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Shortened recurrence relations for Bernoulli numbers

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Abstract

Starting with two little-known results of Saalschütz, we derive a number of general recurrence relations for Bernoulli numbers. These relations involve an arbitrarily small number of terms and have Stirling numbers of both kinds as coefficients. As special cases we obtain explicit formulas for Bernoulli numbers, as well as several known identities.

Keywords: Bernoulli numbers; Stirling numbers; Recurrence relations

1. Introduction

The Bernoulli numbers B_n, $n = 0, 1, 2, \ldots$, can be defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad |x| < 2\pi. \tag{1.1}$$

It is easy to find the values $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, and $B_n = 0$ for all odd $n \geq 3$. Furthermore, $(-1)^{n-1} B_{2n} > 0$ for all $n \geq 1$. These and many other properties can be found, for instance, in [1,9,12], or [15].

Bernoulli numbers have found numerous important applications, most notably in number theory, the calculus of finite differences, and asymptotic analysis. One of the main concerns from the beginning was the efficient calculation of the Bernoulli numbers, and to this end recurrence relations were soon used as the most important tool. Apparently the first such relation was derived by De Moivre in 1830, although Jacob Bernoulli himself in his posthumously published *Ars Conjectandi* of 1713 implicitly used what amounts to a recurrence relation. Later such formulas often appeared in textbooks, starting with Euler. For a survey and relevant references on the earlier history of recurrence relations, see the introduction to [23]. The most basic recurrence relation is

$$\sum_{j=0}^{n} \binom{n+1}{j} B_j = 0, \quad (n \geq 1), \tag{1.2}$$

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with $B_0 = 1$. It can be derived from (1.1) by multiplying both sides by $e^x - 1$, using the Cauchy product with the Maclaurin series for $e^x - 1$, and then equating the coefficients of the powers of x.

Subsequently, *shortened* or *incomplete* recurrence relations of two different types were discovered. The first type consists of lacunary recurrence relations, where B_n is determined not from all the preceding nonzero Bernoulli numbers, but only from every second, or every third one, etc. The first such formula was apparently derived by Knar [13], and further relations were proved, among others, by Ramanujan [17] (see also [22] for a historical survey), and more recently by Chellali [5].

The second type of incomplete recurrence relations requires the knowledge of (typically) only the second-half of all the Bernoulli numbers up to B_n, in order to compute B_{n+1}. Such a formula first appears in a textbook by von Ettingshausen [7] in 1827, and was subsequently rediscovered and extended by several other authors, some of them quite recent. Fairly complete studies of such formulas, and those of the first type mentioned above, can be found in the books [18,14], and in the paper [2]. More recently, the authors [3,4] studied a different type of such formulas. For an extensive bibliography on Bernoulli numbers, see [6].

The present paper is related to the second type of incomplete recurrence relations. In particular, we begin in Section 2 by examining two papers by Saalschütz, [19,20], published in 1903, the second of which is just a summary of the first. To the best of our knowledge they were never cited before, apart from a detailed review in [11] and an entry in [6]. Saalschütz derived a remarkable identity that contains as special cases both a complete recurrence relation and an explicit formula for Bernoulli numbers, and in general incomplete recurrence relations with arbitrarily few terms preceding the highest Bernoulli number B_n. It was not until 80 years later that Todorov [21] obtained a similar formula covering the whole range from full recurrence relation to explicit expansion; this will be discussed in Section 4.

Throughout this paper the Stirling numbers of both kinds play an important role; they will be discussed in Section 3. Finally, in Section 5 we derive other similar formulas for the Bernoulli numbers, using some of Saalschütz's ideas.

2. Saalschütz's identities

In this section we give an account of Saalschütz's main results in [19,20]. The theorems and ideas of proof are basically due to Saalschütz; however, we use different notations and normalizations, in order to simplify the results and their proofs.

We begin with the Laurent expansion of the cosecant function,

$$x \operatorname{cosec} x = \frac{x}{\sin x} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2^{2k} - 2}{(2k)!} B_{2k} x^{2k}; \tag{2.1}$$

see, e.g., [1, p. 75]. Since every second coefficient is 0, the same will be true for all positive integer powers of $x \operatorname{cosec} x$, and we write

$$(x \operatorname{cosec} x)^m = \sum_{j=0}^{\infty} c_{2j}^m x^{2j}, \tag{2.2}$$

where $c_0^m = 1$ for all $m \geq 0$. Using the obvious identity $\operatorname{cosec} x \cdot \operatorname{cosec}^{2n+1} x = \operatorname{cosec}^{2n+2} x$, we get with (2.2) and upon equating the coefficients of x^{2n},

$$\sum_{k=0}^{n} c_{2k}^1 c_{2n-2k}^{2n+1} = c_{2n}^{2n+2}. \tag{2.3}$$

The coefficients c_{2k}^1 are given by (2.1), but the other two coefficient sequences still need to be determined. This is done by using the identity

$$m(m+1)(x \operatorname{cosec} x)^{m+2} = m^2 x^2 (x \operatorname{cosec} x)^m + x^{m+2} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \operatorname{cosec}^m x, \tag{2.4}$$

which is easily obtained by taking the second derivative of $(\sin x)^{-m}$. Now, substituting (2.2) into (2.4) and equating the coefficients of x^{2j}, we get

$$c_{2j}^{m+2} = \frac{m}{m+1} c_{2j-2}^{m} + \frac{(2j-m)(2j-m-1)}{m(m+1)} c_{2j}^{m}. \tag{2.5}$$

If m is even then for $2j = m$ we immediately obtain

$$c_{m}^{m+2} = \frac{m}{m+1} c_{m-2}^{m},$$

and with the initial condition $c_0^2 = 1$ this gives

$$c_{2n}^{2n+2} = \frac{2n(2n-2)\dots 4\cdot 2}{(2n+1)(2n-1)\dots 3\cdot 1} = \frac{4^n n!^2}{(2n+1)!}. \tag{2.6}$$

Next, if we set $m = 2n-1$ and $j = n-k$ in (2.5) and normalize the coefficients as

$$C(n,k) = \frac{(2n)!}{(2k)!} c_{2n-2k}^{2n+1}, \tag{2.7}$$

then (2.5) becomes

$$C(n,k) = (2n-1)^2 C(n-1,k) + C(n-1,k-1), \tag{2.8}$$

with the convention $C(n,-1) = C(n,n+1) = 0$ and the initial conditions

$$C(n,0) = (2n)! c_{2n}^{2n+1} = \prod_{j=1}^{n} (2j-1)^2 = \left(\frac{(2n)!}{2^n n!}\right)^2, \tag{2.9}$$

which follows, once again, from (2.5).

We now show that these numbers have the desired properties.

Lemma 2.1. *For integers $n \geq 0$ we have*

$$\prod_{j=1}^{n} \left((2j-1)^2 + x\right) = \sum_{k=0}^{n} C(n,k) x^k, \tag{2.10}$$

with the usual convention that the empty product is 1.

Proof. We proceed by induction on n. For $n = 0$, both sides of (2.10) are 1, by (2.9). Now suppose that (2.10) holds for some $n-1$. With (2.8) we have

$$\begin{aligned}
\sum_{k=0}^{n} C(n,k)x^k &= (2n-1)^2 \sum_{k=0}^{n-1} C(n-1,k)x^k + \sum_{k=1}^{n} C(n-1,k-1)x^k \\
&= (2n-1)^2 \sum_{k=0}^{n-1} C(n-1,k)x^k + x \sum_{k=0}^{n-1} C(n-1,k)x^k \\
&= \left((2n-1)^2 + x\right) \sum_{k=0}^{n-1} C(n-1,k)x^k,
\end{aligned}$$

and the result follows from the induction hypothesis. □

Saalschütz starts his derivation by considering the sum of all products of k elements from the set of n elements $\{1, 3^{-2}, 5^{-2}, \dots, (2n-1)^{-2}\}$. If we denote this sum by $\widetilde{C}(n,k)$, it is easily seen that these numbers have the generating function

$$\prod_{j=1}^{n} \left(1 + \frac{x}{(2j-1)^2}\right) = \sum_{k=0}^{n} \widetilde{C}(n,k) x^k,$$

and that they are related to $C(n,k)$ by

$$C(n,k) = \left(\frac{(2n)!}{2^n n!}\right)^2 \widetilde{C}(n,k).$$

We are now ready to state Saalschütz's first result; see [19, Eq. (65)], [20, Eq. (9)]. The proof follows immediately from (2.3), with (2.1), (2.6) and (2.7).

Theorem 2.1 (*Saalschütz, 1903*). *For any integer $n \geq 0$ we have*

$$\sum_{k=0}^{n} (-1)^{k+1} \left(2^{2k} - 2\right) C(n,k) B_{2k} = \frac{4^n}{2n+1} n!^2. \tag{2.11}$$

Saalschütz [19, p. 19], [20, p. 100] remarked that (2.11) may be the first recurrence relation for the Bernoulli numbers with only positive terms. Indeed, it is well known that $(-1)^{k+1} B_{2k} > 0$ for $k \geq 0$, so that (2.11) can be rewritten in the slightly simplified form

$$1 + \sum_{k=1}^{n} \left(2^{2k} - 2\right) C(n,k) |B_{2k}| = \frac{4^n}{2n+1} n!^2. \tag{2.12}$$

It should also be pointed out the the modern notation for the Bernoulli numbers, as used here, differs from that used by Saalschütz in [19,20], and also in his classical book [18]. In [19, p. 5] the Bernoulli numbers in that notation, which we shall denote by $\mathbf{B}_n$, are defined through the expansion of the cotangent function,

$$\cot x = \frac{\cos x}{\sin x} = \frac{1}{x} - 2^2 \mathbf{B}_1 \frac{x}{2!} - 2^4 \mathbf{B}_2 \frac{x^3}{4!} - \cdots - 2^{2m} \mathbf{B}_m \frac{x^{2m-1}}{(2m)!} - \cdots,$$

so that $\mathbf{B}_n = (-1)^{n-1} B_{2n} = |B_{2n}|$ for $n \geq 1$; see the modern cotangent expansion, e.g., in [1, p. 75].

To derive Saalschütz's second recurrence relation, we introduce another family of combinatorial numbers.

Definition 2.1. For integers $0 \leq k \leq m \leq n$ consider the set

$$\left\{(2n-2m+1)^2, (2n-2m+3)^2, \ldots, (2n-2k+1)^2\right\},$$

and let $C^{(n)}(m,k)$ be the sum of all products of any k elements of it, with repetitions allowed.

Lemma 2.2. *The numbers $C^{(n)}(m,k)$ satisfy the recurrence relation*

$$C^{(n)}(m+1,k) = (2n-2m-1)^2 C^{(n-1)}(m,k-1) + C^{(n)}(m,k), \tag{2.13}$$

with the convention $C^{(n)}(m,-1) = C^{(n)}(m,m+1) = 0$.

Proof. By definition the sum $C^{(n)}(m+1,k)$ is determined by the set

$$\left\{(2n-2m-1)^2, (2n-2m+1)^2, \ldots, (2n-2k+1)^2\right\}.$$

Now $C^{(n)}(m+1,k)$ can be split into a sum whose summands do not contain $(2n-2m-1)^2$ as a factor, which gives $C^{(n)}(m,k)$, and a sum all of whose summands have $(2n-2m-1)^2$ as a factor. In this latter case we factor out $(2n-2m-1)^2$, which reduces the products to those of $k-1$ elements. The defining set is then rewritten as

$$\left\{(2(n-1)-2m+1)^2, (2(n-1)-2m+3)^2, \ldots, (2(n-1)-2(k-1)+1)^2\right\},$$

which gives the sum of products $C^{(n-1)}(m,k-1)$. This completes the proof. □

A generating function for the numbers $C^{(n)}(m,k)$, analogous to (2.10), can be found in Lemma 3.2 below. We are now ready to state and prove Saalschütz's main result; see [19, p. 26, Eq. (73)] or [20, Eq. (10)].

Theorem 2.2 (*Saalschütz, 1903*). *For any integer* $0 \le m \le n$ *we have*

$$\sum_{k=m}^{n}(-1)^{k+1}\left(2^{2k}-2\right)C(n-m,k-m)B_{2k}=\sum_{j=0}^{m}(-1)^j\frac{4^{n-j}(n-j)!^2}{2(n-j)+1}C^{(n)}(m,j). \tag{2.14}$$

Proof. We proceed by induction on m and first note that for $m = 0$ the identity (2.14) reduces to (2.11) since $C^{(n)}(m,0) = 1$. Now fix an integer m, $0 \le m \le n-1$, and suppose that (2.14) is true for this m and for all $n \ge m+1$. We replace n by $n-1$ in (2.14), multiply both sides of the resulting identity by $(2n-2m-1)^2$, and subtract this from (2.14). On the left-hand side the relevant terms inside the sum are

$$C(n-m,k-m)-(2n-2m-1)^2C(n-1-m,k-m)=C(n-(m+1),k-(m+1)),$$

which follows directly from (2.8). Following the same procedure on the right-hand side, we get

$$\begin{aligned}&\sum_{j=0}^{m}(-1)^j\frac{4^{n-j}(n-j)!^2}{2(n-j)+1}C^{(n)}(m,j)-(2n-2m-1)^2\sum_{j=0}^{m}(-1)^j\frac{4^{n-1-j}(n-1-j)!^2}{2(n-1-j)+1}C^{(n-1)}(m,j)\\ &\quad=\sum_{j=0}^{m}(-1)^j\frac{4^{n-j}(n-j)!^2}{2(n-j)+1}C^{(n)}(m,j)+(2n-2m-1)^2\sum_{j=1}^{m+1}(-1)^j\frac{4^{n-j}(n-j)!^2}{2(n-j)+1}C^{(n-1)}(m,j-1).\end{aligned}$$

Using (2.13), we obtain an expression that is just the right-hand side of (2.14), with m replaced by $m+1$. This completes the proof by induction. □

3. Stirling numbers

The identities (2.8) and (2.13) indicate that there may be a relationship between the coefficients $C(n,k)$, $C^{(n)}(m,k)$ and the Stirling numbers.

The *Stirling numbers of the first kind* can be defined by the generating function

$$x(x-1)\cdots(x-n+1)=\sum_{k=0}^{n}s(n,k)x^k; \tag{3.1}$$

for a combinatorial interpretation see, e.g., [9]. Among their many properties we mention only that $s(0,0) = s(n,n) = 1$, $s(n,0) = 0$ for $n \ge 1$, and $s(n,k) = 0$ for $k < 0$ and for $k > n$. The most basic recurrence relation is

$$s(n+1,k)=s(n,k-1)-n\,s(n,k), \quad 1\le k\le n. \tag{3.2}$$

Further properties can be found in several of the books listed in the references, e.g., [1]. The reader should be aware of the different systems of notation; a summary of notational conventions is given in [1].

We use the Stirling numbers of the first kind to give an expression for the numbers $C(n,k)$ defined in Section 2.

Lemma 3.1. *For any integers* $0 \le k \le n$ *we have*

$$C(n,k)=(-1)^{n-k}4^{n-k}\sum_{j=0}^{2n-2k}\binom{2k+j}{2k}s(2n,2k+2j)\left(n-\frac{1}{2}\right)^j. \tag{3.3}$$

Proof. If we denote the left-hand side of (2.10) by $p_n(x)$, set $x = -y^2$, and factor each of the n terms in $p_n(x)$, we get

$$(-1)^n p_n(-y^2)=(y+(2n-1))\,(y+(2n-3))\cdots(y+1)(y-1)\cdots(y-(2n-1))\,.$$

If we set, for a moment, $z := y+2n-1$, we get with (3.1) and a binomial expansion,

$$\begin{aligned}(-1)^n p_n(-y^2) &= z(z-1)\cdots(z-2n+1)\\ &= \sum_{j=0}^{2n} s(2n,j)z^j\\ &= \sum_{j=0}^{2n} s(2n,j)\sum_{k=0}^{j}\binom{j}{k}(2n-1)^{j-k}y^k.\end{aligned}$$

Writing $y = \mathrm{i}\sqrt{x}$ (the sign is irrelevant) and changing the order of summation, we get

$$p_n(x) = (-1)^n \sum_{k=0}^{2n}(i\sqrt{x})^k \sum_{j=0}^{2n-k} s(2n, j+k)2^{-k}\binom{j+k}{k}(2n-1)^j. \tag{3.4}$$

Since the coefficients on the left are all real, we take only the real part on the right, and obtain

$$p_n(x) = (-1)^n \sum_{k=0}^{n}(-x)^k \sum_{j=0}^{2n-2k} s(2n, 2k+j)2^{-2k}\binom{2k+j}{2k}(2n-1)^j.$$

Now the coefficient of x^k immediately gives (3.3). $\square$

We note in passing that by considering the *imaginary* parts in (3.4) we would obtain a sequence of identities involving the Stirling numbers of the first kind and binomial coefficients. A large number of similar identities can be found in [9] and in [10].

With Lemma 3.1 we can now rewrite Theorem 2.1 as follows.

Corollary 3.1. *For any $n \geq 0$ we have*

$$\sum_{k=0}^{n}\left(1-2^{1-2k}\right)\left(\sum_{j=0}^{2n-2k}\binom{2k+j}{2k}s(2n, 2k+2j)\left(n-\frac{1}{2}\right)^j\right)B_{2k} = \frac{(-1)^{n+1}(n!)^2}{2n+1}. \tag{3.5}$$

The second extreme case of Theorem 2.2 is the case where $m = n$, so that the left-hand side of (2.14) reduces to just one term. We use the fact that $C(0,0) = 1$.

Corollary 3.2. *For any $n \geq 0$ we have*

$$B_{2n} = \frac{1}{1-2^{1-2n}}\sum_{j=0}^{n}(-1)^{n-j+1}\frac{(n-j)!^2}{2^{2j}(2n-2j+1)}C^{(n)}(n,j). \tag{3.6}$$

We can express this last formula in terms of *Stirling numbers of the second kind* $S(n,k)$. These numbers can be defined by the generating function

$$\prod_{j=1}^{k}\frac{x}{1-jx} = \sum_{n=k}^{\infty} S(n,k)x^n; \tag{3.7}$$

for a combinatorial interpretation see again, e.g., [9]. An important companion relation to (3.1) is

$$x^n = \sum_{k=0}^{n} S(n,k)x(x-1)\cdot\cdots\cdot(x-k+1), \tag{3.8}$$

and the triangular recurrence relation analogous to (3.2) is

$$S(n+1,k) = S(n,k-1) + k\,S(n,k). \tag{3.9}$$

In analogy to Lemma 3.1 we now have

Lemma 3.2. (a) *For* $0 \le j \le m \le n$, *the number* $C^{(n)}(m, j)$ *is the coefficient of* x^j *in the Maclaurin expansion of*

$$\prod_{k=0}^{m-j} \frac{1}{1-(2n-2m+2k+1)^2 x}. \tag{3.10}$$

(b) *For* $0 \le j \le n$ *we have*

$$C^{(n)}(n, j) = \sum_{r=0}^{2n+2} (-1)^r \binom{2n+1}{r} 2^{2j-r} S(2n+2-r, 2n+2-2j)(2n+3-2j)^r. \tag{3.11}$$

Proof. (a) Expand each factor in (3.10) as a geometric series, multiply these series and collect the coefficients of x^j. Comparing this with Definition 2.1 gives the result. (This is a much used standard method; see, e.g., [16], Ch. 1.)

(b) We have $m = n$ in (3.10), and to simplify notation we set $d := n - j$. Letting $x = y^2$, we get

$$\begin{aligned}\frac{1}{(2y)^{2d+2} q(y^2)} &= \frac{y^{-1}+(2d+1)}{2} \cdot \frac{y^{-1}+(2d-1)}{2} \cdots \frac{y^{-1}+1}{2} \cdot \frac{y^{-1}-1}{2} \cdots \frac{y^{-1}-(2d+1)}{2} \\ &= \frac{1}{z^{2d+2}}(1-z)(1-2z)\cdots(1-(2d+2)z),\end{aligned}$$

where we have set

$$\frac{y^{-1}+(2d+1)}{2} = \frac{1}{z} - 1, \quad \text{i.e., } z = 2\left(\frac{1}{y} + (2d+3)\right)^{-1}.$$

We denote the expression in (3.10) by $q(x)$, where the dependence on $m = n$ and j is implied. Then

$$\begin{aligned} q(x) &= \frac{1}{(4x)^{d+1}} \frac{1}{(1-z)(1-2z)\cdots(1-(2d+2)z)} \\ &= \frac{1}{(4x)^{d+1}} \sum_{k=2d+2}^{\infty} S(k, 2d+2) z^k \\ &= \frac{1}{(4x)^{d+1}} \sum_{k=2d+2}^{\infty} S(k, 2d+2) 2^k \sum_{r=0}^{\infty} \binom{-k}{r} (2d+3)^r (\sqrt{x})^{r+k};\end{aligned}$$

here we have used (3.7) and a binomial expansion for z. If we change the order of summation by setting $\nu = r + k$ as index of summation, we obtain

$$q(x) = \frac{1}{(4x)^{d+1}} \sum_{k=2d+2}^{\infty} \left(\sum_{r=0}^{\nu} \binom{r-\nu}{r} 2^{\nu-r} S(\nu - r, 2d+2)(2d+3)^r \right) (\sqrt{x})^{\nu}.$$

Finally, by considering only even ν we see that the coefficient of x^j is the right-hand side of (3.11), after small rearrangements of the sum, and using the binomial identity $\binom{r-\nu}{r} = (-1)^r \binom{\nu-1}{r}$. $\square$

Combining (3.11) with (3.6) and rearranging, we obtain the following expression of Bernoulli numbers in terms of Stirling numbers.

Corollary 3.3. *For any* $n \ge 0$ *we have*

$$B_{2n} = \frac{1}{2^{2n}-2} \sum_{j=1}^{n+1} (-1)^j \frac{(j-1)!}{2j-1} \times \sum_{r=2j}^{2n+2} (-1)^r \binom{2n+1}{r-1} 2^{r-2} S(r, 2j)(2j+1)^{2n+2-r}. \tag{3.12}$$

4. Todorov's result

Leading up to our own results, we quote an identity of Todorov [21] which, to the best of our knowledge, is the first result after Saalschütz's to contain arbitrarily few Bernoulli numbers.

Theorem 4.1 (*Todorov, 1984*). *For any integers* $0 \le m \le n$ *we have*

$$\sum_{j=m}^{n}(-1)^{n+j}s(n+1-m,j+1-m)B_j=(n-m)!\sum_{j=0}^{m}(-1)^j\frac{j!S(m,j)}{n-m+1+j}. \tag{4.1}$$

We have slightly rewritten Todorov's result as it appears in [21, p. 339], to make it consistent with our notational convention that has the sums on the left-hand sides range from B_m to B_n (or B_{2m} to B_{2n} in Saalschütz's case). In his proof Todorov uses the explicit expression

$$B_n=\sum_{r=0}^{\nu}(-1)^{\nu-r}r!S(\nu+1,r+1)\sum_{k=0}^{n-\nu}(-1)^k\frac{k!S(n-\nu,k)}{r+k+1}$$

(true for any integer $0 \le \nu \le n$) which he derived in [21], along with many other similar identities. He then multiplies this with a certain Stirling number of the first kind, and sums over a certain range. The identity equivalent to (4.1) is then obtained by using one of the "orthogonality relations" (sometimes called inversion formulas) connecting the Stirling numbers of the first and the second kind, namely,

$$\sum_{\nu=0}^{\max(n,m)}s(n,\nu)S(\nu,m)=\delta_{nm},$$

where δ_{nm} is the Kronecker symbol. There is also a companion formula, with the order of s, S interchanged.

As two extreme cases of (4.1) we get, with $m=0$, the full recurrence relation

$$\sum_{j=0}^{n}(-1)^{n+j}s(n+1,j+1)B_j=\frac{n!}{n+1}, \tag{4.2}$$

and with $m=n$ the explicit expression

$$B_n=\sum_{j=0}^{n}(-1)^j\frac{j!}{j+1}S(n,j). \tag{4.3}$$

This is a well-known expression; see [8] for different proofs and a historical perspective. Any such expression involving Stirling numbers of the second kind is indeed an explicit expression since we have the well known and important finite sum

$$S(n,k)=\frac{1}{k!}\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}(k-j)^n. \tag{4.4}$$

5. New formulas for Bernoulli numbers

If we analyze Saalschütz's proof of Theorem 2.1 we see that the most important ingredients are the generating function (2.1), the trivial identity $\operatorname{cosec} x \cdot \operatorname{cosec}^m x = \operatorname{cosec}^{m+1} x$, and the numbers $C(n,k)$ with their triangular recurrence relation (2.8).

We can take a similar approach by using the left-hand side of (1.1) instead of the cosecant; this will simplify any eventual identity since the right-hand side of (1.1) is obviously easier than that of (2.1). But also, we will get a more direct approach to Stirling numbers through the exponential generating function

$$\frac{1}{k!}(\mathrm{e}^x-1)^k=\sum_{n=k}^{\infty}S(n,k)\frac{x^n}{n!}, \tag{5.1}$$

which is quite obviously related to (1.1) and suggests that we use

$$\left(\frac{\mathrm{e}^x-1}{x}\right)^k\frac{x}{\mathrm{e}^x-1}=\left(\frac{\mathrm{e}^x-1}{x}\right)^{k-1}. \tag{5.2}$$

If we rewrite (5.1) as

$$\left(\frac{e^x-1}{x}\right)^k=\sum_{n=0}^{\infty}\frac{S(n+k,k)}{\binom{n+k}{k}}\frac{x^n}{n!}, \tag{5.3}$$

take the Cauchy products of the right-hand sides of (5.3) and (1.1) and equate the coefficients of x^n via (5.2), we obtain after some easy manipulations the following identity.

Theorem 5.1. *For any integers $k\geq 1$ and $n\geq 0$ we have*

$$\sum_{j=0}^{n}\binom{n+k}{j}S(n-j+k,k)B_j=\frac{n+k}{k}S(n+k-1,k-1). \tag{5.4}$$

For $k=1$ and using the fact that $S(n,1)=1$ for all $n\geq 0$ and $S(n,0)=0$ for all $n\geq 1$, we immediately obtain (1.2). We list the next two special cases; they immediately follow from (5.4) with (4.4).

Corollary 5.1. *For any $n\geq 0$ we have*

$$\sum_{j=0}^{n}\binom{n+2}{j}\left(2^{n+1-j}-1\right)B_j=\frac{n+2}{2}, \tag{5.5}$$

$$\sum_{j=0}^{n}\binom{n+3}{j}\left(3^{n+2-j}-2^{n+3-j}+1\right)B_j=\frac{2}{3}(n+3)(2^{n+1}-1). \tag{5.6}$$

These are, of course, "full" recurrence relations, and not of the Saalschütz or Todorov type. However, such relations can be obtained by suitably modifying the approach that led to Theorem 5.1. Indeed, we will be able to obtain shortened recurrence relations by using instead of (1.1) the mth derivative, namely

$$\frac{d^m}{dx^m}\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_{n+m}\frac{x^n}{n!}. \tag{5.7}$$

To do this, we use an expression from [3] for the left-hand side of (5.7):

Lemma 5.1. *For any $m\geq 0$ we have*

$$\frac{d^m}{dx^m}\frac{1}{e^x-1}=(-1)^m\sum_{j=1}^{m+1}(j-1)!\frac{S(m+1,j)}{(e^x-1)^j}, \tag{5.8}$$

$$\frac{d^m}{dx^m}\frac{x}{e^x-1}=(-1)^m\sum_{j=1}^{m+1}(j-1)!\frac{S(m+1,j)x-mS(m,j)}{(e^x-1)^j}, \tag{5.9}$$

We are now ready to prove our next result.

Theorem 5.2. *For any integers $0\leq m\leq n$ and $k\geq 1$ we have*

$$\sum_{j=m}^{n}\binom{n+k}{j-m}S(n-j+k+m,k+m)B_j=\frac{(-1)^m}{k+m}\sum_{j=1}^{m+1}\frac{N_1(n,m,k,j)}{\binom{k+m-1}{j-1}}, \tag{5.10}$$

where the numerator $N_1(n,m,k,j)$ is

$$(n+k)S(m+1,j)S(n+k-1,k+m-j)-mS(m,j)S(n+k,k+m-j).$$

Proof. Replace k by $k+m$ in (5.3) and multiply the resulting identity with (5.7). Then we have

$$\left(\frac{e^x-1}{x}\right)^{k+m}\frac{d^m}{dx^m}\frac{x}{e^x-1}=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\frac{S(n-j+k+m,k+m)}{\binom{n-j+k+m}{k+m}}B_{j+m}\right)\frac{x^n}{n!}. \tag{5.11}$$

On the other hand, let $A(x)$ denote the left-hand side of (5.11). Then we get with (5.9),

$$A(x)=\frac{(-1)^m}{x^{k+m}}\sum_{j=1}^{m+1}(j-1)!\,(S(m+1,j)x-mS(m,j))\,(e^x-1)^{k+m-j},$$

and with (5.1) we obtain

$$\begin{aligned}A(x)&=\frac{(-1)^m}{x^{k+m}}\sum_{j=1}^{m+1}(j-1)!\,(S(m+1,j)x-mS(m,j))\times(k+m-j)!\sum_{n=k+m-j}^{\infty}S(n,k+m-j)\frac{x^n}{n!}\\&=\frac{(-1)^m}{x^{k+m}}\sum_{j=1}^{m+1}(j-1)!(k+m-j)!\left(\sum_{n=k+m-j}^{\infty}S(m+1,j)S(n,k+m-j)\frac{x^{n+1}}{n!}\right.\\&\qquad\left.-m\sum_{n=k+m-j}^{\infty}S(m,j)S(n,k+m-j)\frac{x^n}{n!}\right).\end{aligned}$$

Now, doing some easy changes in the summation, we obtain

$$A(x)=(-1)^m\sum_{j=1}^{m+1}(j-1)!(k+m-j)!\sum_{n=-j}^{\infty}N_2(n,m,k,j)\frac{x^n}{(n+k+m)!}, \tag{5.12}$$

where $N_2(n,m,k,j)$ denotes the expression

$$(n+k+m)S(m+1,j)S(n+k+m-1,k+m-j)-mS(m,j)S(n+k+m,k+m-j).$$

Finally we equate the coefficients of x^n in (5.11) and (5.12), replace n by $n-m$ and change the summation in the inner sum to $j=m,\dots,n$. After collecting the various binomial coefficients and factorials on both sides, e.g., by using the identity

$$\binom{n-m}{j-m}\binom{n-j+m+k}{k+m}^{-1}=\binom{n+k}{j-m}\binom{n+k}{k+m}^{-1},$$

we obtain (5.10), as desired. $\square$

As a first special case of Theorem 5.2, with $m=0$, we clearly get Theorem 5.1. At the other extreme, for $m=n$, we have the following corollary if we use the fact that $S(n,n)=1$:

Corollary 5.2. *For any $n\ge 0$ and $k\ge 1$ we have*

$$B_n=\frac{(-1)^n}{k+n}\sum_{j=1}^{n+1}\frac{1}{\binom{k+n-1}{j-1}}\left((n+k)S(n+1,j)S(n+k-1,n+k-j)-nS(n,j)S(n+k,n+k-j)\right) \tag{5.13}$$

Thus, for each n we have an infinite class of expressions for B_n. Compare this also with Corollary 5.3.

For a different, perhaps somewhat simpler, family of expressions we now use the identity (5.8) instead of (5.9). The analogue to (5.7) is then

$$\frac{d^{m-1}}{dx^{m-1}}\frac{1}{e^x-1}=(-1)^{m-1}\frac{(m-1)!}{x^m}+\sum_{n=0}^{\infty}\frac{B_{n+m}}{n+m}\frac{x^n}{n!}. \tag{5.14}$$

Leaving the details to the reader, we proceed exactly as in the proof of Theorem 5.2: On the one hand take the Cauchy product of the right-hand sides of (5.14) and of (5.3), with $k+m$ instead of k. On the other hand, multiply the left-hand side of (5.3), again taking $k+m$ instead of k, with the right-hand side of (5.8), where m is replaced by $m-1$; then use (5.3) again. Finally, equate the coefficients of x^n and simplify. We then have the following result.

Theorem 5.3. *For any integers $1 \le m \le n$ and $k \ge 0$ we have*

$$\sum_{j=m}^{n} \binom{n+k}{j-m} S(n-j+k+m, k+m) \frac{B_j}{j} = (-1)^m \frac{(n+k)!(m-1)!}{(n+k+m)!} S(n+k+m, k+m) + \frac{(-1)^{m-1}}{k+m} \sum_{j=1}^{m} \frac{S(m,j)S(n+k,k+m-j)}{\binom{k+m-1}{j-1}}. \tag{5.15}$$

Let us once again consider the extreme cases. For $m=1$ we get Theorem 5.1, upon some simplification. For $m=n$ we easily obtain the following explicit expansions.

Corollary 5.3. *For any $n \ge 1$ and $k \ge 0$ we have*

$$B_n = (-1)^n \left(\frac{S(2n+k, n+k)}{\binom{2n+k}{n}} - \frac{n}{n+k} \sum_{j=1}^{n} \frac{S(n,j)S(n+k,n+k-j)}{\binom{k+n-1}{j-1}} \right), \tag{5.16}$$

and in particular,

$$B_n = (-1)^n \left(\frac{S(2n,n)}{\binom{2n}{n}} - \sum_{j=1}^{n} \frac{S(n,j)S(n,n-j)}{\binom{n-1}{j-1}} \right). \tag{5.17}$$

This last identity indicates that many of the results in this section can also be seen as convolution identities for Stirling numbers.

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