ACADEMIC PRESS

J. Math. Anal. Appl. 270 (2002) 590–601

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.academicpress.com

On the stability of a quadratic Jensen type functional equation

Young Whan Lee

Department of Mathematics, Daejeon University, Daejeon, 300-716, Republic of Korea

Received 9 January 2001

Submitted by T.M. Rassias

Abstract

In this paper we obtain the general solution of the quadratic Jensen type functional equation

$$9f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z)$$
$$=4\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right]$$

and prove the stability of this equation in the spirit of Hyers, Ulam, Rassias, and Găvruta.

Keywords: Hyers–Ulam–Rassias stability; Quadratic functional equation

1. Introduction

In 1940, Ulam [1] raised the following question concerning the stability of homomorphisms: given a group G_1, a metric group G_2 with metric $d(\cdot,\cdot)$ and $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) \leqslant \delta$ for all $x, y \in G_1$, then a homomorphism $g : G_1 \to G_2$ exists with $d(f(x), g(x)) \leqslant \epsilon$ for all $x \in G_1$?

E-mail address: ywlee@dju.ac.kr.

PII: S0022-247X(02)00093-8

The case of approximately additive mappings was solved by Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. Rassias [3] proved a substantial generalization of the result of Hyers and also Găvruta obtained a further generalization of the Hyers–Rassias theorem (see also [4,5]). Later, many Rassias and Găvruta type theorems concerning the stability of different functional equations were obtained by numerous authors (see, for instance, [3,6–20]).

In this paper we deal with the quadratic Jensen type functional equation

$$9f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z) = 4\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right]. \tag{1}$$

Trif [21] solved the Jensen type functional equation

$$3f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z) = 2\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right] \tag{2}$$

and investigated the Hyers–Ulam–Rassias stability of this equation. Equations (1) and (2) have different solutions and further the methods of the proofs of the stability theorems are also different.

In Section 2, we solve the quadratic Jensen type functional equation (1). In Section 3, we prove the stability of Eq. (1) in the spirit of Hyers, Ulam, Rassias, and Găvruta.

2. Solutions of Eq. (1)

Theorem 2.1. *Let X and Y be real linear spaces. A function $f: X \to Y$ satisfies* (1) *for all $x, y, z \in X$ if and only if there exist an element $B \in Y$, an additive function $A: X \to Y$, and a quadratic function $Q: X \to Y$ such that*

$$f(x) = Q(x) + A(x) + B$$

for all $x \in X$.

Proof. (Necessity) Let $Q(x) := (1/2)(f(x) + f(-x)) - f(0)$, $A(x) := (1/2) \times (f(x) - f(-x))$, and $B = f(0)$ for all $x \in X$. Then $A(0) = 0$, $A(-x) = -A(x)$, $Q(0) = 0$, $Q(-x) = Q(x)$,

$$9A\left(\frac{x+y+z}{3}\right)+A(x)+A(y)+A(z) = 4\left[A\left(\frac{x+y}{2}\right)+A\left(\frac{y+z}{2}\right)+A\left(\frac{z+x}{2}\right)\right] \tag{3}$$

and

$$9Q\left(\frac{x+y+z}{3}\right)+Q(x)+Q(y)+Q(z) \\ =4\left[Q\left(\frac{x+y}{2}\right)+Q\left(\frac{y+z}{2}\right)+Q\left(\frac{z+x}{2}\right)\right] \tag{4}$$

for all $x, y, z \in X$.

First we claim that A is additive. Putting $z=-y$ in (3) yields

$$9A\left(\frac{x}{3}\right)+A(x)=4\left[A\left(\frac{x+y}{2}\right)+A\left(\frac{x-y}{2}\right)\right] \tag{5}$$

for all $x, y \in X$. Replacing y by x in (5) we have

$$9A\left(\frac{x}{3}\right)=3A(x) \tag{6}$$

for all $x \in X$. Thus we have $A(3x)=3A(x)$ for all $x \in X$. Replacing x by $3y$ in (5) we get

$$A(2y)=2A(y) \tag{7}$$

for all $y \in X$. Replacing $z=-x-y$ in (3) and using (7) we get

$$A(x)+A(y)-A(x+y)=2\left[A(x+y)-A(x)-A(y)\right]$$

for all $x, y \in X$. Thus

$$A(x+y)=A(x)+A(y)$$

for all $x, y \in X$.

Secondly we claim that Q is quadratic. That is, $Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$ for all $x, y \in X$. Putting $z=-x$ in (4) yields

$$9Q\left(\frac{y}{3}\right)+2Q(x)+Q(y)=4\left[Q\left(\frac{x+y}{2}\right)+Q\left(\frac{y-x}{2}\right)\right] \tag{8}$$

for all $x, y \in X$. Replacing y by 0 and x by $2x$ in (8) we get

$$Q(2x)=4Q(x) \tag{9}$$

for all $x \in X$. Putting $x=0$ in (8) we obtain

$$9Q\left(\frac{y}{3}\right)+Q(y)=2Q(y) \tag{10}$$

for all $y \in X$. By (8)–(10), we have

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$$

for all $x, y \in X$.

(Sufficiency) This is obvious. □

3. Hyers–Ulam–Rassias stability of Eq. (1)

Throughout this section X and Y will be a real normed linear space and a real Banach space, respectively. Given a function $f: X \to Y$, we set

$$
\begin{aligned}
Df(x,y,z) := {} & 9f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) \\
& - 4\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right]
\end{aligned}
$$

for all $x, y, z \in X$. Let $\varphi: X\backslash\{0\} \times X\backslash\{0\} \times X\backslash\{0\} \to [0, \infty)$ be a mapping satisfying one of the conditions (a), (b) and one of the conditions (c), (d):

$$\Phi_1(x,y,z) = \sum_{n=1}^{\infty} \frac{1}{9^n} \varphi(3^n x, 3^n y, 3^n z) < \infty, \tag{a}$$

$$\Phi_2(x,y,z) = \sum_{n=0}^{\infty} 9^n \varphi\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) < \infty, \tag{b}$$

$$\Phi_3(x,y,z) = \sum_{n=1}^{\infty} \frac{1}{3^{n+1}} \varphi(3^n x, 3^n y, 3^n z) < \infty, \tag{c}$$

$$\Phi_4(x,y,z) = \sum_{n=0}^{\infty} 3^{n-1} \varphi\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) < \infty \tag{d}$$

for all $x, y, z \in X\backslash\{0\}$.

One of the conditions (a) , (b) will be needed to derive a quadratic function and one of the conditions (c), (d) will be needed to derive an additive function in the following theorem.

Theorem 3.1. *If the function $f: X \to Y$ satisfies*

$$\left\| Df(x,y,z) \right\| \leqslant \varphi(x,y,z) \tag{11}$$

for all $x, y, z \in X\backslash\{0\}$, then there exist a unique quadratic function $Q: X \to Y$, a unique additive function $A: X \to Y$, and a unique element $B \in Y$ such that

$$
\begin{aligned}
&\left\| f(x) - Q(x) - A(x) - B \right\| \leqslant \epsilon(x) + \delta(x), \\
&\left\| \frac{f(x) + f(-x)}{2} - Q(x) - B \right\| \leqslant \epsilon(x),
\end{aligned}
$$

and

$$\left\| \frac{f(x) - f(-x)}{2} - A(x) \right\| \leqslant \delta(x)$$

for all $x \in X \backslash \{0\}$, *where* $\epsilon(x) = (1/2)[\Phi_i(x,x,-x) + \Phi_i(-x,-x,x)]$, $i = 1$ *or* 2, *and* $\delta(x) = (1/2)[\Phi_j(x,x,-x) + \Phi_j(-x,-x,x)]$, $j = 3$ *or* 4. *The functions* Q, A, *and the element* B *are given by*

$$Q(x) = \begin{cases} \lim_{n\to\infty} \frac{f(3^n x)+f(-3^n x)}{2\cdot 9^n} & \text{if } \varphi \text{ satisfies (a)},\\ \lim_{n\to\infty} \frac{9^n}{2}\left[f\left(\frac{x}{3^n}\right) + f\left(\frac{-x}{3^n}\right) - 2f(0)\right] & \text{if } \varphi \text{ satisfies (b)}, \end{cases}$$

$$A(x) = \begin{cases} \lim_{n\to\infty} \frac{f(3^n x)-f(-3^n x)}{2\cdot 3^n} & \text{if } \varphi \text{ satisfies (c)},\\ \lim_{n\to\infty} \frac{3^n}{2}\left[f\left(\frac{x}{3^n}\right) - f\left(\frac{-x}{3^n}\right)\right] & \text{if } \varphi \text{ satisfies (d)}, \end{cases}$$

for all $x \in X$ *and* $B = f(0)$.

Proof. Let $f_1 : X \to Y$ be the function defined by $f_1(x) := (f(x) + f(-x))/2 - f(0)$ for all $x \in X$. Then $f_1(0) = 0$, $f_1(x) = f_1(-x)$, and

$$\begin{aligned}\|Df_1(x,y,z)\| &= \left\| 9f_1\left(\frac{x+y+z}{3}\right) + f_1(x) + f_1(y) + f_1(z) \right.\\ &\quad \left. - 4\left[f_1\left(\frac{x+y}{2}\right) + f_1\left(\frac{y+z}{2}\right) + f_1\left(\frac{z+x}{2}\right)\right]\right\| \\ &\leqslant \frac{1}{2}\left[\varphi(x,y,z) + \varphi(-x,-y,-z)\right] \end{aligned} \tag{12}$$

for all $x, y, z \in X \backslash \{0\}$. Putting $z = -x$ in (12) yields

$$\begin{aligned}&\left\| 9f_1\left(\frac{y}{3}\right) + 2f_1(x) + f_1(y) - 4\left[f_1\left(\frac{x+y}{2}\right) + f_1\left(\frac{x-y}{2}\right)\right]\right\| \\ &\quad \leqslant \frac{1}{2}\left[\varphi(x,y,-x) + \varphi(-x,-y,x)\right] \end{aligned} \tag{13}$$

for all $x, y \in X \backslash \{0\}$. Putting $y = x$ in (13) we get

$$\left\| 9f_1\left(\frac{x}{3}\right) - f_1(x)\right\| \leqslant \frac{1}{2}\left[\varphi(x,x,-x) + \varphi(-x,-x,x)\right] \tag{14}$$

for all $x \in X \backslash \{0\}$. Replacing x by $3x$ in (14) and dividing by 9 we have

$$\left\| f_1(x) - \frac{f_1(3x)}{9}\right\| \leqslant \frac{1}{2}\cdot\frac{1}{9}\left[\varphi(3x,3x,-3x) + \varphi(-3x,-3x,3x)\right] \tag{15}$$

for all $x \in X \backslash \{0\}$.

Assume that φ satisfies the condition (a). Replacing x by $3^{n-1}x$ and dividing by 9^{n-1} in (15) we obtain

$$\begin{aligned}&\left\| \frac{f_1(3^{n-1}x)}{9^{n-1}} - \frac{f_1(3^n x)}{9^n}\right\| \\ &\quad \leqslant \frac{1}{2}\cdot\frac{1}{9^n}\left[\varphi(3^n x, 3^n x, -3^n x) + \varphi(-3^n x, -3^n x, 3^n x)\right] \end{aligned} \tag{16}$$

for all $n \in N$ and $x \in X\backslash\{0\}$. An induction argument implies

$$\left\| f_1(x) - \frac{f_1(3^n x)}{9^n} \right\| \leqslant \frac{1}{2}\sum_{i=1}^{n}\frac{1}{9^i}\left[\varphi(3^i x, 3^i x, -3^i x) + \varphi(-3^i x, -3^i x, 3^i x)\right] \leqslant \frac{1}{2}\left[\Phi_1(x, x, -x) + \Phi_1(-x, -x, x)\right] \tag{17}$$

for all $n \in N$ and $x \in X\backslash\{0\}$. Hence

$$\left\| \frac{f_1(3^n x)}{9^n} - \frac{f_1(3^m x)}{9^m} \right\| \leqslant \frac{1}{2}\sum_{i=m+1}^{n}\frac{1}{9^i}\left[\varphi(3^i x, 3^i x, -3^i x) + \varphi(-3^i x, -3^i x, 3^i x)\right] \tag{18}$$

for all $n, m \in N$ with $n > m$ and $x \in X\backslash\{0\}$. This shows that $\{f_1(3^n x)/9^n\}$ is a Cauchy sequence for all $x \in X\backslash\{0\}$ and thus converges. Therefore we can define a function $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty}\frac{f_1(3^n x)}{9^n} \quad \text{for all } x \in X. \tag{19}$$

Note that $Q(0) = 0$, $Q(-x) = Q(x)$, and $Q(3x) = 9Q(x)$ for all $x \in X$. By (13) we have

$$9Q\left(\frac{y}{3}\right) + 2Q(x) + Q(y) - 4\left[Q\left(\frac{x+y}{2}\right) + Q\left(\frac{x-y}{2}\right)\right] = 0 \tag{20}$$

for all $x, y \in X\backslash\{0\}$. Putting $y = 3x$ in (20) we obtain

$$Q(2x) = 4Q(x) \tag{21}$$

for all $x \in X\backslash\{0\}$. Replacing x by $x + y$ and y by $x - y$ in (20) we get

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y) \tag{22}$$

for all $x, y \in X\backslash\{0\}$ with $x + y \neq 0$, $x - y \neq 0$. Since $Q(0) = 0$ and $Q(2x) = 4Q(x)$, it follows that

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

for all $x, y \in X$. Taking the limit in (17) as $n \to \infty$, we obtain

$$\left\| f_1(x) - Q(x) \right\| \leqslant \frac{1}{2}\left[\Phi_1(x, x, -x) + \Phi_1(-x, -x, x)\right] \tag{23}$$

for all $x \in X\backslash\{0\}$.

If Q' is another quadratic function satisfying (23), then $Q'(0) = 0$, $Q'(2x) = 4Q'(x)$, and $Q'(-x) = Q'(x)$ for all $x \in X$. Replacing y by $2x$ in $Q'(x+y) +$

$Q'(x-y)=2Q'(x)+2Q'(y)$ we have $Q'(3x)+Q'(-x)=2Q'(x)+2Q'(2x)$ and so $Q'(3x)=9Q'(x)$ for all $x\in X$. Thus we have

$$\begin{aligned}\|Q(x)-Q'(x)\| &\leqslant \left\|\frac{Q(3^nx)}{9^n}-\frac{f_1(3^nx)}{9^n}\right\|+\left\|\frac{f_1(3^nx)}{9^n}-\frac{Q'(3^nx)}{9^n}\right\| \\ &\leqslant \frac{1}{9^n}\big[\Phi_1(3^nx,3^nx,-3^nx)+\Phi_1(-3^nx,-3^nx,3^nx)\big] \\ &= \sum_{i=n+1}^{\infty}\frac{1}{9^i}\big[\varphi(3^ix,3^ix,-3^ix)+\varphi(-3^ix,-3^ix,3^ix)\big]\end{aligned}$$

for all $n\in N$ and $x\in X$. Therefore we can conclude that $Q(x)=Q'(x)$ for all $x\in X$.

If φ satisfies the condition (b), the proof is analogous to that of case (a). Indeed, replacing x by $x/3^{n-1}$ and multiplying by 9^{n-1} in (14) we get

$$\begin{aligned}&\left\|9^nf_1\left(\frac{x}{3^n}\right)-9^{n-1}f_1\left(\frac{x)}{3^{n-1}}\right)\right\| \\ &\quad\leqslant \frac{1}{2}\cdot 9^{n-1}\left[\varphi\left(\frac{x}{3^{n-1}},\frac{x}{3^{n-1}},\frac{-x}{3^{n-1}}\right)+\varphi(\frac{-x}{3^{n-1}},\frac{-x}{3^{n-1}},\frac{x}{3^{n-1}}\right)\right]\end{aligned} \tag{24}$$

for all $n\in N$ and $x\in X\backslash\{0\}$. An induction argument implies

$$\begin{aligned}&\left\|f_1(x)-9^nf_1\left(\frac{x}{3^n}\right)\right\| \\ &\quad\leqslant \frac{1}{2}\sum_{i=0}^{n-1}9^i\left[\varphi\left(\frac{x}{3^i},\frac{x}{3^i},\frac{-x}{3^i}\right)+\varphi\left(\frac{-x}{3^i},\frac{-x}{3^i},\frac{x}{3^i}\right)\right]\end{aligned} \tag{25}$$

for all $n\in N$ and $x\in X\backslash\{0\}$. Hence

$$\begin{aligned}&\left\|9^nf_1\left(\frac{x}{3^n}\right)-9^mf_1\left(\frac{x}{3^m}\right)\right\| \\ &\quad\leqslant \frac{1}{2}\sum_{i=m}^{n-1}9^i\left[\varphi\left(\frac{x}{3^i},\frac{x}{3^i},\frac{-x}{3^i}\right)+\varphi\left(\frac{-x}{3^i},\frac{-x}{3^i},\frac{x}{3^i}\right)\right]\end{aligned} \tag{26}$$

for all $n,m\in N$ with $n>m$ and $x\in X\backslash\{0\}$. This shows that $\{9^nf_1(x/3^n)\}$ is a Cauchy sequence for all $x\in X\backslash\{0\}$ and thus converges. Therefore we can define a function $Q:X\to Y$ by

$$Q(x)=\lim_{n\to\infty}9^nf_1\left(\frac{x}{3^n}\right)$$

for all $x\in X$. Then $Q(0)=0$, $Q(-x)=Q(x)$, and $Q(3x)=9Q(x)$ for all $x\in X$. By the same proof as that of case (a), we have

$$9Q\left(\frac{y}{3}\right)+2Q(x)+Q(y)-4\left[Q\left(\frac{x+y}{2}\right)+Q\left(\frac{x-y}{2}\right)\right]=0$$

for all $x, y \in X \backslash \{0\}$, and so

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

for all $x, y \in X$. Taking the limit in (25) as $n \to \infty$ we obtain

$$\|f_1(x) - Q(x)\| \leqslant \frac{1}{2}\big[\Phi_2(x, x, -x) + \Phi_2(-x, -x, x)\big] \tag{27}$$

for all $x \in X \backslash \{0\}$. Also we easily have that Q is unique.

Now let $f_2 : X \to Y$ be the function defined by $f_2(x) := (1/2)(f(x) - f(-x))$ for all $x \in X$. Then $f_2(0) = 0$, $f_2(-x) = -f_2(x)$, and

$$\begin{aligned} \|D_2 f(x, y, z)\| &= \left\| 9 f_2\left(\frac{x+y+z}{3}\right) + f_2(x) + f_2(y) + f_2(z) \right. \\ &\quad \left. - 4\left[f_2\left(\frac{x+y}{2}\right) + f_2\left(\frac{y+z}{2}\right) + f_2\left(\frac{z+x}{2}\right)\right] \right\| \\ &\leqslant \frac{1}{2}\big[\varphi(x, y, z) + \varphi(-x, -y, -z)\big] \end{aligned} \tag{28}$$

for all $x, y, z \in X \backslash \{0\}$. Putting $z = -x$ in (28) yields

$$\begin{aligned} &\left\| 9 f_2\left(\frac{y}{3}\right) + f_2(y) - 4\left[f_2\left(\frac{x+y}{2}\right) + f_2\left(\frac{y-x}{2}\right)\right] \right\| \\ &\quad \leqslant \frac{1}{2}\big[\varphi(x, y, -x) + \varphi(-x, -y, x)\big] \end{aligned} \tag{29}$$

for all $x, y \in X \backslash \{0\}$. Putting $y = x$ in (29) we get

$$\left\| 9 f_2\left(\frac{x}{3}\right) - 3 f_2(x) \right\| \leqslant \frac{1}{2}\big[\varphi(x, x, -x) + \varphi(-x, -x, x)\big] \tag{30}$$

for all $x \in X \backslash \{0\}$.

Assume that φ satisfies the condition (c). Replacing x by $3x$ and dividing by 9 in (30) we have

$$\left\| f_2(x) - \frac{f_2(3x)}{3} \right\| \leqslant \frac{1}{2} \cdot \frac{1}{3^2}\big[\varphi(3x, 3x, -3x) + \varphi(-3x, -3x, 3x)\big]$$

for all $x \in X \backslash \{0\}$. Replacing x by $3^{n-1}x$ and dividing by 3^{n-1} we obtain

$$\begin{aligned} &\left\| \frac{f_2(3^{n-1}x)}{3^{n-1}} - \frac{f_2(3^n x)}{3^n} \right\| \\ &\quad \leqslant \frac{1}{2} \cdot \frac{1}{3^{n+1}}\big[\varphi(3^n x, 3^n x, -3^n x) + \varphi(-3^n x, -3^n x, 3^n x)\big] \end{aligned}$$

for all $n \in N$ and $x \in X \backslash \{0\}$. An induction argument implies

$$\begin{aligned}
&\left\| f_2(x) - \frac{f_2(3^n x)}{3^n} \right\| \\
&\quad \leqslant \frac{1}{2} \sum_{i=1}^{n} \frac{1}{3^{i+1}} \left[\varphi(3^i x, 3^i x, -3^i x) + \varphi(-3^i x, -3^i x, 3^i x) \right] \\
&\quad = \frac{1}{2} \left[\Phi_3(x, x, -x) + \Phi_3(-x, -x, x) \right]
\end{aligned} \tag{31}$$

for all $n \in N$ and $x \in X \backslash \{0\}$. Hence

$$\begin{aligned}
&\left\| \frac{f_2(3^n x)}{3^n} - \frac{f_2(3^m x)}{3^m} \right\| \\
&\quad \leqslant \frac{1}{2} \sum_{i=m+1}^{n} \frac{1}{3^{i+1}} \left[\varphi(3^i x, 3^i x, -3^i x) + \varphi(-3^i x, -3^i x, 3^i x) \right]
\end{aligned} \tag{32}$$

for all $n, m \in N$ with $n > m$ and $x \in X \backslash \{0\}$. This shows that $\{f_2(3^n x)/3^n\}$ is a Cauchy sequence for all $x \in X \backslash \{0\}$ and thus converges. Therefore we can define a function $A : X \to Y$ by

$$A(x) = \lim_{n \to \infty} \frac{f_2(3^n x)}{3^n} \quad \text{for all } x \in X.$$

Note that $A(0) = 0$, $A(-x) = -A(x)$, and $A(3x) = 3A(x)$ for all $x \in X$. By (29) we have

$$9A\left(\frac{y}{3}\right) + A(y) - 4\left[A\left(\frac{x+y}{2}\right) + A\left(\frac{y-x}{2}\right)\right] = 0$$

for all $x, y \in X \backslash \{0\}$. Putting $y = 3x$ we obtain $A(2x) = 2A(x)$ for all $x \in X \backslash \{0\}$. Replacing $(x + y)/2$ by x and $(y - x)/2$ by y we get $A(x + y) = A(x) + A(y)$ for all $x, y \in X \backslash \{0\}$ with $x + y \neq 0$, $y - x \neq 0$. Since $A(0) = 0$, $A(-x) = -A(x)$, and $A(2x) = 2A(x)$, it follows that

$$A(x + y) = A(x) + A(y)$$

for all $x, y \in X$. Taking the limit in (31) as $n \to \infty$, we obtain

$$\left\| f_2(x) - A(x) \right\| \leqslant \frac{1}{2} \left[\Phi_3(x, x, -x) + \Phi_3(-x, -x, x) \right] \tag{33}$$

for all $x \in X \backslash \{0\}$.

If A' is another additive mapping satisfying (33), then we have

$$\begin{aligned}
\left\| A(x) - A'(x) \right\| &\leqslant \left\| \frac{A(3^n x)}{3^n} - \frac{f_2(3^n x)}{3^n} \right\| + \left\| \frac{f_2(3^n x)}{3^n} - \frac{A'(3^n x)}{3^n} \right\| \\
&\leqslant \frac{1}{3^n} \left[\Phi_3(3^n x, 3^n x, -3^n x) + \Phi_3(-3^n x, -3^n x, 3^n x) \right] \\
&= 3 \sum_{i=n}^{\infty} \frac{1}{3^{i+1}} \left[\varphi(3^i x, 3^i x, -3^i x) + \varphi(-3^i x, -3^i x, 3^i x) \right]
\end{aligned}$$

for all $n \in N$ and $x \in X$. Therefore we can conclude that $A(x) = A'(x)$ for all $x \in X$.

Assume that φ satisfies the condition (d). Dividing by 3 in (30) we get

$$\left\| f_2(x) - 3f_2\left(\frac{x}{3}\right)\right\| \leqslant \frac{1}{2}\cdot\frac{1}{3}\bigl[\varphi(x,x,-x)+\varphi(-x,-x,x)\bigr] \tag{34}$$

for all $x \in X\backslash\{0\}$. Replacing x by $x/3^{n-1}$ and multiplying by 3^{n-1} we have

$$\begin{aligned}&\left\|3^{n-1}f_2\left(\frac{x}{3^{n-1}}\right) - 3^n f_2\left(\frac{x}{3^n}\right)\right\| \\ &\quad\leqslant \frac{1}{2}\cdot 3^{n-2}\left[\varphi\left(\frac{x}{3^{n-1}},\frac{x}{3^{n-1}},\frac{-x}{3^{n-1}}\right)+\varphi\left(\frac{-x}{3^{n-1}},\frac{-x}{3^{n-1}},\frac{x}{3^{n-1}}\right)\right]\end{aligned}$$

for all $n \in N$ and $x \in X\backslash\{0\}$.

The rest of the proof is similar to the corresponding part of the proof of the case (c). Thus there is a unique additive mapping $A: X \to Y$ such that $\|f_2(x) - A(x)\| \leqslant (1/2)[\Phi_4(x,x,-x) + \Phi_4(-x,-x,x)]$ for all $x \in X\backslash\{0\}$. Let $B = f(0)$. Since $f(x) = f_1(x) + f_2(x) + f(0)$ for all $x \in X$, it follows that

$$\begin{aligned}&\bigl\|f(x) - Q(x) - A(x) - B\bigr\| \\ &\quad\leqslant \bigl\|f_1(x) - Q(x)\bigr\| + \bigl\|f_2(x) - A(x)\bigr\| \\ &\quad\leqslant \frac{1}{2}\bigl[\Phi_i(x,x,-x) + \Phi_i(-x,-x,x) + \Phi_j(x,x,-x) + \Phi_j(-x,-x,x)\bigr] \\ &\quad= \epsilon(x) + \delta(x)\end{aligned}$$

for all $x \in X\backslash\{0\}$, $i = 1$ or 2 and $j = 3$ or 4. We complete the proof. □

Corollary 3.2. *Let $p \neq 1, 2$ and $\theta > 0$ be real numbers. Suppose that the function $f: X \to Y$ satisfies*

$$\bigl\|Df(x,y,z)\bigr\| \leqslant \theta\bigl(\|x\|^p + \|y\|^p + \|z\|^p\bigr)$$

for all $x, y, z \in X\backslash\{0\}$. Then there exist a unique quadratic function $Q: X \to Y$, a unique additive mapping $A: X \to Y$, and a unique element $B \in Y$ such that

$$\bigl\|f(x) - Q(x) - A(x) - B\bigr\| \leqslant 3^p\theta\|x\|^p\left(\frac{3}{|9-3^p|} + \frac{1}{|3^p-3|}\right),$$

$$\left\|\frac{f(x)+f(-x)}{2} - Q(x) - B\right\| \leqslant 3^p\theta\|x\|^p\frac{3}{|9-3^p|},$$

and

$$\left\|\frac{f(x)-f(-x)}{2} - A(x)\right\| \leqslant 3^p\theta\|x\|^p\frac{1}{|3^p-3|}$$

for all $x \in X\backslash\{0\}$.

Proof. Let $\varphi(x,y,z)=\theta(\|x\|^p+\|y\|^p+\|z\|^p)$ for all $x,y,z\in X\backslash\{0\}$. Then $\varphi(x,x,-x)=\varphi(-x,-x,x)=3\theta\|x\|^p$ for all $x\in X\backslash\{0\}$. If $p<2$, we have

$$\sum_{n=1}^{\infty}\frac{1}{9^n}\varphi(3^n x,3^n x,-3^n x)=\sum_{n=1}^{\infty}3\theta\|x\|^p\cdot 3^{n(p-2)}=3\theta\|x\|^p\frac{3^p}{9-3^p}$$

for all $x\in X\backslash\{0\}$. If $p>2$, we get

$$\sum_{n=0}^{\infty}9^n\varphi\left(\frac{x}{3^n},\frac{x}{3^n},\frac{-x}{3^n}\right)=\sum_{n=0}^{\infty}3\theta\|x\|^p\cdot 3^{n(2-p)}=3\theta\|x\|^p\frac{3^p}{3^p-9}$$

for all $x\in X\backslash\{0\}$. If $p<1$, then we have

$$\sum_{n=1}^{\infty}\frac{1}{3^{n+1}}\varphi(3^n x,3^n x,-3^n x)=\sum_{n=1}^{\infty}\theta\|x\|^p\cdot 3^{n(p-1)}=\theta\|x\|^p\frac{3^p}{3-3^p}$$

for all $x\in X\backslash\{0\}$. If $p>1$, then we get

$$\sum_{n=0}^{\infty}3^{n-1}\varphi\left(\frac{x}{3^n},\frac{x}{3^n},\frac{-x}{3^n}\right)=\sum_{n=0}^{\infty}\theta\|x\|^p\cdot 3^{n(1-p)}=\theta\|x\|^p\frac{3^p}{3^p-3}$$

for all $x\in X\backslash\{0\}$. Thus

$$\epsilon(x)+\delta(x)=\begin{cases}3^p\theta\|x\|^p\left(\frac{3}{3^p-9}+\frac{1}{3^p-3}\right) & \text{if } p>2,\\ 3^p\theta\|x\|^p\left(\frac{3}{9-3^p}+\frac{1}{3^p-3}\right) & \text{if } 1<p<2,\\ 3^p\theta\|x\|^p\left(\frac{3}{9-3^p}+\frac{1}{3-3^p}\right) & \text{if } p<1,\end{cases}$$

for all $x\in X\backslash\{0\}$. □

Corollary 3.3. *Let $\theta>0$ be a real number. If the function $f:X\to Y$ satisfies*

$$\|Df(x,y,z)\|\leqslant\theta$$

for all $x,y,z\in X\backslash\{0\}$. Then there exist a unique quadratic function $Q:X\to Y$, a unique additive mapping $A:X\to Y$, and a unique element $B\in Y$ such that

$$\|f(x)-Q(x)-A(x)-B\|\leqslant\frac{7}{24}\theta,$$
$$\left\|\frac{f(x)+f(-x)}{2}-Q(x)-B\right\|\leqslant\frac{1}{8}\theta,$$

and

$$\left\|\frac{f(x)-f(x)}{2}-A(x)\right\|\leqslant\frac{1}{6}\theta$$

for all $x\in X\backslash\{0\}$.

References

[1] S.M. Ulam, in: Problems in Modern Mathematics, Wiley, New York, 1964, Chapter VI.
[2] D.H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. USA 27 (1941) 222–224.
[3] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978) 297–300.
[4] G.L. Forti, Hyers–Ulam stability of functional equations in several variables, Aequationes Math. 50 (1995) 143–190.
[5] D.H. Hyers, G. Isac, Th.M. Rassias, Stability of Functional Equations in Several Variables, Birkhäuser, Boston, 1998.
[6] D.H. Hyers, Th.M. Rassias, Approximate homomorphisms, Aequationes Math. 44 (1992) 125–153.
[7] K.W. Jun, G.H. Kim, Y.W. Lee, Stability of generalized gamma and beta functional equations, Aequationes Math. 60 (2000) 15–24.
[8] S.-M. Jung, Hyers–Ulam–Rassias stability of functional equations, Dynamic Systems Appl. 6 (1997) 541–566.
[9] S.-M. Jung, Hyers–Ulam–Rassias stability of Jensen's equation and its application, Proc. Amer. Math. Soc. 126 (1998) 3137–3143.
[10] S.-M. Jung, On the Hyers–Ulam stability of the functional equations that have the quadratic property, J. Math. Anal. Appl. 222 (1998) 126–137.
[11] S.-M. Jung, On the Hyers–Ulam–Rassias stability of a quadratic functional equation, J. Math. Anal. Appl. 232 (1999) 384–393.
[12] Y.W. Lee, The stability of derivations on Banach algebras, Bull. Inst. Math. Acad. Sinica 28 (2000) 113–116.
[13] Y.H. Lee, K.W. Jun, A generalization of the Hyers–Ulam–Rassias stability of Jensen's equation, J. Math. Anal. Appl. 238 (1999) 305–315.
[14] Th.M. Rassias, On a problem of S.M. Ulam and the asymptotic stability of the Cauchy functional equation with applications, in: General Inequalities 7, Internat. Ser. Numer. Math., Vol. 123, Birkhäuser, Basel, 1997, pp. 297–309.
[15] Th.M. Rassias, On the stability of the quadratic functional equation and its applications, Studia Univ. Babes-Bolyai XLIII (1998) 89–124.
[16] Th.M. Rassias, The problem of S.M. Ulam for approximately multiplicative mappings, J. Math. Anal. Appl. 246 (2000) 352–378.
[17] Th.M. Rassias, On the stability of functional equations in Banach spaces, J. Math. Anal. Appl. 251 (2000) 264–284.
[18] Th.M. Rassias, On the stability of functional equations and a problem of Ulam, Acta Appl. Math. 62 (2000) 23–130.
[19] Th.M. Rassias, P. Šemrl, On the behavior of mappings that do not satisfy Hyers–Ulam stability, Proc. Amer. Math. Soc. 114 (1992) 989–993.
[20] P. Găvruta, A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994) 431–436.
[21] T. Trif, Hyers–Ulam–Rassias stability of a Jensen type functional equation, J. Math. Anal. Appl. 250 (2000) 579–588.