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Unbounded Toeplitz Operators in the Segal–Bargmann Space, II*

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The paper mostly deals with the questions of closedness and essential selfadjointness of Toeplitz operators in the Segal–Bargmann space. General criteria for their closedness are formulated. Examples of unclosed Toeplitz operators are given. Explicit formulas for their adjoints are shown for various classes of symbols. The problem of whether polynomials and exponents form cores for Toeplitz operators is investigated. The results presented here improve and extend the ones from an earlier paper.

INTRODUCTION

The theory of unbounded Toeplitz operators in the Segal–Bargmann space $\mathscr{B}_n$ has been initiated by Berezin [3, 4] and Shapiro and Newman [13]. These operators seem to be interesting in view of their unitary equivalence to some pseudo-differential operators in $L^2(\mathbb{R}^n)$ (see [1 and 9]).

In the present paper we study three main topics: the explicit form of the adjoint, the closedness, and the (essential) selfadjointness of Toeplitz operators T_φ in $\mathscr{B}_n$. The problem of computing the adjoint of T_φ is nontrivial; in particular, the equality $T_\varphi^* = T_{\bar{\varphi}}$ does not hold even for very simple (antiholomorphic) symbols φ (see Example 6.3). The first explicit description of T_φ^* has been given by Newman and Shapiro in the case in which φ is an exponential polynomial (cf. [13]). In this paper we add a few

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contributions to this problem, considering not necessarily entire functions φ; roughly speaking, they are either holomorphic in z and $\bar{z}$ or t-radially symmetric ($t \in \mathbb{R}_+^n$). We obtain especially satisfactory results in the case of one variable. Applying the formula for T_φ^* we are able to answer the related question: when do polynomials or exponents form a core for T_φ?

According to our knowledge of the subject, the question of the closedness of Toeplitz operators T_φ in $\mathscr{B}_n$ has not been studied yet. Here we present the general criteria for their closedness, which turn out to be especially effective for t-radially symmetric symbols φ. In particular, we give examples of t-radially symmetric symbols φ inducing unclosed T_φ. To our surprise the above criteria enable us to prove that T_φ is unclosed for $\varphi(z) = \mathrm{Re}(z^2)$, $z \in \mathbb{C}$ (see Example 6.2). In the case of one variable we also apply the method of Fourier coefficients to solve the question of closedness of T_φ for φ's which are holomorphic in variables z and $\bar{z}$.

The first attempt to find criteria which guarantee the selfadjointness of T_φ has been made by Berezin in [4] for φ bounded from below. The first-named author has also considered this question in [11] for φ not necessarily bounded from below. In the present paper we show that the class of symbols inducing (essentially) selfadjoint Toeplitz operators contains t-radially symmetric functions with $t \in \mathbb{R}_+^n$ and squares of moduli of polynomials in one complex variable. Unfortunately we do not know whether $T_{|p|^2}$ is (essentially) selfadjoint for any polynomial p in several complex variables. Some partial solutions of this problem are presented in Section 5. As a byproduct we obtain examples of selfadjoint Jacobi matrices which do not satisfy known criteria for selfadjointness due to Carleman and Berezanskii [2].

1. Preliminaries

Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces. Throughout this paper, by a *linear operator* from $\mathscr{H}$ into $\mathscr{K}$ we understand a linear mapping $T\colon \mathscr{D}(T) \to \mathscr{K}$ defined on a linear subspace $\mathscr{D}(T)$ of $\mathscr{H}$. $\mathscr{D}(T)$ is called the *domain* of T. As usual, $\bar{T}$ and T^* stand for the closure and the adjoint of T, respectively. We say that a linear subspace $\mathscr{D}$ of $\mathscr{H}$ is a *core* for T if $\mathscr{D} \subseteq \mathscr{D}(T)$ and $T \subseteq (T|_{\mathscr{D}})^-$. An operator T in $\mathscr{H}$ is said to be *essentially selfadjoint* if its closure is selfadjoint.

In what follows we will use the standard multiindex notations:

$$k! = k_1! \cdot \cdots \cdot k_n!,\ |k| = k_1 + \cdots + k_n,\ z^k = z_1^{k_1} \cdot \cdots \cdot z_n^{k_n},$$

$$D^k = D_1^{k_1} \cdot \cdots \cdot D_n^{k_n} \qquad (k = (k_1, ..., k_n) \in \mathbb{N}^n,\ z = (z_1, ..., z_n) \in \mathbb{C}^n),$$

where $D_j = \partial/\partial z_j$ and $\mathbb{N} = \{0, 1, 2, ...\}$. Given a polynomial $p(z) = \sum a_k z^k$, we write $p^*(z) = \sum \bar{a}_k z^k$ and $p(D) = \sum a_k D^k$. The translation operator E_z acts on functions $f\colon \mathbb{C}^n \to \mathbb{C}$ as follows:

$$(E_z f)(w) = f(z+w), \qquad z, w \in \mathbb{C}^n.$$

Let $L^2(\mu)$ be the Hilbert space of all complex Borel functions on $\mathbb{C}^n$ which are square integrable with respect to the Gaussian measure μ defined by $d\mu(z) = \pi^{-n} e^{-\|z\|^2}\, dV(z)$, where $\|z\|^2 = |z_1|^2 + \cdots + |z_n|^2$ and V is the Lebesgue measure on $\mathbb{C}^n$. $L^2(\mu)$ is equipped with the usual norm $\|f\|^2 = \int |f|^2\, d\mu$. Denote by $\mathscr{B} = \mathscr{B}_n$ the Segal–Bargmann space of all entire functions, which belong to $L^2(\mu)$. The canonical orthonormal basis $\{f_k : k \in \mathbb{N}^n\}$ of $\mathscr{B}$ is given by

$$f_k(z) = z^k/\sqrt{k!}, \qquad z \in \mathbb{C}^n, k \in \mathbb{N}^n.$$

The function $e_a(z) = e^{\langle z, a\rangle}$, where $\langle z, a\rangle = z_1\bar{a}_1 + \cdots + z_n\bar{a}_n$, is the reproducing kernel for $\mathscr{B}$. Let $\mathscr{E}$ be the linear span of $\{e_z : z \in \mathbb{C}^n\}$. Denote by $\mathscr{P} = \mathscr{P}_n$ the set of all holomorphic polynomials in n complex variables and by $\mathscr{A}$ the algebra generated by $\mathscr{P} \cup \mathscr{E}$. It is clear that $\mathscr{P} \subseteq \mathscr{A} \subseteq \mathscr{B}$.

The following lemma will be useful in the sequel.

LEMMA 1.1. *If $r\colon \mathbb{C}^n \to \mathbb{C}$ is a Borel function such that $re_z \in L^1(\mu)$ for every $z \in \mathbb{C}^n$, then the function φ defined by*

$$\varphi(z) = \int r\bar{e}_z\, d\mu, \qquad z \in \mathbb{C}^n,$$

is entire and

$$D^k\varphi(z) = \int r(w)\, \overline{w^k e_z(w)}\, d\mu(w), \qquad z \in \mathbb{C}^n.$$

Proof. First we show that for every $k \in \mathbb{N}^n$ and for every $R > 0$, there exist $c > 0$ and $a_1, ..., a_s \in \mathbb{C}^n$ such that

$$|w^k e_z(w)| \leqslant c \sum_{j=1}^{s} |e_{a_j}(w)|, \qquad w \in \mathbb{C}^n, \|z\| \leqslant R. \tag{1.1}$$

Since the linear span of $\{|e_a| : a \in \mathbb{C}^n\}$ is an algebra, we can assume without loss of generality that $n = 1$. Then we have

$$\begin{aligned} |w^k \exp(\bar{z}w)| &\leqslant (|\mathrm{Re}\, w| + |\mathrm{Im}\, w|)^k \exp(\mathrm{Re}\, z\, \mathrm{Re}\, w + \mathrm{Im}\, z\, \mathrm{Im}\, w) \\ &= \sum_{j=0}^{k} \binom{k}{j} |\mathrm{Re}\, w|^j \exp(\mathrm{Re}\, z\, \mathrm{Re}\, w)\, |\mathrm{Im}\, w|^{k-j} \\ &\quad \times \exp(\mathrm{Im}\, z\, \mathrm{Im}\, w), \qquad w, z \in \mathbb{C}. \end{aligned}$$

If $|z| \leqslant R$, then

$$|\mathrm{Re}\, w|^j \exp(\mathrm{Re}\, z\, \mathrm{Re}\, w) \leqslant j! \exp(|\mathrm{Re}\, w| + \mathrm{Re}\, z\, \mathrm{Re}\, w)$$
$$\leqslant j! \exp((1+R)\, |\mathrm{Re}\, w|)$$
$$= \begin{cases} j!\, |\exp((1+R)\, w)| & \text{if } \mathrm{Re}\, w > 0 \\ j!\, |\exp(-(1+R)\, w)| & \text{if } \mathrm{Re}\, w \leqslant 0, \end{cases} \qquad w \in \mathbb{C},$$

so

$$|\mathrm{Re}\, w|^j \exp(\mathrm{Re}\, z\, \mathrm{Re}\, w) \leqslant j!(|e_{1+R}(w)| + |e_{-(1+R)}(w)|), \qquad w \in \mathbb{C},\ |z| \leqslant R,$$

which proves (1.1).

For our r define $f(z, w) := e_z(w)\, r(w)$, $z, w \in \mathbb{C}^n$. Then, by (1.1), we have

$$|D_z^k f(z, w)| = |\bar{w}^k e_z(w)\, r(w)| \leqslant c \sum_{j=1}^{s} |r(w)\, e_{a_j}(w)|, \qquad w \in \mathbb{C}^n,\ \|z\| \leqslant R.$$

Since the right hand side of the above inequality belongs to $L^1(\mu)$, the conclusion follows from the theorem on differentiation under the integral sign. This completes the proof. ∎

Given a Borel function $\varphi: \mathbb{C}^n \to \mathbb{C}$, we denote by M_φ the operator of multiplication by φ in $L^2(\mu)$ (i.e., $\mathscr{D}(M_\varphi) = \{f \in L^2(\mu): \varphi \cdot f \in L^2(\mu)\}$ and $M_\varphi f = \varphi \cdot f$ for $f \in \mathscr{D}(M_\varphi)$). By *Toeplitz* (resp., *Hankel*) *operator with symbol* φ we mean the operator T_φ (resp., H_φ) defined in $\mathscr{B}$ by $\mathscr{D}(T_\varphi) = \mathscr{D}(H_\varphi) = \mathscr{B} \cap \mathscr{D}(M_\varphi)$ and $T_\varphi f = PM_\varphi f$ (resp., $H_\varphi f = (I-P)\, M_\varphi f$) for $f \in \mathscr{D}(T_\varphi)$, where P is the orthogonal projection of $L^2(\mu)$ onto $\mathscr{B}$ and I is the identity operator.

Recall definitions of related operators Π_φ and $\tilde{T}_\varphi$ (cf. [11]). The domain of Π_φ consists of all $f \in \mathscr{B}$ such that the integral $\Pi_\varphi f(z) := \int \varphi(a)\, f(a)\, e^{\langle z, a \rangle}\, d\mu(a)$ exists for each $z \in \mathbb{C}^n$ and $\Pi_\varphi f \in \mathscr{B}$. In turn, the domain of $\tilde{T}_\varphi$ is defined as the set of all $f \in \mathscr{B}$ for which there exist $h \in \mathscr{B}$ and a Borel function r such that $\varphi \cdot f = h + r$, the integral $\int r\bar{p}\, d\mu$ exists and vanishes for every $p \in \mathscr{P}$; then we put $\tilde{T}_\varphi f := h$.

The following two basic facts concerning Toeplitz operators will be used several times in this paper.

THEOREM 1.A. [13, Theorem 3]. *If $p \in \mathscr{P}$ and $f \in \mathscr{D}(T_p)$, then*

$$\|T_p f\|^2 = \sum_{j \geqslant 0} \frac{1}{j!} \|(D^j p^*)(D)\, f\|^2.$$

PROPOSITION 1.B. [11, Prop. 1.4]. *If $\mathscr{P} \subseteq \mathscr{D}(T_\varphi)$ (resp., $\mathscr{E} \subseteq \mathscr{D}(T_\varphi)$), then $(T_\varphi|_{\mathscr{P}})^* = \tilde{T}_{\bar{\varphi}}$ (resp., $(T_\varphi|_{\mathscr{E}})^* = \Pi_{\bar{\varphi}}$).*

Observe that $\mathscr{P} \subseteq \mathscr{D}(T_\varphi)$ (resp., $\mathscr{E} \subseteq \mathscr{D}(T_\varphi)$) if and only if $\varphi(\cdot)\,\|\cdot\|^k \in L^2(\mu)$ for every $k \geqslant 1$ (resp. $\varphi(\cdot)\exp(a\|\cdot\|) \in L^2(\mu)$ for every $a>0$). It may happen that $\mathscr{D}(T_\varphi)$ contains $\mathscr{P}$ but not $\mathscr{E}$ (e.g., $\varphi(z) := \exp((|z|^2 - |\mathrm{Re}\, z| - |\mathrm{Im}\, z|)/2)$, $z \in \mathbb{C}$).

As in the case of bounded Toeplitz operators [7], unbounded ones are preserved by the action of the unitary group $\{W_{U,a}\}$:

$$W_{U,a} f := \exp(-\|a\|^2/2) \cdot e_a \cdot E_{-a}(f \circ U^*), \qquad f \in L^2(\mu),$$

where $a \in \mathbb{C}^n$ and U is a unitary operator on $\mathbb{C}^n$. Below $W_{U,a}$ also stands for the restriction of $W_{U,a}$ to $\mathscr{B}$.

PROPOSITION 1.2. *Assume that $\varphi: \mathbb{C}^n \to \mathbb{C}$ is a Borel function, U is a unitary operator on $\mathbb{C}^n$, $a \in \mathbb{C}^n$, and $A = U + a$. Then*

$$W^*_{U,a} T_\varphi W_{U,a} = T_{\varphi \circ A}. \tag{i}$$

If $\mathscr{E} \subseteq \mathscr{D}(T_\varphi)$, then (i) *holds for operators of the form $(T_\rho|_{\mathscr{E}})^-$ and Π_ρ. If $\mathscr{P} \subseteq \mathscr{D}(T_\varphi)$ and $a=0$, then* (i) *holds for operators of the form $(T_\rho|_{\mathscr{P}})^-$ and $\tilde{T}_\rho$.*

Proof. Note that $W^*_{I,a} = W_{I,-a}$ and $W^*_{U,0} = W_{U^*,0}$. Since $W_{U,a} = W_{I,a} W_{U,0}$ and $W_{U,a}(\mathscr{B}) \subseteq \mathscr{B}$, the space $\mathscr{B}$ reduces $W_{U,a}$ and consequently $PW_{U,a} = W_{U,a}P$. This and the equality

$$W^*_{U,a} M_\varphi W_{U,a} = M_{\varphi \circ A}$$

imply (i) (a simple verification is left for the reader).

Assume now that $\mathscr{E} \subseteq \mathscr{D}(T_\varphi)$. Then $\mathscr{E} \subseteq \mathscr{D}(T_{\bar{\varphi} \circ A})$. Since $W_{U,a}(\mathscr{E}) = \mathscr{E}$ and (i) holds for $\bar{\varphi}$, we have $W^*_{U,a}(T_{\bar{\varphi}}|_{\mathscr{E}})\, W_{U,a} = T_{\bar{\varphi} \circ A}|_{\mathscr{E}}$. Hence taking closures (resp., taking adjoints and applying Proposition 1.B), we obtain (i) with $(T_\rho|_{\mathscr{E}})^-$ (resp., Π_ρ) in place of T_ρ. Since $W_{U,0}(\mathscr{P}) = \mathscr{P}$, similar arguments apply to the case $\mathscr{P} \subseteq \mathscr{D}(T_\varphi)$ and $a=0$. This completes the proof. ∎

It has been proved in [11] that if φ is entire, then $T_\varphi = \Pi_\varphi = \tilde{T}_\varphi$. In general we have only inclusions $T_\varphi \subseteq \Pi_\varphi \subseteq \tilde{T}_\varphi$. Below we show that the last inclusion turns into the equality for a large class of φ's containing $\mathscr{A}$. On the other hand, (contrary to the bounded case) we have only the inclusion $T_{\bar{\varphi}} \subseteq T^*_\varphi$ which can be strict even for very simple symbols.

THEOREM 1.3. *Let φ be a complex Borel function on $\mathbb{C}^n$.*

(i) *If $\mathscr{E} \subseteq \mathscr{D}(T_\varphi)$, then $\mathscr{A} \subseteq \mathscr{D}(T_\varphi)$, $\Pi_\varphi = \tilde{T}_\varphi$, and $(T_\varphi|_{\mathscr{P}})^- = (T_\varphi|_{\mathscr{E}})^- = (T_\varphi|_{\mathscr{A}})^-$.*

(ii) *If $\mathscr{P} \subseteq \mathscr{D}(T_\varphi)$ and $\bar{T}_\varphi = \tilde{T}_\varphi$, then $T^*_\varphi = \bar{T}_{\bar{\varphi}} = (T_{\bar{\varphi}}|_{\mathscr{P}})^-$ and $T^*_{\bar{\varphi}} = \bar{T}_\varphi$.*

(iii) *If* $\mathscr{P} \subseteq \mathscr{D}(T_\varphi)$ *and* $|\varphi(e^{i\vartheta}z)| \leqslant c + d \cdot |\varphi(z)|$ *for all* $\vartheta \in \mathbb{R}$ *and* $\|z\|$ *sufficiently large* $(c, d \geqslant 0)$, *then* $\bar{T}_\varphi = (T_\varphi|_{\mathscr{P}})^-$.

Proof. (i) Assume that $\mathscr{E} \subseteq \mathscr{D}(T_\varphi)$. It follows from (1.1) that $\mathscr{A} \subseteq \mathscr{D}(T_\varphi)$. To prove the equality $\tilde{T}_\varphi = \Pi_\varphi$ take $f \in \mathscr{D}(\tilde{T}_\varphi)$ and put $g := \tilde{T}_\varphi f$. Then there exists a Borel function $r: \mathbb{C}^n \to \mathbb{C}$ such that

$$\varphi f = g + r \tag{1.2}$$

$$\int r(w)\, \bar{w}^k \, d\mu(w) = 0, \qquad k \in \mathbb{N}^n. \tag{1.3}$$

Multiplying both sides of (1.2) by $\bar{e}_z$ we get $r\bar{e}_z = \varphi \bar{e}_z f - g\bar{e}_z \in L^1(\mu)$. Define the function h on $\mathbb{C}^n$ by $h(z) = \int r\bar{e}_z \, d\mu$, $z \in \mathbb{C}^n$. It follows from Lemma 1.1 and (1.3) that

$$D^k h(0) = \int r(w)\, \bar{w}^k \, d\mu(w) = 0, \qquad k \in \mathbb{N}^n.$$

Since h is holomorphic, h must vanish. Consequently,

$$\int r\bar{e}_z \, d\mu = 0, \qquad z \in \mathbb{C}^n. \tag{1.4}$$

Combining (1.2) and (1.4) we obtain that $f \in \mathscr{D}(\Pi_\varphi)$ and $\Pi_\varphi f = g = \tilde{T}_\varphi f$. Consequently $\tilde{T}_\varphi = \Pi_\varphi$.

Now we show that $\Pi_{\bar{\varphi}} = (T_\varphi|_{\mathscr{A}})^*$. Take $f \in \mathscr{D}(\Pi_{\bar{\varphi}})$. Then $\bar{\varphi} f = \Pi_{\bar{\varphi}} f + r$, where $\int r\bar{e}_z \, d\mu = 0$ for all $z \in \mathbb{C}^n$. Differentiating under the integral sign the last equality (apply Lemma 1.1), we infer that $\int r\bar{h} \, d\mu = 0$ for all $h \in \mathscr{A}$. Thus

$$(T_\varphi h, f) = \int h\overline{\bar{\varphi} f} \, d\mu = \int h\overline{\Pi_{\bar{\varphi}} f} \, d\mu + \int h\bar{r} \, d\mu = (h, \Pi_{\bar{\varphi}} f), \qquad h \in \mathscr{A}.$$

This implies that $f \in \mathscr{D}((T_\varphi|_{\mathscr{A}})^*)$ and $(T_\varphi|_{\mathscr{A}})^* f = \Pi_{\bar{\varphi}} f$. Hence $\Pi_{\bar{\varphi}} \subseteq (T_\varphi|_{\mathscr{A}})^*$.

The reverse inclusion follows from Proposition 1.B. Finally, taking adjoints in the equalities $\tilde{T}_{\bar{\varphi}} = \Pi_{\bar{\varphi}} = (T_\varphi|_{\mathscr{A}})^*$ and applying Proposition 1.B, we obtain the remaining part of (i).

(ii) Assume that $\mathscr{P} \subseteq \mathscr{D}(T_\varphi)$ and $\bar{T}_\varphi = \tilde{T}_\varphi$. Then, by Proposition 1.B, we have

$$T_\varphi^* = \tilde{T}_\varphi^* = (T_{\bar{\varphi}}|_{\mathscr{P}})^{**} = (T_{\bar{\varphi}}|_{\mathscr{P}})^- \subseteq \bar{T}_{\bar{\varphi}} \subseteq T_\varphi^*,$$

so $T_\varphi^* = \bar{T}_{\bar{\varphi}}$. The equality $T_{\bar{\varphi}}^* = \bar{T}_\varphi$ follows by taking adjoints.

(iii) Put $\psi = 1 + |\varphi|^2$. By Theorem 7.2 in [4], any entire function $f \in L^2(\psi \, d\mu)$ can be approximated by polynomials in the $L^2(\psi \, d\mu)$-norm.

This is equivalent to $M_\varphi|_{\mathscr{A}} = (M_\varphi|_{\mathscr{P}})^-$. However, $\mathscr{D}(T_\varphi) = \mathscr{D}(M_\varphi|_{\mathscr{A}})$ and $\|T_\varphi f\| \leqslant \|M_\varphi f\|$, so $\bar{T}_\varphi = (T_\varphi|_{\mathscr{P}})^-$. This completes the proof. ∎

Shapiro and Newman have proved in [13] that the equality $T_\varphi^* = \Pi_{\bar{\varphi}}$ holds for $\varphi \in \mathscr{A}$. Our next goal is to show that it holds for some nonholomorphic symbols φ and that $\Pi_{\bar{\varphi}}$ can be described in terms of differentiation and translation operations for $\varphi \in \mathscr{A}$.

Below, $\sum_u p_u^*(D)\, E_u$ (p_u are polynomials vanishing for all but a finite number of u's in $\mathbb{C}^n$) is understood as a linear operator in $\mathscr{B}$ with the domain

$$\mathscr{D}\left(\sum_u p_u^*(D)\, E_u\right) = \left\{f \in \mathscr{B} \colon \sum_u p_u^*(D)\, E_u f \in \mathscr{B}\right\}.$$

THEOREM 1.4. (i) *If φ is an entire function on $\mathbb{C}^n$ and $\mathscr{P} \subseteq \mathscr{D}(T_\varphi)$, then $T_\varphi^* = \bar{T}_{\bar{\varphi}} = (T_{\bar{\varphi}}|_{\mathscr{P}})^-$ and $T_{\bar{\varphi}}^* = T_\varphi$.*

(ii) *If $\varphi = \psi\bar{\chi}$, where $\psi, \chi \in \mathscr{A}$, then $T_\varphi^* = \Pi_{\bar{\varphi}} = \tilde{T}_{\bar{\varphi}}$ and $\bar{T}_\varphi = (T_\varphi|_{\mathscr{E}})^- = (T_\varphi|_{\mathscr{P}})^-$.*

(iii) *If $\varphi = \sum_u p_u e_u \in \mathscr{A}$, then $T_\varphi^* = \Pi_{\bar{\varphi}} = \sum_u p_u^*(D)\, E_u$ and $T_{\bar{\varphi}}^* = T_\varphi = (T_\varphi|_{\mathscr{P}})^-$.*

Proof. (i) Notice first that T_φ is closed and $T_\varphi = \tilde{T}_\varphi$ (cf. [11, Prop. 1.2]). So (i) follows from Theorem 1.3(ii).

(ii) Note that $\rho := \psi\chi \in \mathscr{A}$. It has been proved in [13, p. 369] that $\int \|f \cdot E_w \rho\| \cdot e^{-\|w\|^2/2}\, dV(w) < \infty$ for every $f \in \mathscr{D}(T_\rho) = \mathscr{D}(T_\varphi)$. By repeating the first part of the proof of Theorem 2.2 in [11] (see also [13, p. 367]), we have

$$(T_\varphi f, g) = (f, \Pi_{\bar{\varphi}}\, g), \qquad f \in \mathscr{D}(T_\varphi),\ g \in \mathscr{D}(\Pi_{\bar{\varphi}}).$$

This gives us the desired equality $T_\varphi^* = \Pi_{\bar{\varphi}}$. The equality $\Pi_{\bar{\varphi}} = \tilde{T}_{\bar{\varphi}}$ follows from Theorem 1.3(i).

Taking adjoints in the equalities $T_\varphi^* = \Pi_{\bar{\varphi}} = \tilde{T}_{\bar{\varphi}}$ and applying Proposition 1.B, we obtain the remaining equalities in (ii).

(iii) The relation $T_\varphi^* = \Pi_{\bar{\varphi}}$ follows from (ii). On the other hand, the equality $\Pi_{\bar{\varphi}} = \sum_u p_u^*(D)\, E_u$ can be derived from the following identities

$$\begin{aligned}\int f\overline{\varphi e_z}\, d\mu &= \sum_u \int \bar{p}_u f\bar{e}_{z+u}\, d\mu = \sum_u p_u^*(D) \int f\bar{e}_{z+u}\, d\mu \\ &= \left(\sum_u p_u^*(D)\, E_u f\right)(z), \qquad z \in \mathbb{C}^n,\ f \in \mathscr{B}.\end{aligned}$$

The rest of (iii) is an immediate consequence of (i) and (ii). This completes the proof. ∎

2. Closedness

It follows from Proposition 1.B that operators of the form Π_φ and $\tilde{T}_\varphi$ are closed provided their domains contain reproducing kernels and polynomials, respectively. As we will see below, the criteria for closedness of T_φ are quite involved, even in the case of a single variable. Our main interest in this section is to find necessary and sufficient conditions for T_φ to be closed.

To begin with, let us consider the abstract situation. Suppose we are given a linear operator A from a Hilbert space $\mathscr{H}$ into another space $\mathscr{K}$. Let $\|\cdot\|_A$ be the graph norm of A given by

$$\|f\|_A^2 = \|f\|^2 + \|Af\|^2, \qquad f \in \mathscr{D}(A).$$

It is well-known that A is closed if and only if $(\mathscr{D}(A), \|\cdot\|_A)$ is a Hilbert space.

The following lemma plays the crucial role in this paper.

Lemma 2.1. *Let $\mathscr{H}$, $\mathscr{K}$, and $\mathscr{L}$ be Hilbert spaces and let $A\colon \mathscr{D}(A) \to \mathscr{K}$, $B\colon \mathscr{D}(B) \to \mathscr{L}$ be linear operators defined in $\mathscr{H}$. Assume that $\mathscr{D}$ is a core for A, $\mathscr{D}(A) = \mathscr{D}(B)$, and B is closed. Then A is closed iff there are $\alpha, \beta \geqslant 0$ such that*

$$\|f\|_B \leqslant \beta \|f\|_A, \qquad f \in \mathscr{D}, \tag{2.1}$$

$$\|f\|_A \leqslant \alpha \|f\|_B, \qquad f \in \mathscr{D}. \tag{2.2}$$

If A is closed, then $\mathscr{D}$ is a core for B. If additionally

$$\|Af\| \leqslant \|Bf\|, \qquad f \in \mathscr{D}, \tag{2.3}$$

then A is closed iff (2.1) *holds with some $\beta \geqslant 0$.*

Proof. Put $\mathscr{X} := \mathscr{D}(A) = \mathscr{D}(B)$. Note that the identity operator on $\mathscr{X}$ is closed as an operator from $(\mathscr{X}, \|\cdot\|_A)$ into $(\mathscr{X}, \|\cdot\|_B)$ as well as the one from $(\mathscr{X}, \|\cdot\|_B)$ into $(\mathscr{X}, \|\cdot\|_A)$. Thus if A is closed, then $(\mathscr{X}, \|\cdot\|_A)$ and $(\mathscr{X}, \|\cdot\|_B)$ are Hilbert spaces and consequently—by the closed graph theorem—(2.1) and (2.2) hold.

Assume now that (2.1) and (2.2) hold. We claim that the inequalities in (2.1) and (2.2) hold for every $f \in \mathscr{X}$. Indeed, if $f \in \mathscr{X}$, then there exists a sequence $\{f_k\}_{k=0}^\infty \subseteq \mathscr{D}$ convergent to f in the norm $\|\cdot\|_A$. By (2.1) and the completeness of $(\mathscr{X}, \|\cdot\|_B)$, $\{f_k\}_{k=0}^\infty$ converges to f in the norm $\|\cdot\|_B$. Thus $\|f\|_A = \lim_{k\to\infty} \|f_k\|_A$ and $\|f\|_B = \lim_{k\to\infty} \|f_k\|_B$, which—in virtue of (2.1) and (2.2)—proves our claim. This in turn implies that the identity operator on $\mathscr{X}$ is a linear homeomorphism from $(\mathscr{X}, \|\cdot\|_A)$ onto $(\mathscr{X}, \|\cdot\|_B)$.

Since $(\mathscr{X}, \|\cdot\|_B)$ is complete, so is $(\mathscr{X}, \|\cdot\|_A)$ and consequently A is closed. Moreover $\mathscr{D}$, being $\|\cdot\|_A$-dense in $\mathscr{X}$, is also $\|\cdot\|_B$-dense in $\mathscr{X}$. This means that $\mathscr{D}$ is a core for B.

If (2.3) holds, then the inequality (2.2) is satisfied with $\alpha = 1$, which completes the proof. ▌

There are several possibilities of applying Lemma 2.1 to Toeplitz operators. We start with one of them.

PROPOSITION 2.2. *Let φ be a complex Borel function on $\mathbb{C}^n$ and let $\mathscr{D}$ be a linear subspace of $\mathscr{D}(T_\varphi)$. If T_φ and $T_{\bar\varphi}$ are closed, then $T_\varphi = (T_\varphi|_{\mathscr{D}})^-$ iff $T_{\bar\varphi} = (T_{\bar\varphi}|_{\mathscr{D}})^-$. If φ is entire, $\mathscr{P} \subseteq \mathscr{D}(T_\varphi)$ and $T_{\bar\varphi}$ is closed, then $\mathscr{P}$ is a core for T_φ.*

Proof. The first part follows by applying Lemma 2.1 to $A = T_{\bar\varphi}$ and $B = T_\varphi$ or vice versa. The other is a consequence of the first part and the equality $T_{\bar\varphi} = (T_{\bar\varphi}|_{\mathscr{P}})^-$ (see Theorem 1.4(i)). ▌

The next application plays an essential role in our paper.

PROPOSITION 2.3. *If φ is a complex Borel function on $\mathbb{C}^n$ and $\mathscr{D}$ is a core for T_φ, then the following conditions are equivalent,*

(i) *T_φ is closed,*

(ii) *there exists $c \geq 0$ such that*

$$\|\varphi f\|^2 \leq c(\|f\|^2 + \|T_\varphi f\|^2), \qquad f \in \mathscr{D},$$

(iii) *there exists $d \geq 0$ such that*

$$\|H_\varphi f\|^2 \leq d(\|f\|^2 + \|T_\varphi f\|^2), \qquad f \in \mathscr{D}.$$

Proof. Since $M_\varphi|_{\mathscr{A}}$ is closed, equivalence (i) $\Leftrightarrow$ (ii) follows by applying Lemma 2.1 to $A = T_\varphi$ and $B = M_\varphi|_{\mathscr{A}}$ (note that $\|T_\varphi f\|^2 \leq \|\varphi f\|^2$ for $f \in \mathscr{D}(T_\varphi)$). On the other hand, equivalence (ii) $\Leftrightarrow$ (iii) is a consequence of the equality

$$\|\varphi f\|^2 = \|T_\varphi f\|^2 + \|H_\varphi f\|^2, \qquad f \in \mathscr{D}(T_\varphi).$$

This completes the proof. ▌

Note that by interchanging the operators H_φ and T_φ in Proposition 2.3, we get an analogous criterion for the closedness of Hankel operators.

COROLLARY 2.4. *If H_φ is bounded, then T_φ is closed. In particular, this is the case when there exists a constant $c \geq 0$ such that*

$$|\varphi(z) - \varphi(w)| \leq c e^{\|z-w\|^2/4}, \qquad z, w \in \mathbb{C}^n.$$

Proof. It follows from Proposition 6 in [10] and Proposition 2.3. ∎

Let φ be a polynomial in $z_1, ..., z_n$ and $\bar{z}_1, ..., \bar{z}_n$. It is clear that if φ is of degree 1, then the inequality in Corollary 2.4 holds and consequently T_φ is closed. In general, T_φ is not closed when φ is of degree greater than or equal to 2 (see Example 6.3). The question of closedness of T_φ is open even for $\varphi = |p|^2$, where $p \in \mathscr{P}_n$, $n > 1$. In the sequel we will need the following general criterion for closedness of such T_φ's.

PROPOSITION 2.5. *If $\varphi = |p|^2$, where $p \in \mathscr{P}$, then the following conditions are equivalent:*

(i) *T_φ is closed,*

(ii) *there exists $c > 0$ such that*

$$\|p^2 f\|^2 \leqslant c\,\|p^*(D)(pf)\|^2, \qquad f \in \mathscr{P},$$

(iii) *there exists $d > 0$ such that*

$$\|(D^j p^*)(D)(pf)\|^2 \leqslant d\,\|p^*(D)(pf)\|^2, \qquad f \in \mathscr{P}, j \in \mathbb{N}^n.$$

Proof. Assume that $p \neq 0$. By Theorem 1.4 and Proposition 2.3 ($\mathscr{D} = \mathscr{P}$), the operator T_φ is closed if and only if

$$\|p^2 f\|^2 \leqslant \varepsilon(\|f\|^2 + \|p^*(D)(pf)\|^2), \qquad f \in \mathscr{P}, \tag{2.4}$$

for some $\varepsilon > 0$. On the other hand (cf. [15, Lemma 4 and corollary on p. 523] and also [12]), there exists $\delta > 0$ such that

$$\|p^*(D)(pf)\|^2 \geqslant \delta\,\|f\|^2, \qquad f \in \mathscr{P}. \tag{2.5}$$

Combining (2.4) and (2.5) we get equivalence (i)$\Leftrightarrow$(ii). The other one, (ii)$\Leftrightarrow$(iii), follows from the equality (apply Theorem 1.A)

$$\|p^2 f\|^2 = \sum_{j \geqslant 0} \frac{1}{j!}\,\|(D^j p^*)(D)(pf)\|^2, \qquad f \in \mathscr{P}.$$

This completes the proof. ∎

3. TOEPLITZ OPERATORS IN $\mathscr{B}_1$

In Section 3 we investigate the case of one variable. This enables us to obtain the results which are stronger and not extendable to several variables. We start with the description of T_φ^*.

PROPOSITION 3.1. *If p is a polynomial in one complex variable, then* $T_p^* = T_p = \Pi_{\bar{p}} = \tilde{T}_{\bar{p}}$.

Proof. Since $T_p^* = \Pi_{\bar{p}} = \tilde{T}_{\bar{p}}$ (use Theorem 1.4(ii)), we only have to prove the equality $T_p^* = T_{\bar{p}}$.

Using the induction procedure we will show that $\mathscr{D}(T_p^*) = \mathscr{D}(D^m)$, where m is the degree of p. This is obvious for $m = 0$, 1. Assume that it holds for $m-1$ $(m \geqslant 2)$. Take $f \in \mathscr{D}(T_p^*) = \mathscr{D}(T_{p-p(0)}^*)$. There exists a polynomial q such that $p(z) - p(0) = zq(z)$, $z \in \mathbb{C}$. Since $D(q^*(D)f) = (p^*(D) - p^*(0))f = T_{p-p(0)}^* f \in \mathscr{B}_1$ (use Theorem 1.4(iii)), $q^*(D)f$ must belong to $\mathscr{B}_1$ (this is because $Dh \in \mathscr{B}_1$ implies $h \in \mathscr{B}_1$ for any entire h). Therefore, by the induction assumption, we have $f \in \mathscr{D}(q^*(D)) = \mathscr{D}(T_q^*) = \mathscr{D}(D^{m-1})$. In particular, $Df \in \mathscr{B}_1$. Since $q^*(D)\,Df = D(q^*(D)f) \in \mathscr{B}_1$, we obtain $Df \in \mathscr{D}(q^*(D)) = \mathscr{D}(D^{m-1})$, i.e., $f \in \mathscr{D}(D^m)$. Thus $\mathscr{D}(T_p^*) \subseteq \mathscr{D}(D^m)$. The reverse inclusion is a consequence of the equality $T_p^* = p^*(D)$. By a direct computation one can check that $\mathscr{D}(D^k) = \mathscr{D}(T_{Z^k})$ for $k \in \mathbb{N}$. Hence $\mathscr{D}(T_{\bar{p}}) = \mathscr{D}(T_p) \supseteq \mathscr{D}(T_{Z^m}) = \mathscr{D}(D^m) = \mathscr{D}(T_p^*) \supseteq \mathscr{D}(T_{\bar{p}})$ and consequently $T_p^* = T_{\bar{p}}$. This completes the proof. ∎

We are now in a position to state one of the main results of this section. Note that it cannot be extended to the case of several variables (see Example 6.3).

THEOREM 3.2. *Let p be a polynomial in one complex variable and let φ be an entire function on $\mathbb{C}$. Then*

(i) $$\tilde{T}_{\bar{p}\varphi} = \Pi_{\bar{p}\varphi} = T_{\bar{p}\varphi} = T_{\bar{p}} T_{\varphi}.$$

(ii) *If $\mathscr{P} \subseteq \mathscr{D}(T_{\bar{p}\varphi})$, then $T_{p\varphi}$ is closed and*

$$T_{\bar{p}\varphi}^* = \bar{T}_{p\bar{\varphi}} = (T_{p\bar{\varphi}}|_{\mathscr{P}})^{-}.$$

(iii) *If φ is a polynomial in one complex variable, then*

$$T_{\bar{p}\varphi}^* = T_{p\bar{\varphi}} = (T_{p\bar{\varphi}}|_{\mathscr{P}})^{-}.$$

Proof. (i) First we show that $\mathscr{D}(\tilde{T}_{\bar{p}\varphi}) \subseteq \mathscr{D}(T_{\varphi})$. Take $f \in \mathscr{D}(\tilde{T}_{\bar{p}\varphi})$. Then there exists a Borel function r such that

$$\bar{p}\varphi f = g + r \qquad (g = \tilde{T}_{\bar{p}\varphi} f), \tag{3.1}$$

$$\int r(z)\, \bar{z}^m \, d\mu(z) = 0, \qquad m \in \mathbb{N}. \tag{3.2}$$

Let $g=\sum_{j=0}^{\infty} c_j f_j$, $(\varphi f)(z)=\sum_{j=0}^{\infty} d_j f_j(z)$, and $p(z)=\sum_{j=0}^{N} p_j z^j$ for $z \in \mathbb{C}$ ($c_j, d_j, p_j \in \mathbb{C}$, $p_N \neq 0$, $N \geqslant 1$). Denote by B_R the disc $\{z \in \mathbb{C}: |z| < R\}$. Then for all $j, m \in \mathbb{N}$, we have

$$\begin{aligned}\int_{B_R} f_j(z)\,\overline{p(z)\,z^m}\,d\mu(z) &= \sum_{l=0}^{N} \bar{p}_l \int_{B_R} f_j(z)\,\overline{z^{l+m}}\,d\mu(z)\\ &= \frac{1}{\sqrt{j!}}\,\bar{p}_{j-m} \int_{B_R} |z|^{2j}\,d\mu(z).\end{aligned}$$

Since $\overline{pf_m}\varphi f = (g\bar{f}_m + r\bar{f}_m) \in L^1(\mu)$, the above equalities imply

$$\begin{aligned}\int (\varphi f)(z)\,\overline{p(z)\,z^m}\,d\mu(z) &= \lim_{R \nearrow \infty} \int_{B_R} (\varphi f)(z)\,\overline{p(z)\,z^m}\,d\mu(z)\\ &= \lim_{R \nearrow \infty} \sum_{j=0}^{\infty} d_j \int_{B_R} f_j(z)\,\overline{p(z)\,z^m}\,d\mu(z)\\ &= \lim_{R \nearrow \infty} \sum_{j=m}^{m+N} \frac{d_j}{\sqrt{j!}}\,\bar{p}_{j-m} \int_{B_R} |z|^{2j}\,d\mu(z)\\ &= \sum_{j=m}^{m+N} \frac{d_j}{\sqrt{j!}}\,\bar{p}_{j-m} \int |z|^{2j}\,d\mu(z)\\ &= \sum_{j=m}^{m+N} d_j\,\bar{p}_{j-m}\sqrt{j!}, \qquad m \in \mathbb{N}. \qquad (3.3)\end{aligned}$$

On the other hand, equalities (3.1) and (3.2) lead to

$$\begin{aligned}\int (\varphi f)(z)\,\overline{p(z)\,z^m}\,d\mu(z) &= \int g(z)\,\bar{z}^m\,d\mu(z)\\ &= \sqrt{m!}\,c_m, \qquad m \in \mathbb{N}. \qquad (3.4)\end{aligned}$$

Combining (3.3) and (3.4) we have

$$\sum_{j=m}^{m+N} d_j\,\bar{p}_{j-m}\sqrt{j!} = \sqrt{m!}\,c_m, \qquad m \in \mathbb{N}.$$

It follows that

$$d_{m+N}\,\bar{p}_N\sqrt{(m+N)!} = \sqrt{m!}\,c_m - \sum_{j=0}^{N-1} d_{m+j}\,\bar{p}_j\sqrt{(m+j)!}, \qquad m \in \mathbb{N}.$$

Hence

$$|d_{m+N}| \leqslant \frac{1}{|p_N| \sqrt{m+N}} \left[|c_m| + \left(\sum_{j=0}^{N-1} |p_j| \right) \max_{0 \leqslant j \leqslant N-1} |d_{m+j}| \right], \qquad m \in \mathbb{N}. \tag{3.5}$$

Put $\Gamma := |p_N|^{-1} (\|c\|_\infty + \sum_{j=0}^{N-1} |p_j|)$, where $\|c\|_\infty = \sup\{|c_j| : j \geqslant 0\}$. Let $\gamma_m := \max\{1, |d_m|, |d_{m+1}|, ..., |d_{m+N-1}|\}$ for $m \in \mathbb{N}$. Then (3.5) can be rewritten as follows

$$|d_{m+N}| \leqslant \frac{\Gamma}{\sqrt{m+N}} \gamma_m, \qquad m \in \mathbb{N}. \tag{3.6}$$

Choose m_0 such that $\Gamma \leqslant \sqrt{m+N}$ for $m \geqslant m_0$. Then (3.6) implies that

$$|d_{m+N}| \leqslant \gamma_m, \qquad m \geqslant m_0. \tag{3.7}$$

Now we show, using the induction procedure, that

$$|d_{m+s+N}| \leqslant \frac{\Gamma}{\sqrt{m+s+N}} \gamma_m, \qquad s \in \mathbb{N}, m \geqslant m_0. \tag{3.8}$$

The case $s = 0$ is a consequence of (3.6). Assume that (3.8) holds for a fixed $s \geqslant 0$ and for all $m \geqslant m_0$. The induction assumption and the inequality (3.7) give us

$$|d_{m+s+1+N}| \leqslant \frac{\Gamma}{\sqrt{m+s+1+N}} \gamma_{m+1} \leqslant \frac{\Gamma}{\sqrt{m+s+1+N}} \gamma_m, \qquad m \geqslant m_0,$$

which proves (3.8). In particular, we have

$$|d_m| \leqslant \frac{\tilde{\Gamma}}{\sqrt{m}}, \qquad m \geqslant m_0 + N,$$

where $\tilde{\Gamma} := \gamma_{m_0} \Gamma$. Thus

$$\max_{0 \leqslant j \leqslant N-1} |d_{m+j}| \leqslant \frac{\tilde{\Gamma}}{\sqrt{m}}, \qquad m \geqslant m_0 + N. \tag{3.9}$$

Choose $m_1 \geqslant m_0 + N$ such that $|p_N| \sqrt{m+N} \geqslant 1$ for $m \geqslant m_1$. Combining (3.5) and (3.9) we get

$$\begin{aligned} |d_{m+N}| &\leqslant \frac{1}{|p_N| \sqrt{m+N}} \left[|c_m| + \left(\sum_{j=0}^{N-1} |p_j| \right) \frac{\tilde{\Gamma}}{\sqrt{m}} \right] \\ &\leqslant |c_m| + \tilde{\Gamma} |p_N|^{-1} \left(\sum_{j=0}^{N-1} |p_j| \right) m^{-1}, \qquad m \geqslant m_1. \end{aligned}$$

Therefore the sequence $\{d_j\}_{j=0}^{\infty}$ is square summable, so $\varphi f \in \mathscr{B}_1$ or equivalently $f \in \mathscr{D}(T_\varphi)$. This completes the proof of the inclusion $\mathscr{D}(\tilde{T}_{\bar{p}\varphi}) \subseteq \mathscr{D}(T_\varphi)$. Since $\tilde{T}_{\bar{p}} T_\varphi = \tilde{T}_{\bar{p}\varphi}|_{\mathscr{D}}$, where $\mathscr{D} = \mathscr{D}(\tilde{T}_{\bar{p}\varphi}) \cap \mathscr{D}(T_\varphi)$, we have $\tilde{T}_{\bar{p}} T_\varphi = \tilde{T}_{\bar{p}\varphi}$. However, by Proposition 3.1 $\tilde{T}_{\bar{p}} = T_{\bar{p}}$, so

$$T_{\bar{p}\varphi} \subseteq \Pi_{p\varphi} \subseteq \tilde{T}_{\bar{p}\varphi} = T_{\bar{p}} T_\varphi \subseteq T_{\bar{p}\varphi},$$

which completes the proof of (i).

(ii) The closedness of $T_{\bar{p}\varphi}$ follows from (i) and Proposition 1.B while the equalities of (ii) are simple consequences of (i) and Theorem 1.3(ii).

Finally, (iii) can be easily derived from (ii). This completes the proof. ▮

Now we show that if $\varphi = p + \bar{q}$, where p and q are polynomials of one complex variable and $\deg p < \deg q$, then $T_\varphi = \tilde{T}_\varphi$ (consequently, T_φ is closed). This is no longer true for polynomials p and q of the same degree as well as for polynomials of several variables (even if $p = 0$).

THEOREM 3.3. *If p and q are polynomials in one complex variable and $\deg p < \deg q$, then*

$$\text{(i)} \quad T_{p+\bar{q}} = T_p + T_{\bar{q}} = \Pi_{p+\bar{q}} = \tilde{T}_{p+\bar{q}},$$

$$\text{(ii)} \quad T^*_{p+\bar{q}} = \tilde{T}_{\bar{p}+q} = (T_{\bar{p}+q}|_{\mathscr{P}})^{-}.$$

Proof. First note that (ii) follows from (i). Indeed, taking adjoints in (i), we obtain $T^*_{p+\bar{q}} = (T_{\bar{p}+q}|_{\mathscr{P}})^{-}$. On the other hand, $(T_{\bar{p}+q}|_{\mathscr{P}})^{-} \subseteq \tilde{T}_{\bar{p}+q} \subseteq T^*_{p+\bar{q}}$, so (ii) holds.

Since obviously $T_p + T_{\bar{q}} \subseteq T_{p+\bar{q}} \subseteq \Pi_{p+\bar{q}} \subseteq \tilde{T}_{p+\bar{q}}$, we have only to show that $\tilde{T}_{p+\bar{q}} \subseteq T_p + T_{\bar{q}}$. Let $p(z) = \sum_{s=0}^{N} p_s z^s$ and $q(z) = \sum_{s=0}^{M} q_s z^s$, $z \in \mathbb{C}$, where $N = \deg p$ and $M = \deg q$. Without loss of generality we may assume that $q_M = 1$.

Take $f = \sum_{j \geqslant 0} d_j f_j \in \mathscr{D}(\tilde{T}_{p+\bar{q}})$. The main step of the proof is based on the inequality

$$|d_m| \leqslant \alpha \sqrt{\frac{m!}{(m+\mathrm{M})!}}\, \delta_m, \qquad m \in \mathbb{N}, \tag{3.10}$$

where $\alpha > 0$ and $\{\delta_m\}_{m=0}^{\infty} \in l_2$. Its proof is divided into a few steps.

Since $f \in \mathscr{D}(\tilde{T}_{p+\bar{q}})$, there exists a Borel function $r\colon \mathbb{C} \to \mathbb{C}$ such that

$$(p+\bar{q}) f = g + r \qquad (g = \tilde{T}_{p+\bar{q}} f), \tag{3.11}$$

$$\int r \bar{f}_m \, d\mu = 0, \qquad m \in \mathbb{N}. \tag{3.12}$$

Let $g = \sum_{j \geq 0} c_j f_j$. Then the following equality holds for any $m \in \mathbb{N}$:

$$d_{m+M} = \sqrt{\frac{m!}{(m+M)!}} \left(c_m - \sum_{j=m-N}^{m} d_j p_{m-j} \sqrt{\frac{m!}{j!}} - \sum_{j=m}^{m+M-1} d_j \bar{q}_{j-m} \sqrt{\frac{j!}{m!}} \right). \tag{3.13}$$

Indeed, using (3.11) and (3.12) we get

$$\begin{aligned} c_m &= \int g \bar{f}_m \, d\mu = \int (p + \bar{q}) f \bar{f}_m \, d\mu = \lim_{R \nearrow \infty} \sum_{j \geq 0} d_j \int_{B_R} (p + \bar{q}) f_j \bar{f}_m \, d\mu \\ &= \lim_{R \nearrow \infty} \sum_{j \geq 0} d_j \left(\sum_{s=0}^{N} p_s \sqrt{\frac{(j+s)!}{j!}} \int_{B_R} f_{j+s} \bar{f}_m \, d\mu \right. \\ &\quad \left. + \sum_{s=0}^{M} \bar{q}_s \sqrt{\frac{(m+s)!}{m!}} \int_{B_R} f_j \bar{f}_{m+s} \, d\mu \right) \\ &= \lim_{R \nearrow \infty} \sum_{j \geq 0} d_j \left(p_{m-j} \sqrt{\frac{m!}{j!}} \int_{B_R} |f_m|^2 \, d\mu + \bar{q}_{j-m} \sqrt{\frac{j!}{m!}} \int_{B_R} |f_j|^2 \, d\mu \right) \\ &= \lim_{R \nearrow \infty} \left(\sum_{j=m-N}^{m} d_j p_{m-j} \sqrt{\frac{m!}{j!}} \int_{B_R} |f_m|^2 \, d\mu \right. \\ &\quad \left. + \sum_{j=m}^{m+M} d_j \bar{q}_{j-m} \sqrt{\frac{j!}{m!}} \int_{B_R} |f_j|^2 \, d\mu \right) \\ &= \sum_{j=m-N}^{m} d_j p_{m-j} \sqrt{\frac{m!}{j!}} + \sum_{j=m}^{m+M} d_j \bar{q}_{j-m} \sqrt{\frac{j!}{m!}}, \qquad m \in \mathbb{N}. \end{aligned}$$

This proves (3.13).

Let $\gamma_m := \max\{|d_{m-N}|, |d_{m-N+1}|, ..., |d_{m+M-1}|\}$. Since $M > N$, one can deduce from (3.13) that the inequality

$$|d_{m+M}| \leqslant \alpha_1 \left(\sqrt{\frac{m!}{(m+M)!}} |c_m| + \frac{\gamma_m}{\sqrt{m+M}} \right), \qquad m \in \mathbb{N}, \tag{3.14}$$

with $\alpha_1 = \sum_{j=0}^{N} |p_j| + \sum_{j=0}^{M-1} |q_j| + 1$, holds. This in turn implies that

$$|d_m| \leqslant \alpha_2 \frac{1}{\sqrt{m}}, \qquad m \geqslant M, \tag{3.15}$$

with $\alpha_2 = \alpha_1(\|c\|_\infty + \|d\|_\infty)$.

Now we show that for any $s = 1, 2, ..., M$, there exists a constant $\beta_s > 0$ such that

$$|d_m| \leqslant \frac{\beta_s}{\sqrt{(m+1)(m+2)\cdot \cdots \cdot (m+s)}}, \qquad m \in \mathbb{N}. \tag{3.16}$$

We follow the induction procedure. The case $s = 1$ is an immediate consequence of (3.15). Assume that (3.16) holds for $s \leqslant M - 1$. Based on (3.14) and the induction assumption, we get for $m \geqslant N$

$$\begin{aligned}|d_{m+M}| &\leqslant \alpha_1 \sqrt{\frac{m!}{(m+M)!}}\, |c_m| + \alpha_1 \frac{\gamma_m}{\sqrt{m+M}} \\ &\leqslant \frac{\alpha_1 \|c\|_\infty}{\sqrt{(m+1)\cdot \cdots \cdot (m+s+1)}} \\ &\quad + \frac{\alpha_1 \beta_s}{\sqrt{(m-N+1)\cdot \cdots \cdot (m-N+s+1)}}.\end{aligned}$$

Thus there exists a constant $\beta_{s+1} > 0$ such that

$$|d_m| \leqslant \frac{\beta_{s+1}}{\sqrt{(m+1)\cdot \cdots \cdot (m+s+1)}}, \qquad m \in \mathbb{N}.$$

In particular, (3.16) holds for $s = M$. Thus ($\alpha_3 := \beta_M$)

$$|d_m| \leqslant \frac{\alpha_3}{\sqrt{(m+1)(m+2)\cdot \cdots \cdot (m+M)}}, \qquad m \in \mathbb{N}. \tag{3.17}$$

Applying (3.14) and (3.17) we can find $\alpha_4 > 0$ such that

$$\begin{aligned}|d_{m+M}| &\leqslant \alpha_1 \sqrt{\frac{m!}{(m+M)!}}\, |c_m| + \alpha_1 \frac{\gamma_m}{\sqrt{m+M}} \\ &\leqslant \alpha_1 \sqrt{\frac{m!}{(m+M)!}}\, |c_m| + \frac{\alpha_1 \alpha_3}{\sqrt{(m-N+1)\cdot \cdots \cdot (m-N+M)(m+M)}} \\ &\leqslant \alpha_4 \sqrt{\frac{m!}{(m+M)!}} \left(|c_m| + \frac{1}{\sqrt{m+M}} \right), \qquad m \geqslant N.\end{aligned}$$

Hence for some $\alpha_5 > 0$ we have

$$|d_m| \leqslant \alpha_5 \sqrt{\frac{m!}{(m+M)!}} \left(|c_{m-M}| + \frac{1}{\sqrt{m}} \right), \qquad m \geqslant M.$$

This in turn implies that there exists a constant $\alpha_6 > 0$ such that

$$\gamma_m \leqslant \alpha_5 \sqrt{\frac{(m-N)!}{(m-N+M)!}} \left(\max\{|c_{m-N-M}|, ..., |c_{m-1}|\} + \frac{1}{\sqrt{m-N}} \right)$$

$$\leqslant \alpha_6 \sqrt{\frac{m!}{(m+M)!}} \left(\max\{|c_{m-N-M}|, ..., |c_{m-1}|\} + \frac{1}{\sqrt{m}} \right), \qquad (3.18)$$

for m sufficiently large. Using again (3.14) and (3.18) we can find a constant $\alpha_7 > 0$ such that

$$|d_{m+M}| \leqslant \alpha_1 \left(\sqrt{\frac{m!}{(m+M)!}} |c_m| + \frac{\gamma_m}{\sqrt{m+M}} \right)$$

$$\leqslant \alpha_1 \sqrt{\frac{m!}{(m+M)!}} \left(|c_m| + \frac{\alpha_6 \max\{|c_{m-N-M}|, ..., |c_{m-1}|\}}{\sqrt{m+M}} + \frac{\alpha_6}{\sqrt{m(m+M)}} \right)$$

$$\leqslant \alpha_7 \sqrt{\frac{m!}{(m+M)!}} \left(|c_{m-N-M}| + \cdots + |c_{m-1}| + |c_m| + \frac{1}{m+M} \right),$$

for m sufficiently large. Since $\{c_m\}_{m=0}^{\infty} \in l_2$, (3.10) follows easily.

Now we show that $f \in \mathscr{D}(T_{Z^M})$. Using Taylor's expansion of f we obtain

$$m!\, d_{m-M} = \sqrt{(m-M)!} \int f(z)\, z^M \bar{z}^m \, d\mu(z), \qquad m \geqslant M. \qquad (3.19)$$

On the other hand, basing on the Taylor expansion $\sum_{j \geqslant 0} a_j f_j(z)$ of the entire function $f(z)\, z^M$, we have

$$a_m \sqrt{m!} = \int f(z)\, z^M \bar{z}^m \, d\mu(z), \qquad m \in \mathbb{N}. \qquad (3.20)$$

Combinig (3.19) and (3.20), we conclude that

$$a_m = d_{m-M} \sqrt{\frac{m!}{(m-M)!}}, \qquad m \geqslant M.$$

This and (3.10) implies that $\{a_m\}_{m=0}^{\infty} \in l_2$ and consequently $f \in \mathscr{D}(T_{Z^M})$. Since $\mathscr{D}(T_{Z^M}) \subseteq \mathscr{D}(T_{\bar{q}})$ and $\mathscr{D}(T_{Z^M}) \subseteq \mathscr{D}(T_{Z^N}) \subseteq \mathscr{D}(T_p)$, we infer that $f \in \mathscr{D}(T_p) \cap \mathscr{D}(T_{\bar{q}}) = \mathscr{D}(T_p + T_{\bar{q}})$. This completes the proof. ∎

4. Toeplitz Operators with Radially Symmetric Symbols

Though it seems to be hopeless to find an explicit form of T_φ^* for general symbols φ, we distinguish a class of nonholomorphic symbols for which this can be done. The class consists of t-radially symmetric functions, where $t \in \mathbb{R}_+^n$ ($\mathbb{R}_+ := (0, +\infty)$). The notion of t-radial symmetry has been inspired by an old paper by Fischer [8] in which he has defined t-homogeneous polynomials for $t \in \mathbb{R}_+^n$.

Let $t = (t_1, ..., t_n) \in \mathbb{R}^n \setminus \{0\}$. Recall that a *nonzero* polynomial $p(z) = \sum_k c_k z^k$ is said to be *t-homogeneous* of *t-degree d* if $\sum_{j=1}^n t_j k_j = d$ for every $k \in \mathbb{N}^n$ such that $c_k \neq 0$. Denote by $Z(t)$ the set of all real numbers of the form $\sum_{j=1}^n t_j k_j$ with $k \in \mathbb{N}^n$ and by $\mathscr{H}_{d,t}$, $d \in Z(t)$, the linear space generated by t-homogeneous polynomials of t-degree d. It is easy to see that the set $Z(t)$ is countable. Moreover, $Z(t) \subseteq m\mathbb{N}$ for $t \in \mathbb{N}^n$, where m is the greatest common divisor of $t_1, ..., t_n$. If $t = (1, ..., 1)$, then t-homogeneous polynomials are homogeneous in the usual sense. In this case we simply write $\mathscr{H}_d$ instead of $\mathscr{H}_{d,t}$.

A complex Borel function φ on $\mathbb{C}^n$ is said to be *t-radially symmetric*, if

(R1) for each $\vartheta \in \mathbb{R}$, $\varphi(e^{i\vartheta t_1} z_1, ..., e^{i\vartheta t_n} z_n) = \varphi(z)$ for almost every $z = (z_1, ..., z_n) \in \mathbb{C}^n$ (w.r.t. $[V]$).

We say that φ is *polyradially symmetric*, if

(R2) $\varphi(z) = \varphi(|z_1|, ..., |z_n|)$ for almost every $z \in \mathbb{C}^n$ (w.r.t. $[V]$).

Note that any polyradially symmetric function is automatically t-radially symmetric.

In some cases, the t-radial symmetric coincides with the polyradial one. It depends on whether $t_1, ..., t_n$ are algebraically independent over the field $\mathbb{Q}$ of rational numbers. For simplicity we consider only the case $n = 2$. Let us assume that t_1 and t_2 are algebraically independent over $\mathbb{Q}$. Without loss of generality we can assume that $t = (1, s)$ with $s \in \mathbb{R} \setminus \mathbb{Q}$. Take a t-radially symmetric function φ on $\mathbb{C}^2$. We claim that φ is polyradially symmetric. Without loss of generality we can assume that $\varphi \in L^2(\mu)$. Note that the set $\mathbb{T}_\varphi^2 = \{z \in \mathbb{T}^2 \colon U(z)\,\varphi = \varphi\}$ forms a closed subgroup of the multiplicative group $\mathbb{T}^2$, where $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$ and $\{U(z) \colon z \in \mathbb{T}^2\}$ is a strongly continuous group of unitary operators on $L^2(\mu)$ defined by

$$(U(z) f)(w) = f(z_1 w_1, z_2 w_2), \qquad z = (z_1, z_2),\ w = (w_1, w_2) \in \mathbb{C}^2.$$

It follows from the t-radial symmetry of φ that $(e^{i\vartheta}, e^{is\vartheta}) \in \mathbb{T}_\varphi^2$ for $\vartheta \in \mathbb{R}$. In particular, $(1, e^{i2k\pi s}) \in \mathbb{T}_\varphi^2$ for $k \in \mathbb{Z}$. Because $s \in \mathbb{R} \setminus \mathbb{Q}$, the set $\{e^{i2k\pi s} : k \in \mathbb{Z}\}$ is dense in $\mathbb{T}$, so $\{1\} \times \mathbb{T} \subseteq \mathbb{T}_\varphi^2$. Since $(e^{i\vartheta}, e^{i\omega}) = (e^{i\vartheta}, e^{is\vartheta})(1, e^{-is\vartheta})(1, e^{i\omega})$ for $(\vartheta, \omega) \in \mathbb{R}^2$ and $\mathbb{T}_\varphi^2$ is a subgroup of $\mathbb{T}^2$, we have $\mathbb{T}_\varphi^2 = \mathbb{T}^2$. This proves our claim.

Note that for any $t \in \mathbb{R}^n \setminus \{0\}$, $\mathscr{P}$ is a direct sum of linear subspaces $\mathscr{H}_{d,t}$, $d \in Z(t)$. Moreover, $\{\mathscr{H}_{d,t} : d \in Z(t)\}$ are pairwise orthogonal. Indeed, if $p \in \mathscr{H}_{u,t}$, $q \in \mathscr{H}_{w,t}$ and $u \neq w$, then—by the rotation invariance of μ—we have

$$(p, q) = \int (p \cdot \bar{q})(e^{i\vartheta t_1} z_1, ..., e^{i\vartheta t_n} z_n)\, d\mu(z) = e^{i(u-w)\vartheta}(p, q), \qquad \vartheta \in \mathbb{R},$$

so $(p, q) = 0$. Hence we have the orthogonal decomposition

$$\mathscr{B} = \sum_{d \in Z(t)} \oplus \bar{\mathscr{H}}_{d,t}, \qquad t \in \mathbb{R}^n \setminus \{0\}. \tag{4.1}$$

In the case $t \in \mathbb{R}^n_+$, the subspaces $\mathscr{H}_{d,t}$ being finite dimensional are closed. The decomposition (4.1) forces the corresponding one for Toeplitz operators with t-radially symmetric symbols.

THEOREM 4.1. *Let φ be a t-radially symmetric function on $\mathbb{C}^n$ such that $\mathscr{P} \subseteq \mathscr{D}(T_\varphi)$, $t \in \mathbb{R}^n \setminus \{0\}$. Then*

(i) $(T_\varphi|_{\mathscr{P}})^- = \sum_{d \in Z(t)} \oplus (T_\varphi|_{\mathscr{H}_{d,t}})^-$ *and* $\tilde{T}_\varphi = \sum_{d \in Z(t)} \oplus (T_{\bar{\varphi}}|_{\mathscr{H}_{d,t}})^*$.

Moreover, if $t \in \mathbb{R}^n_+$, then each $\mathscr{H}_{d,t}$ reduces T_φ and

(ii) $$T^*_{\bar{\varphi}} = \bar{T}_\varphi = \Pi_\varphi = \tilde{T}_\varphi = (T_\varphi|_{\mathscr{P}})^- = \sum_{d \in Z(t)} \oplus T_\varphi|_{\mathscr{H}_{d,t}}.$$

Proof. (i) First note that $T_\varphi(\mathscr{H}_{u,t}) \subseteq \bar{\mathscr{H}}_{u,t}$. Indeed, if $p \in \mathscr{H}_{u,t}$, $q \in \mathscr{H}_{w,t}$, and $u \neq w$, then by the rotation invariance of μ and (R1), we have

$$(T_\varphi p, q) = e^{i(u-w)\vartheta}(T_\varphi p, q), \qquad \vartheta \in \mathbb{R},$$

so $(T_\varphi p, q) = 0$. This and (4.1) imply that $T_\varphi(\mathscr{H}_{u,t}) \subseteq \bar{\mathscr{H}}_{u,t}$. Denote by Σ_φ the closed operator $\sum_{d \in Z(t)} \oplus (T_\varphi|_{\mathscr{H}_{d,t}})^-$. Note that $\Sigma_\varphi = (\Sigma_\varphi|_{\mathscr{P}})^-$. Since $T_\varphi|_{\mathscr{P}} = \Sigma_\varphi|_{\mathscr{P}}$, the first equality in (i) holds. The other follows by taking adjoints and applying Proposition 1.B.

(ii) Let $t \in \mathbb{R}^n_+$. First we show that for every $u \in Z(t)$, $Q_u T_\varphi \subseteq T_\varphi Q_u$, where Q_u is the orthogonal projection of $\mathscr{B}$ onto $\mathscr{H}_{u,t}$. Since each $\mathscr{H}_{u,t}$ is finite dimensional, $T_\varphi(\mathscr{H}_{u,t}) \subseteq \mathscr{H}_{u,t}$. By the same reason $T_{\bar{\varphi}}(\mathscr{H}_{u,t}) \subseteq \mathscr{H}_{u,t}$. Both these inclusions imply

$$(Q_u T_\varphi f, g) = (f, T_{\bar{\varphi}} g) = (Q_u f, T_{\bar{\varphi}} g) = (T_\varphi Q_u f, g), \qquad f \in \mathscr{D}(T_\varphi),\ g \in \mathscr{H}_{u,t},$$

which proves $Q_u T_\varphi \subseteq T_\varphi Q_u$. This means that $\mathscr{H}_{u,t}$ reduces T_φ and $T_{\bar{\varphi}}$ (because $\bar{\varphi}$ also satisfies (R1)). Thus we have

$$(T_{\bar{\varphi}}|_{\mathscr{H}_{d,t}})^* = T^*_{\bar{\varphi}}|_{\mathscr{H}_{d,t}} = T_\varphi|_{\mathscr{H}_{d,t}}, \qquad d \in Z(t). \tag{4.2}$$

It follows from (i) that $\Sigma_{\bar\varphi} \subseteq \bar T_{\bar\varphi}$ and consequently $T^*_{\bar\varphi} \subseteq \Sigma^*_{\bar\varphi}$. This and (4.2) imply

$$\Sigma_\varphi = (T_\varphi|_{\mathscr{P}})^- \subseteq \bar T_\varphi \subseteq T^*_{\bar\varphi} \subseteq \Sigma^*_{\bar\varphi} = \sum_{d \in Z(t)} \oplus (T_{\bar\varphi}|_{\mathscr{H}_{d,t}})^* = \Sigma_\varphi.$$

Thus we have proved that $\Sigma_\varphi = \bar T_\varphi = T^*_{\bar\varphi} = (T_\varphi|_{\mathscr{P}})^-$. Taking adjoints in the last equality and using Proposition 1.B, we get $\bar T_{\bar\varphi} = \tilde T_{\bar\varphi}$. Since $\bar\varphi$ also satisfies (R1), $\bar T_\varphi = \tilde T_\varphi$. This completes the proof. ∎

Theorem 4.1(ii) is no longer true if $t \in \mathbb{R}^n \backslash \mathbb{R}^n_+$ (see Example 6.7). The reason is that in this case the spaces $\mathscr{H}_{d,t}$ are not finite dimensional.

It turns out that unbounded Toeplitz operators T_φ may be closed or unclosed; this depends on the choice of φ. In Section 6 we shall see that even for polyradially symmetric symbols both possibilities may appear.

To begin with, we strengthen the general criterion for the closedness of Toeplitz operators with t-radially symmetric symbols.

LEMMA 4.2. *If φ is a t-radially symmetric function on $\mathbb{C}^n$, $t \in \mathbb{R}^n_+$, and $\mathscr{P} \subseteq \mathscr{D}(T_\varphi)$, then T_φ is closed iff there exists $c \geqslant 0$ such that*

$$\text{(i)} \qquad \|\varphi p\|^2 \leqslant c(\|p\|^2 + \|T_\varphi p\|^2), \qquad p \in \mathscr{H}_{d,t}, \quad d \in Z(t).$$

Proof. The necessity is obvious by Proposition 2.3. To prove the sufficiency note first that $\{\mathscr{H}_{u,t} : u \in Z(t)\}$ are pairwise orthogonal in $L^2(\nu)$, where $d\nu = (1 + |\varphi|^2)\, d\mu$ (see the proof of the decomposition (4.1)). Applying Theorem 4.1(ii) and Proposition 2.3(ii) with $\mathscr{D} = \mathscr{P}$, we get the conclusion. ∎

The criterion for closedness of T_φ given in Lemma 4.2 becomes more efficient for polyradially symmetric functions.

THEOREM 4.3. *If φ is a polyradially symmetric function on $\mathbb{C}^n$ and $\mathscr{P} \subseteq \mathscr{D}(T_\varphi)$, then T_φ is closed iff there exists $c \geqslant 0$ such that*

$$\|\varphi f_k\|^2 \leqslant c(1 + |(\varphi f_k, f_k)|^2), \qquad k \in \mathbb{N}^n. \tag{4.3}$$

If moreover either $\operatorname{Re} \varphi \geqslant \varepsilon$ *a.e.* $[V]$ *or* $\operatorname{Im} \varphi \geqslant \varepsilon$ *a.e.* $[V]$ *for some* $\varepsilon > 0$, *then T_φ is closed iff there exists $c \geqslant 0$ such that*

$$\|\varphi f_k\|^2 \leqslant c|(\varphi f_k, f_k)|^2, \qquad k \in \mathbb{N}^n. \tag{4.4}$$

Proof. Take $p \in \mathscr{H}_s$ $(s \in \mathbb{N})$ of the form $p = \sum_{|k|=s} \lambda_k f_k$. Since monomials are pairwise orthogonal in $L^2(|\varphi|^2\, d\mu)$, we have

$$\|\varphi p\|^2 = \sum_{|k|=s} |\lambda_k|^2 \|\varphi f_k\|^2. \tag{4.5}$$

Now we show that

$$(T_\varphi p, f_k) = \lambda_k (\varphi f_k, f_k), \qquad k \in \mathbb{N}^n, \quad |k| = s. \tag{4.6}$$

Indeed, since

$$\int_{[0,2\pi]^n} p(re^{it})\, e^{-i\langle k, t\rangle}\, dV(t) = (2\pi)^n r^k \lambda_k / \sqrt{k!}, \qquad r \in \mathbb{R}^n_+,$$

and φ has the property (R2), we can write

$$\begin{aligned}(T_\varphi p, f_k) &= (\pi^n \sqrt{k!})^{-1} \int_{\mathbb{R}^n_+} \varphi(r)\, r^{k+1} e^{-\|r\|^2} \\ &\quad \times \int_{[0,2\pi]^n} p(re^{it})\, e^{-i\langle k, t\rangle}\, dV(t)\, dV(r) \\ &= \frac{2^n \lambda_k}{k!} \int_{\mathbb{R}^n_+} \varphi(r)\, r^{2k+1} e^{-\|r\|^2}\, dV(r) = \lambda_k (\varphi f_k, f_k).\end{aligned}$$

It follows from Theorem 4.1 that $T_\varphi p \in \mathscr{H}_s$. Thus by (4.6) we have

$$\|p\|^2 + \|T_\varphi p\|^2 = \sum_{|k|=s} |\lambda_k|^2 (1 + |(\varphi f_k, f_k)|^2). \tag{4.7}$$

In virtue of Lemma 4.2 and the equalities (4.5) and (4.7), T_φ is closed if and only if there exists $c \geqslant 0$ such that (4.3) holds. This proves the first part of the conclusion.

To prove the other one, assume that $\operatorname{Re} \varphi \geqslant \varepsilon$ a.e. $[V]$ (the case $\operatorname{Im} \varphi \geqslant \varepsilon$ a.e. $[V]$ can be proved similarly). Then we have

$$|(\varphi f_k, f_k)|^2 = ((\operatorname{Re} \varphi) f_k, f_k)^2 + ((\operatorname{Im} \varphi) f_k, f_k)^2 \geqslant \varepsilon^2 (f_k, f_k)^2 = \varepsilon^2, \qquad k \in \mathbb{N}^n.$$

Now if (4.3) holds, then the above inequality yields

$$\|\varphi f_k\|^2 \leqslant c(1 + |(\varphi f_k, f_k)|^2) \leqslant c(\varepsilon^{-2} + 1) |(\varphi f_k, f_k)|^2, \qquad k \in \mathbb{N}^n.$$

The converse implication (4.4) $\Rightarrow$ (4.3) is obvious. This completes the proof.

5. Selfadjointness

In this section we apply our earlier results to discuss the question of (essential) selfadjointness of Toeplitz operators.

Let φ be a real-valued Borel function on $\mathbb{C}^n$ such $\mathscr{P} \subseteq \mathscr{D}(T_\varphi)$. It is clear that T_φ is symmetric. Moreover, if $\varphi(\bar{z}) = \varphi(z)$ a.e., then the deficiency indices of T_φ (resp., $T_\varphi|_{\mathscr{P}}$) are equal (cf. [11, Prop. 3.1]). In particular, one can check that the deficiency indices of $(T_\varphi|_{\mathscr{P}})^-$ are equal to (3, 3) if $n=1$, $\varphi(z)=\mathrm{Re}(z^3)$, and (∞, ∞) if $n=2$, $\varphi(z, w)=\mathrm{Re}(zw^2)$.

Similarly, the deficiency indices of $(T_\varphi|_{\mathscr{P}})^-$ are equal, provided φ is bounded from below (cf. [16]). In this case Berezin has described the *Friedrichs extension* F_φ of $(T_\varphi|_{\mathscr{P}})^-$ for $\varphi \in L^p(\mu)$, $p>2$ ([4, Theorem 7]). Applying our approach we can assume less, namely $\mathscr{E} \subseteq \mathscr{D}(T_\varphi)$, preserving his description of F_φ. Note that if $\varphi(z)=\exp((\|z\|^2-\|z\|^r)/2)$, then $\mathscr{E} \subseteq \mathscr{D}(T_\varphi)$, though φ does not belong to $L^p(\mu)$ for any $p>2$ $(1<r<2)$.

PROPOSITION 5.1. *Assume that* $\varphi \geqslant 1$ *and* $\mathscr{E} \subseteq \mathscr{D}(T_\varphi)$. *Then*

$$\text{(i)} \qquad F_\varphi \subseteq \Pi_\varphi \quad \text{and} \quad \mathscr{D}(F_\varphi)=\mathscr{D}(\Pi_\varphi) \cap \mathscr{F},$$

where $\mathscr{F}$ *is the closure of* $\mathscr{P}$ *in* $L^2(\varphi\, d\mu)$. *If moreover there exist* $c, r \geqslant 0$ *such that* $\varphi(e^{i\vartheta}z) \leqslant c \cdot \varphi(z)$ *for all* $\vartheta \in \mathbb{R}$ *and* $\|z\| \geqslant r$, *then*

$$\text{(ii)} \qquad \mathscr{F}=\left\{u : u \text{ is entire and } \int \varphi |u|^2\, d\mu<\infty\right\}$$

and F_φ *is the Friedrichs extension of* $\bar{T}_\varphi$.

Proof. Put $A=(T_\varphi|_{\mathscr{P}})^-$. Then by Proposition 1.B and Theorem 1.3(i) we have $A^*=\tilde{T}_\varphi=\Pi_\varphi$. Hence (i) follows from the definition of the Friedrichs extension (cf. [16]).

For the proof of the equality (ii) see ([4, Th. 7.2]). Let G_φ denote the Friedrichs extension of $\bar{T}_\varphi$. Note that $\bar{T}_\varphi \subseteq F_\varphi$. Indeed, if $f \in \mathscr{D}(T_\varphi)$, then $\int \varphi |f|^2\, d\mu \leqslant \int \varphi^2 |f|^2\, d\mu<\infty$, so $f \in \mathscr{D}(F_\varphi)$. Thus $T_\varphi \subseteq F_\varphi$. Since the Friedrichs extension is the greatest nonnegative selfadjoint extension of a given symmetric operator [16, Exercise 7.30(b), p. 200], we have $\mathscr{D}(F_\varphi^{1/2})=\mathscr{D}(G_\varphi^{1/2})$ and $\|F_\varphi^{1/2} f\|=\|G_\varphi^{1/2} f\|$ for $f \in \mathscr{D}(F_\varphi^{1/2})$. Consequently, $F_\varphi=G_\varphi$ [16, Exercise 7.30(a), p. 200], which completes the proof. ∎

In general it is quite difficult to decide when T_φ is essentially selfadjoint. In some cases it is easier to prove the selfadjointness of $(T_\varphi|_{\mathscr{P}})^-$, which in turn forces the essential selfadjointness of T_φ.

PROPOSITION 5.2. *If* $\varphi\colon \mathbb{C}^n \to \mathbb{R}$ *is a Borel function,* $\mathscr{P} \subseteq \mathscr{D}(T_\varphi)$, *and* $\mathscr{D}$ *is a linear subspace of* $\mathscr{B}$ *such that* $\mathscr{P} \subseteq \mathscr{D} \subseteq \mathscr{D}(\bar{T}_\varphi)$, *then the following conditions are equivalent*: (i) $(T_\varphi|_{\mathscr{P}})^-$ *is selfadjoint,* (ii) $\tilde{T}_\varphi$ *is selfadjoint,* (iii) $\tilde{T}_\varphi$ *is symmetric and,* (iv) $\tilde{T}_\varphi=(T_\varphi|_{\mathscr{D}})^-$. *In either of these cases* $\bar{T}_\varphi$ *is selfadjoint.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) can be deduced from Proposition 1.B. In turn, (iii) $\Rightarrow$ (ii) because both $\tilde{T}_\varphi$ and $\tilde{T}_\varphi^* = (T_\varphi|_{\mathscr{P}})^-$ are symmetric. To prove (i) $\Rightarrow$ (iv) note that if $(T_\varphi|_{\mathscr{P}})$ is selfadjoint, then

$$\tilde{T}_\varphi = (T_\varphi|_{\mathscr{P}})^* = (T_\varphi|_{\mathscr{P}})^- \subseteq (T_\varphi|_{\mathscr{D}})^- \subseteq \bar{T}_\varphi \subseteq \tilde{T}_\varphi,$$

so $\tilde{T}_\varphi = \bar{T}_\varphi = (T_\varphi|_{\mathscr{D}})^-$. The converse implication follows from Theorem 1.3(ii) and Proposition 1.B. ∎

It turns out that Toeplitz operators with t-radially symmetric symbols are always essentially selfadjoint. More precisely, if $t \in \mathbb{R}_+^n$, $\mathscr{P} \subseteq \mathscr{D}(T_\varphi)$, and φ is t-radially symmetric, then T_φ is essentially selfadjoint and $\bar{T}_\varphi = \Pi_\varphi = \tilde{T}_\varphi$ (use Theorem 4.1). In particular, if p_j and q_j are t-homogeneous polynomials such that $t\text{-deg}\, p_j = t\text{-deg}\, q_j$ and $s_j \in \mathbb{R}_+$ $(j = 1, ..., k)$, then $\operatorname{Re} \sum p_j \bar{q}_j$ and $\sum |p_j|^{s_j}$ are t-radially symmetric functions. Therefore, if $\varphi = |p|^2$, then T_φ is essentially selfadjoint for any t-homogeneous polynomial p with $t \in \mathbb{R}_+^n$. Moreover, $\bar{T}_\varphi = T_p^* T_p$. Indeed, by Theorem 1.4(iii), we have $T_p^* T_p = \Pi_{\bar{p}} \Pi_p \subseteq \Pi_\varphi$. Since $T_p^* T_p$ and $\Pi_\varphi = \bar{T}_\varphi$ are selfadjoint (use Theorem 4.1), they must coincide.

The question of whether T_φ, $\varphi = |p|^2$, is (essentially) selfadjoint for all $p \in \mathscr{P}_n$ is open for $n > 1$. However, if $p \in \mathscr{P}_1$, then T_φ is selfadjoint! (This follows from Theorem 3.2.) We now introduce a class of symbols inducing selfadjoint Toeplitz operators, closed under the operations of tensor products $\varphi \otimes \psi$ and "tensor sums" $\varphi \otimes 1 + 1 \otimes \psi$.

We say that a real-valued Borel function φ on $\mathbb{C}^n$ satisfies the condtion (C) if $\mathscr{P} \subseteq \mathscr{D}(T_\varphi)$, $T_\varphi(\mathscr{P}) \subseteq \mathscr{P}$, $T_\varphi|_{\mathscr{P}}$ is essentially selfadjoint, and $\|\varphi f\|^2 \leqslant d_\varphi \|T_\varphi f\|^2$ for $f \in \mathscr{P}$, where d_φ is a constant independent on f.

Note that if φ satisfies (C), then $\mathscr{P}$ is a core for T_φ and consequently, by Proposition 2.3, T_φ is selfadjoint.

LEMMA 5.3 *Let φ and ψ be real-valued Borel functions on $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively, satisfying* (C). *Then*

(i) *$\varphi \otimes \psi$ satisfies* (C),

(ii) *$\varphi \otimes 1 + 1 \otimes \psi$ satisfies* (C), *provided φ, $\psi \geqslant 0$.*

Proof. Since $\mathscr{P}_m \otimes \mathscr{P}_n = \mathscr{P}_{m+n}$ and polynomials are preserved by both T_φ and T_ψ, we have, for any $h \in \mathscr{P}_{m+n}$,

$$T_{\varphi \otimes 1} h = (T_\varphi \otimes I) h \quad \text{and} \quad T_{1 \otimes \psi} h = (I \otimes T_\psi) h \tag{5.1}$$

$$T_\varphi(h(\cdot, w)) = (T_{\varphi \otimes 1} h)(\cdot, w), \qquad w \in \mathbb{C}^n, \tag{5.2}$$

$$T_\psi(h(z, \cdot)) = (T_{1 \otimes \psi} h)(z, \cdot), \qquad z \in \mathbb{C}^m, \tag{5.3}$$

$$T_{\varphi \otimes 1} h \in \mathscr{P}_{m+n} \quad \text{and} \quad T_{1 \otimes \psi} h \in \mathscr{P}_{m+n}. \tag{5.4}$$

(i) Applying (5.1), (5.2), (5.3), and (5.4) we can write

$$\|\varphi\otimes\psi h\|^2=\int\|\varphi(\cdot)\,h(\cdot,w)\,\psi(w)\|^2\,d\mu_n(w)$$

$$\leqslant d_\varphi\int\|T_\varphi(h(\cdot,w))\,\psi(w)\|^2\,d\mu_n(w)$$

$$=d_\varphi\int\|\psi(w)(T_{\varphi\otimes 1}h)(\cdot,w)\|^2\,d\mu_n(w)$$

$$=d_\varphi\int\|\psi(\cdot)(T_{\varphi\otimes 1}h)(z,\cdot)\|^2\,d\mu_m(z)$$

$$\leqslant d_\varphi d_\psi\int\|T_\psi(T_{\varphi\otimes 1}h)(z,\cdot)\|^2\,d\mu_m(z)$$

$$=d_\varphi d_\psi\int\|(T_{1\otimes\psi}T_{\varphi\otimes 1}h)(z,\cdot)\|^2\,d\mu_m(z)$$

$$=d_\varphi d_\psi\|T_{\varphi\otimes\psi}h\|^2,\qquad h\in\mathscr{P}_{m+n}.$$

It is clear that $T_{\varphi\otimes\psi}|_{\mathscr{P}_{m+n}}=(T_\varphi|_{\mathscr{P}_m})\otimes(T_\psi|_{\mathscr{P}_n})$. Since both operators $T_\varphi|_{\mathscr{P}_m}$ and $T_\psi|_{\mathscr{P}_n}$ are essentially selfadjoint, so is $T_{\varphi\otimes\psi}|_{\mathscr{P}_{m+n}}$ (cf. [14, Theorem VIII.33]).

(ii) Using (i) and the positiveness of $\varphi\otimes\psi$ we have

$$\begin{aligned}\|(\varphi\otimes 1+1\otimes\psi)\,h\|^2&\leqslant 2(\|(\varphi\otimes 1)\,h\|^2+\|(1\otimes\psi)\,h\|^2)\\&\leqslant d(\|T_{\varphi\otimes 1}h\|^2+\|T_{1\otimes\psi}h\|^2)\\&\leqslant d(\|T_{\varphi\otimes 1}h\|^2+2(T_{\varphi\otimes\psi}h,h)+\|T_{1\otimes\psi}h\|^2)\\&=d(\|T_{\varphi\otimes 1}h\|^2+2(T_{\varphi\otimes 1}h,T_{1\otimes\psi}h)+\|T_{1\otimes\psi}h\|^2)\\&=d\|T_{\varphi\otimes 1+1\otimes\psi}h\|^2,\qquad h\in\mathscr{P}_{m+n}(d=2(d_\varphi+d_\psi)).\end{aligned}$$

Note that $T_{\varphi\otimes 1+1\otimes\psi}|_{\mathscr{P}_{m+n}}=(T_\varphi|_{\mathscr{P}_m})\otimes I+I\otimes(T_\psi|_{\mathscr{P}_n})$, so the essential selfadjointness of $T_{\varphi\otimes 1+1\otimes\psi}|_{\mathscr{P}_{m+n}}$ follows once more from Theorem VIII.33 in [14]. This completes the proof.

It follows from Theorem 3.2(iii) and Proposition 2.5 that any function of the form $|p|^2$, $p\in\mathscr{P}_1$, satisfies (C). Repeated application of Lemma 5.3 enables us to produce new classes of symbols inducing selfadjoint Toeplitz operators. Below we present only two possible applications of this procedure.

Call nonconstant polynomials $\{\xi_j\}_{j=1}^s\subseteq\mathscr{P}$ of degree 1 *strongly independent* if the vectors $((\partial/\partial z_1)\,\xi_j,\ldots,(\partial/\partial z_n)\,\xi_j)$, $1\leqslant j\leqslant s$, are pairwise orthogonal in $\mathbb{C}^n$ (equivalently, the Gaussian variables $\{\xi_j - \xi_j(0)\}_{j=1}^s$ are stochastically independent).

PROPOSITION 5.4. *If* $s \geqslant 1$, $\varphi = |p_1(\xi_1) \cdot \cdots \cdot p_s(\xi_s)|^2$, *and* $\psi = |p_1(\xi_1)|^2 + \cdots + |p_s(\xi_s)|^2$, *where* $\{\xi_j\}_{j=1}^s \subseteq \mathscr{P}_n$ *are strongly independent polynomials of degree* 1 *and* $\{p_j\}_{j=1}^s \subseteq \mathscr{P}_1$, *then* T_φ *and* T_ψ *are selfadjoint.*

Proof. Modifying polynomials ξ_j and p_j if necessary we can assume that $\xi_j(\cdot) = \langle \cdot, a_j \rangle$, where $\{a_j\}_{j=1}^s$ are orthonormal vectors in $\mathbb{C}^n$. Let $\{a_j\}_{j=1}^n$ be the completion of $\{a_j\}_{j=1}^s$ to an orthonormal basis in $\mathbb{C}^n$ and let U be the unitary matrix with rows $\bar{a}_1, ..., \bar{a}_n$. If $p_j \equiv 1$ (resp., $p_j \equiv 0$) for $j > s$ and $q_j(z_1, ..., z_n) = p_j(z_j)$ for all j, then $\varphi = |q_1 \cdot ... \cdot q_n|^2 \circ U$ (resp., $\psi = (|q_1|^2 + \cdots + |q_n|^2) \circ U$). Hence (by Proposition 1.2) we can assume that $\xi_j(z_1, ..., z_n) = z_j$.

Now the conclusion follows from Lemma 5.3, because each $|p_j|^2$ satisfies (C). ∎

6. EXAMPLES

In this section we present several examples concerning the questions of closedness and selfadjointness of Toeplitz operators. We begin with unclosed T_φ whose symbol is radially symmetric.

EXAMPLE 6.1. If $0 < a < 1/2$ and $\varphi_a(z) = \exp(a|z|^2)$, $z \in \mathbb{C}$, then T_{φ_a} is not closed. Indeed otherwise there would exist (by Theorem 4.3) some $c > 0$ such that

$$(1 - 2a)^{-(k+1)} = \|\varphi_a f_k\|^2 \leqslant c(\varphi_a f_k, f_k)^2 = c((1-a)^2)^{-(k+1)}, \qquad k \in \mathbb{N},$$

which is impossible.

Note that $\mathscr{D}(T_{\varphi_a}) = \{0\}$ for $a \geqslant 1/2$, $\mathscr{D}(T_{\varphi_a}) = \mathscr{B}_1$ for $a \leqslant 0$, and T_{φ_a} is essentially selfadjoint for $a \in (0, 1/2)$ (see Theorem 4.1). Moreover, one can show that, for any $f \in \mathscr{P}$, $T_{\varphi_a}(f) \to f$ as $a \to 0+$ and $T_{\varphi_a}(f) \to A(f)$ as $a \to \frac{1}{2}-$, where A is a selfadjoint diagonal operator in $\mathscr{B}_1$ with diagonal elements 2^{k+1} (i.e. $A(f_k) = 2^{k+1} f_k$ for $k \in \mathbb{N}$). ∎

As was shown in Section 3, Toeplitz operators T_φ in $\mathscr{B}_1$ with symbols of the form $\varphi = p + \bar{q}$, $p, q \in \mathscr{P}_1$, are closed provided $\deg p < \deg q$. This is no longer true if $\deg p = \deg q$.

EXAMPLE 6.2. If $s \in \mathbb{N}$ and $\varphi(z) = 2\,\mathrm{Re}(z^s)$, $z \in \mathbb{C}$, then T_φ is closed for $s < 2$ and unclosed for $s \geqslant 2$. Moreover, $\bar{T}_\varphi$ is strictly contained in $\tilde{T}_\varphi$ for $s = 3$ (this follows from Proposition 5.2 and the nonselfadjointness of $(T_\varphi|_{\mathscr{P}})^-$; see Example 3.6 in [11]). If $s < 2$, then the closedness of T_φ follows from Corollary 2.4.

Let $s \geqslant 2$. Suppose, contrary to our claim, that T_φ is closed. Then, by Proposition 2.3, there exists $c \geqslant 1$ such that

$$\|\varphi p\|^2 \leqslant c(\|p\|^2 + \|T_\varphi p\|^2), \qquad p \in \mathscr{P}. \tag{6.1}$$

Applying Theorem 1.A we get

$$\begin{aligned} \|z^s p + \bar{z}^s p\|^2 &= \|z^s p\|^2 + \|z^s p\|^2 + 2\,\mathrm{Re}(z^{2s} p, p) \\ &= s!\,\|p\|^2 + \sum_{k=1}^{s-1} \frac{(s(s-1)\cdots(s-k+1))^2}{k!} \|D^{s-k} p\|^2 \\ &\quad + \|D^s p\|^2 + \|z^s p\|^2 + 2\,\mathrm{Re}(z^{2s} p, p) \\ &= s!\,\|p\|^2 + \sum_{k=1}^{s-1} \frac{(s(s-1)\cdots(s-k+1))^2}{k!} \\ &\quad \times \|D^{s-k} p\|^2 + \|z^s p + D^s p\|^2. \end{aligned}$$

This and (6.1) gives us constants $\alpha_1, ..., \alpha_{s-1}$, $c_1 > 0$ such that

$$\sum_{k=1}^{s-1} \alpha_k \|D^k p\|^2 \leqslant c_1(\|p\|^2 + \|z^s p + D^s p\|^2), \qquad p \in \mathscr{P}. \tag{6.2}$$

Suppose that $p = \sum_{j=0}^{N} \lambda_j f_j$, $N > 2s$. Because $Df_j = \sqrt{j}\, f_{j-1}$ (with $f_{-1} = 0$) and $z f_j = \sqrt{j+1}\, f_{j+1}$, we have

$$\begin{aligned} z^s p + D^s p &= \sum_{j=0}^{s-1} \lambda_{j+s} \sqrt{(j+1)\cdots(j+s)}\, f_j \\ &\quad + \sum_{j=s}^{N-s} (\lambda_{j-s} \sqrt{(j-s+1)\cdots j} \\ &\quad + \lambda_{j+s} \sqrt{(j+1)\cdots(j+s)})\, f_j \\ &\quad + \sum_{j=N-s+1}^{N+s} \lambda_{j-s} \sqrt{(j-s+1)\cdots j}\, f_j. \end{aligned}$$

Hence

$$\begin{aligned} \|z^s p + D^s p\|^2 &= \sum_{j=0}^{s-1} |\lambda_{j+s}|^2 (j+1)\cdots(j+s) \\ &\quad + \sum_{j=0}^{N-2s} |\lambda_j \sqrt{(j+1)\cdots(j+s)} \\ &\quad + \lambda_{j+2s} \sqrt{(j+s+1)\cdots(j+2s)}|^2 \\ &\quad + \sum_{j=N-2s+1}^{N} |\lambda_j|^2 (j+1)\cdots(j+s). \end{aligned} \tag{6.3}$$

Suppose that N is divisible by $2s$. Define the sequence $\{\lambda_j\}_{j=0}^N$ by $\lambda_0=0$, $\lambda_j=0$ if j is not divisible by $2s$ and $\lambda_j=(-1)^{j/2s}j^{-s/2}$ if j is a multiple of $2s$. In what follows $\sum'_j$ stands for the sum over the multiples of $2s$. Using (6.3) and the definition of $\{\lambda_j\}_{j=0}^N$ we can write

$$\|z^s p+D^s p\|^2=\sum_{j=0}^{N-2s}{}' \Delta_j^2+(N+1)\cdots(N+s)\,N^{-s}, \tag{6.4}$$

where $\Delta_0:=(2s)^{-s/2}\sqrt{(s+1)\cdots(2s)}$ and

$$\Delta_j:=\frac{\sqrt{(j+1)\cdots(j+s)}}{j^{s/2}}-\frac{\sqrt{(j+s+1)\cdots(j+2s)}}{(j+2s)^{s/2}}, \qquad j\geqslant 1.$$

Note that $\Delta_j>0$. Applying the identity

$$a^{-1/2}-b^{-1/2}=\frac{b-a}{(\sqrt{a}+\sqrt{b})\sqrt{ab}}, \qquad a,b>0,$$

we can estimate Δ_j as follows

$$\begin{aligned}
\Delta_j &\leqslant \sqrt{(j+s+1)\cdots(j+2s)}\,(j^{-s/2}-(j+2s)^{-s/2})\\
&\leqslant (j+2s)^{s/2}\,(j^{-s/2}-(j+2s)^{-s/2})\\
&=\frac{(j+2s)^s-j^s}{[(j+2s)^{s/2}+j^{s/2}]\,j^{s/2}}\\
&\leqslant \frac{(j+2s)^s-j^s}{2j^s}\leqslant c_2 j^{-1}, \qquad j\geqslant 1,
\end{aligned} \tag{6.5}$$

where the constant c_2 is independent of j. Combining (6.4) and (6.5) we get

$$\|z^s p+D^s p\|^2\leqslant c_3\left(1+\sum_{j=1}^{\infty} j^{-2}\right)<+\infty,$$

where the constant c_3 does not depend on $N=\deg p$. Since $\|p\|^2=\sum'^{N}_{j=1} j^{-s}\leqslant\sum_{j=1}^{\infty} j^{-s}<+\infty$, the right-hand side of (6.2) is uniformly bounded as the function of N. On the other hand, the left-hand side of (6.2) tends to infinity, as $N\to\infty$ because

$$\begin{aligned}
\sum_{k=1}^{s-1}\alpha_k\|D^k p\|^2 &\geqslant \alpha_{s-1}\|D^{s-1}p\|^2=\alpha_{s-1}\sum_{j=s-1}^{N}{}'\frac{j(j-1)\cdots(j-s+2)}{j^s}\\
&\geqslant \alpha_{s-1}\beta\sum_{j=s-1}^{N}{}' j^{-1},
\end{aligned}$$

where β is a constant independent of N. This contradiction proves our claim. ∎

The next two examples show that in general Toeplitz operators with simple antiholomorphic symbols are not closed for $n > 1$.

EXAMPLE 6.3. Let $n = 2$ and $\varphi(z_1, z_2) = \overline{z_1 z_2}$. We claim that T_φ is not closed. If not, then—by Proposition 2.3—there would exist a constant $c > 0$ such that

$$\|\varphi p\|^2 \leqslant c(\|p\|^2 + \|T_\varphi p\|^2), \qquad p \in \mathscr{P}.$$

In particular, taking $p(z_1, z_2) = z_1^k$, $k \in \mathbb{N}$, and applying Theorem 1.4(iii), we would have

$$(k+1)! = \|\varphi p\|^2 \leqslant c\left(k! + \left\|\frac{\partial^2}{\partial z_1 \partial z_2} p\right\|^2\right) = c\, k!, \qquad k \in \mathbb{N}.$$

This contradiction proves the claim. ∎

EXAMPLE 6.4. If $\varphi = e_z$, $z \in \mathbb{C}^n$, then $T_{\bar\varphi}$ is closed. However, if $z, w \in \mathbb{C}^n$ are linearly independent, $\lambda \in \mathbb{C} \setminus \{0\}$ and $\psi = e_z + \lambda e_w$, then $T_{\bar\psi}$ is unclosed.

To prove the closedness of $T_{\bar\varphi}$ note that

$$\int |f|^2 \, |e_z|^2 \, d\mu = e^{\|z\|^2} \int |E_z f|^2 \, d\mu, \qquad f \in \mathscr{B},$$

so $\mathscr{D}(T_{\bar\varphi}) = \mathscr{D}(E_z)$ and consequently $T_{\bar\varphi} = E_z$ (see Theorem 1.4(iii)).

Now we show that $T_{\bar\psi}$ is not closed. Indeed, since $\lambda \neq 0$, there exists $c \in \mathbb{C}$ such that $\lambda = -e^c$. Suppose, contrary to our claim, that $T_{\bar\psi}$ is closed. Thus, according to Proposition 2.3, there exists $M > 0$ such that

$$\|\psi f\|^2 \leqslant M(\|f\|^2 + \|E_z f + \lambda E_w f\|^2), \qquad f \in \mathscr{E}.$$

Take an arbitrary $t > 0$. Since z and w are linearly independent, there exists $x \in \mathbb{C}^n$ such that $\langle z, x\rangle = t$ and $\langle w, x\rangle = t - c$. Put $f = e_x$. Then $E_a f = e^{\langle a, x\rangle} f$ for any $a \in \mathbb{C}^n$ and consequently

$$e^{\|x\|^2} e^{2t} \|e_z - e_w\|^2 = \|\psi f\|^2 \leqslant M(\|f\|^2 + \|E_z f + \lambda E_w f\|^2) = M e^{\|x\|^2},$$

which is impossible because $e_z - e_w \neq 0$. ∎

Now we turn to selfadjointness. First we give examples of selfadjoint Jacobi matrices whose selfadjointness does not follow from general theory.

EXAMPLE 6.5. Consider the polynomial $p(z) = az + z^2$, $z \in \mathbb{C}$, with $a > 0$. Then the matrix representation of $T_{|p|^2}|_{\mathscr{P}}$ (with respect to the canonical basis $\{f_n\}$) is given by the symmetric Jacobi matrix J with the main diagonal $(d_0, d_1, \ldots)$ and the subdiagonal $(c_0, c_1, \ldots)$, where $c_k = a(k+2)\sqrt{k+1}$ and $d_k = (k+1)(a^2 + k + 2)$. Applying Theorem 3.2(iii) we conclude that J is selfadjoint. Note that we cannot apply the Carleman criterion for selfadjointness of Jacobi matrices (cf. [2, Theorem VII.1.3]), because the series $\sum c_k^{-1}$ is convergent. Though in this case $c_{k-1}c_{k+1} \leqslant c_k^2$ for all $k \in \mathbb{N}$, our result does not contradict Theorem VII.1.5 in [2], because the main diagonal of J is unbounded. ∎

It is worthwhile to emphasize that some (but not all) t-radially symmetric functions φ induce essentially selfadjoint T_φ's even for $t \in \mathbb{R}^n \setminus \mathbb{R}^n_+$. A sample of such φ is given below.

EXAMPLE 6.6. If $\varphi(z, w) = \mathrm{Re}(zw)$ for $z, w \in \mathbb{C}$, then $\bar{T}_\varphi$ is selfadjoint. Indeed, since φ is t-radially symmetric with $t = (1, -1)$, it is enough to show (by Theorem 4.1(i) and Proposition 5.2) that $(T_\varphi|_{\mathscr{H}_{d,t}})^-$ is selfadjoint for every $d \in \mathbb{Z}$. Fix $d \in \mathbb{Z}$ and put $h_m := f_{m,m-d}$ for $m \geqslant d^+ := \max\{0, d\}$ and $h_m := 0$ for $m < d^+$. Then $\{h_m : m \geqslant d^+\}$ is the orthonormal basis of $\mathscr{H}_{d,t}$ and $T_\varphi|_{\mathscr{H}_{d,t}}$ is represented in this basis by a symmetric Jacobi matrix, i.e.,

$$2T_\varphi h_m = \sqrt{m(m-d)}\, h_{m-1} + \sqrt{(m+1)(m-d+1)}\, h_{m+1}, \qquad m \geqslant d^+.$$

Thus, by the Carleman criterion (cf. [2, Theorem VII.1.3]), $(T_\varphi|_{\mathscr{H}_{d,t}})^-$ is selfadjoint. ∎

We conclude the paper with an example of a non-selfadjoint operator of the form $(T_\varphi|_{\mathscr{P}})^-$, where φ is a real-valued t-radially symmetric function.

EXAMPLE 6.7. Let $\varphi(z, w) = \mathrm{Re}(zw^2)$ for $z, w \in \mathbb{C}$. Then φ is t-radially symmetric with $t = (2, -1)$. We show that $(T_\varphi|_{\mathscr{P}})^-$ is not selfadjoint. Due to Theorem 4.1(i) it is enough to prove that $(T_\varphi|_{\mathscr{H}_{d,t}})^-$ is not selfadjoint for at least one $d \in Z(t)$. We claim that this is the case for $d = 0$. Let us denote by h_m the vector $f_{m,2m}$ for $m \in \mathbb{N}$. In a way to Example 6.6 one can check that $\{h_m : m \geqslant 0\}$ forms the orthonormal basis of $\mathscr{H}_{0,t}$ and

$$2T_\varphi h_m = c_{m-1} h_{m-1} + c_m h_{m+1}, \qquad m \geqslant 0,$$

where $c_m = \sqrt{(m+1)(2m+1)(2m+2)}$. In this case $c_{m-1}c_{m+1} \leqslant c_m^2$ for $m \in \mathbb{N}$ and the series $\sum_{m=0}^\infty c_m^{-1}$ is convergent, so the Berezanskii theorem (cf. [2, Theorem VII.1.5]) proves the claim. Since $(T_\varphi|_{\mathscr{P}})^-$ is not selfadjoint, Proposition 5.2 implies that $\tilde{T}_\varphi \neq \bar{T}_\varphi$. ∎

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