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When do completion processes give rise to extensive categories?

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Abstract

We consider various (free) completion processes: the exact completion and the regular completion of a category with weak finite limits, the pre-regular completion of a category with finite products and weak finite limits, the exact completion of a regular category, the regular reflection of a pre-regular category, and the filtered-colimit completion of a small category. In each case we give necessary and sufficient conditions for the completion to be extensive; or, in the case of the pre-regular completion, for the completion to satisfy a weakened notion of extensivity which we call pre-extensivity. © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

An extensive category is, roughly speaking, a category with finite coproducts which are "well behaved". This notion was already known implicity in the 1960s – a stronger version involving infinite coproducts is part of Giraud's characterization of Grothendieck toposes – but has recently been isolated as an important one in its own right. It has since found a variety of applications, for example in proof theory [9], categorical Galois theory [11], and in the study of descent morphisms for internal structures [36]. For a systematic study of extensive categories, one may consult the reference paper of Carboni, Walters, and the first author [13], or that of Cockett [18].

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An extensive category which moreover has finite limits is said to be *lextensive*: a lextensive category is precisely a category with finite limits, and finite coproducts which are disjoint and stable under pullback. Thus Grothendieck toposes are lextensive, by Giraud's characterization, as are elementary toposes; but so too are much less well-behaved categories, such as the category Top of topological spaces and continuous maps.

Some interesting examples of extensive categories which fail to be lextensive are the free completion under coproducts of a general category, and the category HTop of topological spaces and homotopy classes of maps. In fact, HTop does have products, formed as in Top, but it lacks equalizers. On the other hand, it does have *weak equalizers*, namely the *homotopy equalizers* [37].

A *regular category* [3] is a category with finite limits, in which every arrow may be factorized as a strong epimorphism followed by a monomorphism, and furthermore the strong epimorphisms are stable under pullback. A regular category is said to be *exact* if every equivalence relation is a kernel pair. The basic properties of regular and exact categories are summarized in [6]. The free exact category on a category with weak finite limits has been studied in [17,26]; we call it the *exact completion* of the category with weak finite limits. If the category with weak finite limits actually has finite limits, then this is simply the "free exact category on a left exact one" of [10]. Exact completions have been used to study geometric morphisms and localizations of algebraic categories [40,41], and also in type theory, where the exact completion of the lextensive category Top is of great interest [16,5,39]. As well as the *exact* completion of a category with weak finite limits, one may form the *regular* completion of a category with weak finite limits; the precise universal properties which define these completions are recalled in Sections 1 and 3 below.

The second author and Carboni gave a two-step construction for the exact completion [17]: first one forms the regular completion of the category with weak finite limits, and then the exact completion of the resulting regular category. Here the exact completion of a regular category $\mathbb{B}$ is quite different from the exact completion of $\mathbb{B}$ seen merely as a category with weak finite limits; once again, the precise universal property is recalled below. The original idea for the construction of the exact completion of a regular category is due to Lawvere [34]; it was made explicit in [19], and further studied in [17]. The first author described a quite different construction in [32], using a category of sheaves on the regular category. The exact completion of a regular category has also attracted interest in the study of quasi-varieties [38].

If the original category with weak finite limits actually has finite products, then the regular completion, which was the first step in the construction of the exact completion given in [17], itself splits into two steps: the *pre-regular* (or Freyd) completion, followed by the regular reflection of the resulting pre-regular category; here a pre-regular category is a category with finite limits, endowed with a proper stable factorization system. The pre-regular completion, used implicitly in [21] to embed the stable homotopy category in an abelian category, has been formalized in [24], under the name of the Freyd-completion; while the regular reflection of a pre-regular category has been studied in [29] in connection with the calculus of relations.

For any completion process, such as those considered above, one may ask under what conditions the resulting category is extensive, and this is the general subject of this paper. Some results of this kind are already known. Free completions under coproducts (finite or infinite) are always extensive; but the free completion under initial objects of a category can never be extensive if the category already has an initial object P, for the sum $P+P$ in the new category will not be disjoint. Usually, however, the situation is more complicated; in fact, the two cases just mentioned differ from all others we shall study here in that the original category is not assumed to have finite coproducts, and that the unit – the functor from the original category to the completion that allows one to express the universal property – does not preserve coproducts. In fact, most of the completion processes studied here can be seen as free completions with respect to certain colimits, with these colimits being of a type that commute in Set with finite products; it is this latter property which ensures that the unit preserves finite coproducts.

In [15], as part of a study of syntactic characterizations of locally presentable categories with various types of extra structure, the notion of *nearly extensive category* was defined, and it was shown that the Cauchy completion of a category with finite coproducts is extensive if and only if the original category is nearly extensive. It was also shown that for a small category $\mathbb{C}$ with finite colimits, the category $\mathrm{Lex}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$ of finite-limit-preserving functors from $\mathbb{C}^{\mathrm{op}}$ to Set is extensive if and only if $\mathbb{C}$ is; the fact that $\mathrm{Lex}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$ is extensive for an extensive category $\mathbb{C}$ with pullbacks had already been proved in [20]. The category $\mathrm{Lex}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$, however, is the completion under filtered colimits of $\mathbb{C}$, and so this result fits into the general subject of the paper, being a partial characterization of those categories with finite coproducts whose completion under filtered colimits is extensive.

It was further proved in [15] that for a small category $\mathbb{C}$ with finite coproducts, the category $\mathrm{FP}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$ of finite-product-preserving functors from $\mathbb{C}^{\mathrm{op}}$ to Set is extensive if and only if $\mathbb{C}$ is nearly extensive; or equivalently, if and only if the Cauchy completion of $\mathbb{C}$ is extensive. In [2], however, Adámek and Rosický characterized $\mathrm{FP}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$ as the free completion of $\mathbb{C}$ under a class of colimits there called the *sifted colimits*, following Lair's terminology *tamisante* [33]; Lair (essentially) characterized the sifted colimits as those which commute in Set with finite products. Putting together the results of [15,2], we see that the sifted colimit completion of a small category $\mathbb{C}$ with finite coproducts is extensive if and only if $\mathbb{C}$ is nearly extensive; that is, if and only if the Cauchy completion of $\mathbb{C}$ is extensive.

The free completion $\mathbb{C}_{\mathrm{rc}}$ under reflexive coequalizers, of a category $\mathbb{C}$ with finite coproducts, was constructed by Andy Pitts; this was reported in [8]. Since for a small category $\mathbb{C}$ with finite coproducts $\mathrm{Lex}(\mathbb{C}_{\mathrm{rc}}^{\mathrm{op}},\mathsf{Set})$ is equivalent to $\mathrm{FP}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$, the results of [15] imply that $\mathbb{C}_{\mathrm{rc}}$ is extensive if and only if $\mathbb{C}$ is nearly extensive.

In order to study the exact completion of HTop, the second author and Gran investigated when the exact completion of a category with weak finite limits is extensive, and gave a complete answer to this question [23], using the notion of a *weakly lextensive category*.

The aim of this paper is to give a detailed account of the stability of the notion of extensivity through various completion processes, including those mentioned above. In Section 1 we recall some definitions, and compare the notion of weakly lextensive category to that of nearly extensive category; this analysis includes necessary and sufficient conditions for a Cauchy completion to be extensive, and for the exact completion of a category with weak finite limits to be extensive. In Sections 2 and 3 we consider the exact completion of a regular category, and the regular completion of a category with weak finite limits, giving in each case necessary and sufficient conditions for the completion to be extensive. In Section 4 we give necessary and sufficient conditions for the regular reflection of a pre-regular category with finite coproducts to be extensive; these involve the new notion of a *pre-extensive* category. In Section 5 we give necessary and sufficient conditions for the pre-regular completion of a category with finite products and weak finite limits to be pre-extensive. Finally in Section 6, we consider completions under filtered colimits of arbitrary small categories, not assumed as in [15] to have finite colimits, and characterize when such completions are extensive.

We consider only finite coproducts, which, since we are generally dealing with extensive categories, we henceforth call *sums*. The legs of the colimit cone of a sum we call *injections*; any arrow called i_S with some subscript S is understood to be an injection; and given $f : X \to A$ and $g : Y \to A$, we write $(f\ g) : X + Y \to A$ for the unique arrow satisfying $(f\ g)i_X = f$ and $(f\ g)i_Y = g$.

1. Nearly extensive categories

Recall [13] that a category $\mathbb{C}$ with sums is said to be extensive if for all objects X and Y the "sum-functor"

$$(\mathbb{C}/X) \times (\mathbb{C}/Y) \xrightarrow{+} \mathbb{C}/(X + Y)$$

taking $(f : A \to X, g : B \to Y)$ to $(f + g : A + B \to X + Y)$, and defined on arrows in the obvious way, is an equivalence of categories.

In fact, there are various equivalent ways of defining extensive categories, as the following proposition shows; those facts not explicitly contained in [13] are easy exercises.

Proposition 1.1. *If* $\mathbb{C}$ *is a category with sums, then*:

(1) $\mathbb{C}$ *is extensive if and only if it has pullbacks along injections, and in a commutative diagram*

$$\begin{array}{ccccc} Z_X & \longrightarrow & Z & \longleftarrow & Z_Y \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow[i_X]{} & X + Y & \xleftarrow[i_Y]{} & Y \end{array}$$

the top row is a sum if and only if the squares are pullbacks;

(2) $\mathbb{C}$ *is extensive if and only if it satisfies the following two conditions*:

(i) *all the squares*

$$\begin{array}{ccc} A & \xrightarrow{i_A} & A+B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f+g} \\ X & \xrightarrow[i_X]{} & X+Y \end{array}$$

are pullbacks;

(ii) *for each arrow* $f : Z \to X + Y$ *there are arrows* $f_X : Z_X \to X$ *and* $f_Y : Z_Y \to Y$ *and an isomorphism* $t : Z_X + Z_Y \to Z$, *giving a commutative diagram*

$$\begin{array}{ccccc} Z_X & \xrightarrow{i_{Z_X}} & Z_X+Z_Y & \xleftarrow{i_{Z_Y}} & Z_Y \\ & & \downarrow{\scriptstyle t} & & \\ {\scriptstyle f_X}\downarrow & & Z & & \downarrow{\scriptstyle f_Y} \\ & & \downarrow{\scriptstyle f} & & \\ X & \xrightarrow[i_X]{} & X+Y & \xleftarrow[i_Y]{} & Y \end{array}$$

(3) *if* $\mathbb{C}$ *has a terminal object* 1 *then it is extensive if and only if it satisfies the conditions of* (1) *above in the case* $X = Y = 1$;

It was observed in [15] that condition (i) in part (2) of the proposition is equivalent to the functor $+ : (\mathbb{C}/X) \times (\mathbb{C}/Y) \to \mathbb{C}/(X + Y)$ being fully faithful, while condition (ii) is equivalent to its being essentially surjective on objects.

We also note for future reference the following easy fact about extensive categories:

Proposition 1.2. *In any extensive category*

(i) *the third diagram below is a pullback if the first two are pullbacks*

$$\begin{array}{ccc} P_X & \xrightarrow{q_X} & B \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[a_X]{} & A \end{array} \qquad \begin{array}{ccc} P_Y & \xrightarrow{q_Y} & B \\ {\scriptstyle p_Y}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[a_Y]{} & A \end{array} \qquad \begin{array}{ccc} P_X+P_Y & \xrightarrow{(q_X\ q_Y)} & B \\ {\scriptstyle p_X+p_Y}\downarrow & & \downarrow{\scriptstyle f} \\ X+Y & \xrightarrow[(a_X\ a_Y)]{} & A \end{array}$$

(ii) *sums of monomorphisms are monomorphisms*;

(iii) *sums of pullbacks are pullbacks*;

(iv) *sums of equivalence relations are equivalence relations*;

(v) *sums are disjoint*, *and initial objects are strict*.

We recall from [15] the notion of *nearly extensive category*. A functor $F : \mathbb{A} \to \mathbb{B}$ is said to be *nearly surjective* if every object B of $\mathbb{B}$ is a retract of FA for some object A of $\mathbb{A}$, and a category $\mathbb{C}$ with sums is said to be *nearly extensive* if the sum-functors $+ : (\mathbb{C}/X)\times(\mathbb{C}/Y) \to \mathbb{C}/(X+Y)$ are fully faithful and nearly surjective; this is clearly the case if and only if $\mathbb{C}$ satisfies condition (i) in Proposition 1.1(2), and a modified condition (ii) in which the arrow t is not required to be invertible, but only to have a section; that is, a map t' with $tt' = 1$.

Recall that the Cauchy-completion $\mathbb{C}_{cc}$ of a category $\mathbb{C}$ is defined to be the "free category in which idempotents split" on the category $\mathbb{C}$; that is, the value at $\mathbb{C}$ of a left biadjoint to the inclusion 2-functor from the 2-category Cat_{cc} of Cauchy-complete categories to the 2-category Cat of categories.

One may give an explicit construction of $\mathbb{C}_{cc}$ by defining an object to be an idempotent in $\mathbb{C}$, and an arrow from $(A, e : A \to A)$ to $(A', e' : A' \to A')$ to be an arrow $f : A \to A'$ in $\mathbb{C}$ satisfying $e'f = f = fe$. The fully faithful functor $y : \mathbb{C} \to \mathbb{C}_{cc}$ taking an object A to the identity (idempotent) on A is the unit for the biadjunction. This allows a characterization of Cauchy completions.

Lemma 1.3. *A functor $y : \mathbb{C} \to \mathbb{D}$ exhibits $\mathbb{D}$ as the Cauchy completion of $\mathbb{C}$ if and only if $\mathbb{D}$ is Cauchy complete, and y is fully faithful and nearly surjective.*

Proof. This may be proved directly from the construction of $\mathbb{C}_{cc}$ given immediately before the lemma, but it is also a consequence of [28, Proposition 5.62]. □

The following result was proved in [15]:

Theorem 1.4. *If $\mathbb{C}$ is a category with sums, then its Cauchy completion $\mathbb{C}_{cc}$ has sums, the inclusion $y : \mathbb{C} \to \mathbb{C}_{cc}$ preserves sums, and $\mathbb{C}_{cc}$ is extensive if and only if $\mathbb{C}$ is nearly extensive.*

Remark 1.5. In fact, more is true. There is a 2-category Sums of categories with sums, sum-preserving functors, and natural transformations; and it has full sub-2-categories Ext, Sums_{cc}, and Ext_{cc}, consisting of the extensive categories, the Cauchy complete categories with sums, and the Cauchy complete extensive categories. The inclusions of Sums_{cc} into Sums and Ext_{cc} into Ext both have left biadjoints, each taking $\mathbb{C}$ to $\mathbb{C}_{cc}$.

The exact completion of a category $\mathbb{C}$ with weak finite limits consists of an exact category $\mathbb{C}_{ex}$ and a functor $z : \mathbb{C} \to \mathbb{C}_{ex}$ with the property that, for any exact category $\mathbb{D}$, composition with z induces an equivalence of categories between the full subcategory of the functor category $[\mathbb{C}_{ex}, \mathbb{D}]$ consisting of the regular functors, and the full subcategory of the functor category $[\mathbb{C}, \mathbb{D}]$ consisting of those functors called *left coverings* in [17] and *flat* in [26]. Recall that for a category $\mathbb{C}$ with weak finite limits and a category $\mathbb{D}$ with finite limits, a functor $F : \mathbb{C} \to \mathbb{D}$ is said to be a left covering if for each finite category $\mathscr{J}$, each functor $K : \mathscr{J} \to \mathbb{C}$, and each weak limit L of K, the canonical comparison from FL to the limit of FK is a strong epimorphism. Equivalently, a functor $F : \mathbb{C} \to \mathbb{D}$ between categories $\mathbb{C}$ and $\mathbb{D}$ as above is a left covering if and only if it is "representably flat" in the sense that for each object D of $\mathbb{D}$ the composite $\mathbb{D}(D, F) : \mathbb{C} \to \mathsf{Set}$ of F and the representable functor $\mathbb{D}(D, -)$ is flat in the classical sense, recalled in Section 6 below. In fact, a functor $F : \mathbb{C} \to \mathsf{Set}$ (where $\mathbb{C}$ has weak finite limits) is a left covering if and only it is flat, but in the absence of weak finite limits in $\mathbb{C}$, not every flat functor from $\mathbb{C}$ to Set need be representably flat. We recall that if $\mathbb{C}$ has finite limits, then F is a left covering if and only if it preserves finite limits.

Although strictly speaking the exact completion consists both of the category $\mathbb{C}_{\text{ex}}$ and the functor z, we may sometimes say simply "$\mathbb{C}_{\text{ex}}$ is the exact completion of $\mathbb{C}$". We recall from [17] that $\mathbb{C}_{\text{cc}}$ has weak finite limits if $\mathbb{C}$ does so, and that the inclusion $y : \mathbb{C} \to \mathbb{C}_{\text{cc}}$ then induces an equivalence between $\mathbb{C}_{\text{ex}}$ and $(\mathbb{C}_{\text{cc}})_{\text{ex}}$.

Recall [23] that a category $\mathbb{C}$ with sums and weak finite limits is said to be *weakly lextensive* if

(i) sums are disjoint and initial objects are strict;
(ii) for each choice of the weak products $X \times Y$ and $X \times Z$, the evident projections exhibit the sum $(X \times Y) + (X \times Z)$ as a weak product of X and $Y + Z$;
(iii) if the first two of the diagrams below are weak equalizers then so is the third.

$$E_X \xrightarrow{e_X} X \overset{f_X}{\underset{g_X}{\rightrightarrows}} Z \qquad E_Y \xrightarrow{e_Y} Y \overset{f_Y}{\underset{g_Y}{\rightrightarrows}} Z$$

$$E_X + E_Y \xrightarrow{e_X + e_Y} X + Y \overset{(f_X\ f_Y)}{\underset{(g_X\ g_Y)}{\rightrightarrows}} Z$$

The following result, although not explicitly stated in [23], provides the motivation for the name "weakly lextensive":

Lemma 1.6. *An extensive category with weak finite limits is weakly lextensive.*

Proof. Let $\mathbb{C}$ be an extensive category with weak finite limits, then condition (i) holds by Proposition 1.2. We shall verify condition (iii), leaving (ii) to the reader. In the notation of (iii), given an arrow $h : A \to X + Y$, we may write A as a sum $A_X + A_Y$, and h as $h_X + h_Y : A_X + A_Y \to X + Y$, where $h_X : A_X \to X$ and $h_Y : A_Y \to Y$. Now $(f_X + f_Y)h = (g_X + g_Y)h$ if and only if $f_X h_X = g_X h_X$ and $f_Y h_Y = g_Y h_Y$, in which case there exist arrows $s_X : A_X \to E_X$ and $s_Y : A_Y \to E_Y$ satisfying $h_X = e_X s_X$ and $h_Y = e_Y s_Y$; thus $h = h_X + h_Y = e_X s_X + e_Y s_Y = (e_X + e_Y)(s_X + s_Y)$, giving the required factorization. □

The main abstract result of [23] is:

Theorem 1.7. *If $\mathbb{C}$ has sums and weak finite limits then $\mathbb{C}_{\text{ex}}$ is extensive if and only if $\mathbb{C}$ is weakly lextensive*; *in this case $z : \mathbb{C} \to \mathbb{C}_{\text{ex}}$ preserves sums.*

Remark 1.8. In fact, by [17, Corollary 34], if $\mathbb{C}$ is merely a category with weak finite limits, then $z : \mathbb{C} \to \mathbb{C}_{\text{ex}}$ preserves any sums which happen to exist in $\mathbb{C}$. We have been unable to determine, however, whether $\mathbb{C}_{ex}$ must have sums for an arbitrary category $\mathbb{C}$ with sums and weak finite limits, not necessarily extensive.

Our main new result in this section is that the notions of weakly lextensive category and nearly extensive category in fact coincide:

Theorem 1.9. *If $\mathbb{C}$ has sums and weak finite limits, then the following conditions are equivalent*:
(i) $\mathbb{C}$ *is nearly extensive*;
(ii) $\mathbb{C}_{cc}$ *is extensive*;
(iii) $\mathbb{C}_{cc}$ *is weakly lextensive*;
(iv) $\mathbb{C}_{ex}$ *is extensive*;
(v) $\mathbb{C}$ *is weakly lextensive*.

Proof. We have (i) ⇒ (ii) by Theorem 1.4, and (ii) ⇒ (iii) by Lemma 1.6; while (iii) ⇒ (iv) follows by Theorem 1.7 and the fact that $(\mathbb{C}_{cc})_{ex} \simeq \mathbb{C}_{ex}$; and (iv) ⇒ (v) follows by Theorem 1.7. Thus we need only prove that every weakly lextensive category is nearly extensive. If $\mathbb{C}$ is weakly lextensive, then the functors $+ : (\mathbb{C}/X) \times (\mathbb{C}/Y) \to \mathbb{C}/(X+Y)$ are fully faithful by [23, Proposition 1.2(3)], thus we need only show that they are nearly surjective. Suppose then that $f : Z \to X + Y$ is given, and form weak pullbacks

$$\begin{array}{ccccc} Z_X & \xrightarrow{x} & Z & \xleftarrow{y} & Z_Y \\ {\scriptstyle f_X}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_Y} \\ X & \xrightarrow[i_X]{} & X+Y & \xleftarrow[i_Y]{} & Y. \end{array}$$

By the weak lextensivity of $\mathbb{C}$ (see, for example, [23, Proposition 1.2(2)]) the square

$$\begin{array}{ccc} Z_X + Z_Y & \xrightarrow{(x\ y)} & Z \\ {\scriptstyle f_X + f_Y}\downarrow & & \downarrow{\scriptstyle f} \\ X+Y & \xrightarrow[1]{} & X+Y \end{array}$$

is also a weak pullback, giving a (not necessarily unique) arrow $t' : Z \to Z_X + Z_Y$ satisfying $(x\ y)t' = 1$ and $f = (f_X + f_Y)t'$; now t' is the desired section for $(x\ y)$ giving the near surjectivity of $+ : (\mathbb{C}/X) \times (\mathbb{C}/Y) \to \mathbb{C}/(X+Y)$. □

Corollary 1.10. *If $\mathbb{C}$ is a Cauchy complete category with sums and weak finite limits then $\mathbb{C}_{ex}$ is extensive if and only if $\mathbb{C}$ is so.*

2. The exact completion of a regular category

The 2-category **Reg** of regular categories, regular functors, and natural transformations has a full sub-2-category **Ex** consisting of the exact categories, and the inclusion has a left biadjoint. The value at a regular category of this biadjoint is called the exact completion of the regular category; for a regular category $\mathbb{B}$ we write $\zeta : \mathbb{B} \to \mathbb{B}_{\mathrm{ex/reg}}$ for the unit of the biadjunction. Note that the exact completion of a regular category $\mathbb{B}$ is quite different to the exact completion of $\mathbb{B}$ seen merely as a category with (weak) finite limits. The context will make it clear which completion is meant when we speak of "the exact completion".

Given a regular category $\mathbb{B}$, one may form the 2-category Rel($\mathbb{B}$) of relations in $\mathbb{B}$; now every equivalence relation in $\mathbb{B}$ is an idempotent in Rel($\mathbb{B}$), and if we form a new 2-category by freely splitting *these* idempotents, then we obtain $\mathbb{B}_{\mathrm{ex/reg}}$ as the subcategory of this new 2-category consisting of all the objects and those arrows which have a right adjoint. This is the original construction, due to Lawvere [34]. A quite different construction was described in [32]; there $\mathbb{B}_{\mathrm{ex/reg}}$ was constructed as the closure of $\mathbb{B}$ in a certain category of sheaves on $\mathbb{B}$ under coequalizers of equivalence relations; this shows in particular that $\mathbb{B}_{\mathrm{ex/reg}}$ may be seen as a full subcategory of the presheaf category on $\mathbb{B}$. In either case, one may develop, as in [17], a more explicit description; but all we really need here are the following two facts. The first is proved in [19]; the second was proved in [32] in the case where $\mathbb{B}$ is small, but is in any case an easy consequence of the first.

Lemma 2.1. (i) *Every object in $\mathbb{B}_{\mathrm{ex/reg}}$ appears as the coequalizer of an equivalence relation in $\mathbb{B}$.*

(ii) *If the following diagram is a pullback in $\mathbb{B}_{\mathrm{ex/reg}}$, then A lies in the image of ζ if B and C do so.*

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ C & \to & D \end{array}$$

We shall also use the following fact about regular categories, whose proof can be found in [27]:

Lemma 2.2. *If the large rectangle and the left square in the diagram*

$$\begin{array}{ccccc} A & \to & B & \to & C \\ \downarrow & & \downarrow & & \downarrow \\ A' & \underset{q}{\to} & B' & \to & C' \end{array}$$

are pullbacks, and q is a strong epimorphism, then the right square is a pullback.

A diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{q} C$$

in an exact category will be called an *exact fork* if q is the coequalizer of f and g, and (f, g) is the kernel pair of q. The functor ζ preserves kernel pairs and coequalizers of kernel pairs, and so a diagram in $\mathbb{B}$ is an exact fork if and only if its image under ζ is an exact fork in $\mathbb{B}_{\mathrm{ex/reg}}$.

The goal of this section is to prove Theorem 2.3.

Theorem 2.3. *If $\mathbb{B}$ is a regular category with sums, then $\mathbb{B}$ is extensive if and only if $\mathbb{B}_{\mathrm{ex/reg}}$ is extensive and $\zeta : \mathbb{B} \to \mathbb{B}_{\mathrm{ex/reg}}$ preserves sums.*

Since a full subcategory of an extensive category is itself extensive if it is closed under sums and pullbacks along injections, and since $\zeta : \mathbb{B} \to \mathbb{B}_{\mathrm{ex/reg}}$ preserves finite limits, $\mathbb{B}$ is immediately seen to be extensive if $\mathbb{B}_{\mathrm{ex/reg}}$ is extensive and $\zeta : \mathbb{B} \to \mathbb{B}_{\mathrm{ex/reg}}$ preserves sums. The other half requires more work, and will be the result of three intermediate propositions, which we separate for future reference. (The fact that $\mathbb{B}_{\mathrm{ex/reg}}$ is extensive if $\mathbb{B}$ is so was stated without proof in [14].)

Recall from Proposition 1.2 that in any extensive category sums of kernel pairs are kernel pairs.

Proposition 2.4. *For any regular category $\mathbb{B}$, if $\mathbb{B}$ has sums, and sums of kernel pairs are kernel pairs, then the functor $\zeta : \mathbb{B} \to \mathbb{B}_{\mathrm{ex/reg}}$ preserves sums; in particular this is the case if $\mathbb{B}$ is extensive.*

Proof. Given a sum $C = \Sigma_i C_i$ in $\mathbb{B}$ with injections $c_i : C_i \to C$, we must show that $\zeta c_i : \zeta C_i \to \zeta C$ exhibits ζC as the sum $\Sigma_i \zeta C_i$ in $\mathbb{B}_{\mathrm{ex/reg}}$. Let F be an object of $\mathbb{B}_{\mathrm{ex/reg}}$, and let $f_i : \zeta C_i \to F$ be a family of arrows. By Lemma 2.1(i) there is an exact fork

$$\zeta S \underset{\zeta l}{\overset{\zeta k}{\rightrightarrows}} \zeta R \xrightarrow{q} F$$

in $\mathbb{B}_{\mathrm{ex/reg}}$, and then by Lemma 2.1(ii) we can form a diagram

$$\begin{array}{ccccc} \zeta A_i & \underset{\zeta n_i}{\overset{\zeta m_i}{\rightrightarrows}} & \zeta B_i & \xrightarrow{\zeta p_i} & \zeta C_i \\ {\scriptstyle \zeta s_i}\downarrow & & \downarrow{\scriptstyle \zeta r_i} & & \downarrow{\scriptstyle f_i} \\ \zeta S & \underset{\zeta l}{\overset{\zeta k}{\rightrightarrows}} & \zeta R & \xrightarrow[q]{} & F \end{array}$$

in $\mathbb{B}_{\mathrm{ex/reg}}$ for each i, in which the top row is an exact fork in $\mathbb{B}_{\mathrm{ex/reg}}$ and so the image under ζ of an exact fork in $\mathbb{B}$, the right square is a pullback, and the left squares are both pullbacks. The sum

$$A \underset{n}{\overset{m}{\rightrightarrows}} B \xrightarrow{p} C$$

of these exact forks in $\mathbb{B}$, will itself be an exact fork in $\mathbb{B}$, and so its image under ζ will be an exact fork. If we write $a_i : A_i \to A$ and $b_i : B_i \to B$ for the injections, the arrows $r_i : B_i \to R$ induce a unique $r : B \to R$ satisfying $rb_i = r_i$; and the arrows $s_i : A_i \to S$ induce a unique $s : A \to S$ satisfying $sa_i = s_i$; since $rma_i = rb_i m_i = r_i m_i = ks_i = ksa_i$ and

$rna_i = rb_in_i = r_in_i = ls_i = lsa_i$ we have $rm = ks$ and $rn = ls$. Thus $q.\zeta r.\zeta m = q.\zeta k.\zeta s = q.\zeta l.\zeta s = q.\zeta r.\zeta n$ and so there is a unique arrow $f : \zeta C \to F$ satisfying $f.\zeta p = q.\zeta r$. Now $f.\zeta c_i.\zeta p_i = f.\zeta p.\zeta b_i = q.\zeta r_i = f_i.\zeta p_i$ and so $f.\zeta c_i = f_i$. One easily confirms that f is unique with this property, so that the arrows $\zeta c_i : \zeta C_i \to \zeta C$ do indeed exhibit ζC as the sum of the ζC_i. □

Proposition 2.5. *Let $\mathbb{B}$ be a regular category with sums; suppose that $\zeta : \mathbb{B} \to \mathbb{B}_{\mathrm{ex/reg}}$ preserves sums, and that we are given a diagram*

$$\begin{array}{ccccc}
\zeta A_i & \overset{\zeta k_i}{\underset{\zeta l_i}{\rightrightarrows}} & \zeta B_i & \xrightarrow{q_i} & F_i \\
{\scriptstyle \zeta a_i}\downarrow & & \downarrow{\scriptstyle \zeta b_i} & & \downarrow{\scriptstyle f_i} \\
\zeta A & \overset{\zeta k}{\underset{\zeta l}{\rightrightarrows}} & \zeta B & \xrightarrow[q]{} & F
\end{array}$$

in which the rows are exact forks, the arrows a_i exhibit A as the sum $\Sigma_i A_i$, the b_i exhibit B as the sum $\Sigma_i B_i$, and $b_ik_i = ka_i$, $b_il_i = la_i$, and $f_iq_i = q.\zeta b_i$. Then the f_i exhibit F as the sum $\Sigma_i F_i$ in $\mathbb{B}_{\mathrm{ex/reg}}$. In particular it follows that $\mathbb{B}_{\mathrm{ex/reg}}$ has sums.

Proof. Given arrows $g_i : F_i \to G$, there is a unique $h : \zeta B \to G$ satisfying $h.\zeta b_i = g_iq_i$; and $h.\zeta k.\zeta a_i = h.\zeta b_i.\zeta k_i = g_iq_i.\zeta k_i = g_iq_i.\zeta l_i = h.\zeta b_i.\zeta l_i = h.\zeta l.\zeta a_i$, giving $h.\zeta k = h.\zeta l$; and so there is a unique $g : F \to G$ satisfying $gq = h$; and now $gf_iq_i = gq.\zeta b_i = h.\zeta b_i = g_iq_i$, and so $gf = g_i$. A straightforward argument shows that g is unique with this property. □

Proposition 2.6. *Let $\mathbb{B}$ be a regular category with sums, and suppose that $\zeta : \mathbb{B} \to \mathbb{B}_{\mathrm{ex/reg}}$ preserves sums; then $\mathbb{B}_{\mathrm{ex/reg}}$ is extensive if $\mathbb{B}$ is so.*

Proof. By the previous proposition $\mathbb{B}_{\mathrm{ex/reg}}$ has sums, and since $\mathbb{B}_{\mathrm{ex/reg}}$ has finite limits it certainly has pullbacks along injections. Given an arrow $p : F \to 1 + 1$, we may form a diagram

$$\begin{array}{ccccc}
\zeta A_1 & \xrightarrow{\zeta a_1} & \zeta A & \xleftarrow{\zeta a_2} & \zeta A_2 \\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\
\zeta B_1 & \xrightarrow{\zeta b_1} & \zeta B & \xleftarrow{\zeta b_2} & \zeta B_2 \\
\downarrow & & \downarrow & & \downarrow \\
F_1 & \xrightarrow{f_1} & F & \xleftarrow{f_2} & F_2 \\
{\scriptstyle p_1}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_2} \\
\zeta 1 & \xrightarrow[\zeta i_1]{} & \zeta(1+1) & \xleftarrow[\zeta i_2]{} & \zeta 1
\end{array}$$

in which the middle and lower squares are pullbacks, the upper squares are pairs of pullbacks, and the upper part of each column is an exact fork. By extensivity of $\mathbb{B}$, the arrows a_1 and a_2 exhibit A as the sum $A_1 + A_2$, and the arrows b_1 and b_2 exhibit

B as the sum $B_1 + B_2$. It follows by the previous proposition that f_1 and f_2 exhibit F as the coproduct $F_1 + F_2$, giving the first half of extensivity.

For the second half, let $F = F_1 + F_2$ be given, with injections f_1 and f_2; then we may form the top two-thirds of the diagram as before. Given p, p_1, and p_2 making the bottom part of the diagram commute, the bottom two squares are pullbacks by Lemma 2.2. □

These three propositions complete the proof of Theorem 2.3.

3. The regular completion of a category with weak finite limits

The regular completion of a category $\mathbb{C}$ with weak finite limits is a regular category $\mathbb{C}_{\mathrm{reg}}$ equipped with a functor $z : \mathbb{C} \to \mathbb{C}_{\mathrm{reg}}$ with the property that for any regular category $\mathbb{D}$, composition with z induces an equivalence of categories between the full subcategory of the functor category $[\mathbb{C}_{\mathrm{reg}}, \mathbb{D}]$ consisting of the regular functors, and the full subcategory of the functor category $[\mathbb{C}, \mathbb{D}]$ consisting of the left covering functors. In this section we seek conditions under which the regular completion of a category with weak finite limits is extensive. We use the following result from [17]:

Proposition 3.1. *If $\mathbb{C}$ has weak finite limits, then an object F of $\mathbb{C}_{\mathrm{ex}}$ lies in $\mathbb{C}_{\mathrm{reg}}$ if and only if there is a finite jointly-monic family $(m_i : F \to zA_i)_{i \in I}$; if $\mathbb{C}$ has a terminal object then the family may be assumed to be non-empty.*

Our first result follows easily from previous theorems.

Proposition 3.2. *If $\mathbb{C}$ has sums and weak finite limits and $\mathbb{C}_{\mathrm{reg}}$ is extensive, then $\mathbb{C}$ is nearly extensive.*

Proof. We have $\mathbb{C}_{\mathrm{ex}} \simeq (\mathbb{C}_{\mathrm{reg}})_{\mathrm{ex/reg}}$, thus if $\mathbb{C}_{\mathrm{reg}}$ is extensive, then so, by Theorem 2.3, is $\mathbb{C}_{\mathrm{ex}}$; hence $\mathbb{C}$ is nearly extensive by Theorem 1.9. □

The other direction is less straightforward, and we are forced to assume that the category has a terminal object.

Proposition 3.3. *If $\mathbb{C}$ is nearly extensive with weak finite limits and a terminal object, then $\mathbb{C}_{\mathrm{reg}}$ and $\mathbb{C}_{\mathrm{ex}}$ are lextensive, and the inclusions $\mathbb{C} \to \mathbb{C}_{\mathrm{reg}}$ and $\mathbb{C}_{\mathrm{reg}} \to \mathbb{C}_{\mathrm{ex}}$ preserve sums.*

Proof. We know by Theorem 1.9 that $\mathbb{C}_{\mathrm{ex}}$ is extensive and that the inclusion $\mathbb{C} \to \mathbb{C}_{\mathrm{ex}}$ preserves sums. Since the inclusion $\mathbb{C}_{\mathrm{reg}} \to \mathbb{C}_{\mathrm{ex}}$ is fully faithful, and $\mathbb{C}_{\mathrm{reg}}$ is closed in $\mathbb{C}_{\mathrm{ex}}$ under finite limits, it will suffice to show that $\mathbb{C}_{\mathrm{reg}}$ is closed in $\mathbb{C}_{\mathrm{ex}}$ under sums. Since the inclusion $\mathbb{C} \to \mathbb{C}_{\mathrm{ex}}$ preserves sums, the inital object of $\mathbb{C}_{\mathrm{ex}}$ is contained in $\mathbb{C}$, and so also in $\mathbb{C}_{\mathrm{reg}}$. Thus we need only show that $\mathbb{C}_{\mathrm{reg}}$ is closed in $\mathbb{C}_{\mathrm{ex}}$ under binary sums.

Let F and G be objects of $\mathbb{C}_{\text{reg}}$; then we may find finite non-empty jointly monic families $(m_i : F \to zA_i)_{i\in I}$ and $(n_j : G \to zB_j)_{j\in J}$; and these give rise to monomorphisms $m : F \to \Pi_i zA_i$ and $n : G \to \Pi_j zB_j$. By Proposition 1.2 the arrow $m+n : F+G \to (\Pi_i zA_i) + (\Pi_j zB_j)$ is also a monomorphism. For each pair $(i, j) \in I \times J$ we write r_{ij} for the composite of the projection $\Pi_i zA_i \to zA_i$ and the injection $zA_i \to zA_i + zB_j$, and similarly we write s_{ij} for the composite of the projection $\Pi_i zB_j \to zB_j$ and the injection $zB_j \to zA_i + zB_j$; the arrows r_{ij} and s_{ij} induce an arrow $t_{ij} : (\Pi_i zA_i) + (\Pi_j zB_j) \to zA_i + zB_j$; finally the arrows t_{ij} induce an arrow $t : (\Pi_i zA_i) + (\Pi_j zB_j) \to \Pi_{i,j}(zA_i + zB_j)$. The fact that t is monomorphic follows by a straightforward but tedious argument using the extensivity of $\mathbb{C}_{\text{ex}}$; then $t(m+n) : F+G \to \Pi_{i,j}(zA_i + zB_j)$ is a monomorphism, which proves that $F+G$ is in $\mathbb{C}_{\text{reg}}$. □

In fact, the assumption of the terminal object in $\mathbb{C}$ is less severe than it might seem, due to the following result:

Proposition 3.4. *Let $\mathbb{C}$ be a Cauchy complete category with weak finite limits. If $\mathbb{C}_{\text{reg}}$ is extensive then $\mathbb{C}$ has a terminal object.*

Proof. Since $\mathbb{C}_{\text{reg}}$ is extensive, the inclusion $\mathbb{C}_{\text{reg}} \to \mathbb{C}_{\text{ex}}$ preserves sums, and so $\mathbb{C}_{\text{reg}}$ is closed in $\mathbb{C}_{\text{ex}}$ under sums. Thus if 1 denotes the terminal object of $\mathbb{C}_{\text{reg}}$, there is a finite jointly monic family $(m_i : 1+1 \to zA_i)_{i\in I}$. If I is non-empty, then we can find an arrow $x : 1 \to zA_i$ by composing m_i with one of the injections $1 \to 1+1$, and now $x : 1 \to zA_i$ and the unique map $u : zA_i \to 1$ provide a splitting of the idempotent xu on zA_i. Since $\mathbb{C}$ is Cauchy complete, it follows that 1 lies in $\mathbb{C}$.

If on the other hand I is empty, then $1+1$ is a subobject of 1, and so the two injections $1 \to 1+1$ are equal. But $\mathbb{C}_{\text{reg}}$ is extensive, and so sums are disjoint and initial objects are strict; now the disjointness of sums and the equality of the injections $1 \to 1+1$ imply that 1 is initial, while the fact that initial objects are strict now implies that $\mathbb{C}_{\text{reg}}$ is equivalent to the terminal category. Since $\mathbb{C}$ is a full subcategory of $\mathbb{C}_{\text{reg}}$, either $\mathbb{C}$ is equivalent to the terminal category, or $\mathbb{C}$ is empty, but since $\mathbb{C}$ has a weak terminal object it cannot be empty, and so we conclude that $\mathbb{C}$ is equivalent to the terminal category, and so has a terminal object. □

Corollary 3.5. *If $\mathbb{C}$ is a Cauchy complete category with sums and weak finite limits, then $\mathbb{C}_{\text{reg}}$ is extensive if and only if $\mathbb{C}$ is extensive and has a terminal object. The inclusion $z : \mathbb{C} \to \mathbb{C}_{\text{reg}}$ then preserves sums.*

4. The regular reflection of a pre-regular category

In this and the next section we are concerned with the *pre-regular categories*, introduced below. Recall that a factorization system $(\mathscr{E}, \mathscr{M})$ on a category $\mathbb{B}$ is said to be *proper* if each arrow in $\mathscr{E}$ is an epi and each arrow in $\mathscr{M}$ is a mono; and said to be *stable* if the pullback of any arrow in $\mathscr{E}$ is in $\mathscr{E}$; of course the pullback of any arrow in $\mathscr{M}$ lies in $\mathscr{M}$ for *any* factorization system. The first section of [12] contains a good introduction to factorization systems; here we recall only that for a proper factorization system $(\mathscr{E}, \mathscr{M})$, and a composite gf, we have $f \in \mathscr{M}$ if $gf \in \mathscr{M}$, and $g \in \mathscr{E}$ if $gf \in \mathscr{E}$; and that $\mathscr{M}$ contains the regular monomorphisms and $\mathscr{E}$ the regular epimorphisms.

We define a *pre-regular category* to be a category with finite limits equipped with a stable proper factorization system. Such structures have been considered in [29,25]. Of course, any regular category has a canonical pre-regular structure, for which $\mathscr{E}$ consists of the strong epimorphisms, and $\mathscr{M}$ consists of the monomorphisms. Regular categories will henceforth be assumed to have this canonical pre-regular structure, unless another factorization system is explicitly specified. An example of a pre-regular category which is not regular is Top, with $\mathscr{E}$ consisting of the epimorphisms (that is, the continuous surjections) and $\mathscr{M}$ consisting of the strong monomorphisms (that is, the subspace inclusions).

A functor between pre-regular categories will be called *pre-regular* if it preserves finite limits, takes chosen epimorphisms to chosen epimorphisms, and takes chosen monomorphisms to chosen monomorphisms. We write Preg for the 2-category of pre-regular categories, pre-regular functors, and natural transformations. This contains Reg as the full sub-2-category of Preg consisting of the regular categories with their canonical pre-regular structure; or, equivalently those pre-regular categories for which $\mathscr{M}$ consists of all the monomorphisms, and so $\mathscr{E}$ consists precisely of the strong epimorphisms. The value at a pre-regular category $\mathbb{B}$ of the left biadjoint to the inclusion of Reg in Preg is what we mean by the regular reflection of the pre-regular category; we call it $\mathbb{B}_{\mathrm{reg/preg}}$, and write $p : \mathbb{B} \to \mathbb{B}_{\mathrm{reg/preg}}$ for the unit.

This regular reflection was studied in [29], and constructed using the calculus of relations: first one forms the 2-category $\mathrm{Rel}\,\mathbb{B}$ of relations in $\mathbb{B}$ with respect to $\mathscr{M}$ – thus an arrow in $\mathrm{Rel}\,\mathbb{B}$ from A to B is an arrow from C to $A \times B$ which lies in $\mathscr{M}$ – and then the regular reflection of $\mathbb{B}$ is the subcategory of $\mathrm{Rel}\,\mathbb{B}$ consisting of all the objects and those arrows which have right adjoints. It now follows that $\mathrm{Rel}\,\mathbb{B}$ is equivalent to the 2-category of relations in $\mathbb{B}_{\mathrm{reg/preg}}$.

One may also, as further observed in [29], form the regular reflection of $\mathbb{B}$ as the category of fractions $\mathbb{B}[\Sigma^{-1}]$ in the sense of Gabriel and Zisman [22], where Σ consists of those arrows in $\mathscr{E}$ which are monomorphisms; it is this approach that we shall follow. We give in Theorem 4.3 below a characterization of those preregular categories $\mathbb{B}$ with sums for which the regular reflection $\mathbb{B}_{\mathrm{reg/preg}}$ is extensive and the functor $p : \mathbb{B} \to \mathbb{B}_{\mathrm{reg/preg}}$ preserves sums.

Suppose then that $\mathbb{B}$ is a pre-regular category, and let Σ be the class of those arrows in $\mathscr{E}$ which are monomorphisms. It was observed in [29], that Σ is a *pullback congruence* in the sense of Bénabou [4]; that is, Σ contains the isomorphisms, is closed under composition, and stable under pullback, and if a composite gf is in Σ then g is in Σ if and only if f is so. It follows by the results of [4] that $p : \mathbb{B} \to \mathbb{B}[\Sigma^{-1}]$ preserves finite limits and that an arrow in $\mathbb{B}$ is inverted by p if and only if it lies in Σ.

In the slice category $\mathbb{B}/Z$ an arrow is an arrow of $\mathbb{B}$ rendering commutative a certain triangle; the arrow in $\mathbb{B}/Z$ will be deemed to lie in Σ if the corresponding arrow in $\mathbb{B}$ does so. Likewise an arrow in $\mathbb{B}/X \times \mathbb{B}/Y$ will be deemed to lie in Σ if the corresponding arrows in $\mathbb{B}/X$ and $\mathbb{B}/Y$ do so.

Now for any objects X and Y the "sum-functor"

$$\mathbb{B}/X \times \mathbb{B}/Y \xrightarrow{+} \mathbb{B}/(X+Y)$$

has a right adjoint ρ which takes an object $(A \to X+Y)$ of $\mathbb{B}/(X+Y)$ to its pullbacks along the injections $i_X : X \to X+Y$ and $i_Y : Y \to X+Y$; and we shall say that $\mathbb{B}$ is *pre-extensive* if for each X and Y the components of the unit and the counit of the adjunction $+ \dashv \rho$ lie in Σ. Explicitly, this says that if we form the pullbacks

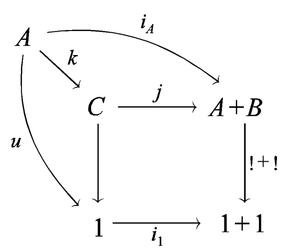
$$\begin{array}{ccccc} A_1 & \xrightarrow{a_1} & A & \xleftarrow{a_2} & A_2 \\ {\scriptstyle u_1}\downarrow & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u_2} \\ X & \xrightarrow[i_X]{} & X+Y & \xleftarrow[i_Y]{} & Y \end{array} \qquad \begin{array}{ccc} C_1 & \xrightarrow{j} & B_1+B_2 \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle v_1+v_2} \\ X & \xrightarrow[i_X]{} & X+Y \end{array}$$

then the induced map $(a_1\ a_2) : A_1 + A_2 \to A$ lies in Σ, as does the unique map $k : B_1 \to C$ satisfying $wk = v_1$ and $jk = i_{B_1}$.

Of course, any extensive pre-regular category is pre-extensive, while in the special case of regular categories (with the canonical pre-regular structure), the notions of pre-extensivity and extensivity coincide.

Proposition 4.1. *In any pre-extensive category injections are monic and initial objects are strict.*

Proof. For an injection $i_A : A \to A+B$ we have a commutative diagram



in which the middle and inner square is a pullback and k is in Σ. Now i_1 is (split) monic, and so its pullback j is monic, while k is monic since it is in Σ; thus the composite $i_A = jk$ is monic. Given an arrow $f : A \to 0$ we have pullback squares

$$\begin{array}{ccccc} A & \xrightarrow{1} & A & \xleftarrow{1} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ 0 & \xrightarrow[1]{} & 0 & \xleftarrow[1]{} & 0 \end{array}$$

and now the codiagonal $\nabla : A + A \to A$ is in Σ and so monic; thus the two injections $A \to A + A$ are equal. It follows that any parallel pair of arrows with domain A must be equal, and so that the unique arrow from 0 to A is inverse to f. □

The next result we wish to prove is that Σ is closed under sums. Of course $\mathscr{E}$ is closed under sums, and so it will suffice to prove that $\sigma + \sigma'$ is monic if σ and σ' are in Σ.

First we introduce some notation. We say that an object P is *pre-initial* if the unique arrow from 0 to P lies in Σ; in fact since the initial object 0 is strict, any arrow with domain 0 is monic, and so it suffices that the arrow be in $\mathscr{E}$. If P is pre-initial and there is an arrow $f : Q \to P$, then

$$\begin{array}{ccc} 0 & \longrightarrow & Q \\ \downarrow & & \downarrow{\scriptstyle f} \\ 0 & \longrightarrow & P \end{array}$$

is a pullback since 0 is strict; thus Q too is pre-initial since the class Σ is stable under pullback. Observe also that any parallel pair of arrows with pre-initial domain must be equal. We write $P_{X,Y}$ for the pullback of the injections $i_X : X \to X + Y$ and $i_Y : X \to X + Y$; by pre-extensivity the object $P_{X,Y}$ must be pre-initial.

We now suppose that $\sigma : A \to B$ and $\sigma' : A' \to B'$ are in Σ and prove that $\sigma + \sigma'$ is monic. We do this in three steps. First, we suppose that $u : C \to A$ and $u' : C \to A'$ satisfy $(\sigma + \sigma')i_A u = (\sigma + \sigma')i_{A'}u'$. Then $i_B\sigma u = (\sigma + \sigma')i_A u = (\sigma + \sigma')i_{A'}u' = i_{B'}\sigma' u'$ and so there is a map from C to $P_{B,B'}$. It follows that C is pre-initial, and so that $i_A u = i_{A'}u'$.

For the second step, we suppose that $u : C \to A$ and $v : C \to A + A'$ satisfy $(\sigma + \sigma')i_A u = (\sigma + \sigma')v$. We can form pullbacks

$$\begin{array}{ccccc} C_1 & \xrightarrow{c_1} & C & \xleftarrow{c_2} & C_2 \\ {\scriptstyle v_1}\downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v_2} \\ A & \xrightarrow[i_A]{} & A + A' & \xleftarrow[i_{A'}]{} & A' \end{array}$$

and now $(\sigma + \sigma')i_A uc_2 = (\sigma + \sigma')vc_2 = (\sigma + \sigma')i_{A'}v_2$ and so $i_A uc_2 = i_{A'}v_2 = vc_2$ by the first step; while $i_B\sigma uc_1 = (\sigma + \sigma')i_A uc_1 = (\sigma + \sigma')vc_1 = (\sigma + \sigma')i_A v_1 = i_B\sigma v_1$ and so $uc_1 = v_1$ since i_B and σ are monic; whence $i_A uc_1 = i_A v_1 = vc_1$. Thus $i_A uc_2 = vc_2$ and $i_A uc_1 = vc_1$, and so $i_A u = v$ since $(c_1\ c_2)$ is in Σ and so epimorphic.

Finally, let $u : C \to A + B$ and $v : C \to A + B$ satisfy $(\sigma + \sigma')u = (\sigma + \sigma')v$, and form pullbacks

$$\begin{array}{ccccc} C_1 & \xrightarrow{c_1} & C & \xleftarrow{c_2} & C_2 \\ {\scriptstyle u_1}\downarrow & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u_2} \\ A & \xrightarrow[i_A]{} & A + A' & \xleftarrow[i_{A'}]{} & A'. \end{array}$$

Then $(\sigma+\sigma')i_A u_1 = (\sigma+\sigma')uc_1 = (\sigma+\sigma')vc_1$ and so $vc_1 = i_A u_1 = uc_1$ by the second step; while $(\sigma+\sigma')i_{A'} u_2 = (\sigma+\sigma')uc_2 = (\sigma+\sigma')vc_2$ and so $vc_2 = i_{A'} u_2 = uc_2$ by the second step once again. Now since $uc_1 = vc_1$ and $uc_2 = vc_2$ and $(c_1\ c_2)$ is in Σ and so epimorphic, it follows that $u = v$, and so that $\sigma+\sigma'$ is monic. This completes the proof. □

Proposition 4.2. *Σ is closed under sums.*

We are now ready to prove the main result of this section.

Theorem 4.3. *If $\mathbb{B}$ is a pre-regular category with sums, then $\mathbb{B}$ is pre-extensive if and only if $\mathbb{B}_{\mathrm{reg/preg}}$ is extensive and $p : \mathbb{B} \to \mathbb{B}_{\mathrm{reg/preg}}$ preserves sums.*

Proof. If $\mathbb{B}$ is pre-extensive then Σ is closed under sums by Proposition 4.2, and now $\mathbb{B}[\Sigma^{-1}]$ has sums and $p : \mathbb{B} \to \mathbb{B}[\Sigma^{-1}]$ preserves sums by Section 3.2 [30] (which is a "worked example" of the more general Theorem 4.2 in the same paper). We have also seen that p preserves finite limits, since Σ is a pullback congruence. Disjointness of sums in $\mathbb{B}[\Sigma^{-1}]$ follows immediately from the fact that the components of the unit of the adjunction $+ \dashv \rho$ are in Σ and so inverted by p; while stability of sums under pullback by an arrow in the image of p follows immediately from the fact that the components of the counit of the adjunction $+ \dashv \rho$ are in Σ and so inverted by p. But of course sums are always stable under pullback by an invertible arrow, and every arrow in $\mathbb{B}[\Sigma^{-1}]$ is the composite of an arrow in the image of p and an invertible arrow, and so sums are stable in $\mathbb{B}[\Sigma^{-1}]$, and $\mathbb{B}[\Sigma^{-1}]$ is extensive.

If on the other hand $\mathbb{B}[\Sigma^{-1}]$ is extensive and p preserves sums, then pre-extensivity of $\mathbb{B}$ is immediate from the fact that p preserves finite limits and that the only arrows inverted by p are those in Σ. □

5. The pre-regular completion

In this final section on the suite of constructions related to exact completions, we consider free pre-regular categories; not in fact on an arbitrary category with weak finite limits, but rather on a category with finite products and weak equalizers; the existence of all weak finite limits is then a consequence.

In [24,25] many interesting examples – mainly triangulated categories and homotopy categories – of categories with finite products and weak equalizers are discussed. For such a category $\mathbb{C}$, a new category $\mathrm{Fr}\,\mathbb{C}$ is constructed; an object of $\mathrm{Fr}\,\mathbb{C}$ is an arrow $a : A \to A'$ in $\mathbb{C}$, while an arrow in $\mathrm{Fr}\,\mathbb{C}$ from $(a : A \to A')$ to $(b : B \to B')$ is an arrow $\check{f} : A \to B'$ for which there exist arrows $f : A \to B$ and $f' : A' \to B'$ satisfying

$bf = \check{f}$ and $f'a = \check{f}$, as in the diagram below.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & \searrow^{\check{f}} & \downarrow{\scriptstyle b} \\ A' & \xrightarrow[f']{} & B' \end{array}$$

(Alternatively one can define an arrow to be an equivalence class of pairs (f, f') satisfying $bf = f'a$, with respect to the evident equivalence relation.) The category $\mathrm{Fr}\,\mathbb{C}$ is pre-regular if we take an arrow to be in $\mathscr{E}$ when it can be represented by (f, f') as above with f invertible, and in $\mathscr{M}$ when it can be represented by (f, f') as above with f' invertible. The evident functor $d : \mathbb{C} \to \mathrm{Fr}\,\mathbb{C}$ taking an object A to $(1_A : A \to A)$ is fully faithful and preserves finite products. Furthermore d is a *left pre-covering*, in the sense that for every finite category $\mathscr{J}$, every functor $K : \mathscr{J} \to \mathbb{C}$, and every weak limit L of K, the canonical comparison from dL to the limit of dK lies in $\mathscr{E}$. We often identify an object of $\mathbb{C}$ with its image under d.

In fact, we shall write $\mathbb{C}_{\mathrm{preg}}$ for $\mathrm{Fr}\,\mathbb{C}$, and call it (along with d) the *pre-regular completion*, for it has the following universal property:

Proposition 5.1. *For any pre-regular category $\mathbb{D}$, composition with d induces an equivalence of categories between the full subcategory of the functor category $[\mathbb{C}_{\mathrm{preg}}, \mathbb{D}]$ consisting of the pre-regular functors, and the full subcategory of the functor category $[\mathbb{C}, \mathbb{D}]$ consisting of those functors which preserve finite products and are left pre-coverings.*

Remark 5.2. In fact, $\mathrm{Fr}\,\mathbb{C}$ has various other universal properties, investigated in some detail in [25], but not needed here; in particular, the above proposition remains true if we assume only that $\mathbb{D}$ has finite limits and a proper factorization system, not necessarily stable.

From the universal properties of the various objects involved one easily deduces the existence of an equivalence between $(\mathbb{C}_{\mathrm{preg}})_{\mathrm{reg/preg}}$ and $\mathbb{C}_{\mathrm{reg}}$.

We say that an object P of a pre-regular category $\mathbb{B}$ is *projective* if it is projective with respect to the class $\mathscr{E}$ of epimorphisms; that is, if $\mathbb{B}(P, -) : \mathbb{B} \to \mathsf{Set}$ sends arrows in $\mathscr{E}$ to surjections (and so is a pre-regular functor). It was observed in [25] that the full subcategory $(\mathbb{C}_{\mathrm{preg}})_{\mathrm{proj}}$ of $\mathbb{C}_{\mathrm{preg}}$ consisting of the projective objects is precisely the closure under retracts of the image of d. It follows that d, seen as a functor from $\mathbb{C}$ to $(\mathbb{C}_{\mathrm{preg}})_{\mathrm{proj}}$, is fully faithful and nearly surjective. Since $\mathbb{C}_{\mathrm{preg}}$ has finite limits, it is Cauchy complete; and so $(\mathbb{C}_{\mathrm{preg}})_{\mathrm{proj}}$ is Cauchy complete since it is closed in $\mathbb{C}_{\mathrm{preg}}$ under retracts. It now follows by Lemma 1.3 that $d : \mathbb{C} \to (\mathbb{C}_{\mathrm{preg}})_{\mathrm{proj}}$ exhibits $(\mathbb{C}_{\mathrm{preg}})_{\mathrm{proj}}$ as the Cauchy completion of $\mathbb{C}$. On the other hand, the inclusion $y : \mathbb{C} \to \mathbb{C}_{\mathrm{cc}}$ induces a functor $y_{\mathrm{preg}} : \mathbb{C}_{\mathrm{preg}} \to (\mathbb{C}_{\mathrm{cc}})_{\mathrm{preg}}$, which is easily seen to be an equivalence, using the construction of $\mathbb{C}_{\mathrm{preg}}$ as $\mathrm{Fr}\,\mathbb{C}$.

Dually, we say that an object I of a pre-regular category $\mathbb{B}$ is *injective* if it is injective with respect to the class $\mathscr{M}$ of monomorphisms; that is if $\mathbb{B}(-,I) : \mathbb{B}^{\mathrm{op}} \to \mathbf{Set}$ sends arrows in $\mathscr{M}$ to surjections. Once again the injective objects are precisely the retracts of those in the image of $\mathbb{C}$.

We use the following facts about $\mathbb{C}_{\mathrm{preg}}$, all proved in [25]:

Proposition 5.3. (i) *For every object B of $\mathbb{C}_{\mathrm{preg}}$, there exist arrows $e_B : \check{B} \to B$ and $m_B : B \to \tilde{B}$ with e_B in $\mathscr{E}$, m_B* in *$\mathscr{M}$, and with $\check{B}$ and $\tilde{B}$ in (the image of) $\mathbb{C}$.*

(ii) *If $\mathbb{C}$ has sums then $\mathbb{C}_{\mathrm{preg}}$ has sums, d preserves sums, and sums of $\mathscr{M}$'s are $\mathscr{M}$'s.*

We are now ready to prove the main result of this section.

Theorem 5.4. *If $\mathbb{C}$ is a category with sums, finite products, and weak equalizers, then the following conditions are equivalent:*
(i) *$\mathbb{C}_{\mathrm{preg}}$ is pre-extensive and sums are stable;*
(ii) *$\mathbb{C}_{\mathrm{preg}}$ is pre-extensive;*
(iii) *$\mathbb{C}$ is nearly extensive;*
(iv) *$\mathbb{C}_{\mathrm{cc}}$ is extensive.*

Proof. Trivially (i) implies (ii); while the fact that (ii) implies (iii) is almost as easy, for if $\mathbb{C}_{\mathrm{preg}}$ is pre-extensive, then $\mathbb{C}_{\mathrm{reg}}$ is extensive by Theorem 1.4 and the equivalence $\mathbb{C}_{\mathrm{reg}} \simeq (\mathbb{C}_{\mathrm{preg}})_{\mathrm{reg/preg}}$, and so $\mathbb{C}$ is nearly extensive by Proposition 3.2. The equivalence of (iii) and (iv) was proved in Theorem 1.4, and so it remains only to prove that (iv) implies (i). Since $\mathbb{C}_{\mathrm{preg}} \simeq (\mathbb{C}_{\mathrm{cc}})_{\mathrm{preg}}$, it will suffice to show that if $\mathbb{D}$ is an extensive category with finite products and weak finite limits, then $\mathbb{D}_{\mathrm{preg}}$ is pre-extensive and has stable sums.

Consider pullbacks

$$\begin{array}{ccccc} A_1 & \xrightarrow{a_1} & A & \xleftarrow{a_2} & A_2 \\ {\scriptstyle u_1}\downarrow & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u_2} \\ X & \xrightarrow[i_X]{} & X+Y & \xleftarrow[i_Y]{} & Y \end{array}$$

in $\mathbb{D}_{\mathrm{preg}}$; we shall show that the induced arrow $(a_1\ a_2) : A_1 + A_2 \to A$ lies both in $\mathscr{E}$ and in $\mathscr{M}$, and so is invertible.

Form $e_X : \check{X} \to X$, $e_Y : \check{Y} \to Y$, and $e_A : \check{A} \to A$; since $e_X + e_Y \in \mathscr{E}$ and $\check{A}$ is projective, there exists an arrow $v : \check{A} \to \check{X} + \check{Y}$ with the property that $(e_X + e_Y)v = ue_A$. Now form the pullbacks

$$\begin{array}{ccccc} B_1 & \xrightarrow{b_1} & \check{A} & \xleftarrow{b_2} & B_2 \\ {\scriptstyle v_1}\downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v_2} \\ \check{X} & \xrightarrow[i_X]{} & \check{X}+\check{Y} & \xleftarrow[i_Y]{} & \check{Y} \end{array}$$

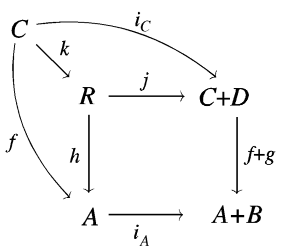
in $\mathbb{D}$. Observe that $ue_Ab_1 = (e_X + e_Y)vb_1 = (e_X + e_Y)i_{\tilde{X}}v_1 = i_Xe_Xv_1$, and so there is a unique arrow $k_1 : B_1 \to A_1$ satisfying $u_1k_1 = e_Xv_1$ and $a_1k_1 = e_Ab_1$. Similarly, there is a unique arrow $k_2 : B_2 \to A_2$ satisfying $u_2k_2 = e_Yv_2$ and $a_2k_2 = e_Ab_2$. Now $(a_1\ a_2)(k_1 + k_2) = (a_1k_1\ a_2k_2) = (e_Ab_1\ e_Ab_2) = e_A(b_1\ b_2)$, and $(b_1\ b_2)$ is invertible since $\mathbb{D}$ is extensive; thus $(a_1\ a_2)(k_1 + k_2)$ is in $\mathscr{E}$, and so too is $(a_1\ a_2)$.

Next form $m_A : A \to \tilde{A}$, $m_X : X \to \tilde{X}$, and $m_Y : Y \to \tilde{Y}$; now m_A and $(m_X + m_Y)u$ induce an arrow $w : A \to \tilde{A}\times(\tilde{X}+\tilde{Y})$; while a_1 and u_1 induce an arrow $j_1 : A_1 \to A\times X$ which lies in $\mathscr{M}$ since it is a regular monomorphism; and likewise a_2 and u_2 induce an arrow $j_2 : A_2 \to A \times X$ in $\mathscr{M}$. Writing $\delta : \tilde{A} \times \tilde{X} + \tilde{A} \times \tilde{Y} \to \tilde{A} \times (\tilde{X} + \tilde{Y})$ for the distributivity isomorphism (in $\mathbb{D}$), one easily verify that the diagram

$$\begin{array}{ccccc} A_1 + A_2 & \xrightarrow{(a_1\ a_2)} & A & \xrightarrow{w} & \tilde{A}\times(\tilde{X}+\tilde{Y}) \\ {\scriptstyle j_1+j_2}\downarrow & & & & \uparrow{\scriptstyle \delta} \\ A\times X + A\times Y & & \xrightarrow[m_A\times m_X + m_A\times m_Y]{} & & \tilde{A}\times\tilde{X}+\tilde{A}\times\tilde{Y} \end{array}$$

commutes. Now δ is invertible, while $j_1 + j_2$ and $m_A \times m_X + m_A \times m_Y$ are in $\mathscr{M}$ by Proposition 5.3, and so $(a_1\ a_2)$ is in $\mathscr{M}$. Since $(a_1\ a_2)$ is already known to be in $\mathscr{E}$, it follows that $(a_1\ a_2)$ is invertible, and so that sums are stable.

Now sums are stable in $\mathbb{D}_{\text{preg}}$, and $\mathbb{D}_{\text{preg}}$ has finite products, thus $\mathbb{D}_{\text{preg}}$ is distributive; it follows as in [13] that injections are monic. Consider a diagram



in which the inner square is a pullback; we have to show that k lies in Σ. First, we observe that jk, being an injection, is monic, so that k is monic; thus we only need to show that k is in $\mathscr{E}$. We may form pullbacks

$$\begin{array}{ccccc} C & \xrightarrow{1} & C & \xleftarrow{c_2} & C_2 \\ {\scriptstyle k_1}\downarrow & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle k_2} \\ R_1 & \xrightarrow{i_1} & R & \xleftarrow{i_2} & R_2 \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j_2} \\ C & \xrightarrow[i_C]{} & C+D & \xleftarrow[i_D]{} & D, \end{array}$$

using the fact that i_C is monic and $jk = i_C$. By stability of sums, $k = k_1 + k_2$, and so it will suffice to show that k_1 and k_2 both lie in $\mathscr{E}$. Now $j_1k_1 = 1$; but also $ji_1k_1j_1 = i_Cj_1k_1j_1 = i_Cj_1 = ji_1$; and j and i_1, being injections, are monic, so that $k_1j_1 = 1$. Thus

k_1 is invertible and so is certainly in $\mathscr{E}$; and now it remains only to prove that k_2 is in $\mathscr{E}$.

The equalities $i_A h i_2 = (f+g) j i_2 = (f+g) i_D j_2 = i_B g j_2$ imply the existence of an arrow from R_2 to $P_{A,B}$; recall from Section 4 that we write $P_{A,B}$ for the pullback of $i_A : A \to A+B$ and $i_B : B \to A+B$. The arrows $A \to 1$ and $B \to 1$ induce an arrow from $P_{A,B}$ to $P_{1,1}$, and so by composition we get an arrow from C_2 to $P_{1,1}$.

But now the inclusion $d : \mathbb{D} \to \mathbb{D}_{\text{preg}}$ preserves sums and terminal objects and is a left pre-covering, while sums in $\mathbb{D}$ are disjoint; thus the unique arrow in $\mathbb{D}_{\text{preg}}$ from 0 to $P_{1,1}$ lies in $\mathscr{E}$. Since $\mathbb{D}_{\text{preg}}$ is distributive, the initial object is strict [13], giving the pullback

$$\begin{array}{ccc} 0 & \xrightarrow{t'} & R_2 \\ \downarrow & & \downarrow \\ 0 & \xrightarrow[t]{} & P_{1,1} \end{array}$$

in $\mathbb{D}_{\text{preg}}$; whence $t' \in \mathscr{E}$ since $t \in \mathscr{E}$. Finally, t' is the composite of the unique arrow from 0 to C_2 and $k_2 : C_2 \to R_2$, and so $k_2 \in \mathscr{E}$. This completes the proof. □

Corollary 5.5. *If $\mathbb{C}$ is Cauchy complete, then $\mathbb{C}_{\text{preg}}$ is pre-extensive if and only if $\mathbb{C}$ is extensive.*

Finally, we observe that although for an extensive category $\mathbb{C}$ the pre-regular completion $\mathbb{C}_{\text{preg}}$ is not just pre-extensive, but has stable sums, we cannot hope that $\mathbb{C}_{\text{preg}}$ will actually be extensive:

Proposition 5.6. *For a category $\mathbb{C}$ with sums, finite products, and weak finite limits, if $\mathbb{C}_{\text{preg}}$ is extensive then $\mathbb{C}$ is equivalent to the terminal category.*

Proof. By disjointness of sums, the unique map $t : 0 \to 1$ in $\mathbb{C}_{\text{preg}}$ is a regular monomorphism, and so in $\mathscr{M}$. But the initial object 0 is in $\mathbb{C}$ and so injective (with respect to $\mathscr{M}$) so that the identity on 0 extends to a map $s : 1 \to 0$ satisfying $st = 1_0$. Since 1 is terminal we also have $ts = 1_1$, whence the initial and terminal objects in $\mathbb{C}_{\text{preg}}$ coincide. An extensive category in which the initial and terminal objects coincide must be equivalent to the terminal category, since the initial object is strict; thus $\mathbb{C}_{\text{preg}}$ is equivalent to the terminal category. But $\mathbb{C}$ is a full subcategory of $\mathbb{C}_{\text{preg}}$, non-empty since it has a terminal object, and so it too must be equivalent to the terminal category. □

6. The filtered colimit completion

The free completion under filtered colimits of a category $\mathbb{C}$ is a category $\mathbb{C}_{\text{filt}}$ with filtered colimits, equipped with a functor $y : \mathbb{C} \to \mathbb{C}_{\text{filt}}$, such that for any category $\mathbb{D}$ with filtered colimits, composition with y induces an equivalence of categories between the category of filtered-colimit-preserving functors from $\mathbb{C}_{\text{filt}}$ to $\mathbb{D}$ and the category of all functors from $\mathbb{C}$ to $\mathbb{D}$. We shall only consider the case where $\mathbb{C}$ is small; then $\mathbb{C}_{\text{filt}}$ is given by the category $\text{Flat}(\mathbb{C}^{\text{op}}, \mathsf{Set})$ of *flat* functors from $\mathbb{C}^{\text{op}}$ to Set. Recall that $F : \mathbb{C}^{\text{op}} \to \mathsf{Set}$ is said to be flat if the left Kan extension $\text{Lan}_Y F : [\mathbb{C}^{\text{op}}, \mathsf{Set}] \to \mathsf{Set}$ of F along the Yoneda embedding preserves finite limits; and that F is flat if and only if the opposite of the category of elements of F is filtered. Representable functors are always flat, and so we have a restricted Yoneda embedding $y : \mathbb{C} \to \text{Flat}(\mathbb{C}^{\text{op}}, \mathsf{Set})$, and it is this y which exhibits $\text{Flat}(\mathbb{C}^{\text{op}}, \mathsf{Set})$ as the filtered-colimit completion of $\mathbb{C}$. We recall also that any functor $F : \mathbb{C}^{\text{op}} \to \mathsf{Set}$, flat or otherwise, is the colimit in $[\mathbb{C}^{\text{op}}, \mathsf{Set}]$ of the the composite Yp_F, where $Y : \mathbb{C} \to [\mathbb{C}^{\text{op}}, \mathsf{Set}]$ is the Yoneda embedding and $p_F : \text{el}(F)^{\text{op}} \to \mathbb{C}$ is the canonical projection functor from the opposite of the category of elements of F to $\mathbb{C}$ which "forgets the element". For more details about filtered colimits, flat functors, and the related finitely presentable objects which arise below, one may consult the recent book [1].

In the special case where $\mathbb{C}$ has finite colimits, and so $\mathbb{C}^{\text{op}}$ has finite limits, a functor $F : \mathbb{C}^{\text{op}} \to \mathsf{Set}$ is flat if and only if it preserves finite limits, and so $\mathbb{C}_{\text{filt}}$ is just $\text{Lex}(\mathbb{C}^{\text{op}}, \mathsf{Set})$. We saw in the Introduction that $\text{Lex}(\mathbb{C}^{\text{op}}, \mathsf{Set})$ is extensive if and only if $\mathbb{C}$ is so; in this section we investigate the extensivity of $\mathbb{C}_{\text{filt}}$ when $\mathbb{C}$ is assumed only to have finite coproducts. Throughout this section $\mathbb{C}$ will be a small category with finite coproducts.

If $\mathbb{C}$ has finite coproducts, then the category $\text{FP}(\mathbb{C}^{\text{op}}, \mathsf{Set})$ of finite-product-preserving functors from $\mathbb{C}^{\text{op}}$ to Set is reflective in $[\mathbb{C}^{\text{op}}, \mathsf{Set}]$ for general reasons [31], but we can in fact describe the reflection explicitly. We use the free completion of the category $\mathbb{C}^{\text{op}}$ with respect to finite products, which may be constructed as $\text{Fam}(\mathbb{C})^{\text{op}}$ – the "Fam" construction is described below, and, in more detail, in [13] – but all we need are certain general facts about such completions, which we now summarize; all are proved in [28]. Composition with the inclusion $Z : \mathbb{C}^{\text{op}} \to \text{Fam}(\mathbb{C})^{\text{op}}$ induces an equivalence of categories, for every category $\mathbb{D}$ with finite products, between the category of finite-product-preserving functors from $\text{Fam}(\mathbb{C})^{\text{op}}$ to $\mathbb{D}$ and the category of all functors from $\mathbb{C}^{\text{op}}$ to $\mathbb{D}$; indeed this is what we mean by the completion with respect to finite products. The finite-product-preserving functor corresponding under this equivalence to a functor $F : \mathbb{C}^{\text{op}} \to \mathbb{D}$ turns out to be the right Kan extension $\text{Ran}_Z F : \text{Fam}(\mathbb{C})^{\text{op}} \to \mathbb{D}$; we also write $R : \text{Fam}(\mathbb{C})^{\text{op}} \to \mathbb{C}^{\text{op}}$ for the essentially unique finite-product-preserving functor satisfying $RZ \cong 1$. A functor $G : \mathbb{C}^{\text{op}} \to \mathbb{D}$ is itself finite-product-preserving if and only if the map $GR \to \text{Ran}_Z G$ induced by the isomorphism $GRZ \cong G$ is invertible.

We shall now describe the reflection. Given an arbitrary functor $F : \mathbb{C}^{\text{op}} \to \mathsf{Set}$ we form $\text{Ran}_Z F$, and then the left Kan extension $\text{Lan}_R \text{Ran}_Z F : \mathbb{C}^{\text{op}} \to \mathsf{Set}$ of $\text{Ran}_Z F$. Now $\text{Ran}_Z F$ preserves finite products, hence so too does $\text{Lan}_R \text{Ran}_Z F$, by a theorem of Borceux and Day [7].

Proposition 6.1. *The reflection into* $\text{FP}(\mathbb{C}^{\text{op}}, \mathsf{Set})$ *of* $F : \mathbb{C}^{\text{op}} \to \mathsf{Set}$ *is given by the finite-product-preserving functor* $\text{Lan}_R \text{Ran}_Z F$.

Proof. For any finite-product-preserving functor $G : \mathbb{C}^{op} \to \mathsf{Set}$, we have natural bijections

$$\begin{aligned} \mathrm{FP}(\mathbb{C}^{op}, \mathsf{Set})(\mathrm{Lan}_R\, \mathrm{Ran}_Z\, F, G) &\cong [\mathbb{C}^{op}, \mathsf{Set}](\mathrm{Lan}_R\, \mathrm{Ran}_Z\, F, G) \\ &\cong [\mathrm{Fam}(\mathbb{C})^{op}, \mathsf{Set}](\mathrm{Ran}_Z\, F, GR) \\ &\cong [\mathrm{Fam}(\mathbb{C})^{op}, \mathsf{Set}](\mathrm{Ran}_Z\, F, \mathrm{Ran}_Z\, G) \\ &\cong (\mathbb{C}^{op}, \mathsf{Set})(F, G), \end{aligned}$$

the last because Ran_Z is fully faithful since Z is so. □

Since $\mathrm{FP}(\mathbb{C}^{op}, \mathsf{Set})$ is reflective in the complete and cocomplete category $[\mathbb{C}^{op}, \mathsf{Set}]$, it is itself complete and cocomplete, but in fact all we really need is the existence of finite coproducts in $\mathrm{FP}(\mathbb{C}^{op}, \mathsf{Set})$, and these we shall compute explicitly in Proposition 6.2 below.

The restricted Yoneda embedding $Y : \mathbb{C} \to \mathrm{FP}(\mathbb{C}^{op}, \mathsf{Set})$ preserves finite coproducts, for given a finite-product-preserving functor $F : \mathbb{C}^{op} \to \mathsf{Set}$ and a finite coproduct $\Sigma_i C_i$ in $\mathbb{C}$, we have natural bijections

$$\begin{aligned} \mathrm{FP}(\mathbb{C}^{op}, \mathsf{Set})(Y(\Sigma_i C_i), F) &\cong F(\Sigma_i C_i) \\ &\cong \Pi_i F C_i \\ &\cong \Pi_i \mathrm{FP}(\mathbb{C}^{op}, \mathsf{Set})(Y C_i, F), \end{aligned}$$

by the Yoneda lemma and the fact that F preserves the finite product $\Sigma_i C_i$ in $\mathbb{C}^{op}$.

A flat functor preserves any finite limits which exist, thus $\mathrm{Flat}(\mathbb{C}^{op}, \mathsf{Set})$ is contained in $\mathrm{FP}(\mathbb{C}^{op}, \mathsf{Set})$. We shall show that $\mathrm{Flat}(\mathbb{C}^{op}, \mathsf{Set})$ is closed in $\mathrm{FP}(\mathbb{C}^{op}, \mathsf{Set})$ under finite coproducts; but to do so, we shall need an explicit description of finite coproducts in $\mathrm{FP}(\mathbb{C}^{op}, \mathsf{Set})$. We could derive such a description from Proposition 6.1, but we find it more convenient to proceed as follows. Given F and G in $\mathrm{FP}(\mathbb{C}^{op}, \mathsf{Set})$, we form their coproduct $F + G$ in $(\mathbb{C}^{op}, \mathsf{Set})$, and then the category of elements of this $F + G$, and the functor $p_{F+G} : \mathrm{el}(F + G)^{op} \to \mathbb{C}$; and we observe that $\mathrm{el}(F + G)^{op}$ is the coproduct of categories $\mathrm{el}(F)^{op} + \mathrm{el}(G)^{op}$, and that p_{F+G} is the functor out of this coproduct induced by p_F and p_G. We now form the free completion under finite coproducts of $\mathrm{el}(F + G)^{op}$: this can be constructed as the category $\mathrm{Fam}(\mathrm{el}(F + G)^{op})$ of finite families in $\mathrm{el}(F + G)^{op}$. An object of $\mathrm{Fam}(\mathrm{el}(F + G)^{op})$ is a finite family $(M_i)_{i \in I}$ of objects of $\mathrm{el}(F + G)^{op}$, while an arrow in $\mathrm{Fam}(\mathrm{el}(F + G)^{op})$ from $(M_i)_{i \in I}$ to $(N_j)_{j \in J}$ consists of a function $\alpha : I \to J$ and an arrow $f_i : M_i \to N_{\alpha i}$ in $\mathrm{el}(F + G)^{op}$ for each $i \in I$; see [13] for details. The functor $p_{F+G} : \mathrm{el}(F + G)^{op} \to \mathbb{C}$ induces a finite-coproduct-preserving functor $p : \mathrm{Fam}(\mathrm{el}(F + G)^{op}) \to \mathbb{C}$, and we may now form the colimit $\mathrm{colim}(Yp)$ in $[\mathbb{C}^{op}, \mathsf{Set}]$.

Proposition 6.2. *The functor* $\mathrm{colim}(Yp)$ *preserves finite products, and is the coproduct in* $\mathrm{FP}(\mathbb{C}^{op}, \mathsf{Set})$ *of* F *and* G.

Proof. Since sifted colimits commute in Set with finite products [2,33], the category $\mathrm{FP}(\mathbb{C}^{\mathrm{op}}, \mathsf{Set})$ is closed in $[\mathbb{C}^{\mathrm{op}}, \mathsf{Set}]$ under sifted colimits; thus in particular sifted colimits of representables lie in $\mathrm{FP}(\mathbb{C}^{\mathrm{op}}, \mathsf{Set})$. But every category with finite coproducts is sifted [2, Example 1.3(3)], whence it follows that $\mathrm{colim}(Yp)$ does indeed lie in $\mathrm{FP}(\mathbb{C}^{\mathrm{op}}, \mathsf{Set})$. □

Now let $H : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$ preserve finite products. To give arrows $F \to H$ and $G \to H$ is equally to give cocones with vertex H over Yp_F and Yp_G; since $\mathrm{el}(F + G)^{\mathrm{op}} = \mathrm{el}(F)^{\mathrm{op}} + \mathrm{el}(G)^{\mathrm{op}}$, to give such cocones is to give a cocone over Yp_{F+G} with vertex H. But now since H preserves finite products, this is the same thing as to give a cocone over Yp with vertex H; that is, an arrow from $\mathrm{colim}(Yp)$ to H.

We now use this construction of finite coproducts in $\mathrm{FP}(\mathbb{C}^{\mathrm{op}}, \mathsf{Set})$ to construct finite coproducts in $\mathrm{Flat}(\mathbb{C}^{\mathrm{op}}, \mathsf{Set})$.

Proposition 6.3. $\mathrm{Flat}(\mathbb{C}^{\mathrm{op}}, \mathsf{Set})$ *is closed in* $\mathrm{FP}(\mathbb{C}^{\mathrm{op}}, \mathsf{Set})$ *under finite coproducts*; *thus* $\mathrm{Flat}(\mathbb{C}^{\mathrm{op}}, \mathsf{Set})$ *has finite coproducts and the inclusions* $\mathrm{Flat}(\mathbb{C}^{\mathrm{op}}, \mathsf{Set}) \to \mathrm{FP}(\mathbb{C}^{\mathrm{op}}, \mathsf{Set})$ *and* $y : \mathbb{C} \to \mathrm{Flat}(\mathbb{C}^{\mathrm{op}}, \mathsf{Set})$ *both preserve finite coproducts.*

Proof. There is no problem with regard to the initial object, which is the representable functor $\mathbb{C}(-, 0)$; thus it will suffice to consider binary coproducts. We shall show that if F and G are in $\mathrm{FP}(\mathbb{C}^{\mathrm{op}}, \mathsf{Set})$, then $\mathrm{Fam}(\mathrm{el}(F+G)^{\mathrm{op}})$ is filtered if $\mathrm{el}(F)^{\mathrm{op}}$ and $\mathrm{el}(G)^{\mathrm{op}}$ are so. The result will then follow since $\mathrm{Flat}(\mathbb{C}^{\mathrm{op}}, \mathsf{Set})$ is closed in $[\mathbb{C}^{\mathrm{op}}, \mathsf{Set}]$ under filtered colimits, and so in particular any filtered colimit of representables is flat.

Since $\mathrm{Fam}(\mathrm{el}(F + G)^{\mathrm{op}})$ has finite coproducts, it certainly has a cocone over any finite discrete diagram; thus we need only prove that there is a cocone over any parallel pair of arrows

$$(M_i)_{i \in I} \underset{(\beta, g_i)}{\overset{(\alpha, f_i)}{\rightrightarrows}} (N_j)_{j \in J}$$

in $\mathrm{Fam}(\mathrm{el}(F + G)^{\mathrm{op}})$. Since I is finite and $(M_i)_{i\in I}$ is the coproduct of the M_i, it will suffice to consider the case where I is a singleton. Furthermore, it will clearly suffice to consider the case where every $j \in J$ is in the image of either α or β. If $\alpha i = \beta i = j$ this reduces to a parallel pair

$$M_i \underset{g_i}{\overset{f_i}{\rightrightarrows}} N_j$$

in $\mathrm{el}(F + G)^{\mathrm{op}}$, while if $\alpha i \neq \beta i$, this reduces to a diagram

$$\begin{array}{ccc} & \overset{f_i}{\nearrow} & N_{\alpha i} \\ M_i & & \\ & \underset{g_i}{\searrow} & N_{\beta i} \end{array}$$

in $\mathrm{el}(F+G)^{\mathrm{op}}$. In either case, since $\mathrm{el}(F+G)^{\mathrm{op}}=\mathrm{el}(F)^{\mathrm{op}}+\mathrm{el}(G)^{\mathrm{op}}$ and the diagram is connected, it must be contained either in $\mathrm{el}(F)^{\mathrm{op}}$ or $\mathrm{el}(G)^{\mathrm{op}}$; but $\mathrm{el}(F)^{\mathrm{op}}$ and $\mathrm{el}(G)^{\mathrm{op}}$ are filtered, and so in either case there is a cocone, hence a cocone in $\mathrm{el}(F+G)^{\mathrm{op}}$, hence a cocone in $\mathrm{Fam}(\mathrm{el}(F+G)^{\mathrm{op}})$.

Recall that an object F of $\mathrm{Flat}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$ is said to be finitely presentable if the hom-functor $\mathrm{Flat}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})(F,-):\mathrm{Flat}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})\to\mathsf{Set}$ preserves filtered colimits (of course, this definition makes sense for objects of any category with filtered colimits), and that if $\mathbb{C}$ is Cauchy complete, then the finitely presentable objects of $\mathrm{Flat}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$ are precisely the representables.

Proposition 6.4. *If $\mathbb{C}$ is Cauchy complete, then* $\mathrm{Flat}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$ *is extensive if and only if $\mathbb{C}$ is so.*

Proof. Suppose that $\mathrm{Flat}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$ is extensive; we shall show that its full subcategory of finitely presentable objects is extensive, hence that $\mathbb{C}$ is so. Since the finitely presentable objects are closed under finite colimits, we need only show that they are closed under summands; that is, if the sum in $\mathrm{Flat}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$ of F and G is finitely presentable, then so too is F.

For this first half of the proof only, we shall use $F+G$ to denote the coproduct of F and G in $\mathrm{Flat}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$. Suppose that $h_i:H_i\to\mathrm{colim}_i H_i$ is the colimit cone of a diagram $H:\mathscr{I}\to\mathrm{Flat}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$ with $\mathscr{I}$ filtered; we shall show that

$$\mathrm{Flat}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})(F,\mathrm{colim}_i H_i)\cong\mathrm{colim}_i\,\mathrm{Flat}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})(F,H_i).$$

Let $H':\mathscr{I}\to\mathrm{Flat}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$ be the evident functor taking an object i of $\mathscr{I}$ to H_i+G. Now $\mathrm{colim}_i\,(H_i+G)\cong\mathrm{colim}_i\,H_i+\mathrm{colim}_i\,G\cong\mathrm{colim}_i H_i+G$, since colimits commute with colimits, and the colimit of a filtered diagram constant at G is G itself; thus $h_i+1_G:H_i+G\to\mathrm{colim}_i H_i+G$ is the (filtered) colimit in $\mathrm{Flat}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$ of H'. Given a morphism $u:F\to\mathrm{colim}_i H_i$ we may form $u+1_G:F+G\to\mathrm{colim}_i H_i+G$; then since $F+G$ is finitely presentable, this factorizes as $u+1_G=(h_i+1_G)v_i$ for some i, and by extensivity of $\mathrm{Flat}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$ the morphism $v_i:F+G\to H_i+G$ is of the form u_i+1_G for a unique $u_i:F\to H_i$, and $u=h_iu_i$.

On the other hand, if $u_i:F\to H_i$ and $u_j:F\to H_j$ satisfy $h_iu_i=h_ju_j$, then $(h_i+1_G)(u_i+1_G)=(h_j+1_G)(u_j+1_G)$; but $F+G$ is finitely presentable, and so there exist a k and arrows $\kappa:i\to k$ and $\lambda:j\to k$ in $\mathscr{I}$ with $(H\kappa+1_G)(u_i+1_G)=(H\lambda+1_G)(u_j+1_G)$, whence, by extensivity of $\mathrm{Flat}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$, we have $H\kappa.u_i=H\lambda.u_j$. This proves that F is finitely presentable, and so that $\mathbb{C}$ is extensive.

Suppose conversely that $\mathbb{C}$ is extensive. As we mentioned in the Introduction, the category $\mathrm{FP}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$ is then extensive – in fact, it is easy to see, as observed by Lawvere [35], that $\mathrm{FP}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$ is the category of sheaves for a Grothendieck topology on $\mathbb{C}$ and so is a (Grothendieck) topos and in particular is extensive – and we know that $\mathrm{Flat}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$ is closed in $\mathrm{FP}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$ under finite coproducts; thus to prove that $\mathrm{Flat}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$ is extensive, we need only prove that it is closed under direct summands. By our construction of coproducts in $\mathrm{FP}(\mathbb{C}^{\mathrm{op}},\mathsf{Set})$, it will suffice to show that for finite-product-preserving functors F and G from $\mathbb{C}^{op}$ to Set, if $\mathrm{Fam}(\mathrm{el}(F+G)^{op})$ is filtered, then so too is $\mathrm{el}(F)^{op}$.

First, we observe that for any finite-product-preserving F the category $\mathrm{el}(F)^{op}$ has finite coproducts: given objects $(C, u \in FC)$ and $(D, v \in FD)$ of $\mathrm{el}(F)^{op}$, since F preserves the product $C+D$ in $\mathbb{C}^{op}$, we have an object $(C+D, (u,v) \in F(C+D))$ of $\mathrm{el}(F)^{op}$ which one easily verifies is the coproduct of (C,u) and (D,v) in $\mathrm{el}(F)^{op}$; the case of the initial object is similar. Thus any finite discrete diagram in $\mathrm{el}(F)^{op}$ has a cocone over it, and we need only check that any parallel pair

$$(C,u) \underset{g}{\overset{f}{\rightrightarrows}} (D,v)$$

has a cocone over it. Such a pair may equally be deemed to live in $\mathrm{el}(F+G)^{op}$; and so, viewing (C,u) and (D,v) as singleton families, in the category $\mathrm{Fam}(\mathrm{el}(F+G)^{op})$. Thus we can find a cocone

$$(D,v) \to (E_i, w_i)_{i \in I}$$

in $\mathrm{Fam}(\mathrm{el}(F+G)^{op})$; but clearly if we can find such a cocone, then we can find a cocone in which I is a singleton, and so a cocone in $\mathrm{el}(F+G)^{op}$. But now, since $\mathrm{el}(F+G)^{op} = \mathrm{el}(F)^{op} + \mathrm{el}(G)^{op}$, and $(D,v) \in \mathrm{el}(F)^{op}$, the cocone must already be in $\mathrm{el}(F)^{op}$. Thus $\mathrm{el}(F)^{op}$ is filtered, F is flat, and $\mathrm{Flat}(\mathbb{C}^{op}, \mathsf{Set})$ is extensive. □

Finally, since $\mathrm{Flat}(\mathbb{C}_{cc}^{op}, \mathsf{Set})$ is equivalent to $\mathrm{Flat}(\mathbb{C}^{op}, \mathsf{Set})$, we may immediately deduce:

Theorem 6.5. *If $\mathbb{C}$ is a small category with finite coproducts, then its filtered colimit completion* $\mathrm{Flat}(\mathbb{C}^{op}, \mathsf{Set})$ *has finite coproducts, the inclusion* $y : \mathbb{C} \to \mathrm{Flat}(\mathbb{C}^{op}, \mathsf{Set})$ *preserves finite coproducts, and the following conditions are equivalent*:
(i) $\mathbb{C}$ *is nearly extensive*;
(ii) $\mathbb{C}_{cc}$ *is extensive*;
(iii) $\mathrm{Flat}(\mathbb{C}^{op}, \mathsf{Set})$ *is extensive.*

Remark 6.6. In fact, if $\mathbb{C}$ is a Cauchy complete small category, then the finitely presentable objects of $\mathrm{Flat}(\mathbb{C}^{op}, \mathsf{Set})$ are precisely the representable functors. Furthermore, the finitely presentable objects are closed under any finite colimits that exist, and so in particular under any finite coproducts that exist. It follows that $\mathbb{C}$ has finite coproducts if $\mathrm{Flat}(\mathbb{C}^{op}, \mathsf{Set})$ does so; from which we deduce that for an arbitrary Cauchy complete small category, not assumed to have finite coproducts, $\mathrm{Flat}(\mathbb{C}^{op}, \mathsf{Set})$ is extensive if and only if $\mathbb{C}$ is so.

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