A theorem on cycle–wheel Ramsey number

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For two given graphs $G_1$ and $G_2$, the Ramsey number $R(G_1, G_2)$ is the smallest integer $N$ such that for any graph $G$ of order $N$, either $G$ contains $G_1$ or the complement of $G$ contains $G_2$. Let $C_n$ denote a cycle of order $n$ and $W_m$ a wheel of order $m + 1$. In this paper, we show that $R(C_n, W_m) = 3n − 2$ for $m$ odd, $n \geq m \geq 3$ and $(n, m) \neq (3, 3)$, which was conjectured by Surahmat, Baskoro and Tomescu.

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1. Introduction

All graphs considered in this paper are finite simple graphs without loops. Let $G = (V(G), E(G))$ be a graph. For $S \subseteq V(G)$, we use $N_S(v)$ to denote the set of the neighbors of a vertex $v$ contained in $S$ and $d_S(v) = |N_S(v)|$. If $S = V(G)$, we write $N(v) = N_G(v)$, $N[v] = N(v) \cup \{v\}$ and $d(v) = d_G(v)$. If $H$ is a subgraph of $G$, we write $N_H(v) = N_{H[v]}(v)$. The minimum degree of $G$ is denoted by $\delta(G)$ and $\varepsilon(G) = |E(G)|$. Let $G_1, G_2$ be two graphs; we use $G_1 + G_2$ to denote a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \text{ and } v \in V(G_2)\}$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by $S$ in $G$. We use $C_n$ and $mK_n$ to denote a cycle of order $n$ and the union of $m$ vertex disjoint complete graph $K_n$, respectively. A Wheel $W_m = K_1 + C_m$ is a graph of $m + 1$ vertices. The lengths of the longest and shortest cycles of $G$ are denoted by $\chi(G)$ and $g(G)$, respectively. We say $G$ is weakly pancyclic if it contains cycles of every length $1, g(G) \leq l \leq \chi(G)$.

For two given graphs $G_1$ and $G_2$, the Ramsey number $R(G_1, G_2)$ is the smallest integer $N$ such that for any graph $G$ of order $N$, either $G$ contains $G_1$ or the complement of $G$ contains $G_2$. For a connected graph $G$ of order $n$, it is well known that $R(G, F)$ satisfies $R(G, F) \geq (n − 1)(\chi(F) − 1) + s(F)$, if $n \geq s(F)$, where $\chi(F)$ is the chromatic number of $F$ and $s(F)$ the minimum number of vertices in some color class under all vertex colorings by $\chi(F)$ colors. A graph $G$ of order $n$ is said to be $F$-good if $R(G, F) = (n − 1)(\chi(F) − 1) + s(F)$. The classical result on $F$-goodness is due to Chvátal [5] that $R(T_n, mK_m) = (n − 1)(m − 1) + 1$, where $T_n$ is a tree of order $n$. For $W_m$-good graphs, it was shown in [4,3] that a star $S_n$ and a path $P_n$ are $W_m$-good when $m$ is odd and $n \geq m − 1 \geq 2$, and $P_n$ is $W_m$-good when $m$ is even and $n \geq m − 1 \geq 3$. Surahmat et al. [13] considered the Ramsey number involving a cycle versus a wheel and established the following.

Theorem 1 (Surahmat et al. [13]). $R(C_n, W_m) = 2n − 1$ for $m$ even and $n \geq 5m/2 − 1$.

Obviously, the result above says that $C_n$ is $W_m$-good when $m$ is even and $n \geq 5m/2 − 1$. In the same paper, they [13] conjectured that $C_n$ is also $W_m$-good when $m$ is odd and posed the following.
Conjecture 1 (Surahmat et al. [13]). \( R(C_n, W_m) = 3n - 2 \) for \( m \) odd, \( n \geq m \geq 3 \) and \( (n, m) \neq (3, 3) \).

For \( m = 3, 5 \), Conjecture 1 was confirmed by Yang et al. [15] and Surahmat et al. [12], respectively. Surahmat et al. [14] proved that Conjecture 1 is true if \( m \geq 5 \) is odd and \( n \geq \frac{5m - 9}{2} \). Shi [11] showed that Conjecture 1 is true in the case when \( n > 70 \) or \( n \geq 3m/2 + 1 \). Recently, Zhang et al. [16] showed that Conjecture 1 is true for \( n \geq 20 \) by discussing the longest cycle in a subgraph of order \( 3n - 6 \). In this paper, by considering the relations between the size and the circumference, the size and weakly pancyclic property of a graph, we give a complete proof of Conjecture 1. The main result of this paper is the following.

**Theorem 2.** \( R(C_n, W_m) = 3n - 2 \) for \( m \) odd, \( n \geq m \geq 3 \) and \( (n, m) \neq (3, 3) \).

2. Several lemmas

In order to prove Theorem 2, we need the following lemmas.

**Lemma 1** (Erdős and Gallai [7]). Let \( G \) be a graph of order \( n \) and \( 3 \leq c \leq n \). If \( e(G) \geq (c - 1)(n - 1)/2 + 1 \), then \( c(G) \geq c \).

**Lemma 2** (Brandt [1]). Every nonbipartite graph \( G \) of order \( n \) and \( e(G) > (n - 1)^2/4 + 1 \) is weakly pancyclic with \( g(G) = 3 \).

**Lemma 3** (Brandt et al. [2]). Every nonbipartite graph \( G \) of order \( n \) with \( \delta(G) \geq (n + 2)/3 \) is weakly pancyclic with \( g(G) = 3 \) or 4.

**Lemma 4** (Dirac [6]). Let \( G \) be a connected graph of order \( n \geq 3 \) with \( \delta(G) = \delta \geq 2 \). Then \( c(G) \geq \delta + 1 \).

**Lemma 5** (Faudree and Schelp [9], Rosta [10]). \( R(C_n, C_m) = 2n - 1 \) for \( n \geq m \geq 3 \), \( m \) is odd and \( (n, m) \neq (3, 3) \).

**Lemma 6** (Faudree et al. [8]). Let \( G \) be a graph of order \( n \geq 6 \). Then \( \max\{c(G), c(G)\} \geq \lceil 2n/3 \rceil \).

3. Proof of Theorem 2

**Proof of Theorem 2.** Let \( G \) be a graph of order \( 3n - 2 \) with \( n \geq 4 \), \( m \) is odd and \( n \geq m \geq 3 \). Since \( G = K_{m-1}n \) contains no \( C_3 \) and \( G \) contains no \( W_m \) for odd \( m \), we have \( R(C_n, W_m) \geq 3n - 2 \). In the following, we need only to show that \( R(C_n, W_m) \leq 3n - 2 \).

Suppose to the contrary neither \( G \) contains a \( C_n \) nor \( G \) contains a \( W_m \).

If there is some vertex \( x \) such that \( d(v) \leq n - 2 \), then \( G - [v] \) is a graph of order at least \( 2n - 1 \). By Lemma 5, \( \delta(G) = n - 1 \). If \( G \) is bipartite, then \( \alpha(G) \geq (3n - 3)/2 \geq m + 1 \) and hence \( G \) contains a \( W_m \), a contradiction. Thus, we may assume \( G \) is nonbipartite. If \( \delta(G) \geq n \), then \( G \) contains a \( C_n \), by Lemmas 3 and 4. Therefore, we have \( \delta(G) = n - 1 \).

Let \( v \in V(G) \) with \( d(v) = \delta(G) = n - 1 \). Set \( H = G - [v] \). Obviously, \( H \) is a graph of order \( 2n - 2 \).

If \( H \) is bipartite, say \( H = (X, Y) \), then \( |X| = |Y| = n - 1 \) for otherwise \( \overline{G}[X \cup \{v\}] \) or \( \overline{G}[Y \cup \{v\}] \) contains a \( W_m \). If \( n \) is even, then \( n \geq m + 1 \), and hence \( G[X \cup \{v\}] \) contains a \( W_m \), a contradiction. Thus, \( n \) is odd. If there is some \( v_i \in N(v) \) such that \( d_X(v_i) \leq n - 4 \) or \( d_Y(v_i) \leq n - 4 \), then \( G[X \cup \{v, v_i\}] \) or \( G[Y \cup \{v, v_i\}] \) contains a \( W_m \), a contradiction. Hence \( d_X(v_i) \geq n - 3 \) and \( d_Y(v_i) \geq n - 3 \) for any \( v_i \in N(v) \). If \( d_X(y) \leq n - 3 \) for some \( y \in Y \), then \( G[X \cup \{v, y\}] \) contains a \( W_m \) with the hub \( v \), and hence \( d_X(y) \geq n - 2 \) for any \( y \). By the symmetry of \( X \) and \( Y \), \( d_Y(x) \geq n - 2 \) for any \( x \in X \). Since for any \( v_i, v_j \in N(v) \), we can choose \( x \in N_C(v_i) \) and \( y \in N_C(v_j) \) such that \( xy \notin E(G) \), which implies that \( v_i, v_j, v_x, v_y \) is a \( C_4 \) in \( G \), we have \( n \geq 7 \). Now, let \( v_1 \in N(v) \). By the similar arguments as above, \( H[N_H(v_1)] \) contains a subgraph \( K_{n-3,n-3} \), where \( K_{n-3,n-3} \) denotes a complete bipartite graph \( K_{n-3,n-3} \) minus a perfect matching. Since \( K_{n-3,n-3} \) has a Hamilton cycle for \( n \geq 6 \), we see that \( v_1 + K_{n-3,n-3} \) has a \( C_4 \). Hence \( H \) is nonbipartite.

If \( H \) is bipartite, say \( H = (X, Y) \), then \( |X| = |Y| = n - 1 \) for otherwise \( G[X] \) or \( G[Y] \) contains a \( C_n \). If there is some \( v_i \in N(v) \) such that \( d_X(v_i) \geq 2 \) or \( d_Y(v_i) \geq 2 \), then \( G[X \cup \{v\}] \) or \( G[Y \cup \{v\}] \) contains a \( C_n \), and hence \( d_X(v_i) \leq 1 \) and \( d_Y(v_i) \leq 1 \) for any \( v_i \in N(v) \). If \( H \) has two edges between \( X \) and \( Y \), then \( H[X] = H[Y] = K_{n-1} \) and \( n \geq 4 \), we see that \( H \) contains a \( C_n \) and hence \( G \) contains a \( C_n \), a contradiction. Thus, noting that \( \delta(G) = n - 1 \), we can see that there exist \( v_1, v_2 \in N(v) \) and \( x_1, x_2 \in X \) such that \( v_1x_1, v_2x_2 \in E(G) \), which implies that \( G \) has a \( C_n \) if \( n \geq 5 \) and hence we have \( n = 4 \). In this case, \( m = 3 \). Assume that \( N(v) = \{v_1, v_2, v_3\} \), \( X = \{x_1, x_2, x_3\} \), \( Y = \{y_1, y_2, y_3\} \) and \( v_1x_1, v_2x_2 \in E(G) \). Since \( G \) has no \( C_4 \), \( v_1v_2 \notin E(G) \). Because \( d_X(v_i) \leq 1 \) and \( d_Y(v_i) \leq 1 \) for any \( v_i \in N(v) \), we may assume \( x_3y_3 \notin N(v_1) \cup N(v_2) \). If \( x_3y_3 \notin E(G) \), then \( G[v_1, v_2, x_3, y_3] = K_4 = W_3 \), which implies that \( x_3y_3 \notin E(G) \). If \( y_1 \notin N(v_1) \cup N(v_2) \) for some \( i \in \{1, 2\} \), then \( G[v_1, v_2, x_3, y_1] = K_4 = W_3 \) and hence we may assume that \( v_1y_1, v_2y_2 \in E(G) \). By the arguments above. If \( v_1, v_2 \in N(v_3) \), then \( v_1v_2 \notin E(G) \) since \( G \) has no \( C_4 \) again a contradiction. Therefore, \( H \) is nonbipartite.
If \( n = 4 \), then \( m = 3 \). If \( \overline{H} \) contains a \( C_3 \), then \( \overline{G} \) has a \( W_4 \) with the hub \( v \), a contradiction. Thus, since \( \overline{H} \) is a nonpartite graph of order \( 2n - 2 = 6 \), \( \overline{H} \) has a \( C_5 \) with no diagonal and the only vertex not in \( C_5 \) has at most two neighbors in \( C_5 \). In this case, it is easy to check that \( H \) contains a \( C_4 \), a contradiction. Thus, since \( H \) is a nonpartite graph of order \( 2n - 2 = 6 \), \( H \) has a \( C_5 \) with no diagonal and the only vertex not in \( C_5 \) has at most two neighbors in \( C_5 \).

We first note that, for \( \|H\| = 2n - 2 \geq 8 \), \( \|H\|\frac{\|H\| - 1}{4} > \frac{\|H\| - 1}{2} + 1 \). Now, if \( e(H) \geq (2n - 2)(2n - 2 - 1)/4 + 1 \), then \( c(H) \geq n \) by Lemma 1. By Lemma 2, \( H \) contains a \( C_n \), a contradiction. Thus we have \( e(H) \leq \frac{(2n - 2)(2n - 2 - 1)}{4} \), which implies that \( e(\overline{H}) \geq \frac{1}{2}(2n - 2)(2n - 2 - 1)/4 \). By symmetry, we have \( e(H) \geq \frac{1}{2}(2n - 2)(2n - 2 - 1)/4 \). By Lemma 6, \( \max\{c(H), c(\overline{H})\} \geq 2\frac{2n - 2}{3} \geq n \). Thus, by Lemma 2, either \( H \) has a \( C_n \) or \( \overline{H} \) contains a \( C_m \) which together with \( v \) form a \( W_m \) in \( G \), a final contradiction.

The proof of Theorem 2 is completed. \( \square \)

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