Discrete Mathematics 312 (2012) 1584–1590

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

# The fractional metric dimension of graphs

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ARTICLE INFO

*Article history:*
Available online 13 July 2011

*Keywords:*
Metric dimension
Resolving set
Basis
Resolving function
Fractional metric dimension

ABSTRACT

A vertex $x$ in a connected graph $G$ is said to resolve a pair $\{u, v\}$ of vertices of $G$ if the distance from $u$ to $x$ is not equal to the distance from $v$ to $x$. A set $S$ of vertices of $G$ is a resolving set for $G$ if every pair of vertices is resolved by some vertex of $S$. The smallest cardinality of a resolving set for $G$, denoted by $dim(G)$, is called the metric dimension of $G$. For the pair $\{u, v\}$ of vertices of $G$ the collection of all vertices which resolve the pair $\{u, v\}$ is denoted by $R\{u, v\}$ and is called the resolving neighbourhood of the pair $\{u, v\}$. A real valued function $g : V(G) \to [0, 1]$ is a resolving function of $G$ if $g(R\{u, v\}) \geq 1$ for any two distinct vertices $u, v \in V(G)$. The fractional metric dimension of $G$ is defined as $dim_f(G) = \min\{|g| : g \text{ is a minimal resolving function of } G\}$, where $|g| = \sum_{v\in V} g(v)$. In this paper we study this parameter.



## 1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of $G$ are denoted by $n$ and $m$ respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [4].

The distance $d(u, v)$ between two vertices $u$ and $v$ in $G$ is the length of a shortest $u - v$ path in $G$. By an ordered set of vertices we mean a set $W = \{w_1, w_2, \dots, w_k\}$ on which the ordering $(w_1, w_2, \dots, w_k)$ has been imposed. For an ordered subset $W = \{w_1, w_2, \dots, w_k\}$ of $V(G)$, we refer to the $k$-vector (ordered $k$-tuple) $r(v|W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$ as the (metric) representation of $v$ with respect to $W$. The set $W$ is called a resolving set for $G$ if $r(u|W) = r(v|W)$ implies that $u = v$ for all $u, v \in V(G)$. Hence, if $W$ is a resolving set of cardinality $k$ for a graph $G$ of order $n$, then the set $\{r(v|W) : v \in V(G)\}$ consists of $n$ distinct $k$-vectors. A resolving set of minimum cardinality for a graph $G$ is called a basis for $G$ and the metric dimension of $G$ is defined to be the cardinality of a basis of $G$ and is denoted by $dim(G)$. A resolving set $W$ of $G$ is a minimal resolving set if no proper subset of $W$ is a resolving set. The upper dimension of $G$ is defined to be the maximum cardinality of a minimal resolving set of $G$ and is denoted by $dim^+(G)$.

A vertex $x \in V(G)$ is said to resolve a pair of vertices $\{u, v\}$ in $G$ if $d(u, x) \neq d(v, x)$. Let $V_p$ denote the collection of all $\binom{n}{2}$ pairs of vertices of $G$. For $u, v \in V(G)$, we define the resolving neighbourhood of the pair $\{u, v\}$ as $R\{u, v\} = \{x \in V(G) : d(u, x) \neq d(v, x)\}$. Also, for each vertex $x \in V(G)$, we define the resolvent neighbourhood of $x$ as $R\{x\} = \{\{u, v\} \in V_p : d(u, x) \neq d(v, x)\}$. The resolving graph [8,9] $R(G)$ of a connected graph $G = (V, E)$ is a bipartite graph with bipartition $(V, V_p)$ where a vertex $x \in V$ is joined to a pair $\{u, v\} \in V_p$ if and only if $x$ resolves $\{u, v\}$ in $G$. Then the minimum cardinality of a subset $S$ of $V$ such that $N(S) = V_p$ in $R(G)$ is the metric dimension of $G$.

Currie and Oellermann [8] have introduced the concept of metric independence number $mi(G)$ of a graph $G$ which is the dual concept of metric dimension $dim(G)$ of $G$. A collection of pairs of vertices of $G$, no two of which are resolved by the same

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doi:10.1016/j.disc.2011.05.039

vertex, is called an independent resolvent set of $G$. The metric independence number $mi(G)$ of $G$ is the maximum cardinality of an independent resolvent set of $G$.

The minimum metric dimension problem is to find a basis of $G$. Garey and Johnson [10] noted that the minimum metric dimension problem is NP-complete for general graphs by a reduction from three-dimensional matching. An explicit reduction from $3 - SAT$ was given by Khuller et al. [12]. Recently, Manuel et al. [13] proved that the minimum metric dimension problem is NP-complete for bipartite graphs by a reduction from $3 - SAT$.

The idea of resolving sets has appeared in the literature previously. In [17] and later in [18], Slater introduced the concept of a resolving set for a connected graph $G$ under the term locating set. He referred to a minimum resolving set as a reference set for $G$. He called the cardinality of a minimum resolving set (reference set) the location number of $G$. Independently, Harary and Melter [11], discovered these concepts as well but used the term metric dimension.

Applications of resolving sets arise in various areas including coin weighing problem [16], drug discovery [3], robot navigation [12], network discovery and verification [1], connected joins in graphs [15] and strategies for the mastermind game [6]. For a survey of results in metric dimension, we refer to Chartrand and Zhang [5].

Chartrand et al. [3], and independently Currie and Oellermann [8] formulated the problem of finding the metric dimension of a graph as an integer programming problem. Further Currie and Oellermann [8] defined fractional metric dimension as the optimal solution of the linear relaxation of the integer programming problem. Fehr et al. [9] proposed the following equivalent formulation of fractional metric dimension.

Suppose $V = \{v_1, v_2, \ldots, v_n\}$ and $V_p = \{s_1, s_2, \ldots, s_{\binom{n}{2}}\}$. Let $A = (a_{ij})$ be the $\binom{n}{2} \times n$ matrix with $a_{ij} = 1$ if $s_i v_j \in E(R(G))$ and 0 otherwise, where $1 \le i \le \binom{n}{2}$ and $1 \le j \le n$. The integer programming formulation of the metric dimension is given by

$$\text{Minimize } f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$$
$$\text{Subject to } A\overline{x} \ge \overline{1}$$

where $\overline{x} = (x_1, x_2, \ldots, x_n)^T$, $x_i \in \{0, 1\}$ and $\overline{1}$ is the $\binom{n}{2} \times 1$ column vector all of whose entries are 1.

The optimal solution of the linear programming relaxation of the above I.P.P, where we replace $x_i \in \{0, 1\}$ by $0 \le x_i \le 1$, gives the fractional metric dimension of $G$, which we denote by $dim_f(G)$. The optimal solution of the dual of this L.P.P is the fractional independence number of $G$, which we denote by $mi_f(G)$. Hence, it follows from the duality and weak duality theorem in linear programming that $mi(G) \le mi_f(G) = dim_f(G) \le dim(G)$.

For a detailed study of fractional graph theory and fractionalization of various graph parameters, we refer to Scheinerman and Ullman [14]. In this paper, we present several fundamental results on fractional metric dimension.

## 2. Graphs with $dim_f(G) = \frac{|V(G)|}{2}$

We start with the formulation of the fractional metric dimension, the upper fractional metric dimension, the lower fractional metric independence number and the fractional metric independence number in terms of minimal resolving functions and maximal metric independence functions.

**Definition 2.1.** Let $G = (V, E)$ be a connected graph of order $n$. A function $f : V \to [0, 1]$ is called a resolving function (RF) of $G$ if $f(R\{u, v\}) \ge 1$ for any two distinct vertices $u, v \in V$, where $f(R\{u, v\}) = \sum_{x \in R\{u,v\}} f(x)$.

A resolving function $g$ of a graph $G$ is minimal (MRF) if any function $f : V \to [0, 1]$ such that $f \le g$ and $f(v) \ne g(v)$ for at least one $v \in V$ is not a resolving function of $G$.

The fractional metric dimension $dim_f(G)$ and the upper fractional metric dimension $dim_f^+(G)$ are given by

$$dim_f(G) = \min\{|g| : g \text{ is a minimal resolving function of } G\} \quad \text{and}$$
$$dim_f^+(G) = \max\{|g| : g \text{ is a minimal resolving function of } G\}.$$

**Definition 2.2.** Let $G = (V, E)$ be a connected graph of order $n$. A function $f : V_p \to [0, 1]$ is called a metric independence function (MIF) of $G$ if $f(R\{v\}) \le 1$ for all $v \in V$.

A metric independence function $g$ of a graph $G$ is maximal (MMIF) if any function $f : V_p \to [0, 1]$ such that $f \ge g$ and $f(s) \ne g(s)$ for at least one $s \in V_p$ is not a metric independence function of $G$.

The lower fractional metric independence number $mi_f^-(G)$ and the fractional metric independence number, $mi_f(G)$ are given by

$$mi_f^-(G) = \min\{|g| : g \text{ is a maximal metric independence function of } G\} \quad \text{and}$$
$$mi_f(G) = \max\{|g| : g \text{ is a maximal metric independence function of } G\}.$$

**Observation 2.3.** *If there exists a minimal resolving function $g$ and a maximal metric independence function $h$ of a graph $G$ with $|g| = |h|$, then $dim_f(G) = |g| = |h| = mi_f(G)$.*

This observation is very useful in computing the fractional metric dimension $dim_f(G)$ of any connected graph $G$.

**Observation 2.4.** *Since the characteristic function of a minimal resolving set is an MRF of G, it follows that* $1 \le dim_f(G) \le dim(G) \le dim^+(G) \le dim_f^+(G) \le n-1$.

**Observation 2.5.** *Since* $dim(G) = 1$ *if and only if* $G = P_n$ *where* $P_n$ *is the path on n vertices* [17,12], *it follows that* $dim_f(P_n) = 1$ *and* $dim(G) \ge 2$ *for all* $G \ne P_n$.

**Theorem 2.6.** *Let G be a connected graph of order n. Then* $dim_f(G) \le \frac{n}{2}$. *Further* $dim_f(G) = \frac{n}{2}$ *if and only if there exists a bijection* $\alpha : V(G) \to V(G)$ *such that* $\alpha(v) \ne v$ *and* $|R\{v, \alpha(v)\}| = 2$ *for all* $v \in V(G)$.

**Proof.** Let $k = \min\{|R\{u, v\}| : u, v \in V(G), u \ne v\}$. Then the constant function $g$ defined on $V(G)$ by $g(v) = \frac{1}{k}$ for all $v \in V$ is a resolving function, so that $dim_f(G) \le \frac{n}{k}$. Since $k \ge 2$, we get $dim_f(G) \le \frac{n}{2}$.

Now, suppose there exists a bijection $\alpha : V(G) \to V(G)$ such that $\alpha(v) \ne v$ and $|R\{v, \alpha(v)\}| = 2$ for all $v \in V(G)$. Let $h$ be any resolving function of $G$. Then $h(R\{u, v\}) \ge 1$ for all $u, v \in V(G)$. In particular, we have $h(R\{u, \alpha(u)\}) \ge 1$ for all $u \in V(G)$. Adding these $n$ inequalities we get $2|h| \ge n$ and so $dim_f(G) \ge \frac{n}{2}$. Hence $dim_f(G) = \frac{n}{2}$.

Conversely, suppose that $dim_f(G) = \frac{n}{2}$. If there exists a vertex $v \in V$ such that $|R\{v, w\}| \ge 3$ for all $w \in V - \{v\}$, then the function $f : V \to [0, 1]$ defined by $f(v) = 0$ and $f(w) = \frac{1}{2}$ for all $w \in V - \{v\}$ is a resolving function of $G$ with $|f| = \frac{n-1}{2}$, which is a contradiction. Hence for each $v \in V(G)$, there exists a vertex $\alpha(v) \ne v$ such that $|R\{v, \alpha(v)\}| = 2$. We now claim that $\alpha : V \to V$ is a bijection. If there exists a vertex $v_i \in V$ such that $|\alpha^{-1}(v_i)| \ge 2$, let $\alpha^{-1}(v_i) = \{v_{i_1}, v_{i_2}, \dots, v_{i_r}\}$, $r \ge 2$. Then $f : V(G) \to [0, 1]$ defined by

$$f(u) = \begin{cases} \dfrac{1}{4} & \text{if } u = v_{i_j},\ j = 1, 2, \dots, r \\ \dfrac{3}{4} & \text{if } u = v_i \\ \dfrac{1}{2} & \text{otherwise} \end{cases}$$

is a resolving function of $G$ and $|f| = \frac{n}{2} + \frac{1}{4} - \frac{r}{4} < \frac{n}{2}$, which is a contradiction. Thus $\alpha: V \to V$ is the required bijection. □

**Corollary 2.7.** $dim_f(G) = \frac{|V(G)|}{2}$ *for each of the following graphs*

(i) $G = K_n$, $n \ge 2$.
(ii) $G = K_n - e$, $n \ge 4$.
(iii) $G = K_{2t} - M$, $t \ge 2$ *and M is a perfect matching in* $K_{2t}$.
(iv) *G is the complete k-partite graph* $K_{n_1,n_2,\dots,n_k}$, *where* $k \ge 2$ *and* $n_i \ge 2$.

**Proof.** We prove the result by exhibiting a bijection $\alpha : V(G) \to V(G)$ satisfying the conditions stated in Theorem 2.6.

If $G = K_n$ and $V(G) = \{v_1, v_2, \dots, v_n\}$, then $\alpha : V(G) \to V(G)$ defined by $\alpha(v_i) = v_{i+1}$, where $v_{n+1} = v_1$ is the required bijection.

If $G = K_n - e$ and $e = v_1v_2$, then $\alpha : V(G) \to V(G)$ defined by

$$\alpha(v_i) = \begin{cases} v_2 & \text{if } i = 1 \\ v_1 & \text{if } i = 2 \\ v_{i+1} & \text{if } 3 \le i \le n-1 \\ v_3 & \text{if } i = n \end{cases}$$

is the required bijection.

If $G = K_{2t} - M$ with $V(G) = \{v_1, v_2, \dots, v_{2t}\}$ and $M = \{v_1v_2, v_3v_4, v_5v_6, \dots, v_{2t-1}v_{2t}\}$, then $\alpha : V(G) \to V(G)$ defined by

$$\alpha(v_i) = \begin{cases} v_{i+1} & \text{if } i \equiv 1 (\text{mod } 2) \\ v_{i-1} & \text{if } i \equiv 0 (\text{mod } 2) \end{cases}$$

is the required bijection.

If $G = K_{n_1,n_2,\dots,n_k}$ with $V(G) = \bigcup_{i=1}^{k} V_i$ where $V_i = \{v_1^i, v_2^i, \dots, v_{n_i}^i\}$, then $\alpha : V(G) \to V(G)$ defined by $\alpha(v_j^i) = v_{j+1}^i$, $j = 1, 2, \dots, n_i$ and $i = 1, 2, \dots, k$, $v_{n_i+1}^i = v_1^i$ is the required bijection. □

**Corollary 2.8.** *Let* $\mathcal{G}$ *denote the collection of all connected graphs G with* $dim_f(G) = \frac{|V(G)|}{2}$. *If* $G_1, G_2 \in \mathcal{G}$, *then* $G_1 + G_2 \in \mathcal{G}$, *where* $G_1 + G_2$ *is the graph obtained from* $G_1$ *and* $G_2$ *by joining every vertex of* $G_1$ *with every vertex of* $G_2$.

**Proof.** If $\alpha_1 : V(G_1) \to V(G_1)$ and $\alpha_2 : V(G_2) \to V(G_2)$ are bijections satisfying the conditions stated in Theorem 2.6, then $\beta : V(G_1 + G_2) \to V(G_1 + G_2)$ defined by

$$\beta(v) = \begin{cases} \alpha_1(v) & \text{if } v \in V(G_1) \\ \alpha_2(v) & \text{if } v \in V(G_2) \end{cases}$$

is the required bijection. □

**Corollary 2.9.** *If $dim_f(G) = \frac{n}{2}$, then $dim_f(G + \overline{K_k}) = \frac{n+k}{2}$.*

**Theorem 2.10.** *Any connected graph $H$ can be embedded as an induced subgraph of a connected graph $G$ with $dim_f(G) = \frac{|V(G)|}{2}$.*

**Proof.** Let $V(H) = \{v_1, v_2, \ldots, v_n\}$. Let $H_1, H_2, \ldots, H_n$ be $n$ disjoint graphs with $dim_f(H_i) = \frac{|V(H_i)|}{2}$. Consider the composition graph $G = H[H_1, H_2, \ldots, H_n]$, which is formed from $H$ by replacing each vertex $v_i$ of $H$ by $H_i$ and joining each vertex of $H_i$ to each vertex of $H_j$ whenever $v_i$ and $v_j$ are adjacent in $H$. By Theorem 2.6, there exists a bijection $\alpha_i : V(H_i) \to V(H_i)$ such that $\alpha_i(v) \neq v$ and $|R\{v, \alpha_i(v)\}| = 2$ for all $v \in V(H_i)$. Now the function $\beta : V(G) \to V(G)$ defined by $\beta(v) = \alpha_i(v)$ if $v \in V(H_i)$ is a bijection with $\beta(v) \neq v$ and $|R\{v, \beta(v)\}| = 2$ for all $v \in V(G)$. Hence $dim_f(G) = \frac{|V(G)|}{2}$ and $H$ is an induced subgraph of $G$. □

## 3. Fractional metric dimension of some standard graphs

In this section we determine the fractional metric dimension of several families of graphs.

**Theorem 3.1.** *For the Petersen graph $P$, we have $dim_f(P) = \frac{5}{3}$.*

**Proof.** The vertex set of $P$ is the set of all 2-element subsets of $\{1, 2, \ldots, 5\}$ and two vertices $x, y$ are adjacent if $x \cap y = \emptyset$. We know that $P$ is a 3-regular graph with $\mathrm{diam}(P) = 2$. Let $u, v \in V(P)$. If $d(u, v) = 1$, then $N(u) \cap N(v) = \emptyset$, $R\{u, v\} = N(u) \cup N(v)$ and hence $|R\{u, v\}| = 6$. Also, if $d(u, v) = 2$ then $|N(u) \cap N(v)| = 1$, $R\{u, v\} = (N[u] \cup N[v]) - (N[u] \cap N[v])$ and hence $|R\{u, v\}| = 6$. Thus, $|R\{u, v\}| = 6$ for all $u, v \in V(P)$. Hence the function $g : V \to [0, 1]$ defined by $g(v) = \frac{1}{6}$ for all $v \in V$, is a minimal resolving function with $|g| = \frac{10}{6} = \frac{5}{3}$.

Now let $v \in V(P)$. Since $\mathrm{diam}(P) = 2$, it follows that $R\{v\} = \{\{x, y\} : x = v\} \cup \{\{x, y\} : x \in N(v) \text{ and } y \in V - N[v]\}$. Hence $|R\{v\}| = 27$. Now the function $h : V_p \to [0, 1]$ defined by $h(\{x, y\}) = \frac{1}{27}$ for all $\{x, y\} \in V_p$ is a maximal metric independence function with $|h| = \frac{\binom{10}{2}}{27} = \frac{5}{3}$. Hence it follows from Observation 2.3 that $dim_f(P) = \frac{5}{3}$. □

**Theorem 3.2.** *For the cycle $C_n$, we have*

$$dim_f(C_n) = \begin{cases} \dfrac{n}{n-1} & \text{if } n \text{ is odd} \\ \dfrac{n}{n-2} & \text{if } n \text{ is even.} \end{cases}$$

**Proof.** Let $C_n = (u_1 u_2 u_3 \ldots u_n u_1)$.

*Case* 1. $n$ is odd.

For any two distinct vertices $\{u_i, u_j\}$, we have $R\{u_i, u_j\} = V - \{u_k\}$, where $u_k$ is the middle vertex of the $u_i - u_j$ section of $C_n$ having even length. Hence $|R\{u_i, u_j\}| = n - 1$. Now the function $g : V \to [0, 1]$ defined by $g(v) = \frac{1}{n-1}$ for all $v \in V$ is a minimal resolving function with $|g| = \frac{n}{n-1}$.

Also for any vertex $u_i$, there exist exactly $\frac{n-1}{2}$ pairs of vertices which are not resolved by $u_i$ and hence $|R\{u_i\}| = \binom{n}{2} - \frac{n-1}{2} = \frac{(n-1)^2}{2}$. Hence the function $h : V_p \to [0, 1]$ defined by $h(s) = \frac{2}{(n-1)^2}$ for all $s \in V_p$ is a maximal metric independence function with $|h| = \binom{n}{2} \frac{2}{(n-1)^2} = \frac{n}{n-1}$. Hence by Observation 2.3, $dim_f(C_n) = \frac{n}{n-1}$.

*Case* 2. $n$ is even.

For any two distinct vertices $u_i$ and $u_j$ the length of both the $u_i - u_j$ sections of $C_n$ have the same parity and hence it follows that

$$|R\{u_i, u_j\}| = \begin{cases} n & \text{if } d(u_i, u_j) \text{ is odd} \\ n-2 & \text{if } d(u_i, u_j) \text{ is even.} \end{cases}$$

Hence the function $g : V \to [0, 1]$ defined by $g(v) = \frac{1}{n-2}$ for all $v \in V$ is a minimal resolving function with $|g| = \frac{n}{n-2}$.

Now let $S = \{\{u_r, u_k\} : d(u_r, u_k) \text{ is even}\}$. We claim that any vertex $u_i$ resolves $\left(\frac{n}{2} - 1\right)^2$ elements of $S$. Let $S_j$ denote the set of elements in $S$ which contain $u_j$ and which are resolved by $u_i$. Then

$$|S_j| = \begin{cases} \dfrac{n}{2} - 1 & \text{if } j = i \text{ or } j = \dfrac{n}{2} + i (\mathrm{mod}\ n) \\ \dfrac{n}{2} - 2 & \text{otherwise.} \end{cases}$$

Hence the number of elements in $S$ which are resolved by $u_i$ is $\frac{1}{2} \sum_{l=1}^{n} |S_l| = \frac{1}{2}\left(2\left(\frac{n}{2} - 1\right) + (n-2)\left(\frac{n}{2} - 2\right)\right) = \left(\frac{n}{2} - 1\right)^2$.
Hence the function $h : V_p \to [0, 1]$ defined by

$$h(x) = \begin{cases} \dfrac{1}{\left(\frac{n}{2}-1\right)^2} & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

is a maximal metric independence function with $|h| = \frac{\frac{n}{2}\left(\frac{n}{2}-1\right)}{\left(\frac{n}{2}-1\right)^2} = \frac{n}{n-2}$. Hence by Observation 2.3, $dim_f(C_n) = \frac{n}{n-2}$. □

While trying to establish the result $mi(Q_n) = 2$, Fehr et al. [9] have proved that $dim_f(Q_n) \leq 2$. In the following theorem we prove that $dim_f(Q_n) = 2$.

**Theorem 3.3.** *For the hypercube $G = Q_n$, $n \geq 2$, we have $dim_f(Q_n) = 2$.*

**Proof.** Consider the 4-cycle $(u_1, u_2, u_3, u_4, u_1)$ in $G$ where $u_1 = (0, 0, \ldots, 0)$, $u_2 = (1, 0, 0, \ldots, 0)$, $u_3 = (1, 1, 0, 0, \ldots, 0)$ and $u_4 = (0, 1, 0, 0, \ldots, 0)$. We have $R\{u_1, u_3\} = \{(0, 0, x_3 \ldots, x_n) : x_i \in \{0, 1\}\} \cup \{(1, 1, x_3 \ldots, x_n) : x_i \in \{0, 1\}\}$ and $R\{u_2, u_4\} = \{(1, 0, x_3, \ldots, x_n) : x_i \in \{0, 1\}\} \cup \{(0, 1, x_3, \ldots, x_n) : x_i \in \{0, 1\}\}$. Thus $R\{u_1, u_3\} \cup R\{u_2, u_4\} = V(G)$ and $R\{u_1, u_3\} \cap R\{u_2, u_4\} = \emptyset$. Now, let $h$ be any minimal resolving function of $G$. Then $h(R\{u_1, u_3\}) \geq 1$ and $h(R\{u_2, u_4\}) \geq 1$. Adding these two inequalities we get $|h| \geq 2$ and hence $dim_f(G) \geq 2$. Therefore $dim_f(G) = 2$. □

**Theorem 3.4.** *Let $G = (V, E)$ be a connected graph of order $n$. Suppose there exists a subset $S$ of $V$ satisfying the following conditions.*

(i) *There exists a bijection $\alpha : V - S \to V - S$ such that $\alpha(v) \neq v$ and $|R\{v, \alpha(v)\}| = 2$ for all $v \in V - S$.*
(ii) *For every $\{x, y\} \in V_p$, there exists $v \in V - S$ such that $R\{v, \alpha(v)\} \subseteq R\{x, y\}$.*

*Then $dim_f(G) = \frac{n-|S|}{2}$.*

**Proof.** The function $g : V \to [0, 1]$ defined by

$$g(v) = \begin{cases} 0 & \text{if } v \in S \\ \dfrac{1}{2} & \text{if } v \in V - S \end{cases}$$

is a minimal resolving function of $G$ with $|g| = \frac{n-|S|}{2}$ and hence $dim_f(G) \leq \frac{n-|S|}{2}$. Now, let $h$ be any *MRF* of $G$. Then $h(R\{v, \alpha(v)\}) \geq 1$ for all $v \in V - S$. Hence $\sum_{v \in V-S} h(R\{v, \alpha(v)\}) \geq |V - S|$ and so $dim_f(G) \geq |h| \geq \sum_{v \in V-S} h(v) \geq \frac{n-|S|}{2}$. Thus $dim_f(G) = \frac{n-|S|}{2}$. □

**Remark 3.5.** For the star $G = K_{1,n}$, $n \geq 2$, we take $S = \{v\}$ where $v$ is the centre of $K_{1,n}$, and applying Theorem 3.4, we get $dim_f(G) = \frac{n}{2}$. Similarly, for the bistar $G = B(r, s)$, we take $S = \{u, v\}$ where $u$ and $v$ are the non-pendant vertices of $G$ and Theorem 3.4 gives $dim_f(G) = \frac{r+s}{2}$.

**Theorem 3.6.** *For the wheel $W_n$, $n \geq 5$, we have*

$$dim_f(W_n) = \begin{cases} 2 & \text{if } n = 5 \\ \dfrac{3}{2} & \text{if } n = 6 \\ \dfrac{n-1}{4} & \text{if } n \geq 7. \end{cases}$$

**Proof.** Let $V(W_n) = \{u_0, u_1, \ldots, u_{n-2}, u\}$ where $u$ is the centre of the wheel and $C_{n-1} = (u_0 u_1 \ldots u_{n-2} u_0)$, $n \geq 4$, is the rim.

*Case* 1. $n = 5$.

In this case $R\{u_0, u_2\} = \{u_0, u_2\}$ and $R\{u_1, u_3\} = \{u_1, u_3\}$. Now let $S = \{u\}$. Then the function $\alpha : V - S \to V - S$ defined by $\alpha(u_0) = u_2, \alpha(u_1) = u_3, \alpha(u_2) = u_0$ and $\alpha(u_3) = u_1$ is a bijection which satisfies the conditions of Theorem 3.4 and hence $dim_f(W_5) = \frac{|V(W_5)|-|S|}{2} = 2$.

*Case* 2. $n = 6$.

In this case $|R\{x, y\}| = 4$ for all $\{x, y\} \in V_p$ and hence the constant function $g : V \to [0, 1]$ defined by $g(v) = \frac{1}{4}$ is a minimal resolving function with $|g| = \frac{3}{2}$. Now, $R\{u_0\} = \{\{u_0, u_1\}, \{u_0, u_2\}, \{u_0, u_3\}, \{u_0, u_4\}, \{u_0, u\}, \{u_1, u_2\}, \{u_1, u_3\}, \{u_2, u_4\}, \{u_2, u\}, \{u_3, u_4\}, \{u_3, u\}\}$ and $R\{u\} = \{\{u_0, u\}, \{u_1, u\}, \{u_2, u\}, \{u_3, u\}, \{u_4, u\}\}$. Clearly, $R\{u\} \cap R\{u_0\} = \{\{u_0, u\}, \{u_2, u\}, \{u_3, u\}\}$. By symmetry we get $|R\{u_i\}| = 11$ and $|R\{u\} \cap R\{u_i\}| = 3$ for $i = 0, 1, \ldots, 4$. Therefore, the function $h : V_p \to [0, 1]$ defined by

$$h(\{x, y\}) = \begin{cases} \dfrac{1}{5} & \text{if } x = u_i, y = u, 0 \leq i \leq 4 \\ \dfrac{1}{20} & \text{otherwise} \end{cases}$$

is a maximal metric independence function with $|h| = \frac{3}{2}$. Hence by Observation 2.3, $dim_f(W_6) = \frac{3}{2}$.

*Case* 3. $n \geq 7$.

In this case $R\{u_i, u_{i+1}\} = \{u_{i-1}, u_i, u_{i+1}, u_{i+2}\}$, $R\{u_i, u_{i+2}\} = \{u_{i-1}, u_i, u_{i+2}, u_{i+3}\}$ where the addition in the suffix is modulo $n-2$. Also, $|R\{x, y\} \cap V(C_{n-1})| \geq 4$ for all $x, y \in V(W_n)$ and $u \notin R\{u_i, u_j\}$, $0 \leq i < j \leq n-2$. Hence the function $g : V \to [0, 1]$ defined by

$$g(u_i) = \begin{cases} \dfrac{1}{4} & \text{if } 0 \leq i \leq n-2 \\ 0 & \text{if } u_i = u \end{cases}$$

is a minimal resolving function of $W_n$ with $|g| = \frac{n-1}{4}$ so that $dim_f(W_n) \leq \frac{n-1}{4}$.

Now, let $h$ be any resolving function of $W_n$. Then $h(R\{u_i, u_{i+1}\}) = h(u_{i-1}) + h(u_i) + h(u_{i+1}) + h(u_{i+2}) \geq 1$, $0 \leq i \leq n-2$. Adding these $(n-1)$ inequalities we get $4\sum_{i=0}^{n-2} h(u_i) \geq n-1$. Hence $dim_f(W_n) \geq |h| = h(u) + \sum_{i=1}^{n-2} h(u_i) \geq \sum_{i=1}^{n-2} h(u_i) \geq \frac{n-1}{4}$. Thus $dim_f(W_n) = \frac{n-1}{4}$ for $n \geq 7$. □

## 4. Graphs with $dim_f(G) = dim(G)$

In this section we present several families of graphs for which $dim_f(G) = dim(G)$.

For the wheel $G = W_5 = K_1 + C_4$ and for the graph $G = K_1 + P_3$ we have $dim_f(G) = dim(G) = 2$. Also, for the path $P_n$, $n \geq 2$, we have $dim_f(P_n) = dim(P_n) = 1$. A graph $G$ with exactly one cut vertex in which each block is $K_3$ is called a friendship graph.

**Theorem 4.1.** *Let $G$ be the friendship graph consisting of $k$ blocks, where $k \geq 2$. Then $dim_f(G) = k = dim(G)$.*

**Proof.** For $1 \leq i \leq k$, let $G_i = (u, v_i, w_i, u)$ be the $k$ blocks of $G$. Then $R\{v_i, w_i\} = \{v_i, w_i\}$, $R\{u, v_i\} = V(G) - \{w_i\}$ and $R\{u, w_i\} = V(G) - \{v_i\}$, $1 \leq i \leq k$. Let $S = \{u\}$. Then the function $\alpha : V(G) - S \to V(G) - S$ defined by $\alpha(v_i) = w_i$ and $\alpha(w_i) = v_i$, $1 \leq i \leq k$ is a bijection which satisfies the conditions of Theorem 3.4 and hence $dim_f(G) = \frac{|V(G)|-|S|}{2} = k$. Clearly $W = \{v_1, v_2, \ldots, v_k\}$ is a minimum resolving set of $G$ so that $dim(G) = k$. □

**Theorem 4.2.** *For the grid graph $G = P_m \square P_n$, we have $dim_f(G) = 2 = dim(G)$.*

**Proof.** Let $V(G) = \{u_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$ where $u_{i,j}$ is the vertex in the $i$th row and $j$th column. Since $dim(G) = 2$ [12], it follows that $dim_f(G) \leq 2$. We now proceed to prove that $dim_f(G) \geq 2$.

We observe that $R\{u_{1,1}, u_{2,2}\} = \{u_{1,1}\} \cup \{u_{i,j} : i \geq 2 \text{ and } j \geq 2\}$ and $R\{u_{1,2}, u_{2,1}\} = \{u_{i,j} : i = 1 \text{ or } j = 1 \text{ but not both}\}$. Thus $R\{u_{1,1}, u_{2,2}\} \cap R\{u_{1,2}, u_{2,1}\} = \emptyset$ and $R\{u_{1,1}, u_{2,2}\} \cup R\{u_{1,2}, u_{2,1}\} = V(G)$.

Let $h$ be any minimal resolving function of $G$. Then $h(R\{u_{1,1}, u_{2,2}\}) \geq 1$ and $h(R\{u_{1,2}, u_{2,1}\}) \geq 1$. Adding these two inequalities we get $|h| \geq 2$ and hence $dim_f(G) \geq 2$. Therefore $dim_f(G) = 2$. □

*Conclusion and Scope*. The following are some interesting problems for further investigation.

1. Characterize graphs $G$ for which $dim_f(G) = \frac{n}{2}$.
2. Characterize graphs for which $dim_f(G) = dim(G)$.
3. Cáceres et al. [2] have proved that $dim(G \square H) \geq \max\{dim(G), dim(H)\}$. Is a similar result true for $dim_f(G)$?

In the study of fractional domination Cockayne et al. [7] have obtained several results about the convexity of the set of all minimal dominating functions of a graph. Similar results regarding the convexity of minimal resolving functions of a connected graph will be reported in a subsequent paper.


## Acknowledgements

We are thankful to the National Board for Higher Mathematics, Mumbai, for its support through the project 48/5/2008/R&D-II/561, awarded to the first author. The second author is thankful to the UGC, New Delhi, India for having been awarded to him, FIP teacher fellowship during the *XI*th plan period. We are thankful to the referees for their helpful suggestions.


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