Ramsey Varieties of Finite Groups

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The Ramsey problem is considered for certain finite varieties of non-abelian groups. It is shown that if these contain non-abelian $p$-groups the variety is not even $(C_p, 2)$-Ramsey. Some positive results for varieties with elementary abelian Sylow subgroups are proved.

1. INTRODUCTION

In recent years several authors (see, for example, Deuber and Rothschild [12], Jezek and Nesetril [6] and Voigt [8]) have considered the following problem:

Let $\mathcal{K}$ be a class of finite algebras closed under the operations of taking quotients, subalgebras and finite direct products. For any algebras $D$ and $E$ in $\mathcal{K}$ let $\binom{E}{D}$ denote the set of subalgebras of $E$ isomorphic to $D$. For what $A$, $B$ in $\mathcal{K}$ and $r$ a natural number does there exist a $C$ in $\mathcal{K}$ such that for any partition of $\binom{E}{D}$ into $r$ classes there exists a member $B$ of $\binom{E}{D}$ such that $(\frac{B}{D})$ is contained in one of the classes?

$\mathcal{K}$ is said to be $(A, r)$-Ramsey for $B$ if the above holds. It is $(A, r)$-Ramsey if it is $(A, r)$-Ramsey for all $B$ and it is a Ramsey variety if it is $(A, r)$-Ramsey for all $A$ and each $r$.

Jezek and Nesetril [6] showed that any $\mathcal{K}$ is 1-Ramsey (where 1 denotes the one element algebra); Deuber and Rothschild [2] showed that if $\mathcal{K}$ is the set of finite abelian groups then $\mathcal{K}$ is not $(A, r)$-Ramsey unless the Sylow subgroups of $A$ are homocyclic, and Voigt [8] completed the result by proving that $\mathcal{K}$ was indeed $(A, r)$-Ramsey for all $A$ which are direct products of homocyclic groups.

In this paper I shall consider the case where $\mathcal{K}$ is the set of finite groups in the variety generated by a finite group, $G$, so that $\mathcal{K} = QSD(G)$ (the finite variety generated by $G$). I shall prove the following results:

THEOREM 1.1. Let $\mathcal{K} = QSD(G)$ where $G$ has non-abelian $p$-Sylow subgroups, then $\mathcal{K}$ is not $(C_p, r)$-Ramsey for $r \geq 2$.

THEOREM 1.2. Let $\mathcal{K} = QSD(G)$, where $G$ is the dihedral group of order $2p$ and $p$ is an odd prime.

(I) If $H \in \mathcal{K}$ has order $2p^k$ then $\mathcal{K}$ is $(C_2, r)$-Ramsey for $H$.

(II) If $H \in \mathcal{K}$ satisfies one of the following conditions:

(i) $H$ has order $2p^k$ and has no proper direct decomposition;

(ii) $H$ has order $p^k$;

then $\mathcal{K}$ is $(C_p, r)$-Ramsey for $H$, $(1 \leq l \leq k)$.

Note that if $G$ is a group with abelian but not elementary abelian $p$-Sylow subgroups then $QSD(G)$ will contain non-homocyclic $p$-groups and so, by Voigt's result, will not be Ramsey. Hence, if there is a Ramsey variety of non-abelian groups it must be sought among those groups having elementary abelian Sylow subgroups for all primes. Theorem 1.2 is a small step in this direction.
2. Non-abelian p-Sylow Subgroups

Before stating the general result on which the proof of Theorem 1.1 is based, I need to recall some nomenclature.

A word, \( w(x_1, \ldots, x_n) \), is an element of \( \text{gp}\{x_1, \ldots, x_n\} \), the free group on countably many generators. If \( g_1, \ldots, g_n \) belong to a group \( G \) then \( w(g_1, \ldots, g_n) \) denotes the image of \( w \) under a homomorphism, taking \( x_i \) to \( g_i \) (loosely speaking, \( w(g_1, \ldots, g_n) \) is the element of \( G \) obtained by substituting \( g_1, \ldots, g_n \) for the ‘variables’ \( x_1, \ldots, x_n \)).

The word subgroup, \( w(G) = \text{gp}\{w(g_1, \ldots, g_n) | g_1, \ldots, g_n \in G\} \) and the marginal subgroup, \( M_w(G) = \{z \in G | \forall g_1, \ldots, g_n \in G \} \). For example, if \( w(x_1, x_2) = x_1x_2x_1^{-1}x_2^{-1} \) then \( w(G) \) is the commutator subgroup of \( G \) and \( M_w(G) \) is the centre of \( G \).

THEOREM 2.1. Let \( G \) be a group and \( w_1, w_2 \) words such that \( w_1(G) \leq M_w_2(G) \). If there exist subgroups \( H_1, H_2 \) of \( G \), both isomorphic to the group \( H \), such that \( H_1 \leq w_1(G) \) and \( H_2 \not\leq M_w_2(G) \) then \( \mathcal{X} = \text{QSD}(G) \) is not \((H, r)\)-Ramsey for any \( r \geq 2 \).

PROOF. We first note that if \( L \) is any group in \( \text{QSD}(G) \) then \( w_1(L) \leq M_w_2(L) \) since this property is clearly preserved by the three operations. In particular, for a subgroup \( A \leq B \), say, we have \( w_1(A) \leq A \cap w_1(B) \leq A \cap M_w_2(B) \leq M_w_2(A) \).

For any group \( L \) in \( \text{QSD}(G) \), partition \( \frac{1}{2} \) into two classes \( \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_1 \) containing all subgroups isomorphic to \( H \) which lie in \( M_w_2(L) \) and \( \mathcal{P}_2 \) containing the remainder. For any subgroup of \( L \) isomorphic to \( G \) we have a subgroup \( H_1 \) isomorphic to \( H \) lying in \( w_1(G) \) and hence in \( w_1(L) \leq M_w_2(L) \). Thus \( H_1 \in \mathcal{P}_1 \). Also we have a subgroup, \( H_2 \), isomorphic to \( H \) outside \( M_w_2(G) \) and hence outside \( M_w_2(L) \). Thus \( H_2 \in \mathcal{P}_2 \) and \( \mathcal{X} \) is not \((H, 2)\)-Ramsey for \( G \). Thus \( \mathcal{X} \) is not \((H, r)\)-Ramsey for any \( r \geq 2 \).

COROLLARY 2.2. If \( G \) is a non-abelian finite p-group then \( \mathcal{X} = \text{QSD}(G) \) is not \((C_p, r)\)-Ramsey.

PROOF. First of all, we note that we can assume \( G \) is monolithic. For, if not, \( G \) is a subdirect product of monolithic quotient groups, \( M_1, \ldots, M_s \), say, and since each monolith is central of order \( p \) we can form the central product of \( M_1, \ldots, M_s \) which is a monolithic group generating the same variety (since it contains each of the \( M_i \)'s as a subgroup and is a quotient of their direct product).

Since \( G \) is monolithic it has a unique central subgroup of order \( p \). We now have two cases to consider according to whether \( p \) is 2 or not.

(I) \( p \neq 2 \). By Theorem 12.5.2 of Hall [5], since \( G \) is not cyclic, \( G \) must contain a non-central group isomorphic to \( C_p \). Suppose \( G \) has class \( c \), so that \( G \) satisfies \([x_1, \ldots, x_{c+1}] = 1\). Take \( w_1 = [x_1, \ldots, x_c], w_2 = [x_1, x_2] \). Then \( 1 \neq w_1(G) \leq M_w_2(G) = Z(G) \). So \( w_1(G) \) contains a subgroup isomorphic to \( C_p \) and \( G \) possesses another subgroup isomorphic to \( C_p \) outside \( Z(G) \). Applying Theorem 2.1 we have the required result.

(II) \( p = 2 \). Again by Theorem 12.5.2, of [5], we have that either \( G \) has a non-central \( C_2 \), in which case the proof proceeds as above, or \( G \) is a generalised quaternion group, that is

\[
G = \text{gp}\{a, b | a^2 = b^{2^n-1}, a^{2^n} = 1, a^{-1}ba = b^{-1}\}, \quad n \geq 2.
\]

However, if we take \( G \times C_4 \), where the latter group is generated by \( c \), and factor out by the normal cyclic group of order 2 generated by \( a^2c^2 \), the subgroup of the quotient group generated by the images of \( ac \) and \( b \) is the dihedral group of order \( 2^{n+1} \). A similar process applied to the dihedral group yields the generalised quaternion group, so these generate...
(2.3) **Proof of Theorem 1.1.** Let \( G \) be a group with non-abelian \( p \)-Sylow subgroups and let \( P \) be a \( p \)-Sylow subgroup of \( G \). Then any \( p \)-group in \( QSD(G) \) in fact belongs to \( QSD(P) \). Thus \( QSD(G) \) is \((C_p, r)\)-Ramsey only if \( QSD(P) \) is. Since from Corollary 2.2 we know that \( QSD(P) \) is not \((C_p, r)\)-Ramsey, Theorem 1.1 follows.

3. **Groups with Elementary Abelian Sylow Subgroups**

The proof of Theorem 1.2 is an application of the Hales-Jewett Theorem [4]. The version given below is based on one in the paper of Deuber, Prömel, Rothschild and Voigt [1].

**Definition 3.1.** Let \( A = \{1, \ldots, t\} \) be a finite set. Let \( I_1, \ldots, I_t \) be non-empty disjoint subsets of \( I = \{1, \ldots, n\} \); then a subset \( C \) of \( A^n \) of the form \( C = \{(x_1, \ldots, x_n) | x_i = x'_i \land (\exists j)(i, i' \in I_j) \land x_i = a_i \land (i \in I \setminus \cup I_j)\} \) is called a combinatorial \( k \)-cube.

**Theorem 3.2.** (Hales-Jewett). For any set \( A \) of size \( t \) and any positive integers \( k \) and \( r \) there exists an integer \( N(t, k, r) \) such that for any \( n > N(t, k, r) \) in any partition of \( A^n \) into \( r \) disjoint subsets, one of the subsets contains a combinatorial \( k \)-cube.

(3.3) **Proof of Theorem 1.2.** Let \( G \) be the dihedral group of order \( 2p \), where \( p \) is an odd prime.

(I) We first note that since a cyclic group of order two contains a unique involution (element of order two) we can work with the set of involutions in the groups considered rather than with the set of subgroups isomorphic to \( C_2 \). Let \( G = gp\{a, b | a^2 = b^p = 1, a^{-1}ba = b^{-1}\} \). Then the set of involutions in \( G \) is \( A = \{a, ab, \ldots, ab^{p-1}\} \). Since the groups in \( QSD(G) \) have elementary abelian Sylow subgroups they split over every normal subgroup [5, Theorem 15.8.6]. Thus \( QSD(G) = SD(G) \). Hence we may assume \( H \) is a subgroup of \( G_1 \times \cdots \times G_s \), where \( G_i \cong \mathbb{Z} \). If we take \( s \) as small as possible, then \( G_i \cap H > 1 \), since otherwise \( H \) is isomorphic to its projection on the other factors. If \( H \cap G_i = C_2 \) then \( H = C_2 \times H_2 \) has a unique subgroup of order 2 and the result is trivial. If \( G_i \leq H \) then \( H \cong H_1 \times H_2 \), where \( H_1 \cong G \) and \( H_2 \) is an elementary abelian \( p \)-group. Otherwise, \( G_i \cap H \cong C_p \) and the projection of \( H \) on \( G_i \) is either \( G_i \) or \( C_p \). Let \( K \) be the direct product of the \( G_i \) on which \( H \) has projection \( G_i \) and \( L \) that on which it has projection \( C_p \). Then again \( H = H_1 \times H_2 \), where \( H_1 = H \cap K \) and \( H_2 = H \cap L \) is an elementary abelian \( p \)-group. Since the result is clearly true for \( H \) if and only if it is true for \( H_1 \), we may assume that either \( H \cong G \) or \( H \) is a subdirect product of \( k \geq 2 \) copies of \( G \) with projection \( G \) and intersection \( C_p \) for each factor. Then \( H \) contains \( 2p^s \) involutions which may be written as

\[
\{(ab^0, \ldots, ab^{p^s}) | 0 \leq x_i \leq p - 1\}.
\]

Let \( C = G^n \), where \( n \geq N(p, k, r) \); then the set of involutions in \( C \) contains \( A^n \) so any partition of the involutions in \( C \) into \( r \) parts induces a partition of \( A^n \). By (3.2) in any such partition one set contains a combinatorial \( k \)-cube, but such a \( k \)-cube will consist precisely of the involutions in a group isomorphic to \( H \). To convince the reader of this statement, I will write out the details for \( p = 3, k = 2 \). The proof in the general case is conceptually no harder, but the details are messy. Thus \( A = \{a, ab, ab^2\} \) and the 9 involutions in \( H \) are \( \{a, a, (a, a), (a, ab), (a, ab^2), \ldots, (ab^2, ab^2)\} \). (Note that the product of any 2 of these has order 3 and the product of any odd number is an involution. A combinatorial 2-cube in \( C^n \) will
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consist of 9 elements of the form \( \{(a, \ldots, a, a, \ldots, a, a, \ldots, a, a, ab, \ldots, ab, \ldots, ab, \ldots, ab, \ldots, ab, a, \ldots, a, a, ab, \ldots, ab, \ldots, ab, \ldots, ab, a, \ldots, a, a, a, a, \ldots, a) \} \), where the \( a \)'s are fixed involutions. (Here I have rearranged the factors so that the places where \( x \) is constant come together and those at which it is fixed come at the end; there is clearly no loss of generality in so doing.) Clearly, the product of any two of these involutions has order 3 and the product of any odd member is an involution. The group they generate is isomorphic to \( H \). Since \( H \) contains only 9 involutions we have the required result.

(II) The results in (II) depend on the Theorem of Graham, Leeb and Rothschild [3, Section 4] (see also Spencer [7]), that a category of finite-dimensional vector spaces has the Ramsey property. Interpreting \( C_p^k \) as a \( k \)-dimensional vector space over \( GF(p) \) we see that this gives us case (ii) immediately since we have an \( n \) (depending only \( k, l \) and \( r \)) such that

\[
\text{any partition of } \left( \begin{array}{c} C_p^n \\ C_p^l \\ C_p \end{array} \right) \text{ into } r \text{ parts yields } B \simeq C_p^k \text{ such that } \\
\left( \begin{array}{c} B \\ C_p^l \end{array} \right) \text{ is contained in one of the parts.}
\]

To deal with case (i) we first note that, using the same arguments as in (I), since \( H \) has no proper direct factor it is the semidirect product of \( C_p^k \) and \( C_2 \), where the 2 cycle inverts all the elements in \( C_p^k \). Let \( n \) be chosen as above and consider \( G^n \). All the \( p \)-subgroups of this lie in the subgroup isomorphic to \( C_p^n \), so

\[
\text{any partition of } \left( \begin{array}{c} G^n \\ C_p^l \end{array} \right) \text{ into } r \text{ parts is a partition of } \left( \begin{array}{c} C_p^n \\ C_p \\ C_p \end{array} \right) \\
\text{and thus we have a } B \simeq C_p^k \text{ such that } \\
\left( \begin{array}{c} B \\ C_p \end{array} \right) \text{ is contained in one of the parts.}
\]

Let \( D \) be the direct product of those factors of \( G^n \) on which \( B \) has non-trivial projection, and let \( d \) be the element of \( D \) whose projection on each factor of \( D \) is \( a \). Then the subgroup of \( D \), generated by \( B \) and \( d \) is a group isomorphic to \( H \) all of whose \( p \)-subgroups lie in \( B \), and the result follows.

\( \square \)

4. CONCLUDING REMARKS

In Theorem 1.2 we obtained some positive results for finite varieties of soluble groups with elementary abelian Sylow subgroups. It would be interesting to see some results for the finite variety generated by \( A_5 \), the smallest simple group (which also has elementary abelian Sylow subgroups). The groups in \( \mathcal{K} = QSD(A_5) \) have a particularly simple structure, being direct products of copies of \( A_5 \) and a soluble group. Thus, if we ask the question: 'Is \( \mathcal{K} (A_5, r) \)-Ramsey for \( A_5 \times A_5 \) ?' we only have to consider the embedding of these groups in direct powers of \( A_5 \). I should indeed be interested to know the answer to this question.

REFERENCES


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