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A residue theorem for Malcev–Neumann series

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Abstract

In this paper, we establish a residue theorem for Malcev–Neumann series that requires few constraints, and includes previously known combinatorial residue theorems as special cases. Our residue theorem identifies the residues of two formal series (over a field of characteristic zero) which are related by a change of variables. We obtain simple conditions for when a change of variables is possible, and find that the two related formal series in fact belong to two different fields of Malcev–Neumann series. The multivariate Lagrange inversion formula is easily derived and Dyson's conjecture is given a new proof and generalized.

Keywords: Totally ordered group; Malcev–Neumann series; Residue theorem; Lagrange inversion

1. Introduction

Let K be a field of characteristic zero. Jacobi [9] used the ring $K((x_1, \dots, x_n))$ of Laurent series, formal series of monomials where the exponents of the variables are bounded from below, to give the following residue formula.

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Theorem 1.1 (Jacobi's Residue Formula). *Let $f_1(x_1,\dots,x_n),\dots,f_n(x_1,\dots,x_n)$ be Laurent series. Let b_{ij} be integers such that $f_i(x_1,\dots,x_n)/x_1^{b_{i1}}\cdots x_n^{b_{in}}$ is a formal power series with nonzero constant term. Then for any Laurent series $\Phi(y_1,\dots,y_n)$, we have*

$$\operatorname*{Res}_{x_1,\dots,x_n}\left|\frac{\partial f_i}{\partial x_j}\right|_{1\leqslant i,j\leqslant n}\Phi(f_1,\dots,f_n)=|b_{ij}|_{1\leqslant i,j\leqslant n}\operatorname*{Res}_{y_1,\dots,y_n}\Phi(y_1,\dots,y_n), \tag{1.1}$$

where $\mathrm{Res}_{x_1,\dots,x_n}$ means to take the coefficient of $x_1^{-1}\cdots x_n^{-1}$.

Note that the convergence of $\Phi(f_1,\dots,f_n)$ is obviously required.

Jacobi's residue formula is a well-known result in combinatorics. It equates the residues of two formal series related by a change of variables. It has many applications and has been studied by several authors, e.g., Goulden and Jackson [6, pp. 19–22], and Henrici [8]. However, Jacobi's formula is rather restricted in application for two reasons: the conditions on the f_i are too strong, and the condition on Φ is not easy to check: given f_i, when does $\Phi(f_1,\dots,f_n)$ converge?

We can obtain different residue formulas by considering different rings containing the ring of formal power series $K[[x_1,\dots,x_n]]$. In obtaining such a formula, we usually embed $K[[x_1,\dots,x_n]]$ into a ring or a field consisting of formal Laurent series, but the embedding is not unique in the multivariate case. Besides Jacobi's residue formula, Cheng et al. [2] studied the ring $K_h((x_1,\dots,x_n))$ of homogeneous Laurent series (formal series of monomials whose total degree is bounded from below), and used homogeneous expansion to give a residue formula. But the above restrictions still exist for the same reason. We will use a more general setting to avoid the above problems.

Let $\mathcal{G}$ be a *totally ordered group*, i.e., a group with a total ordering $\leqslant$ that is compatible with its group structure. Let $K_w[\mathcal{G}]$ be the set of *Malcev–Neumann series* (MN-series for short) on $\mathcal{G}$ over K relative to $\leqslant$: an element in $K_w[\mathcal{G}]$ is a series $\eta=\sum_{g\in\mathcal{G}}a_g g$ with $a_g\in K$, such that the support $\{g\in\mathcal{G}\colon a_g\neq 0\}$ of η is a well-ordered subset of $\mathcal{G}$.

By a theorem of Malcev [10] and Neumann [11] (see also [12, Theorem 13.2.11]), $K_w[\mathcal{G}]$ is a division algebra that includes the group algebra $K[\mathcal{G}]$ as a subalgebra. We study the field of MN-series on a totally ordered abelian group, and show that the field of iterated Laurent series $K\langle\langle x_1,\dots,x_n\rangle\rangle$, which has been studied in [17, Chapter 2], is a special kind of MN-series.

We obtain a residue theorem for $K_w[\mathcal{G}\oplus\mathbb{Z}^n]$, where $x_1,\dots,x_n$ represent the generators of $\mathbb{Z}^n$. This new residue formula includes the previous residue theorems of Jacobi and Cheng et al. as special cases. It is easier to apply and more general: the conditions on the f_i are dropped since we are working in a field; the condition on Φ is replaced with a simpler one and we find that the two related formal series in fact belong to two different fields of MN-series. In particular, our theorem applies to any rational function Φ.

In Section 2 we review some basic properties of MN-series. We give the residue formula in Section 3. Then we talk about the (diagonal and nondiagonal) Lagrange inversion formulas in Section 4, and give a new proof and a generalization of Dyson's conjecture in Section 5.

2. Basic properties of Malcev–Neumann series

A *totally ordered abelian group* or TOA-group is an abelian group $\mathcal{G}$ (written additively) equipped with a total ordering $\leqslant$ that is compatible with the group structure of $\mathcal{G}$; i.e., for all $x, y, z \in \mathcal{G}$, $x < y$ implies $x + z < y + z$. Such an ordering $<$ is also called *translation invariant*. The abelian groups $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are all totally ordered abelian groups under the natural ordering.

Let K be a field. A formal series η on $\mathcal{G}$ over K has the form

$$\eta = \sum_{g \in \mathcal{G}} a_g t^g,$$

where $a_g \in K$ and t^g is regarded as a symbol. Let $\tau = \sum_{h \in \mathcal{G}} b_h t^h$ be another formal series on $\mathcal{G}$. Then the *product* $\eta\tau$ is defined if for every $f \in \mathcal{G}$, there are only finitely many pairs (g, h) of elements of $\mathcal{G}$ such that a_g and b_h are nonzero and $g + h = f$. In this case,

$$\eta\tau := \sum_{f \in \mathcal{G}} t^f \sum_{g+h=f} a_g b_h.$$

The *support* $\operatorname{supp}(\eta)$ of η is defined to be $\{g \in \mathcal{G}\colon\ a_g \neq 0\}$.

For a TOA-group $\mathcal{G}$, a *Malcev–Neumann series* (MN-series for short) is a formal series on $\mathcal{G}$ that has a well-ordered support. Recall that a well-ordered set is a totally ordered set such that every nonempty subset has a minimum. We define $K_w[\mathcal{G}]$ to be the set of all such MN-series.

By a theorem of Malcev and Neumann [12, Theorem 13.2.11], $K_w[\mathcal{G}]$ is a field for any TOA-group. A sketch of the proof will be introduced since we will use some of the facts later.

Let us see some examples of MN-series first.

(1) $K_w[\mathbb{Z}] \simeq K((x))$ is the field of Laurent series.
(2) $K_w[\mathbb{Q}]$ strictly contains the field $K^{\mathrm{fra}}((x))$ of fractional Laurent series [13, p. 161], and is more complicated. When the characteristic of K is a prime number p, it includes as a subfield the generalized Puiseux field [14] with respect to p, which consists all series $f(x)$ such that $\operatorname{supp}(f)$ is a well-ordered subset of $\mathbb{Q}$ and there is an m such that for any $\alpha \in \operatorname{supp}(f)$ we have $m\alpha = n_\alpha / p^{i_\alpha}$ for some integer n_α and nonnegative integer i_α.
(3) Let $\mathbb{Q}^\times$ be the multiplicative group of positive rational numbers. Then $\mathbb{Q}^\times$ is a TOA-group, and $K_w[\mathbb{Q}^\times]$ is a field of MN-series.

The set of MN-series $K_w[\mathcal{G}]$ is clearly closed under addition. The following proposition is the key to showing that $K_w[\mathcal{G}]$ is closed under multiplication, so that $K_w[\mathcal{G}]$ is a ring.

For two subsets A and B of $\mathcal{G}$, we denote by $A + B$ the set $\{a + b\colon\ a \in A, b \in B\}$.

Proposition 2.1 [12, Lemma 13.2.9]. *If $\mathcal{G}$ is a TOA-group and A, B are two well-ordered subsets of $\mathcal{G}$ then $A + B$ is also well ordered.*

For a TOA-group $\mathcal{G}$, $K_w[\mathcal{G}]$ is a maximal ring in the set of all formal series on $\mathcal{G}$: if $\eta = \sum_{g\in\mathcal{G}} a_g t^g$ is not in $K_w[\mathcal{G}]$, then adding η into $K_w[\mathcal{G}]$ cannot yield a ring. For if $\operatorname{supp}(\eta)$ is not well ordered, we can assume that $g_1 > g_2 > \cdots$ is an infinite decreasing sequence in $\operatorname{supp}(\eta)$. Let $\tau = \sum_{n\geqslant 1} a_{g_n}^{-1} t^{-g_n}$. Note that $\tau \in K_w[\mathcal{G}]$, since $-g_1 < -g_2 < \cdots$ is well ordered. But the constant term of $\eta\tau$ equals an infinite sum of 1's, which diverges.

Let $[t^g]\eta$ be the coefficient of t^g in η. Let $\eta_1, \eta_2, \ldots$ be a series of elements in $K_w[\mathcal{G}]$. Then we say that $\eta_1 + \eta_2 + \cdots$ *strictly converges* to $\eta \in K_w[\mathcal{G}]$, if for every $g \in \mathcal{G}$, there are only finitely many i such that $[t^g]\eta_i \neq 0$, and $\sum_{i\geqslant 1}[t^g]\eta_i = [t^g]\eta$. If $\eta_1 + \eta_2 + \cdots$ strictly converges to some $\eta \in K_w[\mathcal{G}]$, then we say that $\eta_1 + \eta_2 + \cdots$ *exists* (in $K_w[\mathcal{G}]$). Note that $\sum_{n\geqslant 1} 2^{-n}$ does not strictly converge to 1.

Let $f(z) = \sum_{n\geqslant 0} b_n z^n$ be a formal power series in $K[\![z]\!]$, and let $\eta \in K_w[\mathcal{G}]$. Then we define the composition $f \circ \eta$ to be

$$f \circ \eta := f(\eta) = \sum_{n\geqslant 0} b_n \eta^n$$

if the sum exists.

If $\eta \neq 0$ belongs to $K_w[\mathcal{G}]$, then it has a nonempty well-ordered support so that we can define the *order* of η to be $\operatorname{ord}(\eta) = \min(\operatorname{supp}(\eta))$. The *initial term* of η is the term with the smallest order. It is clear that $\operatorname{ord}(\eta\tau) = \operatorname{ord}(\eta) + \operatorname{ord}(\tau)$. The order of 0 is treated as ∞.

Theorem 2.2 (Composition Law). *If $f \in K[\![z]\!]$ and $\eta \in K_w[\mathcal{G}]$ with* $\operatorname{ord}(\eta) > 0$, *then $f \circ \eta$ strictly converges in $K_w[\mathcal{G}]$.*

The detailed proof of this composition law can be found in [17, Chapter 3.1]. It consists of two parts: one is to show that for any $g \in \mathcal{G}$, $[t^g]f \circ \eta$ is a finite sum of elements in K; the other is to show that the support of $f \circ \eta$ is well ordered. The following proposition is the key to the proof.

We denote by A^{+n} the set $A + A + \cdots + A$ of n copies of A. A subset A of $\mathcal{G}$ is said to be *positive*, denoted by $A > 0$, if $a > 0$ for all $a \in A$.

Proposition 2.3 [12, Lemma 13.2.10]. *Let $\mathcal{G}$ be a TOA-group. If A is a positive well-ordered subset of $\mathcal{G}$, then $\bigcup_{n\geqslant 0} A^{+n}$ is also well ordered.*

Corollary 2.4. *For any $\eta \in K_w[\mathcal{G}]$ with initial term* 1, $\eta^{-1} \in K_w[\mathcal{G}]$.

Proof. Write $\eta = 1 - \tau$. Then $\tau \in K_w[\mathcal{G}]$ and $\operatorname{ord}(\tau) > 0$. By Theorem 2.2, $\sum_{n\geqslant 0} \tau^n$ strictly converges in $K_w[\mathcal{G}]$. Knowing that $[t^g](1-\tau)\cdot\sum_{n\geqslant 0}\tau^n$ is a finite sum for every g, we can check that $(1-\tau)\cdot\sum_{n\geqslant 0}\tau^n$ reduces to 1 after cancelation. $\square$

For any $\eta \in K_w[\mathcal{G}]$ with initial term f, we write $\eta = f(1-\tau)$ with $\operatorname{ord}(\tau) > 0$. Then the expansion of η^{-1} is given by $f^{-1}\sum_{n\geqslant 0}\tau^n$. This implies that $K_w[\mathcal{G}]$ is a field.

Definition 2.5. If $\mathcal{G}$ and $\mathcal{H}$ are two TOA-groups, then the *Cartesian product* $\mathcal{G} \times \mathcal{H}$ is defined to be the set $\mathcal{G} \times \mathcal{H}$ equipped with the usual addition and the reverse lexicographic order, i.e., $(x_1, y_1) \leqslant (x_2, y_2)$ if and only if $y_1 <_{\mathcal{H}} y_2$ or $y_1 = y_2$ and $x_1 \leqslant_{\mathcal{G}} x_2$.

We define $\mathcal{G}^n$ to be the Cartesian product of n copies of $\mathcal{G}$. It is an easy exercise to show the following.

Proposition 2.6. *The Cartesian product of finitely many TOA-groups is a TOA-group.*

One important example is $\mathbb{Z}^n$ as a totally ordered abelian group.

When considering the ring $K_w(\mathcal{G} \times \mathcal{H})$, it is natural to treat (g, h) as $g + h$, where g is identified with $(g, 0)$ and h is identified with $(0, h)$. With this identification, we have the following.

Proposition 2.7. *The field $K_w[\mathcal{G} \times \mathcal{H}]$ is the same as the field $(K_w[\mathcal{G}])_w\,[\mathcal{H}]$ of Malcev–Neumann series on $\mathcal{H}$ with coefficients in $K_w[\mathcal{G}]$.*

Proof. Let $\eta \in K_w[\mathcal{G} \times \mathcal{H}]$, and let $A = \operatorname{supp}(\eta)$. Let p be the second projection of $\mathcal{G} \times \mathcal{H}$, i.e., $p(g, h) = h$.

We first show that $p(A)$ is well ordered. If not, then we have an infinite sequence $(g_1, h_1), (g_2, h_2), \ldots$ of elements of A such that $p(g_1, h_1) > p(g_2, h_2) > \cdots$, which by definition becomes $h_1 > h_2 > \cdots$. Then in the reverse lexicographic order, this implies that $(g_1, h_1) > (g_2, h_2) > \cdots$ is an infinite decreasing sequence in A, a contradiction. So $p(A)$ is well ordered.

Now η can be written as

$$\eta = \sum_{h \in p(A)} \Bigg(\sum_{g \in \mathcal{G},\, (g,h) \in A} a_{g,h} t^g \Bigg) t^h.$$

Since for each $h \in p(A)$, the set $\{g \in \mathcal{G}\colon (g, h) \in A\}$ is a clearly a well-ordered subset of $\mathcal{G}$, $\sum_{g \in \mathcal{G}, (g,h) \in A} a_{g,h} t^g$ belongs to $K_w[\mathcal{G}]$ for every h, and hence $\eta \in (K_w[\mathcal{G}])_w[\mathcal{H}]$.

Conversely, let $\tau = \sum_{h \in D} b_h t^h \in (K_w[\mathcal{G}])_w[\mathcal{H}]$, where $D = \operatorname{supp}(\tau)$ is a well-ordered subset of $\mathcal{H}$, and $b_h \in K_w[\mathcal{G}]$. Let B_h denote the support of b_h. We need to show that $\bigcup_{h \in D}(B_h \times \{h\})$ is well ordered in $\mathcal{G} \times \mathcal{H}$. Let A be any nonempty subset of $\bigcup_{h \in D}(B_h \times \{h\})$. We show that A has a smallest element. Since $p(A)$ is a subset of the well-ordered set D, we can take h_0 to be the smallest element of $p(A)$. The set $A \cap (B_{h_0} \times \{h_0\})$ is well ordered for it is a subset of the well-ordered set $B_{h_0} \times \{h_0\}$. Let (g_0, h_0) be the smallest element of $A \cap (B_{h_0} \times \{h_0\})$. Then (g_0, h_0) is also the smallest element of A. $\square$

Let K be a field. The field of iterated Laurent series $K\langle\langle x_1, \ldots, x_n\rangle\rangle$ is inductively defined to be the field of Laurent series in x_n with coefficients in $K\langle\langle x_1, \ldots, x_{n-1}\rangle\rangle$, with $K\langle\langle x_1\rangle\rangle$ being the field of Laurent series $K((x_1))$.

Corollary 2.8.

$$K_w[\mathbb{Z}^n] \simeq K\langle\langle x_1, x_2, \dots, x_n\rangle\rangle.$$

The detailed proof of this corollary is left to the reader. We only describe the identification as follows. Let $\{e_i\}_{1\leqslant i\leqslant n}$ be the standard basis of $\mathbb{Z}^n$. Then x_i is identified with t^{e_i}. The field of iterated Laurent series turns out to be the most useful special kind of MN-series [16,17].

We conclude this section with the following remark.

Remark 2.9. MN-series were originally defined on totally ordered groups. It was shown in [17, Chapter 3.1] that the results in this section can be generalized: $\mathcal{G}$ can be replaced with a totally ordered monoid (a semigroup with a unit), and K can be replaced with a commutative ring with a unit.

3. The residue theorem

From now on, K is always a field of characteristic zero. Observe that any subgroup of a TOA-group is still a TOA-group under the induced total ordering. Let $\mathcal{G}$ be a TOA-group and let $\mathcal{H}$ be an abelian group. If $\rho:\mathcal{H}\to\mathcal{G}$ is an injective homomorphism, then $\rho(\mathcal{H})\simeq\mathcal{H}$ is a subgroup of $\mathcal{G}$. We can thus regard $\mathcal{H}$ as a subgroup of $\mathcal{G}$ through ρ. The induced ordering $\leqslant^\rho$ on $\mathcal{H}$ is given by $h_1\leqslant^\rho h_2 \Leftrightarrow \rho(h_1)\leqslant_{\mathcal{G}}\rho(h_2)$. Thus $\mathcal{H}$ is a TOA-group under $\leqslant^\rho$. Clearly a subset A of $(\mathcal{H},\leqslant^\rho)$ is well ordered if and only if $\rho(A)$ is well ordered in $(\mathcal{G},\leqslant_{\mathcal{G}})$.

Let $\mathcal{G}$ be a TOA-group. We can give $\mathcal{G}$ a different ordering so that under this new ordering $\mathcal{G}$ is still a TOA-group. For instance, the total ordering $\leqslant^*$ defined by $g_1\leqslant g_2 \Leftrightarrow g_2\leqslant^* g_1$ is clearly such an ordering. One special class of total orderings is interesting for our purpose. If $\rho:\mathcal{G}\to\mathcal{G}$ is an injective endomorphism, then the induced ordering $\leqslant^\rho$ is also a total ordering on $\mathcal{G}$. We denote the corresponding field of MN-series by $K_w^\rho[\mathcal{G}]$.

For example, if $\mathcal{G}=\mathbb{Z}^n$, then any nonsingular matrix $M\in GL(\mathbb{Z}^n)$ induces an injective endomorphism. In particular, $K_w[\mathbb{Z}^2]\simeq K\langle\langle x,t\rangle\rangle$ is the field of double Laurent series, and $K_w^\rho[\mathbb{Z}^2]\simeq K\langle\langle x^{-1},t\rangle\rangle$, where the matrix corresponding to ρ is the diagonal matrix $\mathrm{diag}(-1,1)$. It is easy to see that $K\langle\langle x_1^{\epsilon_1},\dots,x_n^{\epsilon_n}\rangle\rangle$ with $\epsilon_i=\pm1$ are special fields of MN-series $K^\rho\langle\langle x_1,\dots,x_n\rangle\rangle$, where the corresponding matrix for ρ is the diagonal matrix with entries ϵ_i.

Series expansions in a field of MN-series depend on the total ordering $\leqslant^\rho$. When comparing monomials, it is convenient to use the symbol $\preccurlyeq^\rho$: if $g_1\leqslant^\rho g_2$ then we write $t^{g_1}\preccurlyeq^\rho t^{g_2}$. We shall call attention to the expansions in the following example.

Let ρ be defined by $\rho(x)=x^2y$ and $\rho(y)=xy^2$, and consider $K^\rho\langle\langle x,y\rangle\rangle$. The expansion of $1/(x-y)$ is given by

$$\frac{1}{x-y}=\frac{1}{x}\cdot\frac{1}{1-y/x}=\frac{1}{x}\sum_{k\geqslant0}\frac{y^k}{x^k},$$

since $\rho(y/x) = \rho(y)/\rho(x) = y/x \succ 1$, which implies $1 \prec^{\rho} y/x$.

Now notice the expansion of $1/(x^2 - y)$ is given by

$$\frac{1}{x^2 - y} = -\frac{1}{y} \cdot \frac{1}{1 - x^2/y} = -\frac{1}{y} \sum_{k \geqslant 0} \frac{x^{2k}}{y^k},$$

since $\rho(y/x^2) = \rho(y)/\rho(x^2) = 1/x^3 \prec 1$, which implies $1 \prec^{\rho} x^2/y$.

In order to state the residue theorem, we need more concepts. Consider the following situation. Let $\mathcal{G}$ and $\mathcal{H}$ be groups with $\mathcal{H} \simeq \mathbb{Z}^n$, and suppose that we have a total ordering $\leqslant$ on the direct sum $\mathcal{G} \oplus \mathcal{H}$ such that $\mathcal{G} \oplus \mathcal{H}$ is a TOA-group. We identify $\mathcal{G}$ with $\mathcal{G} \oplus 0$ and $\mathcal{H}$ with $0 \oplus \mathcal{H}$. Let $e_1, e_2, \dots, e_n$ be a basis of $\mathcal{H}$. Let ρ be the endomorphism on $\mathcal{G} \oplus \mathcal{H}$ that is generated by $\rho(e_i) = g_i + \sum_j m_{ij} e_j$ for all i, where $g_i \in \mathcal{G}$, and $\rho(g) = g$ for all $g \in \mathcal{G}$. Then ρ is injective if the matrix $M = (m_{ij})_{1 \leqslant i,j \leqslant n}$ belongs to $GL(\mathbb{Z}^n)$, i.e., $\det(M) \neq 0$.

It is natural to use new variables x_i to denote t^{e_i} for all i. Thus monomials in $K_w[\mathcal{G} \oplus \mathcal{H}]$ can be represented as $t^g x_1^{k_1} \cdots x_n^{k_n}$. Correspondingly, ρ acts on monomials by $\rho(t^g) = t^g$ for all $g \in \mathcal{G}$, and $\rho(x_i) = t^{g_i} x_1^{m_{i1}} \cdots x_n^{m_{in}}$.

Notation. If f_i are monomials, we use $\mathbf{f}$ to denote the homomorphism ρ generated by $\rho(x_i) = f_i$.

An element $\eta \neq 0$ of $K_w[\mathcal{G} \oplus \mathcal{H}]$ can be written as

$$\eta = \sum_{\mathbf{k} \in \mathbb{Z}^n} \sum_{g \in \mathcal{G}} a_{g,\mathbf{k}} t^g x_1^{k_1} \cdots x_n^{k_n} = \sum_{\mathbf{k} \in \mathbb{Z}^n} b_{\mathbf{k}} \mathbf{x}^{\mathbf{k}},$$

where $a_{g,\mathbf{k}} \in K$ and $b_{\mathbf{k}} \in K_w[\mathcal{G}]$. If $b_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \neq 0$, then we call it an *x-term* of η. Since the set $\{\operatorname{ord}(b_{\mathbf{k}} \mathbf{x}^{\mathbf{k}})\colon \mathbf{k} \in \mathbb{Z}^n, b_{\mathbf{k}} \neq 0\}$ is a nonempty subset of $\operatorname{supp}(\eta)$, it is well ordered and hence has a least element. Because of the different exponents in the x's, no two of $\operatorname{ord}(b_{\mathbf{k}} \mathbf{x}^{\mathbf{k}})$ are equal. So we can define the *x-initial* term of η to be the x-term that has the least order.

To define the operators $\frac{\partial}{\partial x_i}$, CT_{x_i}, Res_{x_i}, it suffices to consider the case $\mathcal{H} = \mathbb{Z}$. These operators are defined by:

$$\frac{\partial}{\partial x} \sum_{n \in \mathbb{Z}} b_n x^n = \sum_{n \in \mathbb{Z}} n b_n x^{n-1}, \qquad \operatorname*{CT}_x \sum_{n \in \mathbb{Z}} b_n x^n = b_0, \qquad \operatorname*{Res}_x \sum_{n \in \mathbb{Z}} b_n x^n = b_{-1}.$$

Multivariate operators are defined by iteration. All these operators work nicely in the field of MN-series $K_w[\mathcal{G} \oplus \mathcal{H}]$, because an MN-series has a well-ordered support, and still has a well-ordered support after applying these operators.

There are several computational rules [17, Lemma 3.2.1] for evaluating constant terms in the univariate case, but we are going to concentrate on the residue theorem in the multivariate case.

In what follows, we suppose $F_i \in K_w[\mathcal{G} \oplus \mathcal{H}]$ for all i.

Definition 3.1. The Jacobian determinant (or simply Jacobian) of $\mathbf{F} = (F_1, \dots, F_n)$ with respect to $\mathbf{x}$ is defined to be

$$J(\mathbf{F}|\mathbf{x}) := J\left(\frac{F_1, F_2, \dots, F_n}{x_1, x_2, \dots, x_n}\right) = \det\left(\frac{\partial F_i}{\partial x_j}\right)_{1 \leqslant i,\, j \leqslant n}.$$

When the x's are clear, we write $J(F_1, F_2, \dots, F_n)$ for short.

Definition 3.2. If the x-initial term of F_i is $a_i x_1^{b_{i1}} \cdots x_n^{b_{in}}$, then the Jacobian number of $\mathbf{F}$ with respect to $\mathbf{x}$ is defined to be

$$j(\mathbf{F}|\mathbf{x}) := j\left(\frac{F_1, F_2, \dots, F_n}{x_1, x_2, \dots, x_n}\right) = \det(b_{ij})_{1 \leqslant i,\, j \leqslant n}.$$

Definition 3.3. The log Jacobian of $F_1, \dots, F_n$ is defined to be

$$LJ(F_1, \dots, F_n) := \frac{x_1 \cdots x_n}{F_1 \cdots F_n} J(F_1, \dots, F_n).$$

We call it the log Jacobian because formally it can be written as (see [15])

$$LJ(F_1, \dots, F_n) = J\left(\frac{\log F_1, \dots, \log F_n}{\log x_1, \dots, \log x_n}\right),$$

since

$$\frac{\partial \log F}{\partial \log x} = \frac{\partial \log F}{\partial F} \frac{\partial F}{\partial \log x} = \frac{1}{F} \frac{\partial F}{\partial x} \frac{\partial x}{\partial \log x} = \frac{x}{F} \frac{\partial F}{\partial x}.$$

Remark 3.4. The Jacobian is convenient in residue evaluation, while the log Jacobian is convenient in constant term evaluation.

The following lemma is needed for the proof of our residue theorem. It is also a kind of generalized composition law.

Let Φ be a formal series in $x_1, \dots, x_n$ with coefficients in $K_w[\mathcal{G}]$, and let $F_i \in K_w[\mathcal{G} \oplus \mathcal{H}]$. Then $\Phi(F_1, \dots, F_n)$ is obtained from Φ by replacing x_i with F_i. The following lemma gives a simple sufficient condition for the convergence of $\Phi(F_1, \dots, F_n)$.

Lemma 3.5. *Let Φ and F_i be as above and let f_i be the initial term of F_i for all i. Suppose $j(F_1, \dots, F_n) \neq 0$. Then $\Phi(x_1, \dots, x_n) \in K_w^{\mathbf{f}}[\mathcal{G} \oplus \mathcal{H}]$ if and only if $\Phi(f_1, \dots, f_n)$ exists in $K_w[\mathcal{G} \oplus \mathcal{H}]$, and if these conditions hold then $\Phi(F_1, \dots, F_n)$ exists in $K_w[\mathcal{G} \oplus \mathcal{H}]$.*

Proof. We first show the equivalence. The map $\rho : x_i \to f_i$ induces an endomorphism on $\mathcal{H} \simeq \mathbb{Z}^n$. This endomorphism is injective since $j(f_1, \dots, f_n) \neq 0$, which is equivalent to $j(F_1, \dots, F_n) \neq 0$. Therefore ρ also induces an injective endomorphism on $\mathcal{G} \oplus \mathcal{H}$. We see that $\operatorname{supp}(\Phi(f_1,\dots,f_n))$ is well ordered in $\mathcal{G}\oplus\mathcal{H}$ if and only if $\rho(\operatorname{supp}(\Phi(x_1,\dots,x_n)))$ is well ordered. This, by definition, is to say that $\Phi(x_1,\dots,x_n)\in K_w^{\mathbf{f}}[\mathcal{G}\oplus\mathcal{H}]$.

Now we show the implication. Write each F_i as $f_i(1+\tau_i)$, with $\operatorname{ord}(\tau_i)>0$. Given the convergence of $\Phi(f_1,\dots,f_n)$ we first show that for every $g\in\mathcal{G}$ and $\mathbf{m}\in\mathbb{Z}$, $[t^g\mathbf{x}^{\mathbf{m}}]\Phi(F_1,\dots,F_n)$ is a finite sum.

Write Φ as $\sum_{\mathbf{k}\in\mathbb{Z}^n}a_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}$. Let A be the support of $\Phi(\mathbf{f})$. Then A is the disjoint union of $\operatorname{supp}(a_{\mathbf{k}}\mathbf{f}^{\mathbf{k}})$ for all $\mathbf{k}$. This follows from the first part: ρ is injective.

Now

$$\Phi(F_1,\dots,F_n)=\sum_{\mathbf{k}\in\mathbb{Z}^n}a_{\mathbf{k}}\mathbf{f}^{\mathbf{k}}(1+\tau_1)^{k_1}\cdots(1+\tau_n)^{k_n}. \tag{3.1}$$

We observe that replacing any nonzero element in K by 1 will not reduce the number of summands, so $(1+\tau_i)^{k_i}$ can be replaced with $(1-\tau_i)^{-1}=\sum_{l\geqslant 0}\tau_i^l$. Therefore, the number of summands for the coefficient of $t^g\mathbf{x}^{\mathbf{m}}$ in $\Phi(F_1,\dots,F_n)$ is no more than that in

$$\sum_{\mathbf{k}\in\mathbb{Z}^n}a_{\mathbf{k}}\mathbf{f}^{\mathbf{k}}(1-\tau_1)^{-1}\cdots(1-\tau_n)^{-1}=(1-\tau_1)^{-1}\cdots(1-\tau_n)^{-1}\sum_{\mathbf{k}\in\mathbb{Z}^n}a_{\mathbf{k}}\mathbf{f}^{\mathbf{k}},$$

which is a finite product of elements in $K_w[\mathcal{G}\oplus\mathcal{H}]$. Note that in obtaining the right-hand side of the above equation, we used the fact that the supports of $a_{\mathbf{k}}\mathbf{f}^{\mathbf{k}}$ are disjoint for all $\mathbf{k}$.

The proof of the lemma will be finished after we show that $\Phi(F_1,\dots,F_n)$ has a well-ordered support. Let T_i be the support of τ_i. Then the support of $(1+\tau_i)^{k_i}$ is contained in $\bigcup_{l\geqslant 0}T_i^{+l}$. Thus for every $\mathbf{k}$

$$\operatorname{supp}a_{\mathbf{k}}\mathbf{f}^{\mathbf{k}}(1+\tau_1)^{k_1}\cdots(1+\tau_n)^{k_n}\subseteq A+\bigcup_{l\geqslant 0}T_1^{+l}+\cdots+\bigcup_{l\geqslant 0}T_n^{+l},$$

which is well ordered by Propositions 2.1 and 2.3. So by (3.1), the support of $\Phi(F_1,\dots,F_n)$ is also well ordered. $\square$

Remark 3.6. The implication in Lemma 3.5 is not true when $j(F_1,\dots,F_n)=0$. For instance, let $\Phi=\sum_{k\geqslant 0}x_2^k/x_1^k-\sum_{k\geqslant 0}x_2^{3k}/x_1^{2k}$ and let $F_1=x_1^2$, $F_2=x_1(1+x_1)$. Then it is straightforward to check that $\Phi(f_1,f_2)=0$, but $\Phi(F_1,F_2)$ is not in $K\langle\langle x_1\rangle\rangle$.

Notation. Starting with a TOA-group $\mathcal{G}\oplus\mathcal{H}$ as described above, let Φ be a formal series on $\mathcal{G}\oplus\mathcal{H}$. When we write $\operatorname{CT}_{\mathbf{x}}^{\rho}\Phi(x_1,\dots,x_n)$, we mean both that $\Phi(x_1,\dots,x_n)$ belongs to $K_w^{\rho}[\mathcal{G}\oplus\mathcal{H}]$, and that the constant term is taken in this field. When ρ is the identity map, it is omitted. When we write $\operatorname{CT}_{\mathbf{F}}\Phi(F_1,\dots,F_n)$, it is assumed that $\Phi(x_1,\dots,x_n)\in K_w^{\mathbf{f}}[\mathcal{G}\oplus\mathcal{H}]$, where f_i is the initial term of F_i, and we are taking the constant term of $\Phi(x_1,\dots,x_n)$ in the ring $K_w^{\mathbf{f}}[\mathcal{G}\oplus\mathcal{H}]$. Or equivalently, we always have

$$\operatorname*{CT}_{\mathbf{F}}\Phi(F_1,\dots,F_n)=\operatorname*{CT}_{\mathbf{x}}^{\mathbf{f}}\Phi(x_1,\dots,x_n).$$

This treatment is particularly useful when dealing with rational functions.

Now comes our residue theorem for $K_w[\mathcal{G} \oplus \mathcal{H}]$, in which we will see how an element in one field is related to an element in another field through taking constant terms.

Theorem 3.7 (Residue Theorem). *Suppose for each i, $F_i \in K_w[\mathcal{G} \oplus \mathcal{H}]$ has x-initial term $f_i = a_i x_1^{b_{i1}} \cdots x_n^{b_{in}}$ with $a_i \in K_w[\mathcal{G}]$. If $j(F_1, \dots, F_n) \neq 0$, then for any $\Phi(\mathbf{x}) \in K_w^{\mathbf{f}}[\mathcal{G} \oplus \mathcal{H}]$, we have*

$$\operatorname*{Res}_{\mathbf{x}} \Phi(F_1, \dots, F_n) J(F_1, \dots, F_n) = j(F_1, \dots, F_n) \operatorname*{Res}_{\mathbf{F}} \Phi(F_1, \dots, F_n). \tag{3.2}$$

Equivalently, in terms of constant terms, we have

$$\operatorname*{CT}_{\mathbf{x}} \Phi(F_1, \dots, F_n) L J(F_1, \dots, F_n) = j(F_1, \dots, F_n) \operatorname*{CT}_{\mathbf{F}} \Phi(F_1, \dots, F_n). \tag{3.2$'$}$$

Proof. Replace $\Phi(F_1, \dots, F_n)$ with $F_1 \cdots F_n \Phi(F_1, \dots, F_n)$ in (3.2). Then by a straightforward algebraic manipulation, we will get (3.2′). Similarly we can obtain (3.2) from (3.2′). This shows the equivalence.

By the hypothesis and Lemma 3.5, the left-hand side of (3.2) exists by taking the constant term in $K_w[\mathcal{G} \oplus \mathcal{H}]$, while the right-hand side exists by taking the constant term in $K_w^{\mathbf{f}}[\mathcal{G} \oplus \mathcal{H}]$.

For the remaining part it suffices to show that the theorem is true for monomials Φ by multilinearity. The proof will be completed after we show Lemmas 3.13 and 3.14 below.

Remark 3.8. When $j(F_1, \dots, F_n) = 0$, $\Phi(F_1, \dots, F_n)$ is only well defined in some special cases. In such cases, (3.2) also holds. For example, if $\Phi(x_1, \dots, x_n)$ is a Laurent polynomial, then $\Phi(F_1, \dots, F_n)$ always exists.

Remark 3.9. The theorem holds for any rational function Φ, i.e., $\Phi(x_1, \dots, x_n)$ belongs to the quotient field of $(K_w[\mathcal{G}])[\mathcal{H}]$. This follows from the fact that $K_w^{\mathbf{f}}[\mathcal{G} \oplus \mathcal{H}]$ is a field containing $(K_w[\mathcal{G}])[\mathcal{H}]$ as a subring.

The proof of our residue theorem and lemmas basically comes from [2], except for the proof of Lemma 3.14, which uses the original idea of Jacobi.

The following properties of Jacobians can be easily checked.

Lemma 3.10. *We have*

(1) $J(F_1, F_2, \dots, F_n)$ *is $K_w[\mathcal{G}]$-multilinear.*
(2) $J(F_1, F_2, \dots, F_n)$ *is alternating*; *i.e.*, $J(F_1, F_2, \dots, F_n) = 0$ *if* $F_i = F_j$ *for some* $i \neq j$.
(3) $J(F_1, F_2, \dots, F_n)$ *is anticommutative*; *i.e.*,

$$J(F_1, \dots, F_i, \dots, F_j, \dots, F_n) = -J(F_1, \dots, F_j, \dots, F_i, \dots, F_n).$$

(4) (*Composition rule*) *If $g(z) \in K((z))$ and $g(F_1)$ exists in $K_w[\mathcal{G} \oplus \mathcal{H}]$, then*

$$J\big(g(F_1), F_2, \dots, F_n\big) = \frac{dg}{dz}(F_1) J(F_1, F_2, \dots, F_n).$$

(5) (*Product rule*)

$$J(F_1G_1, F_2, \dots, F_n) = F_1 J(G_1, F_2, \dots, F_n) + G_1 J(F_1, F_2, \dots, F_n).$$

(6) $J(F_2^{-1}, F_2, \dots, F_n) = 0$.

A formal series on $\mathcal{G} \oplus \mathcal{H}$ having only one x-term is called an *x-monomial*.

Lemma 3.11. *If all f_i are x-monomials, then*

$$LJ(f_1, \dots, f_n) = j(f_1, \dots, f_n). \tag{3.3}$$

Equivalently,

$$J(f_1, \dots, f_n) = j(f_1, \dots, f_n)\frac{f_1 \cdots f_n}{x_1 \cdots x_n}. \tag{3.3$'$}$$

Proof. Suppose that for every i, $f_i = a_i x_1^{b_{i1}} \cdots x_n^{b_{in}}$, where a_i is in $K_w[\mathcal{G}]$. Then $\partial f_i/\partial x_j = b_{ij} f_i/x_j$. Factoring f_i from the ith row of the Jacobian matrix for all i and then factoring x_j^{-1} from the jth column for all j, we get

$$J(f_1, f_2, \dots, f_n) = \frac{f_1 \cdots f_n}{x_1 \cdots x_n} \det(b_{ij}).$$

Equations (3.3) and (3.3′) are just rewriting of the above equation. □

Lemma 3.12.

$$\operatorname*{Res}_{\mathbf{x}} J(F_1, \dots, F_n) = 0.$$

Proof. By multilinearity, it suffices to check x-monomials F_i. Suppose $F_i = f_i$ as given in the proof of Lemma 3.11. Then Eq. (3.3′) can be rewritten as

$$J(F_1, \dots, F_n) = \det(b_{ij}) a_1 \cdots a_n x_1^{-1+\sum b_{i1}} \cdots x_n^{-1+\sum b_{in}}.$$

If $\sum b_{i1} = \sum b_{i2} = \cdots = \sum b_{in} = 0$, then the Jacobian number is 0, and therefore the residue is 0. Otherwise, at least one of the x_i's has exponent $\neq -1$, so the residue is 0 by definition. □

Lemma 3.13. *For all integers e_i with at least one $e_i \neq -1$, we have*

$$\operatorname*{Res}_{\mathbf{x}} F_1^{e_1} \cdots F_n^{e_n} J(F_1, \dots, F_n) = 0. \tag{3.4}$$

Proof. The clever proof in [2, Theorem 1.4] also works here.

Permuting the F_i and using (3) of Lemma 3.10, we may assume that $e_1 \neq -1, \dots, e_j \neq -1$, but $e_{j+1} = \cdots = e_n = -1$, for some j with $1 \leqslant j \leqslant n$. Setting $G_i = \frac{1}{e_i+1} F_i^{e_i+1}$ for $i = 1, \dots, j$, we have

$$F_1^{e_1} F_2^{e_2} \cdots F_n^{e_n} J(F_1, F_2, \dots, F_n) = F_{j+1}^{-1} \cdots F_n^{-1} J(G_1, \dots, G_j, F_{j+1}, \dots, F_n).$$

Then applying the formula

$$F_{j+1}^{-1} J(G_1, \dots, G_j, F_{j+1}, \dots, F_n) = J\left(F_{j+1}^{-1} G_1, G_2, \dots, G_j, F_{j+1}, \dots, F_n\right)$$

repeatedly for $j+1, j+2, \dots, n$, we get

$$J\left(F_{j+1}^{-1} \cdots F_n^{-1} G_1, G_2, \dots, G_j, F_{j+1}, \dots, F_n\right).$$

The result now follows from Lemma 3.12. $\square$

For the case $e_1 = e_2 = \cdots = e_n = -1$, we have

Lemma 3.14.

$$\operatorname*{Res}_{\mathbf{x}} F_1^{-1} \cdots F_n^{-1} J(F_1, \dots, F_n) = j(F_1, \dots, F_n). \tag{3.5}$$

The simple proof for this case in [2] does not apply in our situation. The reason will be explained in Proposition 3.15.

Note that Lemma 3.14 is equivalent to saying that

$$\operatorname*{CT}_{\mathbf{x}} L J(F_1, \dots, F_n) = j(F_1, \dots, F_n). \tag{3.6}$$

Proof. Let $f_i := a_i x_1^{b_{i1}} \cdots x_n^{b_{in}}$ be the x-initial term of F_i. Then $F_i = f_i B_i$, where $B_i \in K_w[\mathcal{G} \oplus \mathcal{H}]$ has x-initial term 1. By the composition law, $\log(B_i) \in K_w[\mathcal{G} \oplus \mathcal{H}]$. Now applying the product rule, we have

$$\begin{aligned}
&F_1^{-1} \cdots F_n^{-1} J(F_1, F_2, \dots, F_n) \\
&\quad = f_1^{-1} F_2^{-1} \cdots F_n^{-1} J(f_1, F_2, \dots, F_n) + B_1^{-1} F_2^{-1} \cdots F_n^{-1} J(B_1, F_2, \dots, F_n) \\
&\quad = f_1^{-1} F_2^{-1} \cdots F_n^{-1} J(f_1, F_2, \dots, F_n) + F_2^{-1} \cdots F_n^{-1} J\left(\log(B_1), F_2, \dots, F_n\right).
\end{aligned}$$

From Lemma 3.13, the last term in the above equation has no contribution to the residue in x, and hence can be discarded.

The same procedure can be applied to $F_2, F_3, \dots, F_n$. Finally we will get

$$\operatorname*{Res}_{x} F_1^{-1} \cdots F_n^{-1} J(F_1, F_2, \dots, F_n) = \operatorname*{Res}_{x} f_1^{-1} \cdots f_n^{-1} J(f_1, f_2, \dots, f_n),$$

which is equal to the Jacobian number by Lemma 3.11. $\square$

The proof of our residue theorem is now completed. □

The next result gives a good reason for using the log Jacobian.

Proposition 3.15. *The x-initial term of the log Jacobian $LJ(F_1, \dots, F_n)$ equals the Jacobian number $j(F_1, \dots, F_n)$ when it is nonzero.*

Proof. From the definition,

$$LJ(F_1, \dots, F_n) = \frac{x_1 \cdots x_n}{F_1 \cdots F_n} J(F_1, \dots, F_n) = \frac{x_1 \cdots x_n}{F_1 \cdots F_n} \sum_{\mathbf{g}} J(g_1, \dots, g_n),$$

where the sum ranges over all x-terms g_i of F_i. Applying Lemma 3.11 gives us

$$LJ(F_1, \dots, F_n) = \sum_{\mathbf{g}} \frac{g_1 \cdots g_n}{F_1 \cdots F_n} j(g_1, \dots, g_n).$$

The Jacobian number is always an integer. The displayed summand has the smallest order when g_i equals the x-initial term of F_i for all i. It is clear now that we can write

$$LJ(F_1, \dots, F_n) = j(F_1, \dots, F_n) + \text{higher ordered terms}.$$

To show that $j(F_1, \dots, F_n)$ is the x-initial term, we need to show that all the other terms that are independent of x cancel. (Note that we do not have this trouble when all the coefficients belong to K.) This is equivalent to saying that

$$\operatorname*{CT}_{\mathbf{x}} LJ(F_1, \dots, F_n) = j(F_1, \dots, F_n),$$

which follows from Lemma 3.14. □

Example 3.16. Let $K\langle\langle x, t\rangle\rangle$ be the working field. Let $F = x^2 + xt + x^3 t$. Then the x-initial term of F is x^2, so $j(F|x) = 2$. Now let us see what happens to the log Jacobian $LJ(F|x)$ of F with respect to x.

$$\begin{aligned} LJ(F|x) &= \frac{x}{F}\frac{\partial F}{\partial x} = \frac{x(2x + t + 3x^2 t)}{x^2(1 + t/x + xt)} \\ &= \left(2 + \frac{t}{x} + 3xt\right) \sum_{k \geqslant 0} (-1)^k \left(\frac{t}{x} + xt\right)^k. \end{aligned}$$

Since every other monomial is divisible by t, the initial term of $LJ(F|x)$ is 2. It then follows that the x-initial term of $LJ(F|x)$ must contain 2 and therefore must be the constant term in x.

It is not clear that 2 is the unique term in the expansion of $\mathrm{CT}_x\, LJ(F|x)$, but all the other terms cancel. We check as follows.

$$\begin{aligned}\operatorname*{CT}_x LJ(F|x) &= \operatorname*{CT}_x\left(2+\frac{t}{x}+3xt\right)\sum_{k\geqslant 0}(-1)^k\left(\frac{t}{x}+xt\right)^k\\ &= 2\sum_{k\geqslant 0}\binom{2k}{k}t^{2k} - t\sum_{k\geqslant 0}\binom{2k+1}{k}t^{2k+1} - 3t\sum_{k\geqslant 0}\binom{2k+1}{k+1}t^{2k+1}\\ &= 2+\sum_{k\geqslant 1}\left(2\binom{2k}{k}-4\binom{2k-1}{k}\right)t^{2k}.\end{aligned}$$

Now it is easy to see that the terms, other than 2, not containing x in the expansion of the log Jacobian really cancel.

From Theorem 3.7 and Lemma 3.11, we see directly the following result.

Corollary 3.17. *If f_i are all x-monomials in $K_w[\mathcal{G}\oplus\mathcal{H}]$, $j(f_1,\dots,f_n)\neq 0$, and $\Phi\in K_w^{\mathbf{f}}[\mathcal{G}\oplus\mathcal{H}]$, then*

$$\operatorname*{CT}_{\mathbf{x}}\Phi(f_1,\dots,f_n)=\operatorname*{CT}_{f_1,\dots,f_n}\Phi(f_1,\dots,f_n).$$

In the case that all f_i are monomials in $K[\mathbf{x},\mathbf{x^{-1}}]$ with $j(\mathbf{f})\neq 0$, Φ is in $K[\mathbf{x},\mathbf{x^{-1}}]$ if and only $\Phi(f_1,\dots,f_n)$ is (with possible fractional exponents). Since Φ has a finite support, its series expansion is independent of the working field. In particular, we have

$$\operatorname*{CT}_{f_1,\dots,f_n}\Phi(f_1,\dots,f_n)=\operatorname*{CT}_{x_1,\dots,x_n}\Phi(x_1,\dots,x_n).$$

More generally, we have the following as a consequence of Corollary 3.17 and the above argument.

Corollary 3.18. *Suppose $\mathbf{y}$ is another set of variables. If $\Phi\in K[\mathbf{x},\mathbf{x^{-1}}]\langle\langle\mathbf{y}\rangle\rangle$, and if f_i are all monomials in $\mathbf{x}$ with $j(\mathbf{f})\neq 0$, then*

$$\operatorname*{CT}_{\mathbf{x}}\Phi(f_1,\dots,f_n)=\operatorname*{CT}_{\mathbf{x}}\Phi(x_1,\dots,x_n).$$

The following two examples are illustrative in explaining our residue theorem.

Example 3.19. The following identity follows trivially by replacing x with x^{-1}.

$$\operatorname*{CT}_x\sum_{k\geqslant 0}x^{-k}=\operatorname*{CT}_x\sum_{k\geqslant 0}x^k. \tag{3.7}$$

This identity is not as simple as it might appear at first sight. It equates the constant terms of two elements belonging to two different fields; namely, the left-hand side of (3.7) takes the constant term in $K((x^{-1}))$, while the right-hand side takes the constant term in $K((x))$.

The above cannot be explained by Jacobi's formula, especially when we write it in terms of rational functions:

$$\operatorname*{CT}_x \frac{1}{1-x^{-1}} = \operatorname*{CT}_x \frac{1}{1-x}. \tag{3.8}$$

Now let us explain this identity in two ways: one using our residue theorem, and the other using complex analysis.

Let $f = x^{-1}$. Then the log Jacobian $LJ(f|x) = x/f \cdot \partial f/\partial x = -1$, and the Jacobian number is also -1. Thus

$$\operatorname*{CT}_x \frac{1}{1-x} = \operatorname*{CT}_x \frac{1}{1-f^{-1}} \cdot \bigl(-LJ(f|x)\bigr) = \operatorname*{CT}_f \frac{1}{1-f^{-1}}.$$

So the x on the left-hand side of (3.8) is indeed playing the same role with the variable f defined by $f = x^{-1}$. Now $f^{-1} \succ 1$ since it is the same as $x \succ 1$, and we have the correct series expansion.

Now we sketch the idea in complex analysis, and describe the meaning of Jacobian number in the one variable case. We have

$$\operatorname*{CT}_x \frac{1}{1-x} = \frac{1}{2\pi i} \oint_\gamma \frac{1}{z(1-z)}\, dz,$$

where γ is the counter-clockwise circle $|z| = \epsilon$ for sufficiently small positive ϵ. We can think of ϵ as equal to x.

Now if we make a change of variable by $z = 1/u$, then after simplifying, we get

$$\frac{1}{2\pi i} \oint_{\gamma'} \frac{-1}{u(1-u^{-1})}\, du = \operatorname*{CT}_f \frac{1}{1-f^{-1}},$$

where γ', the image of γ under the map $z \mapsto 1/u$, is the clockwise circle $|u| = 1/\epsilon$. The Jacobian number -1 comes from the different orientation of the circle. Similarly, if we are making the change of variable by $z = u^2$, the new circle will be a double circle, which is consistent with the fact that the Jacobian number is 2.

Example 3.20. Evaluate the following constant term in $K((x))$.

$$\operatorname*{CT}_x \frac{(1-x^{-1})^4}{(x-1)(\pi(1-x^{-1}) + (1-x^{-1})^2)}.$$

Solution. Let $F = 1 - x^{-1}$. Then $LJ(F|x) = x/F \cdot dF/dx = 1/(x-1)$. The x-initial term of F is x^{-1} so that the Jacobian number is -1. Hence by our residue theorem, we have

$$\begin{aligned}\operatorname*{CT}_x \frac{(1-x^{-1})^4}{(x-1)(\pi(1-x^{-1})+(1-x^{-1})^2)} &= \operatorname*{CT}_x \frac{F^4}{\pi F+F^2} LJ(F|x)\\ &= \operatorname*{CT}_F (-1)\cdot \frac{F^4}{\pi F+F^2}.\end{aligned}$$

Now the initial term of F is x^{-1} and the initial term of F^2 is x^{-2} so that $F \succ F^2$. Thus the final solution is

$$\operatorname*{CT}_F \frac{-F^2}{1+\pi F^{-1}} = -\pi^2.$$

Remark 3.21. Suppose the working field is $K((x))$. If the new variable F has a positive Jacobian number $j(F|x)$, the second field as described in our residue theorem is also $K((x))$. In this case, Jacobi's formula also applies. If $j(F|x)$ is a negative number, then we can choose F^{-1} as the new variable to apply Jacobi's formula. This is why the two fields phenomenon as in the above two examples was not noticed before.

The next example is hard to evaluate without using our residue theorem.

Example 3.22. Evaluate the following constant term in $\mathbb{C}\langle\langle x, y, t\rangle\rangle$.

$$\operatorname*{CT}_{x,y} x^3 e^{t/xy}(2t-3xy)\big(x^3ye^{t/xy}-tx-ty\big)^{-1}(x-y)^{-1}\big(-1+x^3e^{t/xy}\big)^{-1}. \tag{3.9}$$

Solution. The x-variables are x and y. Let $F = x^2ye^{t/xy}$, $G = xy^2e^{t/xy}$. It is straightforward to compute the log Jacobian and the Jacobian number. We have

$$LJ(F,G|x,y) = 3-\frac{2t}{xy}, \quad \text{and} \quad j(F,G|x,y)=3.$$

We can check that (3.9) can be written as

$$\operatorname*{CT}_{x,y} \frac{F^3G}{(F^2-(F+G)t)(F-G)(G-F^2)} LJ(F,G|x,y).$$

Thus by the residue theorem, the above constant term equals

$$\operatorname*{CT}_{F,G} \frac{3F^3G}{(F^2-(F+G)t)(F-G)(G-F^2)} = \operatorname*{CT}_{F,G} \frac{3}{\big(1-\frac{(F+G)t}{F^2}\big)\big(1-\frac{G}{F}\big)\big(1-\frac{F^2}{G}\big)}, \tag{3.10}$$

where on the right-hand side of (3.10), we can check that 1 is the initial term of each factor in the denominator.

At this stage, we can use series expansion to obtain the constant term. We use the following lemma instead.

Lemma 3.23. *Suppose that Φ contains only nonnegative powers in x. Then*

$$\operatorname*{CT}_x \Phi(x) \cdot \frac{1}{1-u/x} = \Phi(u),$$

where u is independent of x and $u \succ x$.

This lemma is reduced by linearity to the case when $\Phi(x) = x^k$ for some nonnegative integer k, which is trivial.

We take the constant term in G first by applying Lemma 3.23.

$$\begin{aligned}
\operatorname*{CT}_{F,G} \frac{3}{\left(1-\frac{(F+G)t}{F^2}\right)\left(1-\frac{G}{F}\right)\left(1-\frac{F^2}{G}\right)} &= \operatorname*{CT}_F \frac{3F^3}{(F^2-(F+F^2)t)(F-F^2)} \\
&= \operatorname*{CT}_F \frac{3}{(1-t)(1-F)} \cdot \frac{1}{\left(1-\frac{t}{(1-t)F}\right)} \\
&= \frac{3}{(1-t)\left(1-\frac{t}{1-t}\right)},
\end{aligned}$$

where in the last step, we applied Lemma 3.23 again.

After simplification, we finally get

$$\operatorname*{CT}_{x,y} x^3 e^{t/xy}(2t-3xy)\left(x^3 y e^{t/xy} - tx - ty\right)^{-1}(x-y)^{-1}\left(-1+x^3 e^{t/xy}\right)^{-1} = \frac{3}{1-2t}.$$

4. Another view of Lagrange's inversion formula

Let $F_1, \dots, F_n$ be power series in variables $x_1, \dots, x_n$ of the form $F_i = x_i +$ higher degree terms, with indeterminate coefficients for each i. It is known, e.g., [1, Proposition 5, p. 219], that $\mathbf{F} = (F_1, \dots, F_n)$ has a unique compositional inverse, i.e., there exists $\mathbf{G} = (G_1, \dots, G_n)$ where each G_i is a power series in $x_1, \dots, x_n$ such that $F_i(G_1, \dots, G_n) = x_i$ and $G_i(F_1, \dots, F_n) = x_i$ for all i.

Lagrange inversion gives a formula for the G's in terms of the F's. Such a formula is very useful in combinatorics. A good summary of this subject can be found in [4].

The diagonal (or Good's) Lagrange inversion formula deals with the diagonal case, in which x_i divides F_i for every i, or equivalently, $F_i = x_i H_i$, where $H_i \in K[[x_1, \dots, x_n]]$ with constant term 1. We now derive Good's formula by our residue theorem:

Let $K\langle\langle x_1, x_2, \dots, x_n \rangle\rangle$ be the working field. Then x_i is the initial term of F_i, and the Jacobian number $j(F_1, \dots, F_n)$ equals 1. Let $y_i = F_i(\mathbf{x})$. We will have $x_i = G_i(\mathbf{y})$. Then

$$\left[y_1^{k_1} \cdots y_n^{k_n}\right] G_i(\mathbf{y}) = \operatorname*{Res}_{\mathbf{y}} y_1^{-1-k_1} \cdots y_n^{-1-k_n} G_i(y) \tag{4.1}$$

$$= \operatorname*{Res}_{\mathbf{x}} F_1^{-1-k_1} \cdots F_n^{-1-k_n} x_i J(\mathbf{F}). \tag{4.2}$$

The above argument works the same way by using Jacobi's residue formula.

A similar computation applies to the nondiagonal case by working in $K^\rho\langle\langle x_1,\ldots,x_n\rangle\rangle$, where ρ is the injective homomorphism into $K\langle\langle x_1,\ldots,x_n,t\rangle\rangle$ induced by $\rho: x_i \mapsto x_i t$. This total ordering makes x_i the initial term of F_i for all i, and clearly $K^\rho\langle\langle x_1,\ldots,x_n\rangle\rangle$ contains $K[[x_1,\ldots,x_n]]$ as a subring. This way is equivalent to the homogeneous expansion introduced in [2]. Note that Jacobi's formula does not apply directly, though Gessel [4] showed how the nondiagonal case could be derived from the diagonal case. Note also that we cannot apply the residue theorem in $K\langle\langle x_1,\ldots,x_n\rangle\rangle$, because the Jacobian number might equal 0. For example, if x_n does not divide F_n, then it is easily seen that the exponent of x_n in the initial term of F_i is zero for all i. So the Jacobian number of $F_1,\ldots,F_n$ is 0.

More generally, let $\Phi \in K[[y_1,\ldots,y_n]]$. Then

$$\left[y_1^{k_1}\cdots y_n^{k_n}\right]\Phi\big(\mathbf{G}(\mathbf{y})\big) = \operatorname*{Res}_{\mathbf{x}} F_1^{-1-k_1}\cdots F_n^{-1-k_n}\Phi(\mathbf{x})J(\mathbf{F}).$$

Multiplying both sides of the above equation by $y_1^{k_1}\cdots y_n^{k_n}$, and summing on all nonnegative integers $k_1,k_2,\ldots,k_n$, we get

$$\Phi\big(\mathbf{G}(\mathbf{y})\big) = \operatorname*{Res}_{\mathbf{x}} \frac{1}{F_1-y_1}\cdots\frac{1}{F_n-y_n}J(\mathbf{F})\Phi(\mathbf{x}), \tag{4.3}$$

which is true as power series in the y_i's.

It is natural to ask if we can get this formula directly from our residue theorem. The answer is yes. The argument is given as follows.

The working field is $K^\rho\langle\langle x_1,\ldots,x_n\rangle\rangle\langle\langle y_1,\ldots,y_n\rangle\rangle$. We let $z_i = F_i - y_i$. Then $x_i = G_i(\mathbf{y}+\mathbf{z})$, and the initial term of $F_i - y_i$ is x_i, for y_i has higher order than the x's. Thus the Jacobian number is 1, and the Jacobian determinant $J(\mathbf{z}|\mathbf{x})$ still equals $J(\mathbf{F})$. Applying the residue theorem, we get

$$\operatorname*{Res}_{\mathbf{x}} \frac{1}{F_1-y_1}\cdots\frac{1}{F_n-y_n}J(\mathbf{F})\Phi(\mathbf{x}) = \operatorname*{Res}_{\mathbf{z}}\frac{1}{z_1z_2\cdots z_n}\Phi\big(\mathbf{G}(\mathbf{y}+\mathbf{z})\big).$$

Since $\Phi(\mathbf{G}(\mathbf{y}+\mathbf{z}))$ is in $K[[\mathbf{y},\mathbf{z}]]$, the final result is obtained by setting $\mathbf{z}=\mathbf{0}$ in $\Phi(\mathbf{G}(\mathbf{y}+\mathbf{z}))$.

Note that $J(\mathbf{F}) \in K[[\mathbf{x}]]$ has constant term 1. Therefore $J(\mathbf{F})^{-1}\Phi(\mathbf{x})$ is also in $K[[\mathbf{x}]]$. Hence we can reformulate (4.3) as

$$\operatorname*{Res}_{\mathbf{x}} \frac{1}{F_1-y_1}\cdots\frac{1}{F_n-y_n}\Phi(\mathbf{x}) = \Phi(\mathbf{x})J(\mathbf{F})^{-1}\big|_{\mathbf{x}=\mathbf{G}(\mathbf{y})}.$$

5. Dyson's conjecture

Our residue theorem can be used to prove a conjecture of Dyson.

Theorem 5.1 (Dyson's Conjecture). *Let $a_1,\ldots,a_n$ be n nonnegative integers. Then the following equation holds as Laurent polynomials in $\mathbf{z}$.*

$$\operatorname*{CT}_{\mathbf{z}} \prod_{1\leqslant i\neq j\leqslant n} \left(1-\frac{z_i}{z_j}\right)^{a_j} = \frac{(a_1+a_2+\cdots+a_n)!}{a_1!a_2!\cdots a_n!}. \tag{5.1}$$

For $n=3$ this assertion is equivalent to the familiar Dixon identity:

$$\sum_j (-1)^j \binom{a+b}{a+j}\binom{b+c}{b+j}\binom{c+a}{c+j} = \frac{(a+b+c)!}{a!b!c!}.$$

Theorem 5.1 was first proved by Wilson [15] and Gunson [7] independently. A similar proof was given by Egorychev in [3, pp. 151–153]. These proofs use integrals of analytic functions. A simple induction proof was found by Good [5]. We are going to give a Laurent series proof by using the residue theorem for MN-series. Our new proof uses Egorychev's change of variables, and uses Wilson's argument for evaluating the log Jacobian. This leads to a generalization of Theorem 5.1.

Let $\mathbf{z}$ be the vector $(z_1, z_2, \dots, z_n)$. If $\mathbf{z}$ appears in the computation, we use $\mathbf{z}$ for the product $\mathbf{z}^{\mathbf{1}} = z_1 z_2 \cdots z_n$. We use similar notation for $\mathbf{u}$.

Let $\Delta(\mathbf{z}) = \Delta(z_1, \dots, z_n) = \prod_{i<j}(z_i - z_j) = \det(z_i^{n-j})$ be the Vandermonde determinant in $\mathbf{z}$, and let $\Delta_j(\mathbf{z}) = \Delta(z_1, \dots, \hat{z}_j, \dots, z_n)$, where $\hat{z}_j$ means to omit z_j. We introduce new variables

$$u_j = (-1)^{j-1} z_j^{n-1} \Delta_j(\mathbf{z}).$$

Then they satisfy the equations

$$\Delta(\mathbf{z}) = \sum_{j=1}^n (-1)^{j-1} z_j^{n-1} \Delta_j(\mathbf{z}) = u_1 + u_2 + \cdots + u_n,$$

and

$$u_1 \cdots u_n = \prod_{j=1}^n (-1)^{j-1} z_j^{n-1} \Delta_j(\mathbf{z}) = (-1)^{\binom{n}{2}} \mathbf{z}^{n-1} \big(\Delta(\mathbf{z})\big)^{n-2}.$$

We also have

$$\prod_{i=1, i\neq j}^n \left(1 - \frac{z_i}{z_j}\right) = (-1)^{j-1} \frac{\Delta(\mathbf{z})}{z_j^{n-1}\Delta_j(\mathbf{z})} = \frac{u_1 + u_2 + \cdots + u_n}{u_j}.$$

Thus Eq. (5.1) is equivalent to

$$\operatorname*{CT}_{\mathbf{z}} \frac{(u_1+u_2+\cdots+u_n)^{a_1+a_2+\cdots+a_n}}{u_1^{a_1}\cdots u_n^{a_n}} = \frac{(a_1+a_2+\cdots+a_n)!}{a_1!a_2!\cdots a_n!},$$

which is a direct consequence of the multinomial theorem and the following proposition.

Proposition 5.2. *For any series $\Phi(\mathbf{z}) \in K^{\mathbf{u}}\langle\langle \mathbf{z}\rangle\rangle$, we have*

$$\operatorname*{CT}_{\mathbf{z}} \Phi(u_1, \dots, u_n) = \operatorname*{CT}_{\mathbf{u}} \Phi(u_1, \dots, u_n).$$

In fact, we can prove a more general formula. Let r be an integer and let

$$u_j^{(r)} = (-1)^{j-1} z_j^r \Delta_j(\mathbf{z}).$$

Then $u_1^{(r)} + \cdots + u_n^{(r)}$ equals $h_{r-n+1}(z_1, z_2, \dots, z_n)\Delta(\mathbf{z})$ for $r \geqslant n-1$ and equals 0 for $0 \leqslant r \leqslant n-2$, where $h_k(\mathbf{z}) = \sum_{i_1 \leqslant \cdots \leqslant i_k} z_{i_1} \cdots z_{i_k}$ is the complete symmetric function [13, Theorem 7.15.1]. We have the following generalization.

Theorem 5.3. *If r is not equal to any of $0, 1, \dots, n-2$, or $-\binom{n-1}{2}$, then for any series $\Phi(\mathbf{z}) \in K^{\rho}\langle\langle \mathbf{z}\rangle\rangle$, where $\rho(z_i) = u_i^{(r)}$, we have*

$$\operatorname*{CT}_{\mathbf{z}} \Phi\big(u_1^{(r)}, \dots, u_n^{(r)}\big) = \operatorname*{CT}_{\mathbf{u}^{(r)}} \Phi\big(u_1^{(r)}, \dots, u_n^{(r)}\big).$$

Note that Proposition 5.2 is the special case for $r = n-1$ of Theorem 5.3. If we set $r = n$, the multinomial theorem yields the following:

Corollary 5.4. *Let $a_1, \dots, a_n$ be n nonnegative integers. Then the following equation holds for Laurent polynomials in $\mathbf{z}$.*

$$\operatorname*{CT}_{\mathbf{z}} \frac{(z_1 + \cdots + z_n)^{a_1 + \cdots + a_n}}{z_1^{a_1} \cdots z_n^{a_n}} \prod_{1 \leqslant i \neq j \leqslant n} \left(1 - \frac{z_i}{z_j}\right)^{a_j} = \frac{(a_1 + a_2 + \cdots + a_n)!}{a_1! a_2! \cdots a_n!}. \tag{5.2}$$

By Theorem 3.7 and Proposition 3.15, Theorem 5.3 is equivalent to the assertion that the log Jacobian is a nonzero constant. To show this, we use

Lemma 5.5 [15, Lemma 4]. *Let $G(x_1, \dots, x_n)$ be a ratio of two polynomials in the x's, in which the denominator is $\Delta(x_1, \dots, x_n)$ and*

(1) *G is a symmetric function of $x_1, \dots, x_n$,*
(2) *G is homogeneous of degree 0 in the x's.*

Then G is a constant.

Proof of Theorem 5.3. In order to compute the log Jacobian, we let

$$J = \det(J_{ij}) = \det\left(\frac{\partial \log u_i^{(r)}}{\partial \log z_j}\right).$$

Then $J_{ii} = r$ and $J_{ij} = \sum_{k \neq i} z_i/(z_k - z_j)$ for $i \neq j$. We first show that J is a constant by Lemma 5.5. It is easy to see that J satisfies conditions (1), (2) in Lemma 5.5. Now we show that the denominator of J is $\Delta(\mathbf{z})$, so that we can claim that the Jacobian is a constant, and hence equals the Jacobian number.

Evidently J is the ratio of two polynomials in the $\mathbf{z}$'s, whose denominator is a product of factors $z_i - z_j$ for some $i \neq j$. From the expression of J_{ij}, we see that $z_i - z_j$ only appears in the ith and the jth column. Every 2 by 2 minor of the ith and jth columns is of the following form, in which we assume that k and l are not one of i and j.

$$\begin{vmatrix} J_{ki} & J_{kj} \\ J_{li} & J_{lj} \end{vmatrix} = \begin{vmatrix} \frac{z_k}{z_j - z_i} + \sum_{s \neq i,j} \frac{z_k}{z_s - z_i} & \frac{z_k}{z_i - z_j} + \sum_{s \neq i,j} \frac{z_k}{z_s - z_j} \\ \frac{z_l}{z_j - z_i} + \sum_{s \neq i,j} \frac{z_l}{z_s - z_i} & \frac{z_l}{z_i - z_j} + \sum_{s \neq i,j} \frac{z_l}{z_s - z_j} \end{vmatrix}.$$

In the above determinant, the terms containing $(z_i - z_j)^2$ as the denominator cancel. Therefore, expanding the determinant according to the ith and jth column, we see that $\Delta(\mathbf{z})$ is the denominator of J.

Now the initial term of $z_i - z_j$ is z_i if $i < j$. We see that the initial term of $u_1^{(r)}$ is $z_1^r z_2^{n-2} z_3^{n-3} \cdots z_{n-1}$. Similarly we can get the initial term for $u_j^{(r)}$. The Jacobian number, denoted by $j(r)$, is thus the determinant

$$j(r) = \det \begin{pmatrix} r & n-2 & n-3 & \cdots & 0 \\ n-2 & r & n-3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ n-2 & n-3 & n-4 & \cdots & r \end{pmatrix},$$

where the displayed matrix has diagonal entries r, and other entries in each row are $n-2, n-3, \ldots, 0$, from left to right.

Since the row sum of each row is $r + \binom{n-1}{2}$, it follows that

$$j\left(-\binom{n-1}{2}\right) = 0.$$

We claim that $j(r) = 0$ when $r = 0, 1, \ldots, n-2$. For in those cases, $u_1^{(r)} + \cdots + u_n^{(r)} = 0$. This implies that the Jacobian is 0, and hence $j(r) = 0$. We can regard $j(r)$ as a polynomial in r of degree n, and we already have n zeros. So

$$j(r) = r(r-1)\cdots(r-n+2)\left(r + \binom{n-1}{2}\right)$$

up to a constant. This constant equals 1 through comparing the leading coefficient of r.

In particular, $j(n-1) = \binom{n}{2}(n-1)! = (n-1)n!/2$. Note that in [3, p. 153], the constant was said to be $(2n-3)(n-1)!$, which is not correct. $\square$

Another proof of Dyson's conjecture by our residue theorem is to use the change of variables by Wilson [15].

Let

$$v_j = \prod_{i=1,\, i \neq j}^{n} \left(1 - \frac{z_j}{z_i}\right)^{-1}.$$

Then the initial term of v_j is $z_j^{-(n-j)} z_{j+1} \cdots z_n$ up to a constant. Since the order of v_n is $\mathbf{0}$, we have to exclude v_n from the change of variables, for otherwise, the Jacobian number will be 0. In fact, we have the relation $v_1 + v_2 + \cdots + v_n = 1$, which can be easily shown by Lemma 5.5.

Dyson's conjecture is equivalent to

$$\operatorname*{CT}_{\mathbf{z}} \prod_{j=1}^{n} v_i^{-a_j} = \frac{(a_1 + a_2 + \cdots + a_n)!}{a_1! a_2! \cdots a_n!}. \tag{5.3}$$

Another Proof of Dyson's Conjecture. Using Lemma 5.5 and Wilson's argument, we can evaluate the following log Jacobian. (See [15] for details.)

$$\frac{\partial(\log v_1, \log v_2, \ldots, \log v_{n-1})}{\partial(\log z_1, \log z_2, \ldots, \log z_{n-1})} = (n-1)! v_n.$$

Then by the residue theorem

$$\operatorname*{CT}_{z} \Phi(v_1, \ldots, v_{n-1}, z_n) = \operatorname*{CT}_{v_1, \ldots, v_{n-1}, z_n} (1 - v_1 - \cdots - v_{n-1})^{-1} \Phi(v_1, \ldots, v_{n-1}, z_n).$$

In particular (since the initial term of $1 - v_1 - \cdots - v_{n-1}$ is 1) we have:

$$\begin{aligned}
\operatorname*{CT}_{\mathbf{z}} \prod_{j=1}^{n} v_i^{-a_j} &= \operatorname*{CT}_{v_1, \ldots, v_{n-1}, z_n} (1 - v_1 - \cdots - v_{n-1})^{-a_n - 1} \prod_{j=1}^{n-1} v_i^{-a_j} \\
&= \left[v_1^{a_1} \cdots v_{n-1}^{a_{n-1}}\right] \sum_{m \geqslant 0} \binom{a_n + m}{a_n} (v_1 + \cdots + v_{n-1})^m \\
&= \binom{a_n + a_1 + \cdots + a_{n-1}}{a_n} \binom{a_1 + \cdots + a_{n-1}}{a_1, \ldots, a_{n-1}}.
\end{aligned}$$

Equation (5.3) then follows. □

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