Normal Families of Holomorphic Functions and Mappings on a Banach Space

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Abstract: The authors lay the foundations for the study of normal families of holomorphic functions and mappings on an infinite-dimensional normed linear space. Characterizations of normal families, in terms of value distribution, spherical derivatives, and other geometric properties are derived. Montel-type theorems are established. A number of different topologies on spaces of holomorphic mappings are considered. Theorems about normal families are formulated and proved in the language of these various topologies. Normal functions are also introduced. Characterizations in terms of automorphisms and also in terms of invariant derivatives are presented.

Keywords: holomorphic function, holomorphic mapping, Banach space, normal family, normal function.

Introduction

The theory of holomorphic functions of infinitely many complex variables is about forty years old. Pioneers of the subject were Nachbin [NAC1-NAC17], Gruman and Kiselman [GRK], and Mujica [MUJ]. After a quiet period of nearly two decades, the discipline is now enjoying a rebirth thanks to the ideas of Lempert [LEM1-LEM6]. Lempert has taught us that it is worthwhile to restrict attention to particular Banach spaces, and he has directed our efforts to especially fruitful questions.

The work of the present paper is inspired by the results of [KIK]. That paper studied domains in a Hilbert space with an automorphism group orbit accumulating at a boundary point. As was the case in even one complex variable, normal families played a decisive role in that study. With a view to extending those explorations, it now seems appropriate to lay the foundations for normal families in infinitely many complex variables.

One of the thrusts of the present paper is to demonstrate that normal families may be understood from several different points of view. These include:

MSC Subject Classification: Primary 46B99; secondary 32H02.

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1. Classical function theory
2. Hyperbolic geometry
3. Functional analysis
4. Distribution theory
5. Currents
6. Comparison of different topologies and norms on the space of holomorphic functions

It is our intention to explain these different approaches to the subject and to establish relationships among them.

A second thrust is to relate the normality of a family on the entire space $X$ (or on a domain in $X$) to the normality of the restriction of the family to slices (suitably formulated). This point of view was initiated in [CIK], and it has proved useful and intuitively natural.

Throughout this paper, $X$ is a separable Banach space over the scalar field $\mathbb{C}$, $\Omega$ is a domain (a connected open set) in $X$, $\mathcal{F} = \{f_\alpha\}_{\alpha \in \Lambda}$ is a family of holomorphic functions on $\Omega$, and $D \subseteq \mathbb{C}$ is the unit disc. If $\Omega'$ is another domain in some other separable Banach space $Y$, then we will also consider families $\{F_\alpha\}$ of holomorphic mappings from $\Omega$ to $\Omega'$. Although separability of $X$ is not essential to all of our results, it is a convenient tool in many arguments. Certainly, in the past, the theory of infinite dimensional holomorphy has been hampered by a tendency to shy away from such useful extra hypotheses.

Part of the beauty and utility of studying infinite-dimensional holomorphy is that the work enhances our study of finite-dimensional holomorphy. Indeed, it is safe to say that the present study has caused us to re-invent what a normal family of holomorphic functions ought to be.

One of the interesting features of the present work, making it different from more classical treatments in finite dimensions, is that compact sets now play a different role. If $W$ is a given open set in our space $X$ (say the unit ball in a separable Hilbert space), then $W$ cannot be exhausted by an increasing union of compact sets in any obvious way. Another feature is that, in finite dimensions, all reasonable topologies on the space of holomorphic functions on a given domain are equivalent. In infinitely many variables this is no longer the case, and we hope to elucidate the matter both with examples and results relating the different topologies.

A final note is that there are interesting underlying questions, throughout our study, about the geometry of Banach spaces. We sidestep most of these by concentrating our efforts on separable Banach spaces; most of our deepest results are in separable Hilbert spaces. We intend to study the deeper questions of the geometry of Banach space, and their impact on normal families, in a future work.

It is a pleasure to thank John McCarthy for helpful conversations about various topics in this paper. Eric Bedford also pointed us in some interesting directions.
1 Basic Definitions

We will now define holomorphic functions and mappings and normal families. We refer the reader to the paper [KIK] and the book [MUJ] for background on complex analysis in infinite dimensions.

Definition 1.1 A domain $\Omega \subseteq X$ is a connected open set.

Definition 1.2 Let $\Omega \subseteq X$ be an open set. Let $u : \Omega \to Y$ be a mapping, where $Y$ is some other separable Banach space. For $q \in \Omega$ and $v_1, \ldots, v_k \in X$, we define the derivatives

$$du(q; v_j) = \lim_{\varepsilon \to 0} \frac{u(q + \varepsilon v_j) - u(q)}{\varepsilon}$$

and

$$Du(q; v) = \frac{du(q; v) + idu(q; iv)}{2}.$$

In what follows, we use the word "function" to refer to a (complex) scalar-valued object and "mapping" to refer to a Banach-space-valued object as in Definition 1.2.

A function (mapping) $f$ on $\Omega$ is said to be continuously differentiable, or $C^1$, if $df(q; v)$ exists for every point $q \in \Omega$ and every vector $v$, and if the resulting function $(q, v) \mapsto df(q; v)$ is continuous.

Definition 1.3 Let $\Omega \subseteq X$ be an open set and $f$ a $C^1$-smooth function or mapping defined on $\Omega$. We say that $f$ is holomorphic on $\Omega$ if $Df \equiv 0$ on $\Omega$.

The definition just given of "holomorphic function" or "holomorphic mapping" is equivalent, in the $C^1$ category, to requiring that the restriction of the function or mapping to every complex line be holomorphic in the classical sense of the function theory of one complex variable. We shall have no occasion, in the present paper, to consider functions that are less than $C^1$ smooth, but holomorphicity can, in principle, be defined for rougher functions.

Definition 1.4 Let $\mathcal{F} = \{f_\alpha\}_{\alpha \in A}$ be a family of holomorphic functions on a domain $\Omega \subseteq X$. We say that $\mathcal{F}$ is a normal family if every subsequence $\{f_{j_n}\} \subseteq \mathcal{F}$ either

1.4.1 (normal convergence) has a subsequence that converges uniformly on compact subsets of $\Omega$;

or

1.4.2 (compact divergence) has a subsequence $\{f_{j_n}\}$ such that, for each compact $K \subseteq \Omega$ and each compact $L \subseteq \mathbb{C}$, there is a number $N$ so large that $f_{j_n}(K) \cap L = \emptyset$ whenever $k \geq N$. 
It is convenient at this juncture to define a type of topology that will be of particular interest for us. If $\Omega \subseteq X$ is a domain and $\mathcal{O}(\Omega)$ the space of holomorphic functions on $\Omega$, then we let $\mathcal{B}$ denote the topology on $\mathcal{O}(\Omega)$ of uniform convergence on compact sets. Of course a sub-basis for the topology $\mathcal{B}$ is given by the sets (with $\epsilon > 0$, $g \in \mathcal{O}(\Omega)$ arbitrary, and $K \subseteq \Omega$ a compact set)

$$B_{g,K,\epsilon} = \{f \in \mathcal{O}(\Omega) : \sup_{z \in K} |f(z) - g(z)| < \epsilon\}.$$

It is elementary to verify (or see [MUJ]) that the limit of a sequence of holomorphic functions on $\Omega$, taken in the topology $\mathcal{B}$, will be another holomorphic function on $\Omega$. Indeed, it is clear that the limit of such a sequence is holomorphic on any finite-dimensional slice (this property is commonly called "G-holomorphic"). The limit function is clearly locally bounded. Now it follows (see [MUJ, p. 74]) that the limit is holomorphic.

**Definition 1.5** Let $\Omega \subseteq X$ be a domain. Let $\mathcal{U} \equiv \{U_\alpha\}_{\alpha \in A}$ be a semi-norm topology on the space $\mathcal{O}(\Omega)$ of holomorphic functions on $\Omega$. We say that $\mathcal{U}$ is a \textbf{Montel topology} on $\mathcal{O}(\Omega)$ if the mapping

$$\text{id} : [\mathcal{O}(\Omega), \mathcal{U}] \longrightarrow [\mathcal{O}(\Omega), \mathcal{B}]$$

$$f \longmapsto f$$

is a compact operator.

**Example 1.6** (1) Let $\Omega \subseteq \mathbb{C}^n$ be a domain in finite-dimensional complex space. The topology $\mathcal{B}$ is a Montel topology. This is the content of the classical Montel theorem on normal families (see [MUJ]).

(2) Let $\Omega \subseteq \mathbb{C}$ be a domain in one-dimensional complex space. Let $k$ be a positive integer, and let superscript $(k)$ denote the $k$th derivative. The topology $\mathcal{C}_k$ with sub-basis given by the union of the sets (with $\epsilon > 0$, $g \in \mathcal{O}(\Omega)$ arbitrary, and $K \subseteq \Omega$ a compact set)

$$M_{g,K,\epsilon}^{(j)} = \{f \in \mathcal{O}(\Omega) : \sup_{z \in K} |f^{(j)}(z) - g^{(j)}(z)| < \epsilon\},$$

for $j = 0, 1, \ldots, k$, is a Montel topology. Of course, by integration (and using the Cauchy estimates), the topology $\mathcal{C}_k$ is equivalent to the topology $\mathcal{B}$. [A similar topology can be defined for holomorphic functions on a domain in the finite-dimensional space $\mathbb{C}^n$.]

(3) Let $\Omega \subseteq \mathbb{C}$ be a domain in one-dimensional complex space. The topology $\mathcal{D}$ with sub-basis given by the sets (with $\epsilon > 0$, $g \in \mathcal{O}(\Omega)$ arbitrary, and $\gamma \subseteq \Omega$ the compact image of a closed curve $\gamma : [0, 1] \rightarrow \Omega$)

$$M_{g,\gamma,\epsilon} = \{f \in \mathcal{O}(\Omega) : \sup_{z \in \gamma} |f(z) - g(z)| < \epsilon\}$$

is a Montel topology, as the reader may verify by using the Cauchy estimates. Of course, once again, the maximum principle may be used to check that the topology $\mathcal{D}$ is equivalent to the topology $\mathcal{B}$. 
In the reference [NAC18], Leopoldo Nachbin defined the concept of a seminorm that is "ported" by a compact set. We review the notion here. Let $X$ and $Y$ be separable Banach spaces as usual. Let $\Omega \subseteq X$ be a domain, and let $K \subseteq \Omega$ be a fixed compact subset. We consider the family $\mathcal{H}(\Omega, Y)$ of holomorphic mappings from $\Omega$ to $Y$. A seminorm $\rho$ on $\mathcal{H}(\Omega, Y)$ is said to be ported by the set $K$ if, given any open set $V$ with $K \subseteq V \subseteq \Omega$, we can find a real number $c(V) > 0$ such that the inequality

$$\rho(f) \leq c(V) \cdot \sup_{x \in V} \|f(x)\|$$

holds for every $f \in \mathcal{H}(\Omega, Y)$.

We note that the holomorphic mapping $f$ here need not be bounded on $V$. What is true, however (and we have noted this fact elsewhere in the present paper), is that once $\Omega$ and $K$ are fixed then there will exists some open set $V$ as above on which $f$ is bounded. So that, for this choice of $V$, the inequality (*) will be non-trivial.

Now we use the notion of "seminorm ported by $K$" to define a topology on $\mathcal{H}(\Omega, Y)$ as follows: we consider the topology induced by all seminorms that are ported by compact subsets of $\Omega$. It is to be noted that, in finite dimensions, this new topology is no different from the standard compact-open topology. But in infinite dimensions it is quite different. As an example, let $X = Y = \ell_2$, which is of course a separable Hilbert space. Let a typical element of $\ell_2$ be denoted by $\{a_j\}_{j=1}^{\infty}$, and let the $j$th coordinate be $z_j$. Let $\Omega \subseteq X$ be a domain and let $K \subseteq \Omega$ be a compact set. Consider holomorphic functions $f : \Omega \to \mathbb{C}$. Define a semi-norm by

$$\rho^*(f) = \sum_{j=1}^{\infty} \sup_{z \in K} \left| \frac{\partial f}{\partial z_j} \right| .$$

Then it is clear, by the Cauchy estimates, that $\rho^*$ is ported.

But it is also clear that a typical open set defined by $\rho^*$ will not contain any non-trivial open set from the compact-open topology. Thus this topology is not Montel. Of course it is now a simple matter to generate many other interesting examples of ported seminorms.

Now we have

**Theorem 1.7** Let $\mathcal{F} = \{f_\alpha\}_{\alpha \in A}$ be a family of holomorphic functions on a domain $\Omega \subseteq X$. Assume that there is a finite constant $M$ such that $|f_\alpha(z)| \leq M$ for all $f_\alpha \in \mathcal{F}$ and all $z \in \Omega$. Let $K$ be a compact subset of $\Omega$. Then every sequence in $\mathcal{F}$ has itself a subsequence that converges uniformly on $K$.

**Proof:** Of course the hypothesis of uniform boundedness precludes compact divergence. So we will verify 1.4.1. Fix a compact subset $K \subseteq \Omega$. Then there is a number $\eta > 0$ such that if $k \in K$ then $B(k, 3\eta) \subseteq \Omega$. Select $f_\alpha \in \mathcal{F}$. Now if $k \in K$ and $\ell$ is any point such that $\|k - \ell\| < \eta$ then we may apply the Cauchy estimates (on $B(k, 2\eta)$) to the restriction
of $f_{\alpha}$ to the complex line through $k$ and $\ell$. We find that the $f_{\alpha}$ have bounded directional derivatives. Therefore they are (uniformly) Lipschitz and form an equicontinuous family of functions.

As a result of these considerations, the Arzela-Ascoli theorem applies to the family $\mathcal{F}$ restricted to $K$. Thus any sequence in $\mathcal{F}$ has a subsequence convergent on $K$. \hfill \Box

In practice, it is useful to have a version of Theorem 1.7 that hypothesizes only uniform boundedness on compact sets. This is a tricky point in the infinite-dimensional setting for the following reason: Classically (in finite dimensions), one derives this new result from (the analog of) Theorem 1.7 by taking a compact set $K \subseteq \Omega$ and fattening it up to a slightly larger compact $L \subseteq \Omega$. Since the family $\mathcal{F}$ is uniformly bounded on $L$, an analysis similar to the proof of 1.7 may now be performed. In the infinite-dimensional setting this attack cannot work, since there is no notion of fattening up a compact set to a larger compact set.

Nonetheless, we have several different ways to prove a more general, and more useful, version of Montel’s theorem. The statement is as follows.

**Theorem 1.8 (Montel)** Let $\mathcal{F} = \{f_{\alpha}\}_{\alpha \in \Lambda}$ be a family of holomorphic functions on a domain $\Omega$ in a separable Banach space $X$. Assume that $\mathcal{F}$ is uniformly bounded on compact sets, in the sense that for each compact $L \subseteq \Omega$ there is a constant $M_L > 0$ such that $|f_{\alpha}(z)| \leq M_L$ for every $z \in L$ and every $f_{\alpha} \in \mathcal{F}$. Then every sequence in $\mathcal{F}$ has itself a subsequence that converges uniformly on each compact set $K \subseteq \Omega$. [Note that we are saying that there is a single sequence that works for every set $K$.] Thus $\mathcal{F}$ is a normal family.

**Remark 1.9** We may rephrase Montel’s theorem by saying that the topology $\mathcal{B}$ is a Montel topology.

**Proof of the Theorem:**

Fix a compact set $K \subseteq \Omega$. Of course the family $\mathcal{F}$ is bounded on $K$ by hypothesis. We claim that $\mathcal{F}$ is bounded on some neighborhood $U$ of $K$. To this end, and seeking a contradiction, we suppose instead that for each integer $N > 0$ there is a point $x_N \in U$ such that $|f_{\alpha}(x_N)| > N$. Then the set

$$L = K \cup \{x_N\}_{N=1}^\infty$$

is compact. So the family $\mathcal{F}$ is bounded on $L$. But that contradicts the choice of the $x_N$.

We conclude that, for some $N$, $x_N$ does not exist. That means that there is a number $N_0 > 0$ such that the family $\mathcal{F}$ is uniformly bounded on $U \equiv \{z \in \Omega : \text{dist}(z,K) < 1/N_0\}$. As a result, we may imitate the proof of Theorem 1.7, merely substituting $U$ for $\Omega$. \hfill \Box

**Remark 1.10** We thank Laszlo Lempert for the idea of the proof of 1.8 just presented.

We now indulge in a slight digression, partly for interest’s sake and partly because the argument will prove useful below. In fact we will provide a proof of Theorem 1.8
that depends on the Banach-Alaoglu theorem. This is philosophically appropriate, for it validates in yet another way that a normal families theorem is nothing other than a compactness theorem. After that we will sketch a proof that depends on the theory of currents.

Alternative (Banach-Alaoglu) Proof of Theorem 1.8:

For clarity and simplicity, we begin by presenting this proof in the complex plane \( \mathbb{C} \). The reader who has come this far will have no trouble adapting the argument to \textit{finitely many} complex variable space \( \mathbb{C}^n \). We provide a separate argument below for the infinite dimensional case.

Now fix a domain \( \Omega \subseteq \mathbb{C} \). Let \( \mathcal{F} = \{ f_\alpha \}_{\alpha \in A} \) be a family of holomorphic functions on \( \Omega \) which is bounded on compact sets. Fix a piecewise \( C^1 \) closed curve \( \gamma : [0, 1] \rightarrow \Omega \). Let \( \bar{\gamma} \) denote the \textit{image} of \( \gamma \), which is of course a compact set in \( \Omega \). Consider the functions

\[
g_\alpha = f_\alpha|_{\bar{\gamma}}.
\]

Then each \( g_\alpha \) is smooth on \( \bar{\gamma} \) and the family \( G \equiv \{ g_\alpha \}_{\alpha \in A} \) is bounded by some constant \( M \). So we may think of \( G \subseteq L^\infty(\bar{\gamma}) \) as a bounded set. Since \( L^\infty(\bar{\gamma}) \) is the dual of \( L^1(\bar{\gamma}) \), we may apply the Banach-Alaoglu theorem to extract a subsequence (which we denote by \( \{ g_j \} \) for convenience) that converges in the weak-* topology. Call the weak-* limit function \( g \).

Now fix a point \( z \) that lies in the interior, bounded component of the complement of \( \bar{\gamma} \). Of course the function

\[
t \mapsto \frac{\gamma'(t)}{\gamma(t) - z}
\]

lies in \( L^1(\bar{\gamma}) \). So, by weak-* convergence and the Cauchy integral formula, we know that

\[
g_j(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{g_j(\zeta)}{\zeta - z} d\zeta \rightarrow \frac{1}{2\pi i} \int_{\gamma} \frac{g(\zeta)}{\zeta - z} d\zeta = G(z).
\]

Here the last equality defines the function \( G \).

So we see that the functions \( g_j \), which of course must agree with \( f_j \) at points inside the curve \( \gamma \), tend pointwise to the function \( G \); and the function \( G \) is perforce holomorphic inside the image curve \( \bar{\gamma} \). We will show that in fact the convergence is uniform on compact sets inside of \( \bar{\gamma} \).

So fix a compact set \( K \) that lies in the bounded open set interior to \( \bar{\gamma} \). Fix a piecewise \( C^1 \), simple, closed curve \( \gamma^* \) whose image is disjoint from, and lies inside of, \( \bar{\gamma} \), and which surrounds \( K \). Let \( \eta > 0 \) be the distance of \( K \) to \( \gamma^* \), the image of \( \gamma^* \). Now fix a small \( \epsilon > 0 \) (here \( \epsilon \) should be smaller than the length of \( \gamma^* \)). Choose a set \( E \subseteq \gamma^* \) such that \( E \) has linear measure less than \( \epsilon \) and so that (by Lusin’s theorem)

\[
|g_j(\zeta) - g(\zeta)| < \epsilon
\]

when \( j \) is sufficiently large \( (j > N, \text{ let us say}) \) and \( \zeta \in \gamma^* \setminus E \).
Then, for \( j, k > N \) and \( z \in K \) we have

\[
|g_j(z) - g_k(z)| \leq \frac{1}{2\pi} \int_{E} \left| g_j(\zeta) - g_k(\zeta) \right| \frac{d|\zeta|}{\zeta - z} + \frac{1}{2\pi} \int_{E} \left| g_j(\zeta) - g_k(\zeta) \right| \frac{d|\zeta|}{\zeta - z} = \frac{1}{2\pi} \text{length}(\gamma) \frac{\varepsilon}{\eta} + \frac{1}{2\pi} \cdot \varepsilon \cdot \frac{2M}{\eta}.
\]

Since \( \varepsilon > 0 \) may be chosen to be arbitrarily small, we conclude that \( g_j \to g \) uniformly on the compact set \( K \). That is what we wished to prove for the single compact set \( K \).

We note that this proof may be performed when \( \gamma \) is a positively oriented curve describing any square inside \( \Omega \) with sides parallel to the axes, rational center, and rational side length. Of course it is always possible to produce the curve \( \gamma^* \) as the union of finitely many such curves. As a result, the usual diagonalization procedure may be formed over these countably many curves, producing a single subsequence that converges uniformly on any compact set in \( \Omega \) to a limit function \( G \).

Our next proof depends on the theory of currents. For background in this important technique of geometric analysis, we refer the reader to [FED], [FEF], [MAT], [BLO], [KLI], [LEL], [LEG].

Alternative (Currents) Proof of Theorem 1.8 for Separable Banach Spaces:

We refer to the very interesting paper [ALM] of Almgren. That paper gives a characterization of the dual of the space of all \( k \)-dimensional, real, rectifiable currents in \( \mathbb{R}^N \). Remarkably, Almgren's proof uses both the Continuum Hypothesis and the Axiom of Choice. An examination of Almgren's proof reveals that the arguments are also valid when \( \mathbb{R}^N \) is replaced by any separable Banach space. We take that result for granted, and leave it to the reader to check the details in [ALM].

Accepting that assertion, we see that the hypothesis of uniform boundedness of a family \( \mathcal{F} \) of holomorphic functions on compact subsets of a domain \( \Omega \) in a separable Banach space \( X \) can be interpreted as a boundedness statement about one-dimensional holomorphic currents. Specifically, let \( \mathcal{F} \) be a family of holomorphic functions on a domain \( \Omega \subseteq X \), and assume that \( \mathcal{F} \) is bounded on compact subsets of \( \Omega \). As we have seen (proof of Theorem 1.8), it follows that if \( K \subseteq \Omega \) is any compact set then there is a small neighborhood \( U \) of \( K \), with \( K \subset U \subseteq \Omega \), such that \( \mathcal{F} \) is bounded on \( U \). As a result, we may apply Cauchy estimates to see that if \( \mathcal{F} = \{ f_\alpha \}_{\alpha \in A} \) then \( \mathcal{F}' = \{ \partial f_\alpha \}_{\alpha \in A} \) is bounded on \( K \). But then, by the generalization of Almgren's theorem to infinite dimensions, we may think of \( \mathcal{F}' \) as a bounded family in the dual of the space of 1-dimensional (complex) currents on \( \Omega \). By the Banach-Alaoglu theorem, we may therefore extract from any sequence in \( \mathcal{F}' \) a weak-* convergent subsequence. Call it, for convenience, \( \{ f_j \} \).

But now it is possible to imitate the first alternative proof of Theorem 1.8 as follows. Fix a closed, piecewise \( C^1 \) curve \( \gamma : [0, 1] \to \Omega \) that bounds an analytic disc \( d \) in \( \Omega \). Think of the elements \( \partial f_j \) restricted to the image \( \tilde{\gamma} \) of this curve. They form a bounded family in \( L^\infty(\tilde{\gamma}) \). Thus the first alternative proof may be imitated, step by step, to produce a limit holomorphic function on the analytic disc \( d \). In fact we may even take the argument a
step further. We may look at any $k$-dimensional slice of $\Omega$ and use the Bochner-Martinelli kernel instead of the 1-dimensional Cauchy kernel to find that there is a uniform limit on any compact subset of any $k$-dimensional slice of $\Omega$. This produces the required limit function $G$ for the subsequence $f_j$. [Note that, because we are assuming the space to be separable, we can go further and even extract a subsequence that converges on every compact subset. More will be said about this point in the next remark and in what follows.]

**Remark 1.11** This last is still not the optimal version of what we usually call Montel’s theorem. In the classical, finite-dimensional formulation of Montel’s result we usually derive a single subsequence that converges uniformly on every compact set. The question of whether such a result is true in infinite dimensions is complicated by the observation that it is no longer possible, in general, to produce a sequence of sets $K_1 \subset K_2 \subset \cdots \subset X$ for our Banach space $X$ with the property that each compact subset of $X$ lies in some $K_j$. In fact the full-bore version of Montel’s theorem, as just described, is false. The next example of Y. Choi illustrates what can go wrong, at least in a non-separable Banach space. [We note in passing that most of our examples are in separable spaces—which is the proper venue for the present study. But in some instances we only have examples in the non-separable case.]

**Example 1.12** Consider the Banach space $X = \ell^\infty$. Let

$$e_j = (0, \ldots, 0, 1, 0, \ldots),$$

in which all components except the $j^{th}$ are zero. Let $e_j^*: X \to \mathbb{C}$ be defined by

$$e_j^* \left( \sum_{k=1}^{\infty} a_k e_k \right) = a_j.$$

This function is obviously holomorphic. However, the sequence $\{e_j^*\}$ does not have a subsequence that converges uniformly on compact subsets. To see this, let us assume to the contrary that $\{e_j^*\}_{j=1}^{\infty}$ is a subsequence that converges uniformly on compact subsets. Then in particular it should converge on singleton set consisting of the point $p$ that is given by

$$p = \sum_{m=1}^{\infty} (-1)^m e_{j_m}.$$

But, $e_{j_m}^*(p) = (-1)^m$, and this sequence of scalars does not converge.

It should be noted that this example can be avoided if we demand in advance that the Banach space $X$ be separable. One simply produces a countable, dense family of open balls, extracts a convergent sequence for each such ball, and then diagonalizes as usual. Mujica [MUJ], in his treatment of normal families, achieves the full result by adding a hypothesis of pointwise convergence.

**Proposition 1.13** (p. 74, [MUJ]) Let $U$ be a connected open subset of an arbitrary Banach space $X$ and let $\{f_n : U \to \mathbb{C}\}_{n=1,2,\ldots}$ be a bounded sequence of holomorphic
functions in the compact-open topology. Suppose also that there exists a non-empty open subset $V$ of $U$ such that the sequence $\{f_n(x)\}_n$ converges in $\mathbb{C}$ for every $x \in V$. Then, the sequence $\{f_n\}_n$ converges to a holomorphic function of $U$ uniformly on every compact subset of $U$.

Now we turn our attention to characterizations of normal families that depend on invariant metrics. In what follows, we shall make use of the Kobayashi metric on a domain $\Omega \subseteq X$. It is defined as follows: If $p \in \Omega$ and $\xi \in X$ is a direction vector then we set

$$F^R_\Omega(p; \xi) = \inf \left\{ \|\xi\| : \varphi : D \to \Omega, \varphi \text{ holomorphic, } \varphi(0) = p, \varphi'(0) = \lambda \xi \text{ for some } \lambda \in \mathbb{R} \right\}.$$ 

Here $\|\eta\|$ is the norm of the vector $\eta \in X$.

One of the most useful characterizations of normal families, and one that stems naturally from invariant geometry, is Marty's criterion. We now establish such a result in the infinite dimensional setting.

**Proposition 1.14** Let $X$ be a Banach space. Let $\Omega \subseteq X$ be a domain and let $\mathcal{F} = \{f_\alpha\}_{\alpha \in \mathcal{A}}$ be a family of holomorphic functions. The family $\mathcal{F}$ is normal if and only if there is a constant $C$ such that, for each (unit) direction $\xi$,

$$\left| Df_\alpha(z; \xi) \right| \leq C \cdot F^R_\Omega(z; \xi).$$

Here $Df_\alpha(z; \xi)$ denotes the directional derivative of the function $f_\alpha$ at the point $z$ in the direction $\xi$.

**Proof:** The proof follows standard lines. See the proof of Proposition 1.3 in [CIK, p. 306].

We next present a rather natural characterization of normal families that relates the situation on the ambient space to that on one-dimensional slices (more aptly, one-dimensional analytic discs):

**Proposition 1.15** Let $X$ be a Banach space. Let $\Omega \subseteq X$ be a domain and let $\mathcal{F} = \{f_\alpha\}_{\alpha \in \mathcal{A}}$ be a family of holomorphic functions. The family $\mathcal{F}$ is normal if and only if the following condition holds:

$$\text{For each sequence } \varphi_j : D \to \Omega \text{ of holomorphic mappings and each sequence of indices } \alpha_j \in \mathcal{A}, j = 1, 2, \ldots, \text{ the family } f_{\alpha_j} \circ \varphi_j \text{ is normal on the unit disc } D.$$ 

**Proof:** The implication "$\mathcal{F}$ normal $\Rightarrow$ (*)" is immediate from Marty's characterization of normal families.
For the converse, notice that if Condition (\(\ast\)) holds then, for each sequence \(\varphi_j\) of mappings and each collection \(f_{\alpha_j}\), the compositions \(f_{\alpha_j} \circ \varphi_j\) satisfy the conclusion of Marty’s theorem:
\[
\left| f_{\alpha_j} \circ \varphi_j \right|'(\zeta) \leq C\sqrt{1 + \left| f_{\alpha_j} \circ \varphi_j \right|^2} \cdot \frac{1}{1 - |\zeta|^2}.
\]
Here the constant \(C\) depends in principle on the choice of \(\varphi_j\) and also on the choice of \(f_{\alpha_j}\). But in fact a moment’s thought reveals that the choice of \(C\) can be taken to be independent of the choice of these mappings, otherwise there would be a sequence for which \((\ast)\) fails (this is just an exercise in logic).

But then, using the chain rule, we may conclude that Marty’s Criterion for holomorphic families on a Banach space holds for the family \(\mathcal{F}\) (see also the proof of Proposition 1.4 in [CIK, p. 307]). As a result, \(\mathcal{F}\) is normal. 

2 Other Characterizations of Normality

It is an old principle of Bloch, enunciated more formally by Abraham Robinson and actually recorded in mathematical notation by L. Zalcman (see [ZAL1]), that any “property” that would tend to make an entire function constant would also tend to make a family of functions normal. Zalcman’s formulation, while incisive, is rather narrowly bound to the linear structure of Euclidean space. The paper [ALK] finds a method for formulating these ideas that will even work on a manifold. Unfortunately, we must note that the paper [ALK] has an error, which was kindly pointed out to us by the authors of [HTT]. We shall include their correct formulation of the theorem, and also provide an indication of their proof.

Proposition 2.1 Let \(X\) be a separable Banach space and let \(\Omega \subseteq X\) be a hyperbolic domain (i.e., a domain on which the Kobayashi metric is non-degenerate). Let \(Y\) be another separable Banach space. Let \(\mathcal{F} = \{f_{\alpha}\}_{\alpha \in A} \subseteq \text{Hol}(\Omega, Y)\). The family \(\mathcal{F}\) is not normal if and only if there exists a sequence \(\{p_j\} \subseteq \Omega\) with \(p_j \to p_0 \in \Omega\), a sequence \(f_j \in \mathcal{F}\), and \(\{\rho_j\} \subseteq \mathbb{R}\) with \(\rho_j > 0\) and \(\rho_j \to 0\) such that

\[
g_j(\xi) = f_j(p_j + \rho_j \xi), \quad \xi \in X
\]

satisfies one of the following assertions:

(i) The sequence \(\{g_j\}_{j \geq 1}\) is compactly divergent on \(\Omega\);

(ii) The sequence \(\{g_j\}_{j \geq 1}\) converges uniformly on compact subsets of \(\Omega\) to a non-constant holomorphic mapping \(g : \Omega \to Y\).

Remark 2.2 The error in [ALK] is that the authors did not take into account the compactly divergent case in the theorem. Consider the example (also from [HTT]) of the family \(\mathcal{F}\) of mappings \(f_j : D \to \mathbb{C}^2\) given by

\[
f_j(\zeta) = (\alpha_j, \zeta),
\]

where \(1 > \alpha_j > 0\) and \(\alpha_j \to 0\). Then the family \(\mathcal{F}\) is not normal, but \(\mathcal{F}\) also does not satisfy the conclusions of part (ii) of Proposition 2.1 above, which is the sole conclusion of the Aladro/Krantz theorem.
Sketch of the Proof of Proposition 2.1:

We first need a definition. We say that a non-negative, continuous function $E$ defined on the tangent bundle $TY$ is a length function if it satisfies

(a) $E(v) = 0$ iff $v = 0$;

(b) $E(\alpha v) = |\alpha|E(v)$ for all $\alpha \in \mathbb{C}$ and all $v \in TX$.

Now we have: Let $F \subseteq \text{Hol}(\Omega, Y)$. Then

(1) If $F$ is normal then, for each length function $E$ on $Y$, and for each compact subset $K$ of $\Omega$, there is a constant $c_K > 0$ such that

$$E(f(z), df(z)\xi) \leq c_K \cdot \|f\| \quad \text{for all } z \in K, \xi \in X\{0\}, f \in F; \quad (\ast)$$

(2) If $Y$ is complete and the family $F$ is not compactly divergent and satisfies $(\ast)$ then $F$ is normal.

This result is standard and can be found in [WU] or [HTT]. Now we treat the result by cases:

Necessity

Case 1. The family $F$ is compactly divergent. We treat this case in some details since it is new and does not appear in [ALK]. There is a sequence $\{f_j\} \subseteq F$ that is compactly divergent. Take $p_0 \in \Omega$ and $r_0 > 0$ such that $B(p_0, r_0) \subseteq \Omega$. Take $p_j = p_0$ for all $j \geq 1$ and $\rho_j > 0$ for all $j \geq 1$ such that $\rho_j \to 0^+$ and define

$$g_j(\xi) = f_j(p_j + \rho_j\xi), \quad \text{all } j \geq 1.$$

Observe that each $g_j$ is defined on

$$S_j = \left\{ \xi \in X : \|\xi\| \leq R_j = \frac{1}{\rho_j} \text{dist}(p_0, \partial \Omega) \right\}.$$

If $K \subseteq X$ is compact and $L$ is a compact subset of $Y$ then there is an index $j_0 \geq 1$ such that $p_0 + \rho_j K \subseteq B(p_0, r_0)$ for all $j \geq j_0$. This implies that $g_j(K) \subseteq f_j(\bar{B}(p_0, r_0))$ for each $j \geq j_0$. Since the sequence $\{f_j\}$ is compactly divergent, there is an index $j_1 > j_0$ such that $f_j(\bar{B}(p_0, r_0)) \cap L = \emptyset$ for all $j \geq j_1$. Thus $g_j(K) \cap L = \emptyset$ for all $j \geq j_1$. This means that the family $\{g_j\}$ is compactly divergent.

Case 2. The family $F$ is not compactly divergent. This follows standard lines, as indicated in [ALK].
Sufficiency

Case 1. The sequence $g_j \to g$ with $g$ not a constant function. By direct estimation, one shows that
\[
\lim_{j \to \infty} E(g_j(\xi), dg_j(\xi)(t)) = E(g(\xi), dg(\xi)(t)) = 0
\]
for $\xi, t \in X$. Hence $g' \equiv 0$ and so $g$ is constant, a clear contradiction. So the family $\mathcal{F}$ cannot be normal.

Case 2. The sequence \{\{g_j\}\} is compactly divergent. We may assume that $\{f_j\} \subseteq \mathcal{F}$ and $f_j \to f$. For $\xi \in X$ we then have
\[
g_j(\xi) = f_j(p_j + \rho_j \xi) \to f(p_0) \in Y
\]
since $\rho_j \to 0$. This implies that the family \{\{g_j\}\} is not compactly divergent, a clear contradiction.

That completes our outline of the proof of Proposition 2.1. ☐

Constantin Carathéodory produced a geometric characterization of normal families that is quite appealing (see [SCH, p. 68]). It has never been adapted even to finitely many complex variables. We take the opportunity now to offer an infinite dimensional version (which certainly specializes down to any finite number of dimensions).

We begin with a little terminology. Let $\Omega_j$ be domains in a separable Banach space $X$. If some Euclidean ball $B(0, r), r > 0$, is contained in all the domains $\Omega_j$, then $\ker\{\Omega_j\}$ is the largest domain containing 0 and so that every compact subset of $\ker\{\Omega_j\}$ lies in all but finitely many of the $\Omega_j$. We say that $\{\Omega_j\}$ converges to $\Omega_j = \ker\{\Omega_j\}$, written $\Omega_j \to \Omega_j$, if every subsequence $\{\Omega_{j_k}\}$ of these domains has the property that $\ker\{\Omega_{j_k}\} = \Omega_0$.

**Theorem 2.3** Fix a separable Banach space $X$. Let $\{f_n\}$ be a sequence of univalent, holomorphic mappings from the unit ball $B \subset X$ to another separable Banach space $Y$ with the properties that
\begin{enumerate}
  \item $f_n(0) = 0$;
  \item $\langle df_n(0)1, 1 \rangle > 0$.
\end{enumerate}

[Here $1$ is the unit vector $(1, 0, 0, \ldots)$.] Set $\Omega_n \equiv f_n(B), n = 1, 2, \ldots$. Then the $f_n$ converge normally in $B$ to a univalent function $f$ if and only if
\begin{enumerate}
  \item $\Omega_0 = \ker\{\Omega_n\}$ is hyperbolic and is not $\{0\}$.
  \item $\Omega_n \to \Omega_0$
  \item $\Omega_0 = f(B)$.
\end{enumerate}
Sketch of Proof: We first establish that it is impossible for \( \ker\{\Omega_n\} = \{0\} \). Consider the Kobayashi metric ball \( B = B_{\mathbb{E}}(0,1) \). Then

\[
f_n : B \rightarrow B_{\Omega_n}(0,1),
\]

since of course each \( f_n \) will be a Kobayashi isometry onto its image. Assume that \( f_n \rightarrow f \) normally (i.e., uniformly on compact sets) in \( B \). Clearly, under the hypothesis that \( \ker\{\Omega_n\} = \{0\} \), there is no \( \epsilon > 0 \) such that \( b(0,\epsilon) \subseteq \Omega_n \) for \( n \) large. Here \( b \) denotes a Euclidean ball. Thus \( B_{\Omega_n} \) must shrink to a set with no interior. It follows that the sequence \( \{f_n\} \) collapses any compact subset of \( B \) to a set without interior. Thus \( df \equiv 0 \) on \( B \) hence \( f \) is identically constant. Since \( f(0) = 0 \), \( f \equiv 0 \) (a clear contradiction).

Now we begin proving the theorem proper. Suppose that \( f_n \rightarrow f \) normally on \( B \) with \( f \) univalent. Since, by the preceding paragraph, \( f \) is not identically 0, we may conclude that \( \{\Omega_n\} \) is non-empty.

CLAIM: \( f(B) = \Omega_0 \).

It would follow from this claim that \( \Omega_0 \neq X \), for if \( \Omega_0 = X \) then \( f^{-1} : X \rightarrow B \) univalently, violating Liouville's theorem [MUJ, p. 39]. [It would also contradict the distance-decreasing property of the Kobayashi metric.] Since every subsequence of \( \{f_n\} \) converges to \( f \), it follows that every subsequence of \( \{\Omega_n\} \) has kernel \( \Omega_0 \). We write \( \Omega_n \rightarrow \Omega_0 \).

SUBCLAIM I: \( f(B) \subseteq \Omega_0 \).

For consider any closed metric ball \( \overline{B}(0,R) \subseteq f(B) \). We may restrict attention to any finite-dimensional slice \( L \) of this ball, which will of course be compact. Then \( f_n|_L \rightarrow f|_L \).

Thus \( f_n|_L \rightarrow f|_L \). As a result, for \( n \) large, we apply the argument principle to any curve in \( L \cap \partial B \) to see that each value in \( f(\overline{B}(0,R)) \) is attained just once by \( f_n \) for \( n \) large. But this just says that \( f(B) \subseteq \Omega_0 \).

SUBCLAIM II: \( \Omega_0 \subseteq f(B) \).

For consider \( \Omega_0 \neq \{0\} \), and assume \( \Omega_0 \) is hyperbolic. Let \( \Omega_n \rightarrow \Omega_0 \). If \( b(0,\epsilon) \subseteq \Omega_n \) for all \( n \) large, then

\[
b(0,\epsilon) \subseteq B_{\Omega_n}(0,R) \subseteq \Omega_n \quad \text{for \( n \) large.}
\]

So

\[
f_n : B_{\mathbb{E}}(0,R) \rightarrow B_{\Omega_n}(0,R) \supset b(0,\epsilon).
\]

Hence we have a bound from below on the eigenvalues of \( df_n \).

Obversely, we also claim that the eigenvalues of \( df_n \) are bounded above. If not, then there exist (Euclidean) unit vectors \( \xi_n \) such that

\[
df_n(\xi_n) \rightarrow \infty.
\]
After a rotation and passing to a subsequence, we can assume that the \( \xi_n \) all point in the direction 1. The result would then be that \( \Omega_0 \) cannot be hyperbolic, a contradiction.

Thus the \( \{f_n\} \) are locally bounded and \( \{\hat{f}_n\} \) forms a normal family, as required. Thus some subsequence converges to a univalent \( f \) such that \( f(0) = 0 \). This last follows from the argument principle (see any standard complex analysis text).

\[ \]

3 A Budget of Counterexamples

We interrupt our story to provide some examples that exhibit the limitations of the theory of normal families in infinitely many variables.

Example 3.1 There is no Montel theorem for holomorphic mappings of infinitely many variables. Indeed, let \( B \) be the open unit ball in the Hilbert space \( \ell_2 \). Define

\[
\varphi_j(\{a_m\}) = \left( \frac{\sqrt{3/4a_1}}{1 - a_j/2}, \frac{\sqrt{3/4a_2}}{1 - a_j/2}, \ldots, \frac{\sqrt{3/4a_{j-1}}}{1 - a_j/2}, \frac{a_j - 1/2}{1 - a_j/2}, \frac{\sqrt{3/4a_{j+1}}}{1 - a_j/2}, \ldots \right).
\]

Then each \( \varphi_j \) is an automorphism of \( B \).

Now fix an index \( j \). Let \( K = K_j \) be the compact set \( \{(0, 0, \ldots, 0, \zeta, 0, \ldots, 0) : |\zeta| < c\} \), where the non-zero entry is in the \( j \)th position and \( 1/2 < c < 1 \) is a constant. Define the point \( p \in K \) to be \( p = (0, 0, \ldots, 0, c, 0, \ldots) \), where the non-zero entry is in the \( j \)th position. Then

\[
\sup_k \|\varphi_j - \varphi_k\| \geq \|\varphi_j(p) - \varphi_k(p)\| \geq \left| \frac{c - 1/2}{1 - c/2} - 0 \right| = \left| \frac{2c - 1}{2} \right| > 0.
\]

As a result, we see that the sequence \( \{\varphi_j\} \) can have no convergent subsequence. It also cannot have a compactly divergent subsequence.

Example 3.2 There are no taut domains in infinite dimensional space. First we recall H. H. Wu's notion of "taut". Let \( N \) be a complex manifold. We say that \( N \) is taut if, for every complex manifold \( M \), the family of holomorphic mappings from \( M \) to \( N \) is normal. We now demonstrate that there are no such manifolds in infinite dimensions.

We begin by studying the ball \( B \) in the Hilbert space \( \ell_2 \). We let \( N = B \) and \( M = D \), the disc in \( \mathbb{C} \) (in fact it is easy to see that, when testing tautness, it always suffices to take \( M \) to be the unit disc). Consider the mappings

\[
\varphi_j(\zeta) = \left( 0, 0, \ldots, 0, \frac{1}{3} + \frac{\zeta}{4}, 0, \ldots, 0 \right).
\]

Here the non-zero entry is in the \( j \)th position. Then

\[
|\text{image}(\varphi_j) - \text{image}(\varphi_k)| \geq \sqrt{\left( \frac{1}{12} \right)^2 + \left( \frac{1}{12} \right)^2} = \sqrt{\frac{1}{72}} > 0.
\]
Also
\[ \text{dist}(\text{image}(\varphi_j), \partial D) = \frac{5}{12} > 0. \]
As a result, the sequence \( \{\varphi_j\} \) has no convergent subsequence and no compactly divergent subsequence.

Of course the same argument shows that there is no taut domain in Hilbert space, nor is there any taut Hilbert manifold.

It should be noted that the Arzela-Ascoli theorem will fail for families of functions (mappings) taking values in an infinite dimensional space. For example, if \( X \) is the separable Hilbert space \( \ell_2 \) and \( f_j : X \to X \) is given by \( f_j(\{x_j\}) = x_j \) then the \( f_j \) are equicontinuous and equibounded on bounded sets, yet no compact set supports a uniformly convergent subsequence. Thus the preceding examples do not come as a great surprise.

It is worth noting that there are results for weak or weak-* normal families that can serve as a good substitute when the regular (or strong) Montel theorem fails. We explore some of these in Section 6.

4 Normal Functions

Normal functions were created by Lehto and Virtanen in [LEV] as a natural context in which to formulate the Lindelöf principle. Recall that the Lindelöf principle says this

**Theorem 4.1 (Lindelöf)** Let \( f \) be a bounded holomorphic function on the disc \( D \). If \( f \) has radial limit \( \ell \) at a point \( \zeta \in \partial D \) then \( f \) has non-tangential limit \( \ell \) at \( \zeta \).

Lehto and Virtanen realized that boundedness was too strong a condition, and not the natural one, to guarantee that Lindelöf's phenomenon would hold. They therefore defined the class of normal functions as follows:

**Definition 4.2** Let \( f \) be a holomorphic (meromorphic) function on the disc \( D \subseteq \mathbb{C} \). Suppose that, for any family \( \{\varphi_j\} \) of conformal self-maps of the disc it holds that \( \{f \circ \varphi_m\} \) is a normal family. Then we say that \( f \) is a normal function.

Clearly a bounded holomorphic function, a meromorphic function that omits three values, or a univalent holomorphic function (all in one complex dimension) will be normal according to this definition.

Unfortunately, the original definition given by Lehto and Virtanen is rather limited. One-connected domains in \( \mathbb{C}^1 \) have compact automorphism groups; finitely connected domains in \( \mathbb{C}^1 \), of connectivity at least two, have finite automorphism group. Generic domains in \( \mathbb{C}^n \), \( n \geq 2 \), even those that are topologically trivial, have automorphism group consisting only of the identity (such domains are called rigid). Thus, for most domains in most dimensions, there are not enough automorphisms to make a working definition of "normal function" possible. In [CIK], Cima and Krantz addressed this issue and developed a new definition of normal function. We now adapt that definition to the infinite dimensional case.
Definition 4.3 Let $X$ be a Banach space and let $\Omega$ be a domain in $X$. A holomorphic function $f$ on $\Omega$ is said to be *normal* if

$$\frac{|Df(z;\xi)|}{1 + |f(z)|^2} \leq C \cdot F^\Omega_K(z;\xi)$$

for all $z \in \Omega, \xi \in X$.

Proposition 4.4 Let $f$ be a holomorphic function on a domain $\Omega$ in a Banach space $X$. The function $f$ is normal if and only if $f \circ \varphi$ is normal for each holomorphic $\varphi : D \to \Omega$.

Proof: The proof is just the same as that in Section 1 of [CIK].

Remark 4.5 It is a straightforward exercise, using for example Proposition 4.4 (or Marty's characterization of normality), to see that a holomorphic or meromorphic function on the unit ball $B$ in a separable Hilbert space $H$ is normal if and only if, for every family $\{\varphi_a\}_{a \in A}$ of biholomorphic self maps of $B$, it holds that $\{f \circ \varphi_a\}$ is a normal family.

Now let $B \subseteq X$ be the unit ball in a separable Banach space $X$. We define a holomorphic function $f$ on $B$ to be *Bloch* if

$$\|Df(z;\xi)\| \leq C \cdot F^B_K(p;\xi)$$

for every $z \in B$ and every vector $\xi$. Then it is routine, following classical arguments, to verify

Proposition 4.6 If $f$ on $B$ is a Bloch function then $f$ is normal.

5 Different Topologies on Spaces of Holomorphic Functions

One way to view a "normal families" theorem is that it is a compactness theorem. But another productive point of view is to think of these types of results as relating different topologies on spaces of holomorphic functions. We begin our discussion of this idea by recalling some of the standard topologies, as well as a few that are more unusual.

The Compact-Open Topology In the language of analysis, this is the topology of uniform convergence on compact sets. Certainly in finite-dimensional complex analysis this is, for many purposes, the most standard topology on general spaces of holomorphic functions. In infinite dimensions this topology is often too coarse (just because compact sets are no longer very "fat").

The Topology of Pointwise Convergence Here we say that a sequence $f_j$ of functions or mappings converges if $f_j(x)$ converges for each $x$ in the common domain $X$ of the $f_j$. 

The Weak Topology for Distributions Here we think of a space of holomorphic functions as a subspace of the space $E$ of testing functions for the compactly supported distributions. We say that a sequence $f_j$ of holomorphic functions converges if $\psi(f_j)$ converges for each such distribution $\psi$. Of course a similar definition can (and should) be formulated for nets.

The Topology of Currents Let $f_j$ be holomorphic functions and consider the 1-forms $\partial f_j$. Then we may think of these forms as currents lying in the dual of the space of rectifiable 1-chains; we topologize the $\partial f_j$ accordingly.

6 A Functional Analysis Approach to Normal Families

In the classical setting of the unit disc $D \subseteq \mathbb{C}$, it is straightforward to prove that

$$H^\infty(D) = \left( L^1(D)/H^1(D) \right)^*. \quad (*)$$

Thus $H^\infty$ is a dual space in a natural way. Properly viewed, the classical Montel theorem is simply weak-* compactness (i.e., the Banach-Alaoglu theorem) for this dual space. Using the Cauchy integral formula as usual, one can see that convergence in the dual norm certainly dominates uniform convergence on compact subsets of the disc.

Alternatively, one can think of the elements of $H^\infty(D)$, with $D$ the disc, as the collection of all operators (by multiplication) on $H^2$ that commute with multiplication by $z$. This was Beurling’s point of view. The operator topology turns out to be equivalent (although this is non-trivial to see) to the weak-* topology as discussed in the last paragraph, and this in turn is equivalent to the classical sup-norm topology on $H^\infty$.

The classical arguments go through to show that there is still a Beurling theorem on the unit ball in Hilbert space. It is a purely formal exercise to verify that $(*)$ still holds on the unit ball in $\ell_2$, our usual separable Hilbert space. As a result, one can think of the Montel theorem even in infinite dimensions either in the operator topology or as an application of the Banach-Alaoglu theorem to $H^\infty$, thought of as a dual space.

Now we would like to present an effective weak-normal family theorem in the context. Let $Z$ be a Banach space and let $Y = Z^*$ be its dual Banach space. Let $\Omega$ be an open subset of a Banach space $X$ and let $O(\Omega, Y)$ be the set of all holomorphic mappings from $\Omega$ into $Y$. Then we consider the topology on $O(\Omega, Y)$ generated by the sub-basic open sets given by

$$G(K, U) = \{ f \in O(\Omega, Y) | f(K) \subset U \}$$

where $K$ is a compact subset of $\Omega$ and $U$ a weak-* open subset of $Y$. Let us call this topology the compact-weak*-open topology.

Theorem 6.1 Let $\Omega$ be a domain in a separable Banach space $X$. Let $Z$ be a separable Banach space with a countable Schauder basis, and let $Y = Z^*$. Further, let $W$ be a bounded domain in $Y$. Then the compact-weak*-open topology is a Montel topology. In particular, the family $O(\Omega, Y)$ is normal with respect to the compact-weak*-open topology.
Proof: Let \( \{f_j \mid j = 1, 2, \ldots \} \subset \mathcal{O}(\Omega, W) \) be given. We would like to show that there exists a subsequence that converges in the compact-weak* open topology.

Let \( \{e_j \mid j = 1, 2, \ldots \} \) be a Schauder basis for \( Z \). For \( z \in Y \), we define the linear functional \( \psi_k : Y \to \mathbb{C} \) by \( \psi_k(z) = z(e_k) \). Now we define

\[
\Psi_{k,j} = \psi_k \circ f_j.
\]

Then, we see for each \( k \) that the sequence \( \{\Psi_{k,j}\}_j \) is normal by Theorem 1.8. Therefore we may select subsequences inductively so that

1. \( \{\Psi_{1, j(k)}\}_j \) is a subsequence of \( \Psi_{1, j} \) which converges in the compact-weak* open topology, and
2. \( \{f_{k+1(j)}\}_j \) is a subsequence of \( \{f_{j(k)}\}_j \) for every \( k = 1, 2, \ldots \).

Notice that the diagonal sequence \( \Psi_{k, j(k)} = \psi_k \circ f_{j(k)} \) converges in the compact-weak* open topology. Since the weak-* topology separate points, we may denote the weak-* limit of the sequence \( f_{j(k)}(z) \) by \( f(z) \) for each \( z \in \Omega \). Then the map \( f : \Omega \to Y \) is Gateaux holomorphic. Since the range of \( f \) is bounded, it follows that \( f \) is in fact holomorphic. Thus the proof is complete. \( \square \)

Notice that this theorem works for the mappings from the spaces \( \ell^p \) or \( c_0 \) into the space \( \ell^\infty \), for each \( p \) with \( 1 \leq p < \infty \). Therefore, this may be useful for a characterization problem of infinite dimensional polydisc by its automorphism group in the space \( c_0 \) of sequences of complex numbers converging to zero, for instance. On the other hand, not only is this theorem a generalization of the weak-normal family theorems in the works of Kim/Krantz and Byun/Gaussier/Kim, it also provides an easier and shorter proof even in the case of separable Hilbert spaces. See [KIK] and [BGK].

We conclude this section with some examples, due to Jisoo Byun [BYU], that suggest some of the limitations of normal families in the infinite dimensional setting. These examples all relate to the failure of convexity.

Let \( \Omega_1 \) and \( \Omega_2 \) be bounded domains in a Banach space \( X \). We point out that for the holomorphic weak-* limit mapping \( \tilde{f} : \Omega \to X \) of a sequence of holomorphic mappings \( f_j : \Omega_1 \to \Omega_2 \) may in general show a surprising behavior in contrast with the finite dimensional cases. In the finite dimensional cases, \( \tilde{f}(\Omega_1) \) should be contained in the closure of \( \Omega_2 \). Here we demonstrate that weak-* closure is about the best one can do, even with the nicest candidates such as sequences of biholomorphic mappings from the ball.

Example 6.2 Let \( B \) be the unit open ball in \( \ell^2 \). Let \( \{e_j \mid j = 1, 2, \ldots \} \) be the standard orthonormal basis for \( \ell^2 \). Let \( f_k : B \to \ell^2 \) be defined by

\[
f_k(z) = \sum_{j=1}^{k-1} z_j e_j + (z_k + z_{k+1}^2) e_k + \sum_{j=k+1}^{\infty} z_j e_j
\]
where \( z = z_1e_1 + \ldots \). Notice that none of \( f_k(B) \) is convex. In fact, the ball centered at \( \frac{1}{100}e_1 \) with radius 1/100 never meets \( f_k(B) \), while it is obvious that the origin and the point \( e_1 \) are clearly in the norm closure of the union of \( f_k(B) \). Moreover, the weak limit \( \hat{f} \) of the sequence \( f_k \) is the identity map. Hence \( \hat{f}(B) = B \), which is convex. This shows that the weak limit can gain in its image more than the norm closure of the union of the images of \( f_k(B) \).

**Example 6.3** In general the weak-* limit does not make the range convex, automatically. If one considers \( g_k : B \to \ell^2 \) defined by

\[
g_k(z)(z_1 + z_2^2)e_1 + \sum_{j=2}^{k-1} z_j e_j + \left( z_k + \frac{1}{2} z_2^2 \right) e_k + \sum_{j=k+1}^{\infty} z_j e_j,
\]

for \( k = 3, 4, \ldots \). Then each \( g_k \) and the weak limit

\[
\hat{g}(z) = (z_1 + z_2^2)e_1 + \sum_{j=2}^{\infty} z_j e_j
\]

are biholomorphisms of the ball \( B \) onto its image. Notice that \( \hat{g}(B) \) is not convex.

**7 Many Approaches to Normal Families**

It is natural to try to relate the infinite-dimensional case to the well-known case of finite dimensions. In particular, let \( \mathcal{F} \) be a family of holomorphic functions on a domain \( \Omega \) in a Banach space \( \mathcal{X} \). Is it correct to say that \( \mathcal{F} \) is normal if and only if the restriction of \( \mathcal{F} \) to any finite-dimensional subspace is normal? Obversely, if the post-composition of the elements of \( \mathcal{F} \) with each finite-dimensional subspace projection operator is normal then can we conclude that \( \mathcal{F} \) is normal? We would like to treat some of these questions here.

**Example 7.1** Suppose that if \( \mathcal{F} \) is a family of maps of a domain \( \Omega \) in a separable Hilbert space \( \mathcal{H} \), and assume that

\[
\{ \pi_j \circ f : f \in \mathcal{F} \}
\]

is normal for each \( \pi_j : \mathcal{H} \to \mathcal{H}_j \) the projection of \( \mathcal{H} \) to the one-dimensional subspace \( \mathcal{H}_j \) spanned by the unit vector in the \( j^{th} \) direction. Then it does not necessarily follow that \( \mathcal{F} \) is a normal family.

To see this, let \( \mathcal{H} = \ell_2 \), and let \( f_j(\{x_j\}) = x_j \). Consider each \( f_j \) as a map from the unit ball \( \mathcal{B} \subset \mathcal{H} \) to itself. Then, for each fixed \( k \), \( \{ \pi_k \circ f_j \}^\infty_{j=1} \) is a normal family, yet the family \( \mathcal{F} = \{ f_j \}^\infty_{j=1} \) is definitely not normal.

**Example 7.2** Suppose that if \( \mathcal{F} \) is a family of maps of a domain \( \Omega \) in a separable Hilbert space \( \mathcal{H} \). Suppose that, for each \( k \), the collection

\[
\{ f \circ \mu_k : f \in \mathcal{F} \}
\]

is normal for each \( \mu_j : \mathbb{C} \to \mathcal{H} \) the injection of \( \mathbb{C} \) to \( \mathcal{H} \) in the \( j^{th} \) variable. Then it does not necessarily follow that \( \mathcal{F} \) is a normal family.
For this result, again consider $H = \ell_2$, and let $f_j(\{x_k\}) = x_j$. Consider each $f_j$ as a map from the unit ball $B \subseteq H$ to itself. Then, for each fixed $k$, the family $\{f \circ \mu_k\}_{f \in \mathcal{F}}$ is clearly normal. Yet the entire family $\mathcal{F}$ is plainly not normal—as we discussed in Example 2.1. The reader should compare this example to Proposition 1.14, which gives a positive result along these lines.

One of the main lessons of the classic paper [WU] by H. H. Wu is that the normality or non-normality of a family of mappings depends essentially on the target space (this is the provenance of the notion of taut manifold). With this point in mind, we now formulate a counterpoint to Example 2.1:

**Proposition 7.3** Let $C = \{\{x_j\}_{j=1}^{\infty} : |x_j| \leq 1/j\}$ be the Hilbert cube. Let $H = \ell_2$ be the canonical separable Hilbert space. Then any family $\mathcal{F}$ from a domain $\Omega \subseteq H$ to $C$ will be normal.

**Proof:** It suffices to prove that the correct formulation of the Arzela-Ascoli theorem holds. In particular, we establish this result:

If $\mathcal{G} = \{g_\alpha\}_{\alpha \in \mathcal{A}}$ is a family of functions from a domain $\Omega \subseteq H$ into $C$ which is (i) equibounded and (ii) equicontinuous, then $\mathcal{G}$ has a uniformly convergent subsequence.

In fact the usual proof of Arzela-Ascoli, that can be found in any text (see, for instance, [KRA, p. 284]), goes through once we establish this basic fact: If $g_\alpha : \Omega \to C$ and $x_0 \in \Omega$ is fixed then $\{g_\alpha(x_0)\}$ has a convergent subsequence. Of course this simple assertion is the consequence of a standard diagonalization argument. 

\[\Box\]

**Acknowledgement.** Both authors were guests at the American Institute of Mathematics (AIM) during a portion of this work. AIM sponsored a workshop on holomorphic mappings that was particularly useful to these studies. First author supported in part by grant R01-1999-00005 from The Korean Science and Engineering Foundation. Second author supported in part by grant DMS-9988854 from the National Science Foundation.

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Received: 19.09.2002
Revised: 29.01.2003