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A theorem on decomposition into compact-scattered subspaces and cardinality of topological spaces *

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Abstract

The connections between the problem of decomposability of a topological space X into two subspaces containing no unscattered compacta and the problem of estimation of cardinality of X are studied. The developed technique which uses essentially these connections allows to show that: *if each subspace X' of a topological space X can be represented as a union of λ Hausdorff compacta, then $|X| \leqslant \lambda$ (λ is a cardinal)*, as well as some other assertions.

Key words: Scattered space; Compact-scattered space; Decomposition; Cardinality; rc-compact set; rc-closed set

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Attention of some specialists working in the area of set-theoretic topology was drawn to the problem of decomposition of topological spaces into subsets which do not contain spaces from a given fixed class. These problems originate from the classic work by Bernstein [5] who has proved as early as 1908 that each complete separable metric space can be decomposed into a union of two subspaces $X = X_1 \cup X_2$ in such a way that neither X_1 nor X_2 contains the Cantor set $D^{\aleph_0}$. On the other side, at the beginning of 70's Prague mathematicians Nesetřil, Rödl and Pelant [15] for every T_1-space Y (in particular, for $Y = D^{\aleph_0}$) constructed a T_1-space X such that for each decomposition $X = X_1 \cup X_2$ either X_1 or X_2

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contains the space Y. We emphasize that the constructed space X is essentially non-Hausdorff. The problem of the existence of a Hausdorff space X having these properties (Frolík was probably the first to pose this problem in such a form) remains unsolved in the full measure up till now, in spite of efforts of some authors (see [6–8,12,14,15,18, ...]). As a final, in a known sense, can be considered the following result [7, p. 27].

Theorem A. *For each space X there exists a decomposition $X = X_1 \cup X_2$ such that neither X_1 nor X_2 contains unscattered compacta closed in X.*

(It is to be emphasized, that in case $|X| > \mathfrak{c}_{w_0}$ this result is proved under some additional axiomatic assumptions (and hence in this sense it is not final: see in this connection also [12]).)

In the sequel we shall need the following

Definition 1. A subspace Y of a topological space X is called *compact-scattered*, or *k-scattered*, in X, if each closed in X compactum H which is contained in Y, is scattered.

This definition allows us to reformulate Theorem A as follows:

Theorem A′. *Every topological space is representable as a union of two of its k-scattered subspaces.*

It turns out that an unexpected consequence of this result and its modifications is an estimation of cardinality of a space, which gives a positive answer to the following Arhangel'skiĭ's question:

Assume that each subspace of X is a union of $\leqslant \mu$ compacta. Is it true that $|X| \leqslant \mu$?

Here and in the sequel small Greek letters are used to denote cardinal numbers and the corresponding initial ordinal numbers. No separation axiom is assumed unless it is explicitly stated.

The exact relations between decomposition of a space into k-scattered subspaces and the corresponding estimation of its cardinality gives the next

Theorem 2. *Let the space X be Hausdorff and $X = \bigcup\{X_\alpha\colon \alpha < \nu\}$ where all X_α are k-scattered, $\nu < \lambda$ and λ is a regular cardinal. If*

$$\text{each subspace of } X \text{ is a union of } < \lambda \text{ compacta}, \qquad (*)$$

then $|X| < \lambda$.

Proof. It is obvious that $\nabla \mathrm{hl}(X) \leqslant \lambda$, where $\nabla \mathrm{hl}(X) = \min\{\tau\colon$ for each family $\mathscr{P}$ of open subsets of X there exists a subfamily $\mathscr{P}' \subset \mathscr{P}$ such that $\bigcup \mathscr{P} = \bigcup \mathscr{P}'$ and $|\mathscr{P}'| < \tau\}$. The inequality $\nabla \mathrm{hl}(X) \leqslant \lambda$ in case of a regular cardinal λ implies that $|H| < \lambda$ for each scattered subspace H (see e.g. [2]). On the other hand H can be represented as $H = \bigcup\{H_\beta: \beta < \mu\}$ where $\mu < \lambda$ and all H_β are scattered compacta. (Indeed for each $\alpha < \nu$ the space X_α is k-scattered and hence, according to $(*)$ it is representable as a union of $< \lambda$ (scattered) compacta.) Regularity of the cardinal number λ allows to conclude now that $|X| < \lambda$. $\square$

In case $\lambda = \mu^+$ from Theorem 2 follows

Theorem 2′. *If a T_2-space X can be represented as a union of $\leqslant \mu$ of its k-scattered subspaces and if, besides, each subspace of X is a union of $\leqslant \mu$ compacta, then* $|X| \leqslant \mu$.

In the sequel we shall need the following notions:

Definition 3. A topological space X is called *pseudoparacompact* (*τ-pseudoparacompact*), if each its open cover $\mathscr{P}$ has a refinement $\mathscr{P}'$ such that $\mathscr{P}' = \bigcup\{\mathscr{P}'_\alpha: \alpha < w_0\}$ (respectively $\mathscr{P}' = \bigcup\{\mathscr{P}'_\alpha: \alpha < \tau\}$) and each $\mathscr{P}'_\alpha$ is discrete in itself (i.e., $\mathscr{P}'_\alpha$ is disjoint and besides for each $P \in \mathscr{P}'_\alpha$ there exists an open (in X) set U_p such that $U_p \supset P$ and $U_p \cap (\bigcup \mathscr{P}'_\alpha) = P$.)

Notice that if we assume additionally in the definition of pseudoparacompactness that all $\mathscr{P}'_\alpha$: $\alpha < w_0$ are discrete in X and consist of closed sets, then, according to Choban and Burke's theorem (see [9,10]) we come to the notion of a σ-paracompact space (Arhangel'skiĭ [1]). Basing on this observation, a τ-pseudoparacompact space will be called τ-paracompact, if all $\mathscr{P}'_\alpha$, $\alpha < \tau$ in its definition can be chosen discrete in X and consisting of closed sets.

A subset A of X will be called τ-discrete, if it is a union of τ discrete (in itself) subspaces. In case A is a union of τ discrete in the whole X subspaces, it will be called strictly discrete in X.

Obviously each τ-discrete space is τ-pseudoparacompact. Moreover, the following statement holds

Proposition 4. *If a space X satisfies the condition*

each nonempty closed set F in X contains a nonempty τ-discrete open in F subset, $(**)$

then X is hereditarily τ-pseudoparacompact iff X is τ-discrete.

Proof. Let $\mathscr{P} = \{U: U$ is an open τ-discrete subset of $X\}$. According to $(**)$ the family $\mathscr{P}$ is not empty. Besides, the union $\bigcup \mathscr{P}$ is τ-discrete, since $\bigcup \mathscr{P}$ is τ-pseudoparacompact and $\mathscr{P}$ is its cover by τ-discrete open sets. To complete the proof it is sufficient to notice now that $\bigcup \mathscr{P} = X$: otherwise according to $(**)$ there would exist a nonempty τ-discrete open in $X \setminus \bigcup \mathscr{P}$ subset U_0, and hence $U_0 \cup (\bigcup \mathscr{P})$ would be an open τ-discrete subset of X. However, this contradicts the definition of the family $\mathscr{P}$. $\square$

Since each scattered space satisfies, obviously, the condition (* *), Proposition 4 implies immediately

Proposition 4′. *A scattered space X is τ-discrete iff it is hereditarily τ-pseudoparacompact.*

Corollary 5. *A scattered space X is σ-discrete iff it is hereditarily pseudoparacompact.*

Noticing that the condition $l(X) \leqslant \tau$ in an obvious way implies that X is τ-pseudoparacompact, we can derive from Proposition 4′ also the following

Corollary 6. *If X is a scattered space, then* $\mathrm{hl}(X) = |X|$ (see e.g. [2]).

(We use the standard notation to denote cardinal properties of a topological space: $l(X)$ is the Lindelöf number of X, $\mathrm{hl}(X)$ is its hereditary Lindelöf number, i.e., $\mathrm{hl}(X) = \sup\{l(X')\colon X' \subset X\}$, $\psi(X)$ is the pseudocharacter of X, $d(X)$ is the density of X and $\mathrm{hd}(X)$ is the hereditary density of X, i.e., $\mathrm{hd}(X) = \sup\{d(X')\colon X' \subset X\}$.)

Assuming additionally regularity of the space X one can obtain the following modifications of Proposition 4′:

Proposition 4″. *If X is a regular scattered space and $\psi(X) \leqslant \tau$, then X is τ-discrete iff it is τ-pseudoparacompact.*

Proof. Let $\mathscr{P} = \{U\colon U$ is an open τ-discrete subset of $X\}$; we are to show that $\bigcup \mathscr{P} = X$. Assume that $X \setminus \bigcup \mathscr{P} = F \neq \emptyset$. Then there exists a point x_0 isolated in F and a family $\{B_\alpha\colon \alpha < \tau\}$ of open (in X) neighborhoods of the point x_0 such that $\{x_0\} := \bigcap\{B_\alpha\colon \alpha < \tau\}$ and besides $\overline{B_0} \cap F = \{x_0\}$. For each $\alpha < \tau$ let $H_\alpha = \overline{B_0} \setminus B_\alpha$. It is clear that $H_\alpha \subset \bigcup \mathscr{P}$ and H_α is τ-pseudoparacompact as a closed subset of X. Since $\mathscr{P}$ is an open cover of H_α by τ-discrete sets, it follows from here that H_α is τ-discrete itself. Noticing that $\overline{B_0} = (\bigcup\{H_\alpha\colon \alpha < \tau\}) \cup \{x_0\}$ we conclude that $\overline{B_0}$ is τ-discrete. However this contradicts the definition of $\mathscr{P}$, because $x_0 \in B_0 \setminus \bigcup \mathscr{P}$. The obtained contradiction completes the proof. □

Proposition 4‴. *A regular scattered space X is strictly τ-discrete iff it is τ-pseudoparacompact and $\psi(X) \leqslant \tau$.*

Proof. If X is strictly τ-discrete, then $\psi(X) \leqslant \tau$. Let $\mathscr{P} = \{U\colon U$ is an open strictly τ-discrete subset of $X\}$. One can prove now the proposition quite analogously to the proof of Proposition 4″ substituting everywhere the condition of strict τ-discreteness for the condition of τ-discreteness and substituting the condition of τ-paracompactness for the condition of τ-pseudoparacompactness. □

Corollary 7. *If X is a regular scattered space, then* $|X| \leqslant \psi(X) \cdot l(X)$ [3, Theorem 9].

We pass now the proof of results on decomposition of a space into k-scattered subspaces; as Theorem 2′ shows, these results give a key to the corresponding estimations of the cardinality of a space. An important role here, as well as in [7], will play the following concept:

Definition 8 [7]. Subset A of X is called *rc-compact*, if for each countable $N \subset A$ the following implication holds:

(i) N has a cluster point in A if (and only if) N has a cluster point in X.

(A point $x \in X$ is called cluster for N if $(U_x \setminus \{x\}) \cap N \neq \emptyset$ for each of its neighborhoods U_x.)

Clearly, a subset A of X is rc-compact if (and only if) implication (i) holds for each discrete in itself subset $N \subset A$.

Proposition 9. *Let $f: X \to Y$ be a closed continuous mapping. If A is an rc-compact subset in T_1-space X, then $f(A)$ is rc-compact in space Y.*

Proof. Let M be a countable discrete (in itself) set, $M \subset f(A)$ and assume that M has a cluster point in Y. We shall show that there exists a point $y_0 \in f(A)$ which is a cluster point for M.

Indeed, for every $y \in M$ having chosen a point $x_y \in f^{-1}(y) \cap A$, we get a set $N = \{x_y \colon y \in M\} \subset A$. It is clear that N is not closed (since f is a closed mapping and $f(N) = M \neq \overline{M}$). Therefore, since A is rc-compact, there exists a point $x_0 \in A \setminus \bigcup \{f^{-1}(y) \colon y \in M\}$ which is a cluster point for N. Then $y_0 = f(x_0)$ is a cluster point for the set $M = f(N)$ and besides $y_0 \in f(A)$. However this just means that $f(A)$ is rc-compact. □

Remark. A set A in X is called *rc-closed* [7], if for each countable $N \subset A$ the following implication holds

(i′) N is closed in X if (and only if) N is closed in A.

Notice that the concepts of rc-compactness and rc-closedness, which are generally different, coincide, in particular, in case of a countably compact space X, and in all statements of [7] they are interchangeable. The only exception in this respect is [7, Proposition 3.11], the proof of which was in fact reproduced above (see Proposition 9); as it has been noticed by V. Tkacuk, this proof is valid only for rc-compact sets.

In the sequel we shall need also the following obvious fact [7, Proposition 3.13].

Proposition 10. *For a set A in a sequential T_2-space X the following conditions are equivalent:*

(a) *A is rc-compact in X;*

(b) *A is rc-closed in X;*

(c) *A is closed in X.*

Indeed, if A is not closed in X, then, since X is sequential, there exists a sequence $\{x_n: n < w_0\}$ converging to a point $x \notin A$ and hence A is not rc-compact. It remains to notice only that the implications (c) $\Rightarrow$ (b) $\Rightarrow$ (a) are obvious for each space X.

From Propositions 9 and 10 easily follows

Proposition 10′. *If $f: X \to Y$ is a closed continuous mapping, Y is a sequential T_2-space and A is an rc-compact set in X, then the set $f(A)$ is closed in Y.*

Proposition 11. *If A is rc-compact in X and B is closed in X, then $A \cap B$ is rc-compact in B and hence also in X.*

We shall need also the following simple statement [7, Proposition 3.15]:

Proposition 12. *If $A \subset X$, then there exists a set $\tilde{A}$ which is rc-closed in X (and hence also rc-compact in X) and besides $A \subset \tilde{A}$ and $|\tilde{A}| \leqslant |A|^{\aleph_0}$.*

Let's recall that a family $\{A_\alpha: \alpha < \nu\}$ of sets of X is said to be a chain, if $A_\alpha \subset A_\beta$ for every $\alpha < \beta < \nu$.

Proposition 13. *Let $\mathscr{A} = \{A_\alpha: \alpha < \nu\}$ be a chain in a space X such that $|A_\alpha| < \tau$ for all $\alpha < \nu$. Then:*

(1) $|\bigcup \mathscr{A}| \leqslant \tau$, *and moreover*

(2) *if* $\mathrm{cf}(\tau) \neq \mathrm{cf}(\nu)$, *then* $|\bigcup \mathscr{A}| < \tau$;

(3) *if* $\bigcup \mathscr{A} \neq A_\alpha$ *and* $A_\alpha = \overline{A}_\alpha$ *for all* $\alpha < \tau$, *then* $\mathrm{cf}(\nu) \leqslant d(\bigcup \mathscr{A})$;

(4) *if* $d(\bigcup \mathscr{A}) < \mathrm{cf}(\tau)$ *and* $A_\alpha = \overline{A}_\alpha$ *for all* $\alpha < \nu$, *then* $|\bigcup \mathscr{A}| < \tau$.

Proof. (1) Assume that $|\bigcup \mathscr{A}| > \tau$, then there exists a set $M \subset \bigcup \mathscr{A}$ such that $|M| = \tau$. For each $x \in M$ fix an index $\alpha(x)$ in such a way that $x \in A_{\alpha(x)}$, and let $\beta = \sup\{\alpha(x): x \in M\}$. It is clear that $\beta = \nu$ (otherwise $M \subset A_\beta$ and hence $|A_\beta| \geqslant \tau$). However, this means that the set of ordinals $\{\alpha(x): x \in M\}$ is cofinal in ν, i.e., $\mathrm{cf}(\nu) \leqslant |M| \leqslant \tau$, and therefore $|\bigcup \mathscr{A}| = |\bigcup\{A_{\alpha(x)}: x \in M\}| \leqslant |M| \cdot \sup\{|A_{\alpha(x)}|: x \in M\} \leqslant \tau$.

(2) Assume that $|\bigcup \mathscr{A}| = \tau$ and for each $\lambda < \tau$ find a cardinal $\alpha(\lambda) < \nu$ such that $|A_{\alpha(\lambda)}| \geqslant \lambda$ (According to point (1) it would mean that $|\bigcup \mathscr{A}| \leqslant \lambda < \tau$, if such a cardinal $\alpha(\lambda)$ does not exist!) Having chosen a set $\{\lambda_\beta: \beta < \mathrm{cf}(\tau)\}$ of cardinals in such a way that $\lambda_\beta < \tau$ and $\sum\{\lambda_\beta: \beta < \mathrm{cf}(\tau)\} = \tau$, we obtain $|\bigcup\{A_{\alpha(\lambda_\beta)}: \beta < \mathrm{cf}(\tau)\}| = \tau$ and hence the family $\{A_{\alpha(\lambda_\beta)}: \beta < \mathrm{cf}(\tau)\}$ is cofinal in $\mathscr{A}$. Thus, $\mathrm{cf}(\nu) \leqslant \mathrm{cf}(\tau)$.

To prove the converse inequality consider a subset $L \subset \nu$ which is cofinal in ν and such that $|L| = \mathrm{cf}(\nu)$. It is clear that the corresponding family $\{A_\alpha: \alpha \in L\}$ is cofinal in the chain $\mathscr{A}$, and hence the set of cardinals $\{|A_\alpha|: \alpha \in L\}$ is cofinal for τ (since $|A_\alpha| < \tau$ for all α) and therefore $\mathrm{cf}(\tau) \leqslant \mathrm{cf}(\nu)$.

(3) Choose $M \subset \bigcup \mathscr{A}$ in such a way that $\overline{M} \supset \bigcup \mathscr{A}$ and $|M| = d(\bigcup \mathscr{A})$ and let $\beta = \sup\{\alpha(x): x \in M\}$, where $\alpha(x)$ is taken in such a way that $x \in A_{\alpha(x)}$. It is clear

that $\beta = \nu$ (otherwise $A_\beta \supset M$ and hence $A_\beta = \overline{A}_\beta \supset \bigcup \mathscr{A}$), and therefore $\mathrm{cf}(\nu) \leqslant |M| = d(\bigcup \mathscr{A})$.

(4) Assuming that $\bigcup \mathscr{A} \neq A_\alpha$ for all $\alpha < \nu$ (otherwise the statement is obvious) we conclude by point (3) that $\mathrm{cf}(\nu) \leqslant d(\bigcup \mathscr{A})$, and hence $\mathrm{cf}(\nu) < \mathrm{cf}(\tau)$. In accordance with point (2) we conclude from here that $|\bigcup \mathscr{A}| < \tau$. □

In case $\tau = \aleph_1$ from Proposition 13 in an obvious way follows

Proposition 13′. *If $\mathscr{A}$ is a chain of countable closed subsets of X and X is hereditarily separable, then the set $\bigcup \mathscr{A}$ is also countable*

Lemma 14 (Main). *Let $\mathscr{A} = \{A_\alpha\colon \alpha < \nu\}$ be a chain of rc-compact subsets of X, $\bigcup \mathscr{A} = X$ and assume that for each $A'_\alpha = A_\alpha \setminus \bigcup\{A_\gamma\colon \gamma < \alpha\}$ ($A'_0 = A_0$) there exists a family $\{G^\beta_\alpha\colon \beta < \mu\}$ such that:*

(1) *$A'_\alpha = \bigcup\{G^\beta_\alpha\colon \beta < \mu\}$;*

(2) *each G^β_α is contained in some scattered (in X) compactum $C^\beta_\alpha \subset X$.*

Then $X_\beta = \bigcup\{G^\beta_\alpha\colon \alpha < \mu\}$ is k-scattered in X for all $\beta < \mu$ and, besides, $X = \{X_\beta\colon \beta < \mu\}$.

Proof. Let H be a closed subset of X, for which there exists a closed mapping $f: H \to I = [0, 1]$ such that $|f(H)| > \aleph_0$. Then by letting $A_\nu = X$ and $\bar{\alpha} = \min\{\alpha \leqslant \nu\colon |f(H \cap A_\alpha)| \geqslant \aleph_1\}$ we get a chain $\{f(H \cap A_\alpha)\colon \alpha < \bar{\alpha}\}$ of countable closed subsets of I. (Indeed, according to Proposition 11 the sets $H \cap A_\alpha$ are rc-compact, and hence, by Proposition 10′ the sets $f(H \cap A_\alpha)$ are closed in I.) Applying Proposition 13′ we conclude from here that the set $\bigcup\{f(H \cap A_\alpha)\colon \alpha < \bar{\alpha}\} = f(H \cap (\bigcup\{A_\alpha\colon \alpha < \bar{\alpha}\}))$ is countable and hence

$$|f(H \cap A'_{\bar{\alpha}})| \geqslant \aleph_1 \text{ (and moreover, } |f(H \cap A'_{\bar{\alpha}})| \geqslant \mathfrak{c}). \qquad (*)$$

(Indeed, $f(H \cap A'_{\bar{\alpha}}) = f(H \cap A_{\bar{\alpha}} \setminus H \cap (\bigcup\{A_\alpha\colon \alpha < \bar{\alpha}\})) \supset f(H \cap A_{\bar{\alpha}}) \setminus f(H \cap (\bigcup\{A_\alpha\colon \alpha < \bar{\alpha}\}))$.)

On the other hand, for each $\beta < \mu$ the set $f(C^\beta_\alpha \cap H)$ is a scattered compactum (this follows from the fact that the mapping f has an irreducible restriction—see e.g. [11]) and hence, according to Corollary 6 or Proposition 4‴, it is countable.

It follows now, that $|f(G^\beta_\alpha \cap H \cap A_{\bar{\alpha}})| \leqslant \aleph_0$ for each $\beta < \mu$ and hence, according to $(*)$,

$$f(H \setminus X_\beta) \neq \emptyset \text{ (and even } |f(H \setminus X_\beta)| \geqslant \mathfrak{c}, \qquad (**)$$

and moreover $H \setminus X_\beta \neq \emptyset$. (Indeed, since $A'_{\bar{\alpha}} \cap X_\beta = G^\beta_{\bar{\alpha}}$, it follows that

$$f(H \setminus X_\beta) \supset f(A'_{\bar{\alpha}} \cap H \setminus A'_{\bar{\alpha}} \cap X_\beta \cap H)$$
$$= f(A'_{\bar{\alpha}} H \setminus G^\beta_{\bar{\alpha}} \cap H) \supset (f(A'_{\bar{\alpha}} \cap H) \setminus f(G^\beta_{\bar{\alpha}} \cap H)) \neq \emptyset.)$$

To complete the proof notice that for each unscattered compactum F there exists a surjective mapping $f: F \to I$ (see e.g. [7, Theorem 4.2, or Theorem 4.5]) and hence, if besides F is closed in X, then $F \setminus X_\beta \neq \emptyset$ for each $\beta < \mu$. However, this means that all X_β are k-scattered.

Finally, the equality $X = \bigcup\{X_\beta\colon \beta < \mu\}$ in an obvious way follows from condition (1). □

It is clear that in case $|A_\beta| \leqslant \mu$ we have $|A'_\alpha| \leqslant |A_\alpha| \leqslant \mu$ and $A'_\alpha = \bigcup\{\{x_\beta\}\colon \beta < \mu\}$ and hence from Lemma 14 immediately follows:

Lemma 14′. *Let $\mathscr{A} = \{A_\alpha\colon \alpha < \nu\}$ be a chain of rc-compact subsets, $\bigcup \mathscr{A} = X$ and $|A_\alpha| \leqslant \mu$ for all $\alpha < \nu$. Then X is a union of $\leqslant \mu$ compact-scattered subspaces.*

Lemma 15. *Let $\nu^{1/\aleph_0} = \nu$ (i.e., if $\lambda < \nu$ then also $\lambda^{\aleph_0} < \nu$). Then for each X such that $|X| = \nu$ there exists a chain $\mathscr{A} = \{A_\alpha\colon \alpha < \nu\}$ of rc-compact subsets of X such that $|A_\alpha| < \nu$ for all $\alpha < \nu$ and $\bigcup \mathscr{A} = X$.*

Proof. Let $X = \{x_\alpha\colon \alpha < \nu\}$. Assume that for all $\alpha < \alpha'$ where $\alpha' < \nu$ we have constructed the sets A_α such that

(0) $A_0 = \{x_0\}$;

(1) $\{A_\alpha\colon \alpha < \alpha'\}$ is a chain in X and $x_\alpha \in A_\alpha$ for all $\alpha < \alpha'$;

(2) $|A_\alpha| \leqslant |\alpha|^{\aleph_0}$ (if $\alpha > 1$);

(3) A_α is rc-compact in X.

Consider the set $C_{\alpha'} = \{x_{\alpha'}\} \cup (\bigcup\{A_\alpha\colon \alpha < \alpha'\})$, then $|C_{\alpha'}| \leqslant \sum\{|A_\alpha|\colon \alpha < \alpha'\} \leqslant |\alpha'|^{\aleph_0} \cdot |\alpha'| = |\alpha'|^{\aleph_0}$ (if $\alpha' > 1$), and hence, according to Proposition 12 there exists an rc-compact (in X) set $A_{\alpha'} \supset C_{\alpha'}$ such that $|A_{\alpha'}| \leqslant |C_{\alpha'}|^{\aleph_0} = |\alpha'|^{\aleph_0}$. Obviously the family $\{A_\alpha\colon \alpha < \nu\}$ thus obtained satisfies the conditions (0)–(3) and hence it is the desired chain. □

If $\nu = (\mu^{\aleph_0})^+$, then, obviously, for each $\lambda < \nu$ we have $\lambda^{\aleph_0} \leqslant \mu^{\aleph_0} < \nu$, i.e., $\nu^{1/\aleph_0} = \nu$. Thus, from Lemma 14′ and 15 follows:

Theorem 16. *If $\mu^{\aleph_0} = \mu$ and $|X| \leqslant \mu^+$, then X can be decomposed into $\leqslant \mu$ compact-scattered subspaces, i.e., $X = \bigcup\{X_\beta\colon \beta < \mu\}$ where all X_β are compact-scattered.*

The main result of the paper follows now easily from Theorems 2′ and 16:

Theorem 17. *If each subspace X' of a space X is a union of $\leqslant \lambda$ compacta (i.e., $X' = \bigcup\{C_\beta\colon \beta < \lambda\}$, where all C_β are compacta), then $|X| \leqslant \lambda^{\aleph_0}$.*

Proof. Consider a set $X' \subset X$ such that $|X'| \leqslant (\lambda^{\aleph_0})^+$, By setting $\lambda^{\aleph_0} = \mu$ we get from Theorem 16 that $X' = \bigcup\{X_\beta\colon \beta < \mu\}$ where all X_β are compact-scattered. Theorem 2′ allows to conclude from here that $|X'| \leqslant \mu$, and hence $|X| \leqslant \mu = \lambda^{\aleph_0}$. □

Corollary 18. *If $\mu^{\aleph_0} = \mu$, then $|X| \leqslant \mu$ iff each subspace of X can be represented as a union of $\leqslant \mu$ compacta.*

(Notice that the first result in this direction was obtained in [4].) At the same time the problem whether the statement of Corollary 18 is true "naively" for each infinite cardinal μ (i.e., without the assumption $\mu^{\aleph_0} = \mu$) remains open. On the other hand, under additional axiomatic assumptions, the authors of this paper have earlier proved an essentially more strong (than it is necessary, as Theorem 2′ shows, for the estimation of the cardinality) theorem on decomposition [7, Theorems 8.3, 9.7] (cf. Theorem A), from which by virtue of Theorem 2′ immediately follows

Theorem 19 (see [17]). *Let each subspace of a space X be representable as a union of* $\leqslant \mu$ *compacta, where* μ *is an infinite cardinal. Then* $|X| \leqslant \mu$ *in each one of following cases:*

(1) $\mu < \mathfrak{c}_{w_0}$;
(2) $\mu < \mathfrak{c}_{w_1}$, *under assumption* $\aleph_1 < \mathfrak{c}$;
(3) μ *is arbitrary under assumption* $(ACP^{\#})$ *or* $(V = L)$.

(Here ACP$^{\#}$ = (ACP) & ($\aleph_1 < \mathfrak{c}$); ACP = "For each cardinal $\mu \geqslant \mathfrak{c}$ there exists a cardinal $\tau < \mathfrak{c}$ such that $\mu^{\aleph_0} \leqslant \mu_\tau$" and $V = L$ is the Hödel axiom of constructibility.)

We shall mention here also the following result; it can be proved similarly to the proof of Theorem 19 having applied a modification of our Lemma 19 which makes a more complete use of the technique developed in [7, Lemma 6.11(d)]:

Theorem 19′. *Let each subspace of X be representable as a union of* $\leqslant \mu = \mu^{\aleph_0}$ *compacta. Then also* $|X| \leqslant \mu$ *in each one of the following cases:*

(1) $|X| < \mu_{w_0}$;
(2) $|X| < \mu_{w_1}$, *and* $\aleph_1 < \mathfrak{c}$.

(Here $\mu_1 = \mu^+$ and $\mu_\alpha = \min\{\nu\colon \nu \geqslant \mu_\beta, \forall \beta < \alpha\}$ (see [7]).)

Theorem 20. *If each subspace of X is representable as a union of* $\leqslant \mu$ *compacta, where* $\mu \geqslant \aleph_0$, *then* $|X| < 2^\mu$ *and* $\operatorname{hd}(X) \leqslant \mu$.

Proof. It is clear that each space has an everywhere dense left subspace (see e.g. [13]). On the other hand each left compactum is scattered [3,16], and hence each space Z contains an everywhere dense k-scattered subspace $Z' \subset Z$.

On the conditions of the theorem $|X'| \leqslant \mu$ for each k-scattered subspace X' of X (see the proof of Theorem 2). It follows from here that $\operatorname{hd}(X) \leqslant \mu$. Therefore by setting $\mathscr{F} = \{F \subset X\colon F \text{ closed in } X,\ |F| \geqslant 2^\mu\}$ we conclude that $|\mathscr{F}| \leqslant 2^\mu$ and hence, by virtue of [7, Proposition 2.3] there exist subspaces X_0 and X_1 of X such that $X = X_0 \cup X_1$ and for each closed in X subset $H \subset X_i$ $(i = 0, 1)$ it holds $|H| < 2^\mu$. Now taking into account that $\operatorname{cf}(2^\mu) > \mu$, and representing X as a union of $\leqslant \mu$ compacta H_β: $X_i = \bigcup\{H_\beta\colon \beta < \mu\}$, $i = 0, 1$, we conclude in case of a Hausdorff space X that $|X_i| < 2^\mu$, $i = 0, 1$ and hence $|X| < 2^\mu$. To complete the proof it is sufficient to notice that the assumption of Hausdorffness of the space X is unessential; because on the conditions of the theorem it is sufficient to show that $|H| < 2^\mu$ for each compactum $H \subset X$. □

Note added in proof

Professor I. Juhász has kindly informed me that recently J. Gerlits, A. Hajnal and Z. Szentmiklossy have proved that for Hausdorff spaces the statement of Theorem 19 holds without any set-theoretic assumptions. Namely *if each subspace of a Hausdorff space X is the union of μ compact subsets, then* $|X| \leqslant \mu$ (see J. Gerlits, A. Hajnal, Z. Szentmiklossy, On the cardinality of certain Hausdorff spaces, Discrete Math. 108 (1992) 31–35).

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