \(x \in \mathbb{R}^n \setminus \{0\}\). Moreover, for any \(r > 0\)
\[
(-\Delta)^iy^*(x) \geq \inf_{\partial B_r(0)} (-\Delta)^iy^* > 0,
\]
\[
(-\Delta)^zv^*(x) \geq \inf_{\partial B_r(0)} (-\Delta)^zv^* > 0,
\]
for \(x \in B_r(0) \setminus \{0\}, i = 1, 2, \ldots, m - 1\).

Proof. By Lemma 3.2, we are lead to prove (3.5) in \(B_1(0) \setminus \{0\}\). Without loss of
generality we may assume that (3.5) is true on the boundary of \(B_1(0)\). To proceed, we follow the
similar arguments as those used in [9]. Let \(\varphi \in C_0^\infty(B_1)\) be a non-negative function, we
prove that
\[
\int_{B_1(0)} \Delta \varphi (-\Delta)^{m-1}u^* \geq 0.
\]
(3.7)

For any \(\epsilon > 0\), we take \(\eta_\epsilon \in C_0^\infty\) such that \(\eta_{\epsilon}(x) \equiv 1\) for \(|x| \geq 2\epsilon\) and \(\eta_{\epsilon} \equiv 0\) for \(|x| \leq \epsilon\),
and \(|D^j\eta_{\epsilon}(x)| \leq c/\epsilon^j\), then
\[
0 \leq \int_{B_1} \varphi \eta_{\epsilon}|x|^{(N-2m)\alpha-(N+2m)}(v^*)^\alpha(x)\,dx = \int_{B_1} \varphi \eta_{\epsilon} (-\Delta)^mu^*\,dx
\]
\[
= \int_{B_1} (-\Delta)(\varphi \eta_{\epsilon})(-\Delta)^{m-1}u^*\,dx.
\]

On the other hand, we see that
\[
\int_{B_1} (-\Delta)^{m-1}u^* \Delta (\varphi \eta_{\epsilon}) = \int_{B_1} (-\Delta)^{m-1}u^* \Delta \varphi \eta_{\epsilon} + \int_{B_1} (-\Delta)^{m-1}u^*(2\nabla \varphi \nabla \eta_{\epsilon} + \varphi \Delta \eta_{\epsilon})\,dx.
\]

Let \(\Psi = 2\nabla \varphi \nabla \eta_{\epsilon} + \varphi \Delta \eta_{\epsilon}\). Then \(\Psi \equiv 0\) for \(|x| \leq \epsilon\) and for \(|x| \geq 2\epsilon\) and \(|\Delta^j\Psi| \leq c\epsilon^{-j-2}\). So
\[ \left| \int_{B_1} (-\Delta)^{m-1} u^*(x) \Psi \, dx \right| \leq \left| \int_{B_1} u^*(x)(-\Delta)^{m-1} \Psi \, dx \right| \leq c \epsilon^{-2m} \int_{\epsilon \leq |x| \leq 2\epsilon} u^*(x) \, dx \leq c \epsilon^{-2m} \left( \int_{\epsilon \leq |x| \leq 2\epsilon} |x|^{(N-2m)\beta-(N+2m)} (u^*)^\beta(x) \, dx \right)^{\frac{1}{\beta}} \left( \int_{\epsilon \leq |x| \leq 2\epsilon} |x|^s \, dx \right)^{1-\frac{1}{\beta}}, \]

where \( (N-2m)\beta-(N+2m) \beta + s(1-\frac{1}{\beta}) = 0 \). Hence

\[ \left| \int_{B_1} (-\Delta)^{m-1} u^*(x) \Psi \, dx \right| \leq c \epsilon^{-2m} + 2m(1+\frac{1}{\beta}) \left( \int_{|x| \geq 1} |x|^{N-2m} |u(x)|^\beta \, dx \right)^{\frac{1}{\beta}} \left( \int_{|x| \geq 1} |x|^{N-2m} u^\beta(x) \, dx \right)^{\frac{1}{\beta}} \to 0, \quad \text{as } \epsilon \to 0, \]

therefore

\[ \int_{B_1} (-\Delta)^{m-1} u^*(x) \Delta \varphi \, dx \geq 0, \quad \forall \varphi \in \mathcal{C}_0^\infty(B_1). \]

This implies that \((-\Delta)^{m-1} u^*(x) \geq 0\). Suppose that \((-\Delta)^{m-k-1} u^*(x) \geq 0\), for some \(k\), then

\[ \int_{B_1} (-\Delta)^{m-k} u^*(x) \Delta (\varphi \eta_\epsilon) \, dx = \int_{B_1} (-\Delta)^{m-k-1} u^*(x) \varphi \eta_\epsilon \, dx \geq 0, \]

by a similar argument, we have

\[ \int_{B_1} (-\Delta)^{m-k} u^*(x) \Psi \, dx = \int_{B_1} u^*(x)(-\Delta)^{m-k} \Psi \, dx \to 0, \quad \text{as } \epsilon \to 0. \]

This shows that \((-\Delta)^{m-k} u^*(x) \geq 0\), by mathematics induction method, we get \((-\Delta)^k \times u^*(x) \geq 0\) for \(k = 1, 2, \ldots, m-1\). Similarly \((-\Delta)^k v^*(x) \geq 0\) for \(k = 1, 2, \ldots, m-1\).

Now we turn to the proof of (3.7). Indeed, since \(u^*(x) > 0\), by the strong Maximum principle, \((-\Delta)^i u^*(x) > 0\), \(i = 1, 2, \ldots, m\). Let

\[ \gamma_i(r) = \inf \{ (-\Delta)^i u^*(x), |x| = r \}, \]
then
\[
\int_{B_r} ((-\Delta)^i u^* (x) - \gamma_i(r)) \Delta \phi \geq 0 \quad \forall \phi \in C_0^\infty (B_r)
\]
and
\[
(-\Delta)^i u^* (x) - \gamma_i(r) \geq 0 \quad \text{on} \ |x| = r.
\]
Again, by Maximum principle, we have
\[
(-\Delta)^i u^* (x) \geq \gamma_i(r), \quad \text{for} \ |x| \leq r.
\]
The proof is completed. \(\square\)

4. Non-existence of radial solutions, the proof of Theorems 1.1 and 1.3

In this section we will make use of the Rellich-type identity to prove non-existence of radial solution to system (1.1). This kind of Rellich-type identity was initially obtained by Mitidieri in [7]. Suppose that \(u, v \in C^{2m}(\Omega)\) with \(\Omega \subset \mathbb{R}^N\) being a smooth bounded domain, we consider the function
\[
R_m(u, v) = \int_{\Omega} \Delta^m u (x, \nabla v) + \Delta^m v (x, \nabla u),
\]
then Mitidieri proved the following identity (see [7, Lemma 2.2]):
\[
R_m(u, v) = \sum_{i=0}^{m-2} R_1(\Delta^i u, \Delta^{m-2-i} v) + R_1(\Delta^i u, \Delta^{m-1+i} v)
- \sum_{i=0}^{m-1} R_1(\Delta^i u, \Delta^{m-1-i} v) - \sum_{i=0}^{m-2} B(\Delta^i u, \Delta^{m-2-i} v), \quad m \geq 3, \quad (4.1)
\]
with
\[
B(u, v) = \int_{\partial \Omega} \Delta u \Delta v (x, n) \, ds - N \int_{\Omega} \Delta u \Delta v \, dx, \quad (4.2)
\]
\[
R_1(u, v) = \int_{\partial \Omega} \left( \frac{\partial u}{\partial n} (x, \nabla v) + \frac{\partial v}{\partial n} (x, \nabla u) - (\nabla u, \nabla v)(x, n) \right) \, ds
+ (N-2) \int_{\Omega} (\nabla u, \nabla v) \, dx,
\]
where \(n\) is the outward unit normal to \(\partial \Omega\) and \((\cdot, \cdot)\) is the inner product in \(\mathbb{R}^N\).
Proof of Theorem 1.1. Suppose \((u, v)\) is a non-negative, radial, \(C^{2m}(\mathbb{R}^N)\) solution of system (1.1). By Theorem 2.2, we have
\[
(-\Delta)^i u \geq 0, \quad (-\Delta)^i v \geq 0, \quad i = 1, 2, \ldots, m, \quad \text{in} \ \mathbb{R}^N.
\]
Set \(u_{i+1} = -\Delta u_i, \ v_{i+1} = -\Delta v_i\) with \(i = 0, 1, \ldots, m - 1, \ u = u_0, \ v = v_0\). Then we can rewrite the system (1.1) as
\[
-(r^{N-1}u')' = r^{N-1}u_1,
-(r^{N-1}u_1')' = r^{N-1}u_2,
\ldots
-(r^{N-1}u_{m-1}')' = r^{N-1}v^\alpha,
-(r^{N-1}v^\alpha)' = r^{N-1}v_1,
\ldots
-(r^{N-1}v_{m-1}')' = r^{N-1}u^\beta
\]
and \(u'(0) = u_1'(0) = \cdots = u'_{m-1}(0) = 0, \ v'(0) = v_1'(0) = \cdots = v'_{m-1}(0) = 0\). Then we can follow the same arguments as we did in Section 3 (see also [2,7]) to obtain the following a priori estimates:
\[
u(r) \leq Cr^{-\frac{2m(\alpha + 1)}{\alpha^2 - 1}}, \quad v(r) \leq Cr^{-\frac{2m(\beta + 1)}{\beta^2 - 1}},

(-\Delta)^i u = u_i \leq Cr^{-2i}r^{-\frac{2m(\alpha + 1)}{\alpha^2 - 1}},
\]
\[
(-\Delta)^i v = v_i \leq Cr^{-2i}r^{-\frac{2m(\beta + 1)}{\beta^2 - 1}}, \quad i = 1, 2, \ldots, m - 1,
\]
\[
|r^{N-1}((\Delta)^ju)((\Delta)^jv)| \leq Cr^{N-2r^{-2(j+s)}r^{-\frac{2m(\alpha + \beta + 2)}{\alpha^2 - 1}}}, \quad 1 \leq j, s \leq m - 1,
\]
\[
|r^{N-1}((\Delta)^ju)((\Delta)^jv)'| \leq Cr^{N-2r^{-2(j+s)}r^{-\frac{2m(\alpha + \beta + 2)}{\beta^2 - 1}}}, \quad 1 \leq j, s \leq m - 1,
\]
\[
|r^N((\Delta)^ju)'((\Delta)^jv)'| \leq Cr^{N-2r^{-2(j+s+2)}r^{-\frac{2m(\alpha + \beta + 1)}{\alpha^2 - 1}}}, \quad 1 \leq j, s \leq m - 1. \quad (4.3)
\]
In what follows, we need to consider the two cases for \(m\) being even and odd, respectively. We only prove the case when \(m\) is even. The other case is similar. Since \(m\) is even, we may write \(m = 2n\). Applying (4.1) to our system (1.1) on \(B_r(0)\) reduce that
\[
\left(\frac{N - 4n}{2} - \frac{N}{\alpha + 1}\right) \int_0^r v^{\alpha + 1}(s)s^{N-1}ds + \left(\frac{N - 4n}{2} - \frac{N}{\beta + 1}\right) \int_0^r u^{\beta + 1}(s)s^{N-1}ds
\]
\[
= -\frac{r^N}{\alpha + 1}v^{\alpha + 1}(r) - \frac{r^N}{\beta + 1}u^{\beta + 1}(r) + \left(\frac{N - 4n}{2}\right)
\]
\[
\times \{r^{N-1}(\Delta^n u)'(r)(\Delta^{n-1}v)(r) - r^{N-1}(\Delta^n v)'(r)(\Delta^{n-1}u)(r)\}\]
\[ + \frac{4 - N - 4n}{2} r^{N-1} \sum_{k=1}^{n-1} \left\{ (\Delta^{2n-k} u)'(r)(\Delta^{k-1} v)(r) + (\Delta^{2n-k} v)'(r)(\Delta^{k-1} u)(r) \right\} \]

\[ + r^N \sum_{k=0}^{n-1} \left\{ (\Delta^k u)'(r)(\Delta^{2n-k-1} v)(r) - (\Delta^k v)(r)(\Delta^{2n-k} u)(r) \right\} \]

\[ + r^N \sum_{k=0}^{n-1} \left\{ (\Delta^k v)'(r)(\Delta^{2n-k-1} u)(r) - (\Delta^k u)(r)(\Delta^{2n-k} v)(r) \right\} \]

\[ + r^{N-1} \sum_{k=1}^{n-1} \sum_{i=1}^{n-k} (\Delta^{2n-k-i} u)'(r)(\Delta^{k+i-1} v)'(r) + (\Delta^{2n-k-i} v)'(r)(\Delta^{k+i-1} u)'(r) \]

\[ + r^{N-1} \sum_{k=1}^{n-1} \sum_{i=1}^{n-k} (\Delta^{k+i-1} u)'(r)(\Delta^{2n-k-i} v)(r) + (\Delta^{k+i-1} v)'(r)(\Delta^{2n-k-i} u)(r) \]

\[ + (N - 2) r^{N-1} \sum_{k=1}^{n-1} (\Delta^{2n-k-1} u)'(r)(\Delta^k u)(r) + (\Delta^{2n-k-1} v)'(r)(\Delta^k v)(r) \]

\[ + (N - 2) r^{N-1} \left\{ (\Delta^{2n-k} u)'(r)v(r) + (\Delta^{n-1} u)'(r)(\Delta^n v)(r) \right\} \]

\[ - (\Delta^{n+1} u)'(r)(\Delta^{n-2} v)(r) \right\} + (N - 2) r^{N-1} \left\{ (\Delta^{2n-k} v)'(r)u(r) \right\} \]

\[ + (\Delta^{n-1} v)'(r)(\Delta^n u)(r) - (\Delta^{n+1} v)'(r)(\Delta^n-2 u)(r) \]

\[ - \frac{1}{2} r^N (\Delta^n u)(\Delta^n v)(r). \]  

(4.4)

Then the a priori estimates (4.3) imply that the right-hand side of (4.4) converges to zero as \( r \to \infty \). Hence

\[
\left( \frac{N - 4n}{2} - \frac{N}{\alpha + 1} \right) \int_0^r v^{\alpha+1}(s)s^{N-1} ds + \left( \frac{N - 4n}{2} - \frac{N}{\beta + 1} \right) \int_0^r u^{\beta+1}(s)s^{N-1} ds \to 0,
\]

as \( r \to \infty \).

On the other hand, multiplying the first equation of system (1.1) by \( v \) and integrating in \( B_r(0) \), for \( r > 0 \), we obtain

\[
\int_{B_r(0)} v^{\alpha+1} \, dx
\]

\[
= \int_{B_r(0)} (-\Delta)^m uv \, dx = - \int_{B_r(0)} (\nabla ((-\Delta)^{m-1} u), \nabla v) \, dx + \int_{\partial B_r(0)} \frac{\partial ((-\Delta)^{m-1} u)}{\partial n} v \, d\sigma.
\]
\[ \int_{B_r(0)} (-\Delta)^{m-1}u \Delta v \, dx - \int_{\partial B_r(0)} \frac{\partial v}{\partial n}(-\Delta)^{m-1}u \, d\sigma + \int_{\partial B_r(0)} \frac{\partial((-\Delta)^{m-1}u)}{\partial n}v \, d\sigma. \]

Hence

\[ \int_0^r v^{\alpha+1}(s)s^{N-1} \, ds = \int_0^r ((-\Delta)^{m-1}u)(s)(\Delta v)(s)s^{N-1} \, ds \]

\[ -((-\Delta)^{m-1}u(r)v'(r)r^{N-1} + ((-\Delta)^{m-1}u)'(r)v(r)r^{N-1}). \]

Similarly,

\[ \int_0^r u^{\beta+1}(s)s^{N-1} \, ds = \int_0^r ((-\Delta)^{m-1}v)(s)(\Delta u)(s)s^{N-1} \, ds \]

\[ -((-\Delta)^{m-1}v(r)u'(r)r^{N-1} + ((-\Delta)^{m-1}v)'(r)u(r)r^{N-1}). \]

So

\[ \int_0^r v^{\alpha+1}(s)s^{N-1} \, ds = \int_0^r u^{\beta+1}(s)s^{N-1} \, ds + o(1), \quad \text{as } r \to \infty. \]

This combines with (4.4) gives that

\[ \left( N - 2m - \frac{N}{\alpha + 1} - \frac{N}{\beta + 1} \right) \int_0^r u^{\beta+1}(s)s^{N-1} \, ds = o(1), \quad \text{as } r \to +\infty. \]

Since \( N - 2m - \frac{N}{\alpha + 1} - \frac{N}{\beta + 1} > 0 \), passing to the limit we obtain \( u = 0 \) and, as a consequence, also \( v = 0 \).  

**Proof of Theorem 1.3.** Let \( \varphi_1 \in H^1_0(B_1) \) be the positive eigenfunction corresponding to the first eigenvalue \( \lambda_1 \) of \((-\Delta, H^1_0(B_1))\). For any \( R > 0 \), we define \( \varphi_R(x) := \varphi_1(x/R) \), then \( \varphi_R \in H^1_0(B_R) \) is a positive eigenfunction corresponding to the eigenvalue \( \lambda_1/R^2 \).

Multiplying the both sides of the inequality \((-\Delta)^{m}u \geq v^{\alpha} \) by \( \varphi_R \) and integrating the obtained inequality in \( B_R \), we have

\[ \int_{B_R} v^{\alpha} \varphi_R \leq \int_{B_R} (-\Delta)^{m}u \varphi_R = \int_{B_R} (-\Delta)(-\Delta)^{m-1}u \varphi_R \]

\[ = \frac{\lambda_1}{R^2} \int_{B_R} (-\Delta)^{m-1}u \varphi_R + \int_{\partial B_R} (-\Delta)^{m-1}u \frac{\varphi_R}{\partial n}. \]
Noting that \((-\Delta)^{m-1} u \geq 0\), and \(\partial \varphi_R / \partial n < 0\) by Hopf’s lemma, we are lead to

\[
\int_{B_R} v^\alpha \varphi_R \leq \frac{\lambda_1}{R^2} \int_{B_R} (-\Delta)^{m-1} u \varphi_R \leq \left( \frac{\lambda_1}{R^2} \right)^m \int_{B_R} u \varphi_R
\]

\[
\leq \left( \frac{\lambda_1}{R^2} \right)^m \left( \int_{B_R} u^\beta \varphi_R \right)^{\frac{1}{\beta}} \left( \int_{B_R} \varphi_1 \right)^{1-\frac{1}{\beta}} \]

\[
= \lambda_1^m \left( \int_{B_R} u^\beta \varphi_R \right)^{\frac{1}{\beta}} \left( \int_{B_R} \varphi_1 \right)^{1-\frac{1}{\beta}} R^{N(1-\frac{1}{p})-2m}. \tag{4.5}
\]

Similarly

\[
\int_{B_R} u^\beta \varphi_R \leq \lambda_1^m \left( \int_{B_R} v^\alpha \varphi_R \right)^{\frac{1}{\alpha}} \left( \int_{B_R} \varphi_1 \right)^{1-\frac{1}{\alpha}} R^{N(1-\frac{1}{p})-2m}. \tag{4.6}
\]

Inserting (4.6) in (4.5) reads as

\[
\int_{B_R} v^\alpha \varphi_R \leq \frac{\lambda_1^{m+m/\beta}}{R^{N-2m\beta-(N/\alpha+2m)}} \]

hence

\[
\left( \int_{B_R} v^\alpha \varphi_R \right)^{1-\frac{1}{\alpha\beta}} \leq c_1 R^{\frac{1}{\beta}(N-2m)\beta-(N/\alpha+2m)},
\]

where \(c_1\) is a constant. Since \((N-2m)\beta-(N/\alpha+2m) < 0\) by taking limits in the above formula we obtain \(v \equiv 0\). In a similar way, we can prove \(u \equiv 0\). \(\square\)

5. Proof of Theorem 1.2

In this section we shall use moving planes method to prove Theorem 1.2. To proceed, we start by considering planes parallel to \(x_1\). For each \(\lambda < 0\), we write \(x = (x_1, x')\) with \(x' = (x_2, \ldots, x_n) \in \mathbb{R}^{N-1}\) and define \(\Sigma_\lambda := \{ x \mid x_1 < \lambda \}\), \(T_\lambda := \partial \Sigma\). For \(x = (x_1, x') \in \Sigma_\lambda\), let \(x^\lambda = (2\lambda - x_1, x')\) be the reflected point with respect to \(T_\lambda\). To start the process of moving planes, we begin with some auxiliary facts.

**Definition 5.1.** Let \(l\) be an integer. We say that a \(C^2\) function in \(f\) has a harmonic asymptotic expansion at infinity in a neighborhood of infinity if:
\[ f(x) = \frac{1}{|x|^l} \left( a_0 + \sum_{i=1}^{N} \frac{a_i x_i}{|x|^2} \right) + O \left( \frac{1}{|x|^{l+2}} \right), \]
\[ f_{x_i}(x) = -l a_0 \frac{x_i}{|x|^{l+2}} + O \left( \frac{1}{|x|^{l+2}} \right), \quad i = 1, 2, \ldots, N, \]
\[ f_{x_i, x_j}(x) = O \left( \frac{1}{|x|^{l+2}} \right), \quad i, j = 1, 2, \ldots, N, \quad (5.1) \]

where \( a_i \in \mathbb{R} \), for \( i = 1, 2, \ldots, N \).

We observe by Lemma 3.2 that both \( u^* \) and \( v^* \) have harmonic asymptotic expansions at infinity, with \( l = N - 2m \) and \( a_0 > 0 \). Also \( -\Delta u^* \) and \( -\Delta v^* \) have harmonic asymptotic expansions at infinity with \( l = N - 2m + 2 \) and \( a_0 > 0 \).

**Lemma 5.2.** Let \( f \) be a function in a neighborhood at infinity satisfying the asymptotic expansion (5.1). Then there exist \( \lambda_0 < 0 \) and \( R > 0 \) such that if \( \lambda \leq \lambda_0 \),
\[ f(x) < f(x^\lambda), \quad \text{for } x_1 < \lambda, \ x \notin B_R(e_\lambda), \]
where \( (e_\lambda) := (2\lambda, 0) \).

**Lemma 5.3.** Let \( f \) be a \( C^2 \) positive solution of \( -\Delta f = F(x) \) for \( |x| > R \), where \( f \) has a harmonic asymptotic expansion (5.1) at infinity with \( a_0 > 0 \). Suppose that, for some negative \( \lambda_0 \) and for every \( (x, x') \) with \( x_1 < \lambda_0 \),
\[ f(x_1, x') < f(2\lambda_0 - x_1, x') \quad \text{and} \quad F(x_1, x') \leq F(2\lambda_0 - x_1, x'). \]
Then there exist \( \varepsilon > 0, S > R \) such that

(i) \( f(x_1, x') > 0 \) in \( |x_1 - \lambda_0| < \varepsilon, \ |x| > S \),
(ii) \( f(x_1, x') < f(2\lambda - x_1, x') \) in \( x_1 < \lambda_0 - \frac{1}{2}\varepsilon < \lambda, \ |x| > S \),

for all \( x \in \Sigma_\lambda, \lambda \geq \lambda_1 \) with \( |\lambda_1 - \lambda_0| < c_0\varepsilon \), where \( c_0 \) is a positive number depending on \( \lambda_0 \) and \( f \).

We refer the reader to [5, Lemmas 2.3 and 2.4] for the proof of Lemmas 5.2 and 5.3.

Let \( \tilde{\Sigma}_\lambda = \Sigma_\lambda \setminus \{ e_\lambda \} \), we define
\[ U_\lambda(x) = u^*(x^\lambda) - u^*(x), \quad V_\lambda(x) = v^*(x^\lambda) - v^*(x). \]

Let \( \lambda \leq 0 \), noting the invariance of the Laplacian under a reflection together with the mean value theorem and the fact that \( |x^\lambda| \leq |x| \), we have
\[ \Delta^m U_\lambda \geq c(x, \lambda) V_\lambda(x), \quad \Delta^m V_\lambda \geq \hat{c}(x, \lambda) U_\lambda(x), \quad (5.2) \]
for $x \in \tilde{\Sigma}_\lambda$, where $c(x, \lambda) = |x|^{\alpha(N-2m)-(N+2m)}(\psi(x, \lambda))^{\alpha-1}$, with $\psi(x, \lambda)$ a real number between $v^*(x^\lambda)$ and $v^*(x)$, and similarly $\hat{c}(x, \lambda) = |x|^{\beta(N-2m)-(N+2m)}(\hat{\psi}(x, \lambda))^{\beta-1}$, with $\hat{\psi}(x, \lambda)$ a real number between $u^*(x^\lambda)$ and $u^*(x)$. Since $u, v > 0$ we conclude that both $c(x, \lambda)$ and $\hat{c}(x, \lambda)$ are positive.

**Proposition 5.4.** There exists $\lambda_1 < 0$ such that if $\lambda \leq \lambda_1$, then $(-\Delta)^i U_\lambda(x) > 0$, $(-\Delta)^i V_\lambda(x) > 0$, for $i = 1, 2, \ldots, m - 1$, and $U_\lambda(x) > 0$ and $V_\lambda(x) > 0$ in $\tilde{\Sigma}_\lambda$.

**Proof.** From Lemmas 3.2 and 5.2, there exist $\tilde{\lambda}_1 < 0$ and $R_1 > 0$ such that $(-\Delta)^i U_{\tilde{\lambda}_1}(x) > 0$, $(-\Delta)^i V_{\tilde{\lambda}_1}(x) > 0$ ($i = 1, 2, \ldots, m - 1$), $U_{\lambda}(x) > 0$ and $V_{\lambda}(x) > 0$ in $\tilde{\Sigma}_\lambda \setminus B_{R_1}(e_\lambda)$, for all $\lambda \leq \tilde{\lambda}_1$.

By Proposition 3.3, $(-\Delta)^{m-1} u^*(x) > 0$ in $\mathbb{R}^N \setminus \{0\}$. Since

$$(-\Delta)((-\Delta)^{m-1} u^*)(x) = |x|^{\alpha(N-2m)-(N+2m)}(v^*)^\alpha(x) > 0.$$ 

Again, Proposition 3.3 allows us to conclude that

$$(-\Delta)^{m-1} u^*(x) \geq \gamma(R_1) := \inf \{(-\Delta)^{m-1} u^*(y): |y| = R_1\}, \quad \forall x: 0 < |x| < R_1.$$

If $x \in B_{R_1}(e_\lambda) \setminus \{e_\lambda\}$ then $|x - e_\lambda| = |x^\lambda| < R_1$. So, for $x \in B_{R_1}(e_\lambda) \setminus \{e_\lambda\}$, we have $(-\Delta)^{m-1} u^*(x^\lambda) \geq \gamma(R_1)$. From the fact that $(-\Delta)^{m-1} u^*(x) \to 0$ as $|x| \to +\infty$, we conclude that there exists $R_2 > 0$ such that $(-\Delta)^{m-1} u^*(x) < \gamma(R_1)/2$, for $|x| > R_2$. Let $\tilde{\lambda}_1 := \min\{-R_1, -R_2\}$. Then, for all $\lambda \leq \tilde{\lambda}_1$,

$$(-\Delta)^{m-1} U_{\lambda}(x) = (-\Delta)^{m-1} u^*(x^\lambda) - (-\Delta)^{m-1} u^*(x) > \gamma(R_1) - \gamma(R_1)/2 > 0,$$

for $x \in B_{R_1}(e_\lambda) \setminus \{e_\lambda\}$. Similarly, let $\tilde{\lambda}_2, \tilde{\lambda}_3, \tilde{\lambda}_4 < 0$ be such that

$$(-\Delta V)^{m-1}_{\lambda}(x) = (-\Delta)^{m-1} v^*(x^\lambda) - (-\Delta)^{m-1} v^*(x) > \gamma'(R_1) - \gamma'(R_1)/2 > 0,$$

for $x \in B_{R_1}$ and all $\lambda < \tilde{\lambda}_2$;

$$U_{\lambda}(x) = u^*(x^\lambda) - u^*(x) > m(R_1) - m(R_1)/2 > 0,$$

for $x \in B_{R_1}$ and all $\lambda < \tilde{\lambda}_3$;

$$V_{\lambda}(x) = v^*(x^\lambda) - v^*(x) > m'(R_1) - m'(R_1)/2 > 0,$$

for $x \in B_{R_1}$ and all $\lambda < \tilde{\lambda}_4$, where

$$M'(R_1) := \inf \{(-\Delta)^{m-1} v^*(y): |y| = R_1\},$$

$$m(R_1) := \min\{u^*(y): |y| = R_1\}, \quad m'(R_1) := \min\{v^*(y): |y| = R_1\}.$$

By choosing $\lambda_1 = \min\{\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, \tilde{\lambda}_4, \tilde{\lambda}_1\}$ we get the conclusion. $\Box$

Let $\lambda_0 := \sup\{\lambda < 0: (-\Delta)^i U_\lambda(x) > 0, (-\Delta)^i V_\lambda(x) > 0 (i = 1, 2, \ldots, m - 1), U_\lambda(x) > 0, \text{ and } V_\lambda(x) > 0 \text{ in } \tilde{\Sigma}_\lambda\}$. 
Lemma 5.5. $U_{\lambda_0} \equiv 0$ if and only if $V_{\lambda_0} \equiv 0$.

**Proof.** If $U_{\lambda_0} \not\equiv 0$ and $V_{\lambda_0} \equiv 0$, by (5.2) we have $\hat{c}(x, \lambda_0)U_{\lambda_0}(x) \leq 0$. Since $\hat{c}(x, \lambda_0) > 0$, then $U_{\lambda_0} \leq 0$. Since also $U_{\lambda_0} > 0$, this is a contradiction. By a same way, one can prove if $U_{\lambda_0} \equiv 0$, then $V_{\lambda_0} \equiv 0$. □

**Proposition 5.6.** If $\lambda_0 < 0$ then $U_{\lambda_0} \equiv 0$ and $V_{\lambda_0} \equiv 0$.

**Proof.** Suppose by contradiction that the conclusion of the proposition is not true. By Lemma 5.5 we conclude that $U_{\lambda_0} \not\equiv 0$ and $V_{\lambda_0} \not\equiv 0$. Since $(-\Delta)U_{\lambda_0} \geq 0$ in $\tilde{\Sigma}_{\lambda_0}$, $U_{\lambda_0} \geq 0, \quad U_{\lambda_0} \not\equiv 0$ in $\tilde{\Sigma}_{\lambda_0}$, $U_{\lambda_0} = 0$ on $T_{\lambda_0}$, and since $U_{\lambda_0}(x) \to 0$ when $|x| \to \infty$, by the maximum principle we have that $U_{\lambda_0}(x) > 0$ in $\tilde{\Sigma}_{\lambda_0}$. By the same arguments we can prove that $V_{\lambda_0}(x) > 0$ in $\tilde{\Sigma}_{\lambda_0}$. Thus

$$(-\Delta)^m U_{\lambda_0} \geq c(x, \lambda_0)V_{\lambda_0} > 0 \quad \text{in } \tilde{\Sigma}_{\lambda_0},$$

and consequently

$$(-\Delta)((-\Delta)^{m-1}U_{\lambda_0}) > 0 \quad \text{in } \tilde{\Sigma}_{\lambda_0},$$

$$(-\Delta)^{m-1}U_{\lambda_0} \geq 0 \quad \text{in } \tilde{\Sigma}_{\lambda_0},$$

$$(-\Delta)^{m-1}U_{\lambda_0} = 0 \quad \text{on } T_{\lambda_0}.$$

Since $(-\Delta)^{m-1}U_{\lambda_0}(x) \to 0$ when $|x| \to 0$, by the maximum principle we must have $(-\Delta)^{m-1}U_{\lambda_0}(x) > 0$ in $\tilde{\Sigma}_{\lambda_0}$. Using the Hopf’s maximum principle we obtain

$$\frac{\partial(-\Delta)^{m-1}U_{\lambda_0}}{\partial \nu}(x) > 0 \quad \text{on } T_{\lambda_0},$$

where $\nu$ is the outward unit normal to $\tilde{\Sigma}_{\lambda_0}$. We will prove that this is impossible.

From the definition of $\lambda_0$, there exists a sequence of real numbers $\lambda_n \searrow \lambda_0$ and a sequence of points in $\tilde{\Sigma}_{\lambda_n}$ where $(-\Delta)^{m-1}U_{\lambda_n}$ or $(-\Delta)^{m-1}V_{\lambda_n}$ is negative or $U_{\lambda_n}$ or $V_{\lambda_n}$ is negative.

If $(-\Delta)^{m-1}U_{\lambda_n}(x) < 0$ for some $x \in \tilde{\Sigma}_{\lambda_n}$, then

$$c_1 := \inf_{\tilde{\Sigma}_{\lambda_n}}(-\Delta)^{m-1}U_{\lambda_n} < 0.$$

We shall see that this infimum is attained. Let

$$c_2 := \min_{\partial B_{1/2}(x_0)}(-\Delta)^{m-1}U_{\lambda_0} > 0.$$
It follows from Proposition 3.3,
\[ (-\Delta)^{m-1}U_{\lambda_0}(x) \geq c_2 > 0 \quad \text{in } B_{\lambda_0/2}(e_{\lambda_0}). \]

By continuity, we have for large \( n \),
\[ (-\Delta)^{m-1}U_{\lambda_n}(x) \geq \frac{c_2}{2} > 0 \quad \text{in } B_{\lambda_0/2}(e_{\lambda_n}). \]

As \((-\Delta)^{m-1}U_{\lambda_n}(x) \to 0\) when \(|x| \to +\infty\), there exists \( r_n \) such that, for all \(|x| \geq r_n\),
\[ (-\Delta)^{m-1}U_{\lambda_n}(x) > c_1/2. \]
Thus
\[ \inf_{\tilde{\Sigma}_{\lambda_n}}(-\Delta)^{m-1}U_{\lambda_n} = \inf \left\{ (-\Delta)^{m-1}U_{\lambda_n}(x) : x \in \left( \tilde{\Sigma}_{\lambda_n} \setminus B_{\lambda_0/4}(e_{\lambda_n}) \right) \cap B_{r_n}(0) \right\}. \]

Therefore there exists a sequence \((x_n) \subset \left( \tilde{\Sigma}_{\lambda_n} \setminus B_{\lambda_0/4}(e_{\lambda_n}) \right) \cap B_{r_n}(0)\) such that
\[ \inf_{\tilde{\Sigma}_{\lambda_n}}(-\Delta)^{m-1}U_{\lambda_n} = (-\Delta)^{m-1}U_{\lambda_n}(x_n) < 0. \]

Hence
\[ \nabla((-\Delta)^{m-1}U_{\lambda_n})(x_n) = 0 \quad \text{and} \quad \Delta((-\Delta)^{m-1}U_{\lambda_n})(x_n) \geq 0. \quad (5.3) \]

It follows from Lemma 5.3 that \((x_n)\) is bounded. Thus, up to a subsequence, \( x_n \to x_0 \) with \( x_0 \in \tilde{\Sigma}_{\lambda_0} \). Passing (5.3) to the limit we obtain
\[ \nabla((-\Delta)^{m-1}U_{\lambda_0})(x_0) = 0 \quad \text{and} \quad \Delta((-\Delta)^{m-1}U_{\lambda_0})(x_0) \geq 0. \]

Thus we conclude that \( x_0 \in T_{\lambda_0} \). Since
\[ 0 < \frac{\partial(-\Delta)^{m-1}U_{\lambda_0}}{\partial x_1}(x_0) = \frac{\partial(-\Delta)^{m-1}U_{\lambda_0}}{\partial x_1}(x_0) = 0, \]
we have a contradiction.

The case when \((-\Delta)^{m-1}V_{\lambda_n}\) takes negative values and the cases when \(U_{\lambda_n}\) or \(V_{\lambda_n}\) take positive values are proved similarly. □

**Proof of Theorem 1.2 (Continued).** (I) In applying the moving planes method, we must consider two cases.

(a) If \( \lambda_0 < 0 \), by Proposition 5.6 we have \( U_{\lambda_0} \equiv 0 \) and \( V_{\lambda_0} \equiv 0 \), so \( u^*(x) \) and \( v^*(x) \) are symmetric with respect to the plane \( T_{\lambda_0} \). Since the Laplacian is invariant for dilations, we get a contradiction. So \( u = v = 0 \) in \( \mathbb{R}^N \).

(b) If \( \lambda_0 = 0 \) then \( U_0(x) \geq 0 \) and \( V_0(x) \geq 0 \) in \( \tilde{\Sigma}_0 \), i.e.,
\[ u(-x_1, x') \geq u(x_1, x') \quad \text{and} \quad v(-x_1, x') \geq v(x_1, x'), \quad \text{for } x_1 \leq 0. \quad (5.4) \]
Defining $\bar{u}(x_1, x') := u(-x_1, x)$ and $\bar{v}(x_1, x') := v(-x_1, x)$, we have

$$(-\Delta)^m \bar{u} = \bar{v}^\alpha, \quad (-\Delta)^m \bar{v} = \bar{u}^\beta$$

in $\mathbb{R}^N$.

By performing the latter procedure, we deduce the existence of a corresponding value $\bar{\lambda}_0 \leq 0$. If $\bar{\lambda}_0 < 0$ then $\bar{u} = \bar{v} = 0$ and consequently $u = v = 0$. If $\bar{\lambda}_0 = 0$ then

$$\bar{u}(-x_1, x') \geq \bar{u}(x_1, x') \quad \text{and} \quad \bar{v}(-x_1, x') \geq \bar{v}(x_1, x'), \quad \text{for} \ x_1 \leq 0.$$

By (5.4), we conclude that $u$ and $v$ are radially symmetric with respect to the origin.

Since we can perform the Kelvin’s transform with respect to any point, thus $u$ and $v$ are radially symmetric with respect to any point. This implies that $u$ and $v$ are constant functions. From system (1.1), we get $u = v = 0$.

(II) We proceed the same arguments as in [4].

References