Polytopes of high rank for the symmetric groups

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Abstract

In the Atlas of abstract regular polytopes for small almost simple groups by Leemans and Vauthier, the polytopes whose automorphism group is a symmetric group $S_n$ of degree $5 \leq n \leq 9$ are available. Two observations arise when we look at the results: (1) for $n \geq 5$, the $(n-1)$-simplex is, up to isomorphism, the unique regular $(n-1)$-polytope having $S_n$ as automorphism group and, (2) for $n \geq 7$, there exists, up to isomorphism and duality, a unique regular $(n-2)$-polytope whose automorphism group is $S_n$. We prove that (1) is true for $n \neq 4$ and (2) is true for $n \geq 7$. Finally, we also prove that $S_n$ acts regularly on at least one abstract polytope of rank $r$ for every $3 \leq r \leq n-1$.

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1. Introduction

In [14], E.H. Moore gives for the first time a set of involutions of $S_n$ that corresponds to the $(n-1)$-simplex. He therefore shows that, for every $n \geq 3$, there is an abstract regular polytope of rank $n-1$ whose automorphism group is $S_n$.

More than 100 years after, Sjerve and Cherkassoff show in [17] that $S_n$ is a group generated by three involutions, two of which commute, provided that $n \geq 4$. The examples they give satisfy the intersection condition and therefore are rank three abstract regular polyhedra. So every group

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Table 1
Number of polytopes for $S_n$ ($5 \leq n \leq 9$).

<table>
<thead>
<tr>
<th>$G$</th>
<th>Rank 3</th>
<th>Rank 4</th>
<th>Rank 5</th>
<th>Rank 6</th>
<th>Rank 7</th>
<th>Rank 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_5$</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S_6$</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S_7$</td>
<td>35</td>
<td>7</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S_8$</td>
<td>68</td>
<td>36</td>
<td>11</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$S_9$</td>
<td>129</td>
<td>37</td>
<td>7</td>
<td>7</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

$S_n$ with $n \geq 4$ is the automorphism group of at least one abstract regular polytope of rank 3 and one polytope of rank $n - 1$. Earlier work by Conder in [3] and [4] covers all but a few cases of the results of [17] for the alternating and symmetric groups. As Conder pointed out to us, these days, it takes a few seconds to handle the missing cases for $S_n$ with MAGMA. Moreover, Conder observed that one may just take $x = (1, 2)$ and $y = (1, 2, 3, \ldots, n - 1, n)$ as standard generators of $S_n$, which have been known for at least 100 years. This generating pair is “reflexible” within $S_n$, e.g. by conjugation by $t = (1, 2)(3, n)(4, n - 1) \ldots (n/2, 3 + n/2)(1 + n/2, 2 + n/2)$ if $n$ is even or $(1, 2)(3, n)(4, n - 1) \ldots ((n + 1)/2, n + 5)/2)$ if $n$ is odd. In particular, $S_n$ is generated by the three involutions $xt$, $yt$ and $t$, for any $n > 2$.

In [11], Leemans and Vauthier collected all abstract regular polytopes having a “small” almost simple group as automorphism group. Table 1 summarizes the results obtained for the symmetric groups $S_n$ with $5 \leq n \leq 9$.

Looking at these results, three observations are easily made:

1. the $(n - 1)$-simplex is, up to isomorphism, the unique regular $(n - 1)$-polytope having $S_n$ as automorphism group;
2. for $n \geq 7$, there exists, up to isomorphism and duality, a unique regular $(n - 2)$-polytope whose automorphism group is $S_n$;
3. there are polytopes of rank $r$ with $3 \leq r \leq n - 1$; moreover, there is more than one $r$-polytope provided $r \leq n - 3$.

In this paper, we prove the following three theorems, showing that the above observations stay true when $n$ grows.

**Theorem 1.** For $n \geq 5$, the $(n - 1)$-simplex is, up to isomorphism, the unique polytope of rank $n - 1$ having a group $S_n$ as automorphism group. For $n = 4$, there are, up to isomorphism and duality, two abstract regular polyhedra whose automorphism group is $S_4$, namely the hemicube and the tetrahedron. Finally, for $n = 3$, there is, up to isomorphism, a unique abstract regular polygon whose automorphism group is $S_3$, namely the triangle.

**Theorem 2.** For $n \geq 7$, there exists, up to isomorphism and duality, a unique $(n - 2)$-polytope $P$ having a group $S_n$ as automorphism group. The Schl"afli symbol of $P$ is $\{4, 6, 3^{n-5}\}$.

**Remarks.**

- For $n = 6$, there are, up to isomorphism and duality, four polytopes of rank 4, of respective Schl"afli symbols $\{3, 4, 4\}$, $\{3, 6, 4\}$, $\{4, 4, 4\}$ and $\{4, 6, 4\}$. For $n = 5$, there are also, up to
isomorphism and duality, four polytopes of rank 3, of respective Schläfli symbols \(\{4, 5\} \), \(\{4, 6\} \), \(\{5, 6\} \) and \(\{6, 6\} \). These may be found, for instance, in [11].

- For \(n \geq 8\), there exist at least two non-isomorphic \((n - 3)\)-polytopes with automorphism group \(S_n\), for instance the ones with respective Schläfli symbols \(\{4, 6, 3^{n-6}\} \) and \(\{4, 6, 3^{n-8}, 6, 4\} \).
- The facet of the \((n - 2)\)-polytope with automorphism group \(S_n\) is an \((n - 3)\)-polytope with automorphism group \(S_{n-1}\).

**Theorem 3.** Let \(n \geq 4\). For every \(r \in \{3, \ldots, n - 1\}\), there exists at least one \(r\)-polytope \(P\) having a group \(S_n\) as automorphism group. Its Schläfli symbol is \(\{n - r + 2, 6, 3^{r-3}\}\).

In order to prove these theorems, we use some general theory of permutation groups and more precisely transitivity and primitivity. We also use CPR graphs [15] and some elementary graph theory. Finally, we use the correspondence between abstract regular polytopes and string \(C\)-groups.

The paper is organized as follows. In Section 2, we give the definitions and notation needed to understand this paper. In Section 3, we prove some general results on polytopes with automorphism group \(S_n\). In Section 4 (resp. 5, 6), we prove Theorem 1 (resp. 2, 3). We conclude the paper with some final remarks in Section 7.

### 2. Abstract regular polytopes

It is well known that thin regular residually connected geometries with a linear diagram, abstract polytopes and string \(C\)-groups are the same mathematical objects. The link between these objects may be found for instance in [13]. We take here the viewpoint of string \(C\)-groups because it is the easiest and the most efficient one to define abstract regular polytopes.

According to [13], we define a \(C\)-**group** as a group \(G\) generated by pairwise distinct involutions \(\rho_0, \ldots, \rho_{r-1}\), that satisfy the following property, called the **intersection property**: \[
\forall J, K \subseteq \{0, \ldots, r-1\}, \quad \langle \rho_j \mid j \in J \rangle \cap \langle \rho_k \mid k \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle.
\]

A \(C\)-group \(\Gamma := (G, \{\rho_0, \ldots, \rho_{r-1}\})\) is a string \(C\)-**group** if its generators satisfy the following relations, called the **commuting property**: \[
(\rho_j \rho_k)^2 = 1_G, \quad \forall j, k \in \{0, \ldots, r-1\} \text{ with } |j - k| \geq 2.
\]

Let \(S\) be a family of elements of a group \(G\). As in [18], we say that \(S\) is an **independent set** if \(s \notin \langle S \setminus \{s\}\rangle\), for all \(s \in S\). Moreover, if in addition \(\langle S \rangle = G\) we say that \(S\) is an **independent generating set**. Particularly, if \(\Gamma\) is a string \(C\)-group, \(\{\rho_0, \ldots, \rho_{r-1}\}\) is an independent generating set for \(G\). We use \(\Gamma_{i_1, \ldots, i_m}\) to denote \(\langle \rho_j \mid j \notin \{i_1, \ldots, i_m\}\rangle\).

A **string group generated by involutions** (sggi) is a group generated by a set of involutions satisfying the commuting property. The following theorem gives sufficient conditions to prove that a sggi satisfies the intersection property.

**Proposition 4.** (See [13].) Let \(\Gamma = \langle \rho_0, \ldots, \rho_{r-1}\rangle\) be a sggi and suppose that its subgroup \(\Gamma_{r-1}\) is a string \(C\)-group. If \(\Gamma_0\) is a string \(C\)-group and \(\Gamma_0 \cap \Gamma_{r-1} = \Gamma_0,_{r-1}\), then \(\Gamma\) is a string \(C\)-group.

Let \(\Gamma\) be a string \(C\)-group. A **regular \(r\)**-**polytope** \(P\) with automorphism group \(\Gamma\) is a poset \((P, \preceq)\) where \(P\) is a set of right cosets of \(\Gamma_j\) in \(\Gamma\) (with \(j = 0, \ldots, r - 1\)) and the relation \(\preceq\) is
defined as follows:

\[ \Gamma_j \phi \leq \Gamma_k \psi \quad \text{when } j \leq k \text{ and } \Gamma_j \phi \cap \Gamma_k \psi \neq \emptyset. \]

The elements of \( P \) are called \textit{faces} more precisely the \textit{j-faces} are the elements of the right cosets of \( \Gamma_j \) in \( \Gamma \). The rank of \( P \) is \( r \).

The Schläfli symbol of \( P \) is \{\( p_1, \ldots, p_r \)\} where \( p_j \) is the order of \( \rho_j \). The rank of \( P \) is \( r \).

The dual polytope of \( P \) with automorphism group \( \Gamma \) is obtained by reverting the set of generators of \( \Gamma \).

Let \( G \) be a permutation group of degree \( n \) and let \( \Gamma := (G; \{\rho_0, \ldots, \rho_{r-1}\}) \) be a string \textit{C}-group. As defined by Pellicer in [15], the \textit{CPR graph} \( G \) of \( \Gamma \) is an \( r \)-edge-labeled multigraph with vertex set \{1, \ldots, n\} such that \( \{i, j\} \) is an edge of \( G \) of label \( k \) if \( i\rho_k = j \). The edges of each label \( k \) form a matching on \( G \). If \( \Gamma \) acts transitively on \{1, \ldots, n\} then \( G \) is connected. We will denote by \( G_{i_1, \ldots, i_m} \) the spanning subgraph \( G \) (including all the vertices of \( G \)) whose edge set consists of the edges with labels \{\( i_1, \ldots, i_m \)\}.

\[ \text{Proposition 5. (See [15].)} \] Let \( G = G_0, \ldots, r-1 \) be a CPR graph of a string \textit{C}-group \( \Gamma := (G; \{\rho_0, \ldots, \rho_{r-1}\}) \), and let \( |i - j| \geq 2 \). Then every connected component of \( G_{i, j} \) is either a single vertex, a single edge, a double edge, or an alternating square.

\section{3. Some general results}

The O’Nan–Scott theorem gives a list of the maximal subgroups of \( S_n \), putting them in three categories, namely the primitive groups, the imprimitive but transitive groups and the intransitive groups. Naturally, any proper subgroup of \( S_n \) belongs to one of these three categories as well. Exhaustive lists of primitive subgroups of \( S_n \) exist up to a certain degree (see for instance [1,16]). Imprimitive but transitive subgroups of \( S_n \) are embedded in groups isomorphic to \( S_k \wr S_m \) with \( n = km \). In what follows let \( n \geq 5 \), let \( \Gamma := (S_n; \{\rho_0, \ldots, \rho_{r-1}\}) \) be a string \textit{C}-group of rank \( r \) and let \( G \) be the CPR-graph of \( \Gamma \) defined by the natural action of \( S_n \) on \{1, \ldots, n\}. We start with a result of Whiston that bounds the rank of \( \Gamma \).

\[ \text{Proposition 6. (See [18].)} \] The size of an independent set in \( S_n \) is at most \( n - 1 \), with equality only if the set generates the whole group \( S_n \).

This proposition implies that we may assume \( r < n \).

As mentioned above, for \( i = 0, \ldots, r - 1 \), either \( \Gamma_i \) is transitive and imprimitive, or \( \Gamma_i \) is primitive or \( \Gamma_i \) is intransitive. We now focus on the transitive cases.

\subsection{3.1. The case where \( \Gamma_i \) is imprimitive but transitive}

If \( \Gamma_i \) is imprimitive but transitive, it is embedded in a wreath product \( S_k \wr S_m \) with \( n = km \).

The set of distinguished generators of \( \Gamma_i \) splits into two sets \( N \) and \( M \) with \( N \) being the set of involutions fixing all blocks of size \( k \) and \( M \) being the set of involutions that swap at least a pair of blocks.

The edges of the graph \( G \) without the \( i \)-edges, also split in two sets, namely \( N \)-edges (the edges with labels corresponding to the elements of \( N \)) and \( M \)-edges (the edges with labels corresponding to the elements of \( M \)). An \( N \)-edge joins vertices in a same block, while an \( M \)-edge
either joins vertices in a same block or vertices in two distinct blocks. If a $j$-edge joins vertices in different blocks, there are $k$ parallel $j$-edges between these blocks. As $\Gamma_i$ is transitive, the graph $G$ without the $i$-edges is connected.

3.2. The case where $\Gamma_i$ is primitive

Lemma 7. $\Gamma_i$ is not the alternating group $A_n$ for all $i \in \{0, \ldots, r - 1\}$.

Proof. Suppose $\Gamma_i \cong A_n$ for some $i \in \{0, \ldots, r - 1\}$. Then the polytope $\mathcal{P}$ corresponding to $\Gamma$ has exactly two $i$-faces. Each of them must be incident to all the other elements of $\mathcal{P}$ and the diagram of $\mathcal{P}$ is not connected. Therefore, we get $S_n \cong A_n \times 2$, a contradiction. $\square$

In fact, the result above is a consequence of a more general result that we give here.

Proposition 8. If $\mathcal{P}$ is an abstract regular polytope having $S_n$ as automorphism group, there are no 2’s in the Schläfli symbol of $\mathcal{P}$.

Proof. It is sufficient to observe that if there is a 2 in the Schläfli symbol of $\mathcal{P}$, then, $S_n$ may be written as a direct product of two non-trivial groups, which is clearly impossible. $\square$

Proposition 9. (See [19].) Every non-trivial normal subgroup of a primitive group is transitive.

This proposition implies the following result.

Lemma 10. If $r \geq 3$, then $\Gamma_1$ and $\Gamma_{r-2}$ are not primitive.

Proof. Without loss of generality, we consider $\Gamma_1 \cong \langle \rho_0 \rangle \times \langle \rho_2, \ldots, \rho_{r-1} \rangle$ and $\langle \rho_0 \rangle$ is a cyclic group of order 2. By Proposition 9, $\Gamma_1$ cannot be primitive. $\square$

4. $(n-1)$-polytopes with symmetric group $S_n$

We now focus on the case where $r = n - 1$. There is a well-known regular polytope of rank $n - 1$ that has $S_n$ as automorphism group, namely the $(n-1)$-simplex. The first time this string $C$-group appeared was in a paper by E.H. Moore [14]. To prove that the $(n-1)$-simplex is the unique polytope of rank $n-1$ for $S_n$, we use the following proposition, due to Cameron and Cara, that describes the structure of an independent generating set of maximal rank in $S_n$ [2].

Proposition 11. If $S$ is an independent generating set for $S_n$ of size $n-1$, with $n \geq 7$, then one of the following holds:

1. $S$ is a set of transpositions;
2. There exist a transposition $s \in S$ and a set of transpositions $T$ such that

$$S = \{s\} \cup \{(st)^{\epsilon(t)} \mid t \in T\}, \quad \text{where } \epsilon(t) = \pm 1.$$

Corollary 12. If $\Gamma = (S_n; \{\rho_0, \ldots, \rho_{n-2}\})$ is a string $C$-group, then $\rho_i$ is a transposition for all $i \in \{0, \ldots, n-2\}$. 
Proof. Let us show that case 2 of Proposition 11 cannot occur. Let \( S := \{ \rho_0, \ldots, \rho_{n-2} \} \). Suppose that there exist a transposition \( s \in S \) and a set of transpositions \( T \) such that
\[
S = \{ s \} \cup \{(st)^{\varepsilon(t)} \mid t \in T \}, \quad \text{where } \varepsilon(t) = \pm 1.
\]
As \( \rho_i \) is an involution for all \( i \in \{0, \ldots, n-2\} \) we must have \( \varepsilon(t) = 1 \) and \( \text{supp}(s) \cap \text{supp}(t) = \emptyset \) for every \( t \in T \). Consequently \( \langle S \rangle = \langle T \cup \{s\} \rangle \) is not transitive on \( \{1, \ldots, n\} \) which is a contradiction.

Proof of Theorem 1. Let \( \Gamma = (S_n; \{\rho_0, \ldots, \rho_{n-2}\}) \) be a string \( C \)-group. The cases where \( n = 3 \) and \( n = 4 \) are obvious and well known. We split the proof in three cases, namely the case where \( n \geq 7 \), the case where \( n = 5 \) and the case where \( n = 6 \).

1. The case where \( n \geq 7 \): This is just a consequence of Corollary 12. Indeed the CPR graph of \( \Gamma \) has \( n \) vertices and \( n-1 \) edges. Using the commuting property of the generators there is only one possibility, namely the one given in Fig. 1.

2. The case where \( n = 5 \): In this case, \( S_5 \cong \text{PGL}(2, 5) \) and, as shown in [10], the only rank 4 string \( C \)-group of \( S_5 \) is the 4-simplex.

3. The case where \( n = 6 \): We have \( \Gamma = (S_6; \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4\}) \).

We first show that we may assume that \( \Gamma_0 \) is intransitive.

3.1. Suppose that \( \Gamma_0 \) is transitive but imprimitive. It is then embedded in the wreath product \( S_3 \wr S_2 \) or \( S_2 \wr S_3 \). These groups are of respective orders 72 and 48. But the smallest rank four non-degenerate regular polytope (i.e. without 2’s in its Schlafli symbol) has an automorphism group of order 96 according to the atlas of polytopes for small groups [7]. By Proposition 8, \( \Gamma_0 \) has a Schlafli symbol without 2’s. Therefore \( \Gamma_0 \) cannot be transitive imprimitive.

3.2. Suppose that \( \Gamma_0 \) is primitive, but not \( A_6 \) nor \( S_6 \). Therefore, as given by [1], the group \( \Gamma_0 \cong \text{PGL}_2(5) \cong S_5 \) or \( \Gamma_0 \cong L_2(5) \cong A_5 \). The latter case is impossible because, as shown in [9], the only groups \( L_2(q) \) that are rank 4 string \( C \)-groups are for \( q = 11 \) or 19. If \( \Gamma_0 \cong S_5 \), by [10], we know that \( \Gamma_0 \) is a string \( C \)-group corresponding to the 5-simplex. The group \( S_6 \) has two conjugacy classes of subgroups \( S_5 \), one transitive, and one not transitive, that are fused under the action of \( \text{Aut}(S_5) \). Therefore, since we are interested in string \( C \)-groups up to isomorphism, we may assume that \( \Gamma_0 \) is an intransitive group isomorphic to \( S_5 \).

3.3. We now assume that \( \Gamma_0 \) is intransitive. It acts regularly on a 4-polytope. Necessarily the action of \( \Gamma_0 \) on the set of 6 points has two orbits, namely one with one point and the other with five points, for otherwise, the subgroup \( \Gamma_0 \) would have an order at most 48, leading to a contradiction as in 3.1. Moreover \( \Gamma_0 \cong S_5 \) and the CPR graph \( G_{1,2,3,4} \) is as follows:

\[
\begin{array}{cccccc}
& & & & & \\
& 1 & 2 & 3 & 4 & \\
\end{array}
\]

Now there is only one place for a 0-edge and therefore the CPR graph of a 5-polytope with symmetric group \( S_6 \) is unique, as follows:

\[
\begin{array}{cccccc}
0 & & & & & \\
1 & 2 & 3 & 4 & \\
\end{array}
\]

This finishes the proof of Theorem 1. \( \Box \)
5. \((n-2)\)-polytopes with symmetric group \(S_n\)

In the atlas \([11]\), we find four polyhedra with symmetric group \(S_5\) and also four polytopes of rank 4 with automorphism group \(S_n\). For \(n = 7, 8\) and 9, there is, up to isomorphism and duality, a unique rank \((n-2)\)-polytope with automorphism group \(S_n\). Its Schläfli symbol is \(\{4, 6, 3^{n-5}\}\). It is a generalized petrial of the hypercube \([8]\) and therefore is a polytope. These experimental results led us to the statement of Theorem 2.

From now on, we assume \(n \geq 7\), and \(r = n - 2\). We look for abstract regular polytopes of rank \(n - 2\), that have \(S_n\) as automorphism group. Let \(\Gamma = (S_n; \{\rho_i \mid i \in \{0, \ldots, n - 3\}\})\) be a string \(C\)-group and let \(P\) be the corresponding abstract regular polytope.

We first prove, in Section 5.1 that \(\Gamma_i\) is intransitive for all \(i = 0, \ldots, n - 3\). Then, in Section 5.2 we study all possibilities for the CPR graph \(G\) of \(\Gamma\) using the intransitivity of \(\Gamma_i\) and we show that there exists only one possibility corresponding to the CPR-graph of a regular polytope with Schläfli symbol \(\{4, 6, 3^{n-5}\}\).

5.1. The subgroups \(\Gamma_i\) are intransitive

We first prove, in Lemma 13, that if \(\Gamma_i\) is imprimitive, it is intransitive. Then, in Lemma 18, we prove that \(\Gamma_i\) cannot be primitive.

**Lemma 13.** Let \(i \in \{0, \ldots, n - 3\}\) with \(n \geq 7\). If \(\Gamma_i\) is imprimitive, then it is intransitive.

**Proof.** Suppose \(\Gamma_i\) is imprimitive and transitive. In this case, as we observed in Section 3.1, \(\Gamma_i\) is embedded in a wreath product \(S_k \wr S_m\) with \(n = km\). Denote by \(\phi\) the natural homomorphism from \(S_k \wr S_m\) to \(S_m\). Let us denote by \(\{g_1, \ldots, g_{n-3}\}\) the generators \(\rho_j\) (with \(j \neq i\)) of \(\Gamma_i\), reordered in such a way that \(\phi(\Gamma_i)\) is generated by the action of the first \(l\) generators \(\phi(g_1), \ldots, \phi(g_l)\), where \(\phi(g_1), \ldots, \phi(g_l)\) form an independent set of \(S_m\). In [18], Whiston proves in his Lemma 3 that \(l \leq m - 1\) and gives a way to rewrite the generators to undo the block action in \(g_{l+1}, \ldots, g_{n-3}\) while preserving the independence. Suppose that, for all \(i \geq l + 1\), the block action can be undone by \(g_i \rightarrow g_i w_i\) where \(w_i\) is a word in \(\{g_1, \ldots, g_l\}\) and let \(h_i = g_i\) for \(i \leq l\) and \(h_i = g_i w_i\) for \(i \geq l + 1\). The set \(\{h_1, \ldots, h_{n-3}\}\) is still an independent set. It is not necessarily a sggi of course since some of the \(h_i\) are not necessarily involutions. Observe nevertheless that, when \(k = 2\), it is still a sggi. Now, in his Proposition 4, Whiston explains that there are two cases to consider. The first case, is when there is no numbering of the blocks \(B_1, \ldots, B_m\) such that \(\{h_{l+1}, \ldots, h_{n-3}\}\) acts as \(S_k \times \cdots \times S_k\) (with \(m - 1\) factors) on \(B_1 \cup \cdots \cup B_{m-1}\). In that case, \(\langle h_{l+1}, \ldots, h_{n-3} \rangle\) is a subgroup of \(S_k \times \cdots \times S_k\) (with \(m - 2\) factors). Hence, we have \(km - 3 \leq (m - 1) + (m - 2)(k - 1)\). This implies \(k = 2\). In the second case, Whiston shows that \(km - 3 \leq k + 2m - 3\) giving only three cases to consider, namely the case where \(k = 4\) and \(m = 2\), the case where \(k = 3\) and \(m = 3\) and the case \(k = 2\). These constraints must of course also be satisfied by the starting set of generators. Therefore, we can come back to our initial set of generators and analyse these three cases separately.

Consider the set \(Z\) of elements of \(N\) that commute with all elements of \(M\). For each element \(\rho_j\) of \(M\), there are at most two elements that are not in \(Z\), namely \(\rho_{j-1}\) and \(\rho_{j+1}\), thus \(|Z| \geq km - 3 - 3|M|\).

Since \(Z\) contains elements that commute with all elements of \(M\), by Proposition 5, a \(j\)-edge (with \(\rho_j \in Z\)) and an \(l\)-edge (with \(\rho_l \in M\)) are necessarily on an alternating square. Therefore, if \(\rho_j \in Z\), in every block there is at least one \(j\)-edge. Moreover, every \(\rho_j \in Z\) has exactly the
same number of 2-cycles in each block. Hence $Z$ is an independent set and each element in $Z$ is decomposable into $m$ conjugate independent elements in $S_k$ and therefore $|Z| \leq k - 1$ by Proposition 6.

1. Assume $k = 4$ and $m = 2$: In this case, $n = km = 8$, $|M| = 1$ and $|N| = 4$. From the two inequalities above, we have $2 \leq |Z| \leq 3$. Suppose first that $|Z| = 3$. Then $\langle Z \rangle \cong S_4$, by the second part of Proposition 6. The CPR graph of $\Gamma$ restricted to the $Z$-edges (that is, the set of $j$-edges with $\rho_j \in Z$) has $m$ connected components, each being the CPR graph of a 3-polytope with automorphism group $S_4$. By Theorem 1, there are two possibilities, namely the tetrahedron that is the 3-simplex and the hemicube. So all the possible CPR-graphs of $\Gamma$ restricted to the $Z$-edges are known. Thanks to the commuting property of the generators, we cannot have more edges joining vertices in a same block. Hence $N = Z$, a contradiction. Therefore $|Z| = 2$. We now look at each possibility for $i$ separately, up to duality.

1.1. Suppose $i = 0$. There are three possibilities for $M$, namely $\{\rho_2\}$, $\{\rho_3\}$ and $\{\rho_4\}$.

1.1.1. $M = \{\rho_2\}$: In this case $Z = \{\rho_4, \rho_5\}$. As $\rho_4 \neq \rho_5$, in the CPR graph, there exists a 4-edge meeting a 5-edge in one vertex. As $\rho_1$ commutes with both $\rho_5$ and $\rho_4$ we have two possibilities for $G_{1,2,4,5}$:

![Diagrams](image1)

In the first case, $\rho_1 \rho_1^{\rho_2} = \rho_4 \rho_4^{\rho_5}$, a contradiction with the intersection condition. In the second case $\rho_4 \rho_2 \rho_5 \rho_4 \rho_2 = \rho_1 \rho_2 \rho_1 \rho_5 \rho_2 \neq \langle \rho_2, \rho_5 \rangle$, so the intersection condition is not satisfied.

1.1.2. $M = \{\rho_3\}$: In this case $Z = \{\rho_1, \rho_5\}$ and therefore there are three possibilities for $G_{1,5}$: either it has two connected components (one in each block) being alternating squares, or it has double edges, or 1-edges and 5-edges are all disjoint. As $\rho_4 \neq \rho_5$ there is a 4-edge meeting a 5-edge in one vertex.

In the first case we get three possibilities for $G_{1,3,4,5}$:

![Diagrams](image2)

The first graph gives $\rho_1 = (\rho_4 \rho_4^{\rho_5})^{\rho_3} \rho_4 \rho_4^{\rho_5}$, the second gives $\rho_1 = \rho_4 \rho_4^{\rho_5}$ and the third gives $\rho_1 = (\rho_4 \rho_4^{\rho_5})(\rho_4 \rho_4^{\rho_5})^{\rho_3}$, so none of these give independent generating sets.

In the case that $G_{1,5}$ has double edges, as $\rho_4 \neq \rho_5$ and $\rho_4 \neq \rho_3$ there exists an unique possibility for $G_{1,3,4,5}$ that is:
Again here, \( \rho_1 = (\rho_5^\rho_4 \rho_5)^{\rho_3} \rho_5^\rho_4 \rho_5 \), so we do not have an independent generating set.

The case with all 5-edges and 1-edges disjoint does not occur because as there exists a 1-edge meeting a 2-edge and as \( \rho_2 = \rho_5 \) one gets at least a double \{1, 5\}-edge.

1.1.3. \( M = \{\rho_4\} \): Looking at \( \Gamma_0 \), we see that this is the dual case of case 1.1.1. So here also, there is no possibility.

1.2. Suppose \( i = 1 \). There are two possibilities for \( M \), namely \( \{\rho_3\} \) or \( \{\rho_4\} \).

1.2.1. \( M = \{\rho_3\} \): In this case we can use the same argument as in case 1.1.2, replacing \( \rho_1 \) by \( \rho_0 \) and vice versa.

1.2.2. \( M = \{\rho_4\} \): The particularity of this case is that \( \rho_1 \) is consecutive with the two permutations of \( Z = \{\rho_0, \rho_2\} \). There are three possibilities for \( G_{0,2} \): it has two alternating squares (one in each block), or it has double edges, or 0-edges and 2-edges do not meet. The last case cannot happen because there must be a 3-edge meeting a 2-edge and then \( G_{0,3} \) has an alternating square and \( G_{0,2,3} \) has a double \{0, 2\}-edge.

Consider first that \( G_{0,2} \) has two alternating squares. As \( \rho_5 \) commutes with both \( \rho_0 \) and \( \rho_2 \) we have the following possibilities for \( G_{0,2,4,5} \):

The first graph gives \( \rho_0 = \rho_5 \rho_5^\rho_4 \) and the second graph gives \( \rho_2 = \rho_5 \rho_5^\rho_4 \). So we do not have an independent generating set.

Now suppose that \( G_{0,2} \) has double edges. There is a 3-edge meeting a 2-edge. There is only one possibility for \( G_{0,2,3,4} \) that is:

But then, \( \rho_0 = (\rho_2^\rho_3 \rho_2)(\rho_2^\rho_3 \rho_2)^\rho_4 \), a contradiction.

1.3. Suppose \( i = 2 \). There is just one possibility for \( M \), namely \( M = \{\rho_4\} \). Now \( Z = \{\rho_0, \rho_1\} \).

If \( G_{0,5} \) has a square in one block then as \( \rho_2 \) commutes with both \( \rho_0 \) and \( \rho_5 \) we have \( \rho_2 = \rho_4 \).
which is impossible. Thus 0-edges and 5-edges are not in alternating squares. As there is a 1-edge meeting a 0-edge and \( \rho_5 \) does not commute with \( \rho_4 \), the remaining possibility for \( G_{0,1,4,5} \) is:

\[
|\begin{array}{cc}
0 & 4 \\
4 & 0
\end{array}|
\]

Here, \( \rho_5 \rho_5^\rho = \rho_0 \rho_1 \rho_0 \), hence, \( \Gamma_{2,4,5} \cap \Gamma_{0,1} \neq 1 \), contradicting the intersection condition. This concludes the case where \( k > 3 \).

2. Assume \( k = 2 \): In this case we may assume \( |M| \leq m - 1 \) as Whiston does in his paper. Indeed, if it is not the case, we use Whiston’s construction as in the beginning of the proof of this lemma to obtain a sggi that is also an independent generating set and we prove below that it cannot exist. Since \( |M| \leq m - 1 \), we have \( |N| \geq m - 2 \). The group \( \langle N \rangle \) is a subgroup of an elementary abelian group \( E_{2^m} \) of order \( 2^m \). Thus it is itself abelian and we have \( \rho_j \in N \Rightarrow \{ \rho_j-1, \rho_j+1 \} \cap N = \emptyset \). Therefore \( |N| \leq \frac{(n-3)+1}{2} = m - 1 \). So there are two cases to consider, namely \( |N| = m - 1 \) or \( m - 2 \).

2.1. The case \( |N| = m - 2 \): It implies \( |M| = m - 1 \). Hence, we have that \( Z(\langle M \rangle) \neq 1 \). On the other hand, \( M \) is an independent set acting on the \( m \) blocks of 2 points. Therefore, \( \langle M \rangle \cong S_m \). But \( n \geq 7 \) and \( m = n/2 \geq 4 \), implying \( Z(\langle M \rangle) = 1 \), a contradiction.

2.2. The case \( |N| = m - 1 \): It implies \( |M| = m - 2 = |N| - 1 \). Then \( M \) is isomorphic to one of \( E_{2^{m-2}} \) or \( E_{2^{m-4}} \times D_{2^a} \), where \( D_{2^a} \) is a dihedral group of order \( 2^a \). Moreover, since \( \Gamma_i \) is transitive, \( M \) has to be transitive on the \( m \) blocks of 2 points. Also, because \( \Gamma_i \leq S_2 : S_m \), we have that \( \langle M \rangle \leq H \) for some \( H \) isomorphic to \( S_m \). The biggest subgroup \( E_{2^a} \) in \( S_m \) is obtained by taking as many disjoint transpositions as possible, that is, as many as \( \lfloor \frac{m}{2} \rfloor \).

2.2.1. Suppose first that \( \langle M \rangle \) is nonabelian. So it is isomorphic to \( E_{2^{m-2}} \times D_{2^a} \) which contains elementary abelian groups of order \( 2^{m-3} \). Therefore, \( m - 3 \leq \lfloor \frac{m}{2} \rfloor \). There are three possibilities for \( m \), namely 4, 5 or 6. If \( m = 6 \), we look for subgroups \( 2^2 \times D_{2^a} \) in \( S_6 \) and there is none. If \( m = 5 \), the only subgroups \( 2 \times D_{2^a} \) or \( S_5 \) are isomorphic to \( D_{12} \cong 2 \times D_6 \) and they are not transitive on the blocks. If \( m = 4 \), the only possible transitive subgroups are \( E_4 \) or \( D_8 \), meaning that \( |\Gamma_i| \leq |S_2 : D_8| = 2^4 \cdot 2^3 = 128 \). But \( \Gamma_0 \) has order at least 432, \( \Gamma_1 \) has order at least 2 \cdot 96 = 192 and \( \Gamma_2 \) has order at least 3! \cdot 4! = 144. Indeed, looking at the atlas of polytopes for small groups [7], we see that the smallest order of the automorphism group of an abstract regular polytope of rank 2 (resp. 3, 4, 5) is 6 (resp. 24, 96, 432). So \( M \) cannot be nonabelian.

Observe that Marston Conder recently determined the smallest order of the automorphism group of an abstract regular polytope of rank \( n \) for every integer \( n \geq 2 \) [5].

2.2.2. Suppose \( \langle M \rangle \) is abelian. So it is elementary abelian, and \( m - 2 \leq \lfloor \frac{m}{2} \rfloor \). So \( m = 4 \), and \( \langle M \rangle = E_4 \). Moreover, \( |\Gamma_i| \leq |S_2 : E_4| = 64 \). As seen in 2.2.1, \( \Gamma_i \) must have order at least 144, a contradiction.

3. Assume \( k = 3 \) and \( m = 3 \): In that case, \( n = km = 9 \) and we have three blocks of three points. Similar arguments as above permit to conclude that this case cannot occur. □
Corollary 14. \( \Gamma_1 \) and \( \Gamma_{n-4} \) are intransitive.

Proof. This is a combination of Lemmas 13 and 10. □

The following proposition, due to Maróti, is needed to prove imprimitivity of all of the \( \Gamma_i \)'s.

Proposition 15. (See [12].) If \( G \) is a primitive subgroup of \( S_n \) not containing \( A_n \), then \( |G| < 3^n \). Moreover, if \( n > 24 \), then \( |G| < 2^n \).

Lemma 16. \( \Gamma_0 \) and \( \Gamma_{n-3} \) are not primitive.

Proof. Let us consider \( i \in \{0, n-3\} \). The group \( \Gamma_i \) is the automorphism group of a rank \((n-3)\)-polytope and that polytope has a connected diagram. As mentioned above, if the rank of \( \Gamma_i \) is 2 (resp. 3, 4, 5), \( |\Gamma_i| \geq 6 \) (resp. 24, 96, 432). If the rank of \( \Gamma_i \) is \( \geq 6 \), then it is easy to show that \( |\Gamma_i| \geq 3^{n-5}2^4 \). We now use Proposition 15. For \( n > 24 \), it is clear that \( 3^{n-5}2^4 > 2^n \). Therefore, \( \Gamma_i \) cannot be primitive.

Lemma 17. \( \Gamma_i \) is not primitive for \( i \in \{2, \ldots, n-5\} \).

Proof. Assume \( i \in \{2, \ldots, n-5\} \). Then \( \Gamma_i = \Gamma_{<i} \times \Gamma_{>i} \) where \( \Gamma_{<i} = \{\rho_j \mid j < i\} \) and \( \Gamma_{>i} = \{\rho_j \mid j > i\} \). Now, \( \Gamma_{>i} < \Gamma_i \) and \( \Gamma_i \) is not transitive by Corollary 14. So \( \Gamma_{>i} \) is not transitive and by Proposition 9, \( \Gamma_i \) is not primitive. □

Lemma 18. Let \( i \in \{0, \ldots, n-3\} \). The subgroup \( \Gamma_i \) is not primitive.

Proof. This is a combination of Lemmas 10, 16 and 17. □

Theorem 19. Let \( n \geq 7 \) and \( i \in \{0, \ldots, n-3\} \). The subgroup \( \Gamma_i \) is intransitive.

Proof. This is a direct consequence of Lemmas 13 and 18. □

5.2. The CPR graph of \( \Gamma \) for \( n \geq 7 \)

Now that we have proven in the previous section that all the subgroups \( \Gamma_i \) (\( i = 0, \ldots, n-3 \)) are intransitive, we use this to determine the possible shape of the CPR graph associated to a rank \((n-2)\) regular polytope of \( S_n \) and show that, up to isomorphism and duality, this graph is unique.
Lemma 20. Let \( n \geq 7 \) and \( \Gamma = (S_n; \{ \rho_i \mid i \in [0, \ldots, n-3] \}) \) be a string \( C \)-group. Up to numbering of the vertices, the CPR graph of \( \Gamma \) is unique. It is given below.

\[
\begin{array}{cccccccc}
1 & 0 & 1 & 2 & 3 & \cdots & n-5 & n-4 & n-3
\end{array}
\]

Proof. Let \( G \) be the CPR graph of \( \Gamma \). Consider a subgraph \( T \) of \( G \) with \( n \) vertices and \( n-2 \) edges, one for each \( \rho_i \), such that an \( i \)-edge joins vertices in different \( \Gamma_i \)-orbits. Let \( \{x_i, y_i\} \) be an edge of \( T \) with label \( i \), for \( i = 0, \ldots, n-2 \). In the following figures the dashed lines represent edges of \( G \) that are not edges of \( T \). We use polygonal lines to represent paths with an undetermined length.

1. \( T \) has no circuits and has two connected components: First observe that, by construction, \( T \) has neither loops nor double edges. Suppose that there are two paths between the vertices \( v \) and \( w \) of \( T \). Then there exists an edge \( e_i = \{x_i, y_i\} \) with \( x_i \) and \( y_i \) in the same \( \Gamma_i \)-orbit, a contradiction. The rest holds because \( T \) has \( n \) vertices and \( n-2 \) edges.

2. If \( \rho_i \) is a product of at least two transpositions \( (x_i y_i) \) and \( (a b) \) then \( a \) and \( b \) are in different connected components of \( T \): Suppose that \( a \) and \( b \) are in the same connected component of \( T \). Then there is a path on \( T \) from \( a \) to \( b \) with more than one edge. Let \( e_j \) be an edge in this path with \( j \neq i \). Then \( x_j \) and \( y_j \) are in the same \( \Gamma_j \)-orbit (as shown in the picture below), a contradiction.

![Diagram 1](image1.png)

3. \( \rho_i \) is a product of at most two transpositions: Suppose that there exist three transpositions in the decomposition of \( \rho_i: (x_i y_i), (a b), (c d) \). By 2, one may assume without loss of generality that \( a, c, x_i \) and \( y_i \) are in the same connected component of \( T \) and \( b \) and \( d \) are in the other. Then there exists an edge \( e_j = \{x_j, y_j\} \) with \( x_j \) and \( y_j \) in the same \( \Gamma_j \)-orbit and we have a contradiction as shown by the following two pictures.

![Diagram 2](image2.png)

4. \( G_{i,j} \) cannot have 4 parallel edges: Let \( (a b) \) and \( (c d) \) be transpositions of the decomposition of \( \rho_i \) and \( \rho_j \) respectively in addition to \( (x_i y_i) \) and \( (x_j y_j) \). Without loss of generality we may assume that \( x_i, y_i \) and \( a \) are in one connected component of \( T \) and \( b \) in the other. The same kind of assumption can be done for \( \rho_j: x_j, y_j \) and \( c \) are in one connected component of \( T \) and \( d \) is
in the other. We have only two possible situations according to whether \( b \) and \( d \) are in the same connected component or not. These possibilities are shown in the following two pictures.

In both cases, we can find an edge \( e_k = \{x_k, y_k\} \) with \( x_k \) and \( y_k \) in the same \( \Gamma_k \)-orbit, leading to a contradiction.

5. \( G_{i,j} \) cannot have 4 edges in an alternating square: Let \( \rho_i = (a \, b)(c \, d) \) and \( \rho_j = (a \, c)(b \, d) \). There exists a permutation \( \rho_k \) moving one of the vertices of the square to a point not on the square. If \( |k-j| > 1 \) and \( |k-i| > 1 \) then \( G_{i,j,k} \) is a cube with four parallel edges, contradicting 3. Hence \( \rho_k \) is consecutive with \( i \) or \( j \) or with both. If \( k = i + 1 \) and \( |j - k| > 1 \) then \( G_{i,i,i+1} \) has two squares with three parallel \( j \)-edges, again a contradiction with 3. Now consider the remaining case, \( G_{i-1,i,i+1} \) with \( \rho_{i-1} = (a \, b)(c \, d) \) and \( \rho_{i+1} = (a \, c)(b \, d) \). Then, \( G_{i-1,i,i+1} \) must have either 5 or 6 vertices as shown below.

In both cases no more edges not in \{\( i, i-1, i+1 \)\} can meet these vertices and transitivity fails for \( n > 6 \).

6. 2-Transpositions have one common transposition in their decompositions: Let \( \rho_i = (x_i \, y_i)(a \, b) \). For \( |i-j| \geq 2 \), \( \rho_j \) is either a transposition \( \rho_j = (x_j \, y_j) \) or \( \rho_j = (x_j \, y_j)(a \, b) \) by 4 and 5. For \( |i-j| = 1 \), suppose that \( \rho_j = (x_j \, y_j)(c \, d) \). Then \( (a \, b) \neq (c \, d) \) and \( x_j \) and \( y_j \) are in different \( \Gamma_j \)-orbits. At least one pair of edges of \( G_{i,j} \) must meet. But they cannot meet in a square by 5. Hence \( G_{i,j} \) is linear and has at most 3 connected components. The possible configurations are given below.

Using 1 and 2, it turns out that there exists at least one edge \( e_k \) with \( x_k \) and \( y_k \) in the same \( \Gamma_k \)-orbit, a contradiction.
7. There is exactly one $\rho_i$ that is a 2-transposition: Let $(a \, b)$ be the common transposition to all 2-transpositions of the generating set. Without loss of generality, we may assume that there exists a permutation $\rho_i$ moving $a$ to a vertex $c \neq b$ and this permutation must be a transposition. It does not commute with the 2-transpositions, so there are at most two 2-transpositions. Suppose that we have exactly two 2-transpositions. Then we have $\rho_i = (x_i \, y_i)(a \, b)$, $\rho_{i+1} = (a \, c)$ and $\rho_{i+2} = (x_{i+2} \, y_{i+2})(a \, b)$. But then, the vertices $a, b$ and $c$ are not connected to the rest of the CPR graph.

8. There is only one possibility for the CPR graph of $\Gamma$ up to a renumbering of the vertices: In 7 we proved that $\mathcal{G}$ has $n-1$ edges. By transitivity and the commuting property of the generators, $\mathcal{G}$ is linear. The edges of this graph have labels in the set $I = \{0, \ldots, n-3 \}$. The 2-transposition $\rho$ corresponds to two nonconsecutive edges of the graph with label $i$. The involutions $\rho_0$ and $\rho_{n-3}$ are necessarily transpositions and the two $i$-edges meet only two other edges. Thus the CPR graph is, up to a renumbering of the vertices, as follows:

![Diagram](image)

Proof of Theorem 2. The CPR graph above gives a string $C$-group obtained from the $(n-1)$-simplex by using the generalized Petrie operator defined in [8]. Nevertheless, we now give a proof that it is indeed a string $C$-group, not relying on the latter reference. The subgroup $\Gamma_0$ is a subgroup of $C_2 \times S_{n-2}$. It is obvious that $\Gamma_{0,1} \cong S_{n-3}$ and $\Gamma_{0,1,n-2} \cong S_{n-4}$. Moreover, $\Gamma_{0,n-2}$ is a subgroup of $C_2 \times S_{n-3}$. Since $\Gamma_{0,1,n-2}$ is maximal in $\Gamma_{0,1}$, we clearly have $\Gamma_{0,1,n-2} = \Gamma_{0,1} \cap \Gamma_{0,n-2}$. Hence, $\Gamma_0$ is a string $C$-group. The subgroup $\Gamma_{0,n-2}$ is a subgroup of a $C_2 \times S_{n-3}$, clearly maximal in $\Gamma_0$. So a similar argument as above shows that $\Gamma$ is a string $C$-group provided $\Gamma_{n-2}$ is. Now, $\Gamma_{n-2}$ is a string $C$-group if $n = 7$. This can be checked by hand or using MAGMA. Using induction, we see that $\Gamma_{n-2}$ is a string $C$-group for every $n > 6$. Hence $\Gamma$ is a string $C$-group. It has the Schlafli symbol given in the statement of the theorem. It is shown to be unique by Lemma 20. □

6. Polytopes of rank $3, \ldots, n-1$

Lemma 21. Let $n \geq 5$ and $3 \leq r \leq n-2$; There exists a string $C$-group $\Gamma := (S_n; \{\rho_0, \ldots, \rho_{r-1}\})$ with Schlafli symbol $(n-r+2, 6, 3^{r-3})$.

Proof. Let $3 \leq r \leq n-2$. We construct a graph $\mathcal{G}^{n,r}$ from the CPR graph of the $r$-simplex $S_{0,1,\ldots,r-1}$, by adding to it $n-r$ vertices and $n-r$ edges with labels 0 and 1 alternatively.

![Diagram](image)

Let us prove that this graph corresponds to a string $C$-group generating $S_n$.

Assume $n = 5$ and therefore $r = 3$. It is obvious to see that $\Gamma := (S_5; \{(2, 3), (1, 2)(3, 4), (4, 5)\})$ is a string $C$-group of Schlafli symbol $(4, 6, 3)$.

Suppose, by induction, that the graph $\mathcal{G}^{n,r}$ corresponds to a string $C$-group for every $3 \leq r \leq n-2$. Consider the graph obtained from the $r$-simplex, $3 \leq r \leq n-1$ adding $n+1-r$ vertices and $n+1-r$ edges with labels 0 and 1 alternatively. Let $\Gamma := (S_{n+1}; \{\rho_0, \ldots, \rho_{r-1}\})$. By induction $\Gamma_{r-1}$ is a string $C$-group, hence $\Gamma_{0,r-1}$ is also a string $C$-group. The group $\Gamma_{0,1}$ is isomorphic to $S_{r-2}$ and is the string $C$-group of an $(r-3)$-simplex. It remains to prove that $\Gamma_{0,1} \cap \Gamma_{0,r-1} = \Gamma_{0,1,r-1}$ and $\Gamma_{0} \cap \Gamma_{r-1} = \Gamma_{0,r-1}$. For the first equality, observe that $\Gamma_{0,r-1}$ fixes
vertex \((n + 1)\). The permutations in \(\Gamma_{0,1}\) fixing \((n + 1)\) are in \(\Gamma_{0,1,r-1}\). For the other equality we use the same argument.

To get the Schläfli symbol, it suffices to observe that \(\rho_0\rho_1\) is a cycle of length \(n - r + 2\), \(\rho_1\rho_2\) is a product of transpositions and a 3-cycle and \(\rho_i\rho_{i+1}\) with \(2 < i < r - 2\) is a 3-cycle. \(\square\)

**Proof of Theorem 3.** This is a direct consequence of Theorem 1 and Lemma 21. \(\square\)

7. Final remarks

Theorem 3 might seem obvious but we want to point out that, recently, the authors, together with Mark Mixer, found an example of a group, namely \(A_{11}\), that has polytopes of rank 3 and 6, but none of rank 4 or 5 [6].

As pointed out to us by Mark Mixer, the proof of Theorem 2 never makes any special assumption about \(G\) being \(S_n\), indeed only the fact that the group is not decomposable into a direct product is used. Therefore, it gives the following more general result.

**Theorem 22.** Let \(\Gamma\) be a subgroup of \(S_n\) with \(n \geq 7\). If \(\Gamma\) is the automorphism group of an abstract regular polytope of rank \(n - 2\), with a connected Coxeter diagram, then \(\Gamma\) is isomorphic to \(S_n\).

Particularly, we have that:

**Corollary 23.** The group \(\text{Alt}(n)\) with \(n \geq 7\) has no abstract regular polytope of rank \(n - 2\).

Observe that \(\text{Alt}(5)\) has rank 3 regular polyhedra.

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