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On topological relaxations of chromatic conjectures

Gábor Simonyi [a], Ambrus Zsbán [b]

[a] Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Hungary

[b] Department of Computer Science and Information Theory, Budapest University of Technology and Economics, Hungary

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ABSTRACT

There are several famous unsolved conjectures about the chromatic number that were relaxed and already proven to hold for the fractional chromatic number. We discuss similar relaxations for the topological lower bound(s) of the chromatic number. In particular, we prove that such a relaxed version is true for the Behzad–Vizing conjecture and also discuss the conjectures of Hedetniemi and of Hadwiger from this point of view. For the latter, a similar statement was already proven in Simonyi and Tardos (2006) [41], our main concern here is that the so-called odd Hadwiger conjecture looks much more difficult in this respect. We prove that the statement of the odd Hadwiger conjecture holds for large enough Kneser graphs and Schrijver graphs of any fixed chromatic number.

1. Introduction

There are several hard conjectures about the chromatic number that are still open, while their fractional relaxation is solved, i.e., a similar, but weaker statement is proven for the fractional chromatic number in place of the chromatic number. (For the definition and basic facts about the fractional chromatic number, cf. [39].) Examples include the Behzad–Vizing conjecture [25], the Erdős–Faber–Lovász conjecture [21], Hedetniemi's conjecture [46], a relaxed version of Hadwiger's conjecture [36], as well as a similarly relaxed version of the so-called odd Hadwiger conjecture [23]. (In some of these cases the proven fractional version has an approximative form, nevertheless, it is a statement not known to hold for the chromatic number.)

There are not very many examples of graphs with a large gap between their chromatic and fractional chromatic numbers. To determine the chromatic number of such a graph is usually difficult because no lower bound that also bounds the fractional chromatic number from below can give a tight result. The primary example for such a graph family is that of Kneser graphs. The value of their

E-mail addresses: simonyi@renyi.hu (G. Simonyi), ambrus@math.bme.hu (A. Zsbán).

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chromatic number was conjectured by Kneser [26] in 1955 and proved by Lovász [31] in 1978 thereby developing the topological method for estimating the chromatic number. This method was later successfully applied to other graphs, e.g., generalized Mycielski graphs, cf. [44,17], see [33] for a thorough survey on the later developments.

The above suggests that one could gain further supporting evidence for the above conjectures if one could prove that the topological lower bound for the chromatic number, considered as a graph parameter for its own sake, also satisfies the above statements if it is put in place of the chromatic number. (In fact, "the topological lower bound" is not a well-defined term, as there are more than one such bounds, for further details see the next section.)

Such a result already appears in the last section of [41] concerning a relaxation of Hadwiger's conjecture. Further impetus for such studies was given to us by a conversation of the first author with Claude Tardif and Gábor Tardos at the first CanaDAM Conference in summer 2007 where the idea of considering the topological lower bound(s) as a graph parameter was made more explicit. In particular, Tardif asked, whether a result of the above type would be possible concerning the Erdős–Faber–Lovász conjecture. Though we were not able to make progress in this particular question, we will prove in this paper a similar result about the Behzad–Vizing conjecture and elaborate about some of the others.

The paper is organized as follows. Section 2 contains some basic facts about the topological method. In Section 3 we prove our result concerning the Behzad–Vizing conjecture. In Section 4 we give a simple topological analog of Hedetniemi's conjecture. In Sections 5 and 6 we discuss Hadwiger's conjecture and the odd Hadwiger conjecture. In the latter section we prove that the odd Hadwiger conjecture holds for some of the graphs for which the topological method gives a tight bound on the chromatic number, in particular, large enough Kneser graphs, Schrijver graphs, and generalized Mycielski graphs.

2. About the topological bound(s) on the chromatic number

There are several formally different topological lower bounds on the chromatic number that are all closely related to each other. As we will only use combinatorial consequences of the situation when these parameters achieve certain values rather than using them directly, we will not give full definitions of these bounds. (Most importantly, we are not defining the topological notions used. They can be looked up in several of the references and though they give important background, familiarity with these notions, or even knowing them, is not essential for understanding this paper.) Instead we only hint the definitions and give references for detailed treatments, while list those statements that are to be used in this paper.

The idea behind all versions of the topological lower bound of the chromatic number is to associate a topological space to the graph and use its topological invariants for bounding the chromatic number. Originally Lovász [31] used the connectivity of the associated topological space defined via a simplicial complex, called the neighborhood complex, and showed that this parameter is less than the chromatic number of the graph by at least 3.

Other variants of the same idea appeared over the years that use $\mathbb{Z}_2$-spaces defined by certain box complexes. (In fact, $\mathbb{Z}_2$-maps and the Borsuk–Ulam theorem are also key in Lovász' original proof, the difference is only the more direct use of $\mathbb{Z}_2$-spaces in these later variants.) For a variety of box complexes, see [34]. One of the most basic box complexes, $B(G)$, associated to graph G, is a simplicial complex that has $V(G) \times \{1\} \cup V(G) \times \{2\}$ as its vertex set, and a subset of vertices forms a simplex in it iff it has the form $A \uplus B := A \times \{1\} \cup B \times \{2\}$, the induced subgraph of G on $A \cup B \subseteq V(G)$ contains a complete bipartite graph with color classes A and B, and in case A (or B) is empty, we have that the vertices in B (resp. A) have at least one common neighbor. (In other words, simplices of the form $A \uplus \emptyset$ and $\emptyset \uplus B$ are contained only when they should be by the hereditary nature of simplicial complexes.) The $\mathbb{Z}_2$-space evolving from this simplicial complex is the topological space given by its geometric realization equipped with the involution generated by the simplicial map $\nu : A \uplus B \mapsto B \uplus A$. The most important property of this construction is that whenever G and H are two graphs such that there exists a homomorphism, i.e., an edge preserving map of the vertices, from G to H, then there is also a simplicial $\mathbb{Z}_2$-map (that is, a simplicial map respecting the involution ν) from $B(G)$ to $B(H)$. It is not hard to show that $B(K_n)$ is homotopy equivalent to the sphere $\mathbb{S}^{n-2}$. (As a $\mathbb{Z}_2$-space the sphere $\mathbb{S}^h$ is considered to be equipped with the antipodal map as the involution.) One can define for any $\mathbb{Z}_2$-space $T = (T, \nu)$ its $\mathbb{Z}_2$-index $\mathrm{ind}\,(T)$ as the smallest dimension h for which a $\mathbb{Z}_2$-map exists from T to $\mathbb{S}^h$. The celebrated Borsuk–Ulam theorem (cf. e.g. [33]) states in one of its standard forms, that no $\mathbb{Z}_2$-map exists from $\mathbb{S}^h$ to $\mathbb{S}^{h'}$ if $h' < h$. Putting all this together, and using the fact that a proper coloring with m colors is nothing but a homomorphism to K_m, one obtains that $\chi(G) \geq \mathrm{ind}\,(B(G)) + 2$ should always hold.

A somewhat different box complex $B_0(G)$ can be defined by simply dropping the extra condition about common neighbors for the containment of simplices having the form $A \uplus \emptyset$ or $\emptyset \uplus B$. (Thus $V(G) \uplus \emptyset$, $\emptyset \uplus V(G)$ and all their subsets are simplices in $B_0(G)$.) Csorba [5] proved that $B_0(G)$ is $\mathbb{Z}_2$-homotopy equivalent to the suspension of $B(G)$, cf. also [34] for this and other relations between various box complexes. The latter fact implies $\mathrm{ind}\,(B_0(G)) \leq \mathrm{ind}\,(B(G)) + 1$ thus we have by the foregoing that the inequality $\chi(G) \geq \mathrm{ind}\,(B_0(G)) + 1$ holds, too.

By the above mentioned form of the Borsuk–Ulam theorem, the $\mathbb{Z}_2$-index of a $\mathbb{Z}_2$-space T is bounded from below by the $\mathbb{Z}_2$-coindex, $\mathrm{coind}\,(T)$ which is defined as the largest dimension h for which a $\mathbb{Z}_2$-map exists *from* the sphere $\mathbb{S}^h$ to T. By the suspension relationship we have $\mathrm{coind}\,(B_0(G)) \geq \mathrm{coind}\,(B(G)) + 1$. Thus the $\mathbb{Z}_2$-index and $\mathbb{Z}_2$-coindex of the two box complexes we discussed give the following chain of lower bounds on the chromatic number:

$$\chi(G) \geq \mathrm{ind}\,(B(G)) + 2 \geq \mathrm{ind}\,(B_0(G)) + 1 \geq \mathrm{coind}\,(B_0(G)) + 1 \geq \mathrm{coind}\,(B(G)) + 2.$$

For a more thorough introduction to these notions we refer to [33] or [41].

Seeing the four lower bounds on $\chi(G)$ in the above chain of inequalities one may ask why we do not keep only the strongest one and drop the rest. The reason is that if a weaker lower bound of the above gives the same value as one of the stronger ones, that may have stronger graph theoretic consequences compared to the situation when there is a gap between the two bounds. An example of this phenomenon is demonstrated in [43].

We will use the following results of earlier papers that give graph theoretic consequences of the property that one of the above lower bounds attain a certain value.

The first such theorem we need involves the strongest of the above bounds. It is proven by Csorba et al. in [9] where it is called the $K_{\ell,m}$-theorem.

$K_{\ell,m}$-theorem (*[9]*). *If G is a graph satisfying* $\mathrm{ind}\,(B(G)) + 2 \geq t$, *then for every possible $\ell, m \in \mathbb{N}$ with $\ell + m = t$, the complete bipartite graph $K_{\ell,m}$ appears as a subgraph of G.*

The following result, that was named Zig–zag Theorem in [41], involves the third of the above bounds.

Zig–zag Theorem (*[41], cf. Also [15]*). *If G is a graph satisfying* $\mathrm{coind}\,(B_0(G)) + 1 \geq t$, *then the following holds for every proper coloring $c : V(G) \to \mathbb{N}$. G contains a $K_{\lceil t/2\rceil,\lfloor t/2\rfloor}$ subgraph all t vertices of which receive a different color by c. Furthermore, these t colors, if considered in their natural order as natural numbers, appear alternately on the two sides of the given $K_{\lceil t/2\rceil,\lfloor t/2\rfloor}$ subgraph.*

Note that the number of colors used for the coloring in the Zig–zag Theorem may be much more than $\chi(G)$. In case $\chi(G) = t = \mathrm{coind}\,(B_0(G)) + 1$ a colorful version of the $K_{\ell,m}$-theorem is proven in [42].

We quote another result from [41] that gives a characterization of those graphs for which the fourth of the above lower bounds is above a certain value. This characterization needs the notion of Borsuk graphs defined by Erdős and Hajnal [13].

Definition 1 (*[13]*). The Borsuk graph $B(n, \alpha)$ of parameters n and $0 < \alpha < 2$ is an infinite graph whose vertices are the points of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ and whose edges connect its pairs of vertices with distance at least α.

Theorem B (*Lemma 4.4 in [41]*). *A finite graph G satisfies* $\mathrm{coind}\,(B(G)) \geq n - 1$ *if and only if there is a graph homomorphism from $B(n, \alpha)$ to G for some $\alpha < 2$.*

Note that coind $(B(G)) \geq t-2$ implies $\chi(G) \geq t$ and $\chi(B(t-1,\alpha)) = t$ for large enough $\alpha < 2$ is equivalent to the Borsuk–Ulam theorem, cf. [32]. We remark that graphs satisfying coind $(B(G)) \geq t-2$ are called strongly topologically t-chromatic in [41] as opposed to topologically t-chromatic graphs defined by satisfying coind $(B_0(G)) \geq t-1$.

3. On the Behzad–Vizing conjecture

The Behzad–Vizing conjecture states that one can always color the vertices and the edges of a simple graph G with at most $\Delta(G)+2$ colors in such a way that neither adjacent vertices nor edges with a common endvertex get the same color, furthermore, no edge is colored the same as one of its endpoints. Here $\Delta(G)$ denotes the maximum degree of G. The minimum number of colors needed for such a coloring is called the *total chromatic number* and is often denoted by $\chi''(G)$. It is simply the chromatic number of $T(G)$, the *total graph* of G defined by

$$V(T(G)) = V(G) \cup E(G)$$

and

$$E(T(G)) = \{\{a,b\} : a,b \in V(G), \{a,b\} \in E(G) \text{ or } a \in V(G), b \in E(G), a \in b \text{ or } \\ a,b \in E(G), a \cap b \neq \emptyset\}.$$

This problem is open for more than forty years. Its original appearance seems to be independently [1,2,53], see also [35,19]. It was solved for $\Delta(G) = 3$ by Rosenfeld [38] and Vijayaditya [52] (it is trivial for $\Delta(G) \leq 2$) and for $\Delta(G) = 4$ and 5 by Kostochka [27–29]. The fractional chromatic number $\chi_f(T(G))$ is proven to be at most $\Delta(G)+2$ for any value of $\Delta(G)$ by Kilakos and Reed [25].

Here we prove a topological version, stating that even the strongest of the above topological bounds is at most $\Delta(G)+2$ for $T(G)$.

Theorem 1. *For any simple graph G the inequality*

$$\text{ind}\,(B(T(G))) \leq \Delta(G)$$

holds.

Proof. Let $\varDelta = \Delta(G)$. We prove that $T(G)$ can contain the complete bipartite graph $K_{2,\varDelta+1}$ as a subgraph only if $\varDelta \leq 3$. In the latter case the statement of the theorem follows from the above mentioned result of Rosenfeld [38] and Vijayaditya [52] that verifies the original conjecture in this case. For $\varDelta > 3$ the lack of the above complete bipartite subgraph proves the theorem by the $K_{\ell,m}$-theorem of Csorba et al. [9] quoted in Section 2.

Assume for a contradiction that $T(G)$ does contain a $K_{2,\varDelta+1}$ subgraph, while $\varDelta > 3$. Let the two sides (color classes) of this complete bipartite subgraph be denoted by A and B, where $|A| = 2$ and $|B| = \varDelta + 1$.

Recall that $V(T(G)) = V(G) \cup E(G)$. We will simply denote $V(G)$ by V and $E(G)$ by E and distinguish among a few cases according to the size of the intersections of A and B with V and E, respectively.

First observe that $A \cup B \subseteq V$ is impossible, because then the vertices of G in A should have degree at least $\varDelta + 1$ contradicting the definition of $\varDelta$.

So there is some $e \in E$ that belongs to $A \cup B$. First assume $e \in B$ and $A \subseteq V$. Then the two vertices in A must be the two endpoints of e. Since there is no other edge both of these vertices belong to, we must have $|B \cap V| = |B \setminus \{e\}| = \varDelta$. Then the elements of A are adjacent in G (as vertices of G) to the $\varDelta$ vertices in $|B \cap V|$ plus each other (by e), so their degree in G is at least $\varDelta + 1$, a contradiction.

This proves that $A \cap E$ cannot be empty, so we may assume $e \in A$. If $|A \cap E| = 2$ (that is both elements of A are edges of G), say $A = \{e, f\} \subseteq E$, then $B \cap V \subseteq e \cap f$. If the edges e and f have no common endpoint, then $B \cap V = \emptyset$ and $\varDelta + 1 = |B| = |B \cap E| \leq 4$, as there are at most 4 edges that have a common endpoint with both e and f. This contradicts to $\varDelta > 3$. If, on the other hand, e and f have a common endpoint u, then we still have $|B \cap E| \geq |B| - 1 = \varDelta$ and all these edges except at most one must have u as one of its endpoints. (One exceptional edge can connect the endpoints of e and f different from u.) But then the degree of u is at least $\varDelta + 1$ in G, a contradiction.

So we may assume $|A \cap E| = 1$. Let $A = \{v, e\}$ where $v \in V$ and $e \in E$. If v is an endpoint of e then $|B \cap V| \leq 1$ as there are only 2 vertices e is connected to in $T(G)$ and one of them is $v \notin B$. Thus $|B \cap E| = |B| - 1 = \Delta$ and all these Δ edges have v as one of their endpoints. But v is also an endpoint of $e \notin B$, so v has degree at least $\Delta + 1$ in G, a contradiction again. Finally we have to look at the case when v is not an endpoint of e. Then $|B \cap V| \leq 2$ since e has only two endpoints and $|B \cap E| \leq 2$, because there are at most 2 edges containing v as an endpoint and having the other endpoint at one end of e. Thus $\Delta + 1 = |B| \leq 4$ contradicting the assumption that $\Delta > 3$. □

Remark 1. In the proof above we had two cases where the contradiction was with $\Delta(G) > 3$, i.e., where we relied on Rosenfeld's and Vijayaditya's theorem. The first such case is inessential, there we could continue by simply saying that if $A \cup B \subseteq E$ and the two elements in A are independent edges of G, then getting $|B| = 4$ means that the 6 edges in $A \cup B$ form a K_4 subgraph of G which must be a connected component itself and from this point the argument is easy to complete. The second case when we relied on $\Delta > 3$ is more essential. This is at the very end of the proof and the $K_{2,4}$ produced there can in fact come up in $T(G)$ without forcing the vertices and edges belonging to it to form a separate component of G. ◇

4. On Hedetniemi's conjecture

For two graphs F and G their direct (or categorical) product $F \times G$ is defined on vertex set $V(F) \times V(G)$ such that two vertices (f_1, g_1) and (f_2, g_2) are adjacent if and only if $\{f_1, f_2\} \in E(F)$ and $\{g_1, g_2\} \in E(G)$. Let F and G be simple graphs. It is easy to check that $\chi(F \times G) \leq \min\{\chi(F), \chi(G)\}$ (simply color vertex (f, g) of $F \times G$ with the color of f to obtain a proper coloring with $\chi(F)$ colors), and Hedetniemi's conjecture states that equality holds. This conjecture is wide open, the major special case proven is when the right hand side is 4 [12]. In fact, even that is not known whether the function $f(t) := \min\{\chi(F \times G) : \chi(F) \geq t,\ \chi(G) \geq t\}$ goes to infinity with t or not. (Though rather surprisingly, if it does not, then it remains below 10.) For further information and references we refer the reader to the excellent recent survey by Tardif [47].

Concerning relaxations involving the fractional chromatic number, Tardif proved in [48] that $\chi(F \times G) \geq \frac{1}{2} \min\{\chi_f(F), \chi_f(G)\}$ and in [46] that $\chi_f(F \times G) \geq \frac{1}{4} \min\{\chi_f(F), \chi_f(G)\}$, where χ_f stands for the fractional chromatic number. Note that while the first of these inequalities is a weakening of Hedetniemi's conjecture, the second is only an analog, although if the exact value of the multiplicative constant is ignored then it also implies the first one.

Another relaxation mentioned in Tardif's survey [47] is due to Hell [18]. It already connects Hedetniemi's conjecture to Lovász's topological lower bound on the chromatic number. In particular, in Tardif's formulation, Hell shows that if F and G are two graphs for which Lovász's bound is tight then $\chi(F \times G) = \min\{\chi(F), \chi(G)\}$. This result can also be found in Dochtermann's paper [10].

Along these lines we state the following topological analog of Hedetniemi's conjecture.

Theorem 2.

$$\operatorname{coind}(B(F \times G)) = \min\{\operatorname{coind}(B(F)), \operatorname{coind}(B(G))\}.$$

Proof. It is true for any pair of graphs F and G that a homomorphism from $F \times G$ exists both to F and G (simply by taking projections). Assume $\operatorname{coind}(B(F \times G)) = h$. Then, by Theorem B, there is some Borsuk graph $B(h+1, \alpha)$ which homomorphically maps into $F \times G$. Combining this homomorphism with either of the projection homomorphisms mentioned above, we get a homomorphism from $B(h+1, \alpha)$ to F and to G, respectively. Thus $\operatorname{coind}(B(F)) \geq h$ and $\operatorname{coind}(B(G)) \geq h$ also holds. This proves $\operatorname{coind}(B(F \times G)) \leq \min\{\operatorname{coind}(B(F)), \operatorname{coind}(B(G))\}$.

To prove the reverse inequality let $d = \min\{\operatorname{coind}(B(F)), \operatorname{coind}(B(G))\}$. Then by Theorem B there is some large enough $\alpha < 2$ for which $B(d+1, \alpha)$ admits a homomorphism f to F and also a homomorphism g to G. But then the function which maps every vertex $v \in V(B(d+1, \alpha))$ to $(f(v), g(v))$ is a homomorphism of $B(d+1, \alpha)$ to $F \times G$ and its existence proves $\operatorname{coind}(F \times G) \geq d$. Therefore we have $\operatorname{coind}(B(F \times G)) \geq \min\{\operatorname{coind}(B(F)), \operatorname{coind}(B(G))\}$ completing the proof. □

Remark 2. Dochtermann and Schultz [11] found another class of graphs that satisfy a similar statement as Theorem B if put in place of Borsuk graphs. One could base the above proof also on this result instead of Theorem B. $\diamond$

Remark 3. Péter Csorba [6] observed that a result of Kozlov (equation 2.4.2 in [30]; cf. also [18,10]) combined with results from Csorba [7] implies that $B(F \times G)$ is $\mathbb{Z}_2$-homotopy equivalent to $B(F) \times B(G)$ for every pair of graphs F and G. The product $A \times B$ here is meant to be the product topological space equipped with the involution $\nu : (x, y) \mapsto (\nu_A(x), \nu_B(y))$ where ν_A and ν_B are the respective involutions on the $\mathbb{Z}_2$-spaces A and B. From this observation an alternative proof for Theorem 2 can easily be obtained. $\diamond$

5. On Hadwiger's conjecture

Hadwiger's conjecture states that if $\chi(G) \geq t$, then G contains a K_t minor. It is essentially trivial for $t \leq 3$, relatively easy to prove for $t = 4$, known to be equivalent to the Four Color Theorem for $t = 5$, and proven to be so also for $t = 6$ [37]; see [51] for an excellent survey. For $t \geq 7$ the conjecture is open and is widely considered as one of the most important open problems in graph theory. Even a linear approximation is not proven, that is, it is not known whether there exists a constant c such that $\chi(G) \geq t$ implies that G contains a complete minor on ct vertices. Such a result is proven for the fractional chromatic number in place of the chromatic number with $c = 1/2$ by Reed and Seymour [36]. Stating in the counterpositive form they proved that if a graph contains no complete minor on $m + 1$ vertices then its fractional chromatic number is at most $2m$. In [41] it was observed that an analogous statement immediately follows from the $K_{\ell,m}$-theorem for the topological lower bounds on the chromatic number. Namely, if G contains no K_{m+1} minor, then $\operatorname{ind}(B(G)) + 2 < 2m$.

Although Hadwiger's conjecture is wide open, a strengthening, called the "odd Hadwiger conjecture" received much attention in recent years, see [16,22–24]. To state it we need the concept of an odd K_m minor.

Definition 2. An odd K_m minor of graph G is formed by m vertex disjoint trees $T_1, \ldots, T_m$ in G that have the following additional properties.

The vertices in these trees can be simultaneously 2-colored such that

1. Each tree T_i is properly colored;
2. For every pair $i \neq j$, $1 \leq i, j \leq m$, there is an edge between a vertex of T_i and a vertex of T_j that have the same color.

The odd Hadwiger conjecture was suggested by Gerards and Seymour (cf. [20]) and it states that if $\chi(G) \geq t$ then G must contain an odd K_t minor.

In some cases the known results about this conjecture show surprising similarities with those known about Hadwiger's original conjecture. In particular, Kawarabayashi and Reed [23] have proved an analog of the Reed–Seymour theorem, namely, they showed that if G does not contain an odd K_m minor then the fractional chromatic number of G is at most $2m$. This suggests the question whether one can prove that graphs with no odd K_m minor satisfy $\operatorname{ind}(B(G)) + 2 \leq 2m$. It is clear that now we cannot get this just from the $K_{\ell,m}$-theorem, since its conclusion holds for large complete bipartite graphs that contain no large odd complete minors. Though we also did not succeed to get something similar from the Zig–zag Theorem, we wonder whether its conclusion would already imply such a statement.

Question. *Is there some constant c for which the following is true? If every proper coloring of a graph G satisfies the conclusion of the Zig–zag Theorem (with parameter t) then G contains an odd complete minor on ct vertices.*

In the following section we prove that the odd Hadwiger conjecture holds for some graph families that have their chromatic number equal to its topological lower bound, while the fractional chromatic number is much smaller.

6. The odd Hadwiger conjecture for large enough Kneser graphs and generalized Mycielski graphs

Recall that the Kneser graph $\mathrm{KG}(n, k)$ is defined for $n > 2k$ on all k-subsets of an n-element set as vertices, where two of these form an edge iff they are disjoint. The chromatic number of graph $\mathrm{KG}(n, k)$ is $n - 2k + 2$ [31], while their fractional chromatic number is only n/k, cf. [14,39]. We are going to prove the following result.

Theorem 3. *If $t = n - 2k + 2$ is fixed and n is large enough, then the t-chromatic Kneser graph* $\mathrm{KG}(n, k)$ *contains an odd K_t minor.*

Recall that a topological K_r subgraph in a graph G is a collection of r *branching vertices* together with $\binom{r}{2}$ vertex disjoint paths in G connecting all pairs of the branching vertices. We call a topological K_r subgraph *odd* if all the latter $\binom{r}{2}$ paths are odd, i.e., they contain an odd number of edges. A famous conjecture stronger than Hadwiger's was due to Hajós claiming that every graph of chromatic number t contains a topological K_t subgraph. This was disproved by Catlin [4] for $t \geq 7$, cf. also [51]. (It is known to hold for $t \leq 4$ when it is actually equivalent to Hadwiger's conjecture, and is still open for $t = 5, 6$.) Several other counterexamples can be found in a more recent paper by Thomassen [50].

Since every odd topological K_r subgraph gives rise to an odd K_r minor, and since the odd Hadwiger conjecture is trivial if the chromatic number is less than 4 and is also known to hold when it is equal to 4 (the latter was proven by Catlin [4]), Theorem 3 immediately follows from the following result.

Theorem 4. *If $t = n - 2k + 2 \geq 5$ is fixed and n is large enough then the Kneser graph* $\mathrm{KG}(n, k)$ *contains an odd topological K_t subgraph.*

Proof. Arrange the n points $1, 2, \ldots, n$ on a circle and let their k-subsets be identified with the vertices of $\mathrm{KG}(n, k)$.

A k-subset formed by k cyclically consecutive points on the circle will be called a *short arc*, while a *long arc* is formed by a set of $\ell := \left\lfloor \frac{n-1}{2} \right\rfloor$ cyclically consecutive points. The first point of a long arc is meant to be its first element when the arc is traversed in a clockwise order along the circle. The relative position of a k-subset of a long arc within the long arc will be called its *pattern* if it contains the first element of the long arc. Thus for example, if $k = 3$, then the subset $\{1, 3, 7\}$ has the same pattern in the long arc starting with 1 as the subset $\{n-1, 1, 5\}$ in the long arc starting with $n-1$. Note that a pattern in a given long arc defines a vertex of $\mathrm{KG}(n, k)$, and if we have two different pairs, both consisting of a long arc and a pattern, then these define distinct vertices of $\mathrm{KG}(n, k)$. This is ensured by the condition that the first element of the long arc is always in the k-subset defined by a pattern. Note also that for such vertices it is meaningful to speak about the *pattern of the vertex*, since a k-subset that fits into a long arc defines the long arc with which it has the same starting vertex, i.e., the one in which it has a pattern.

Select $t = n - 2k + 2$ short arcs (there are n altogether) and fix them. These will be the branching vertices of our odd topological K_t. Call a pattern *good* if it is not identical with the first k vertices in the long arc. (In other words, vertices with a good pattern are not short arcs.) Next we select a different good pattern for each pair of the branching vertices. First we will show that this is possible, and then we show that between any two branching vertices there is a path of odd length, all inner vertices of which have the same pattern, namely the one attached to the given pair of vertices. These paths will then be automatically disjoint as their inner vertices have different patterns. They also cannot touch other branching vertices than their endpoints, since the branching vertices form short arcs and they are excluded from the set of good patterns.

The number of good patterns is easily seen to be $\binom{\lfloor (n-3)/2 \rfloor}{k-1} - 1$. This is equal to $\binom{(n-3)/2}{(t-3)/2} - 1$ if n is odd and to $\binom{(n-4)/2}{(t-4)/2} - 1$ if n is even.

We can select a different good pattern for all pairs of branching vertices if the above expression is not less than $\binom{t}{2}$. Since t is fixed, the latter number is constant, while the above expressions go to infinity with n whenever $t \geq 5$. This proves that for large enough n the required inequality holds.

It remains to prove that between any two branching vertices there exists a path of odd length with all inner nodes having an arbitrarily fixed good pattern. To this end, first observe that for any two given long arcs, a and b, one can find a sequence of long arcs $a = s_0, s_1, \ldots, s_r = b$, such that $s_i \cap s_{i+1} = \emptyset$ for all $i = 0, 1, \ldots, r-1$ and r is even. In other words, there is an even length path between any two vertices in $\mathrm{KG}(n, \ell)[L]$, where $\mathrm{KG}(n, \ell)[L]$ is the subgraph of $\mathrm{KG}(n, \ell)$ induced by the set L of vertices that form long arcs. This statement is true because two closest long arcs, i.e., two long arcs with symmetric difference 2 still have a long arc in the complement of their union. Thus with two steps (a step meaning going from one vertex of $\mathrm{KG}(n, \ell)[L]$ to another along an edge) we can shift any long arc along our circle by 1. Therefore we can realize any shift with an even number of steps. Given two branching points, i.e., two short arcs x and y, choose a to be a long arc disjoint from x and b to be a long arc containing y. Consider the above sequence of long arcs between a and b and then substitute each long arc of the sequence by the vertex of $\mathrm{KG}(n, k)$ contained in the given long arc and having the pattern attached to the pair of branching points (x, y) (while b is substituted by y). Adding x to the beginning of this sequence we obtain the required odd length path in $\mathrm{KG}(n, k)$ between vertices x and y completing the proof. □

Schrijver [40] defined a beautiful family of graphs, that appear as induced subgraphs of Kneser graphs and share some of their important properties.

Definition 3 (*[40]*)**.** The Schrijver graph $\mathrm{SG}(n, k)$ is defined as the induced subgraph of the Kneser graph $\mathrm{KG}(n, k)$ on the vertices

$$V(SG(n,k)) = \left\{ a \subseteq \binom{[n]}{k} : \forall i\ \{i, i+1\} \not\subseteq a,\ \{1, n\} \not\subseteq a \right\}.$$

It is proven in [40] that the chromatic number of $\mathrm{SG}(n, k)$ is also $n-2k+2$ as for $\mathrm{KG}(n, k)$, moreover, $\mathrm{SG}(n, k)$ is vertex color critical. Talbot [45] determined the independence number of $\mathrm{SG}(n, k)$ which easily implies that $\chi_f(\mathrm{SG}(n, k)) = \chi_f(KG(n, k))$, too.

It is easy to see, that if n is odd, then choosing the cyclic permutation on our circle at the beginning of the proof of Theorem 4 as $1, 3, 5, \ldots, n, 2, 4, \ldots, n-1$, each long arc will be such that neither a set $\{i, i+1\}$, nor $\{n, 1\}$ will be contained in it. Thus any k-subset of any long arc will be a vertex of $\mathrm{SG}(n, k)$ and the proof goes through for $\mathrm{SG}(n, k)$ just as it did for $\mathrm{KG}(n, k)$. In case n is even, we can simply ignore the point n and fix the circle as above on the elements $1, \ldots, n-1$ only. Observing that the proof would allow more than t branching points, too, we apply the above argument for $t+1$ branching points that goes again through the same way. Thus we obtain the following strengthening of Theorems 3 and 4.

Corollary 5. *If $t = n - 2k + 2 \geq 5$ is fixed and n is large enough, then the t-chromatic Schrijver graph $\mathrm{SG}(n, k)$ contains an odd topological K_t subgraph. In particular, for any fixed $t = n - 2k + 2$ and n large enough $\mathrm{SG}(n, k)$ contains an odd K_t minor.*

Generalized Mycielski graphs form another family of graphs where topological lower bounds on the chromatic number give sharp estimates, while the fractional chromatic number is far below the chromatic number [49].

The r-level generalized Mycielskian $M_r(G)$ of a graph G is defined on the vertex set

$$V(G) \times \{0, 1, \ldots, r-1\} \cup \{z\}$$

with edge set

$$E(M_r(G)) = \Big\{\{(u, i), (v, j)\} : \{u, v\} \in E(G) \text{ and } (|i-j| = 1 \text{ or } i = j = 0)\Big\} \\ \cup \Big\{\{(u, r-1), z\} : u \in V(G)\Big\}.$$

Mycielski graphs are usually meant to be the graphs obtained from K_2 by an iterative use of the above general Mycielski construction with $r = 2$.

Results of Stiebitz [44] (cf. also [17,33]) generalized by Csorba [8] imply that if the box complex $B(G)$ of a graph G is homotopy equivalent to a sphere $\mathbb{S}^h$ (this is the case for complete graphs and more generally for all Schrijver graphs, see [3]), then for any positive integer r, the box complex $B(M_r(G))$ is homotopy equivalent to $\mathbb{S}^{h+1}$, therefore $\text{ind}\,(B(M_r(G))) = \text{ind}\,(B(G)) + 1$ holds. In particular, if the above homotopy equivalence holds and the topological lower bound (in this case the four lower bounds we discussed coincide) of the chromatic number is tight (this also happens for all Schrijver graphs), then it is 1 more and also tight for $M_r(G)$. (Note that there are graphs with $\chi(M_r(G)) = \chi(G)$, an example given in [49] is the complement of the 7-cycle with $r = 3$. Another example is given in [8].)

Concerning the odd Hadwiger conjecture we prove the following.

Proposition 6. *If G contains an odd K_t minor then $M_r(G)$ contains an odd K_{t+1} minor for every $r \geq 1$.*

Proof. We may assume that G is connected and that $r \geq 2$. Consider G as the subgraph induced on vertices $(v, 0)$ of $M_r(G)$ and the t vertex disjoint trees $T_1, \ldots, T_t$ with their 2-coloring that give an odd K_t minor in this induced subgraph G. Notice that if some of these t trees have only one vertex then they are all colored the same, say blue.

Now take an arbitrary spanning tree T_{t+1} on the vertices in the set $\{(v, i) : i > 0\} \cup \{z\}$ and its proper 2-coloring that gives color blue to all vertices of the form $(v, 1)$. (Such a coloring is valid as the vertices $\{(v, i) : i > 0\} \cup \{z\}$ induce a bipartite subgraph in $M_r(G)$ in which the distance between any two vertices $\{v, 1\}, \{v', 1\}$ is even.) It remains to show only that all trees T_i with $i \leq t$ have a blue colored vertex that has a neighbor among the vertices $(v, 1)$. But this is almost obvious: By the connectedness of G every vertex $(u, 0)$ has some neighbor of the form $(v, 1)$ and all the trees $T_1, \ldots, T_t$ either have an edge and then one of its endpoints is necessarily blue or it is a one-point tree, but then it is blue by the above observation. So we have an odd K_{t+1} minor. $\square$

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