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Algorithms for General Monotone Mixed Variational Inequalities

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In this paper, we suggest and analyze some new iterative methods for solving general monotone mixed variational inequalities, which are being used to study odd-order and nonsymmetric boundary value problems arising in pure and applied sciences. These new methods can be viewed as generalizations and extensions of the methods of He, Solodov and Tseng, and Noor for solving monotone (mixed) variational inequalities.

Key Words: Variational inequalities, resolvent equations, iterative methods, convergence, fixed points

1. INTRODUCTION

It is well known that variational principles can be used to interpret the basic principles of mathematical and physical sciences with simplicity and elegance. One of the most fruitful ideas in the calculus of variations is that of variational inequalities, which were introduced by Stampacchia [18] in 1964. During the last three decades, there has been considerable activity in the development of numerical techniques for solving variational inequalities. There are a substantial number of numerical methods, including projection technique and its variant forms, Wiener–Hopf equations, descent linear approximation, the auxiliary principle technique, and the Newton method (see [1–18]). It is worth mentioning that the projection method and the Wiener–Hopf equations cannot be extended and modified to study the existence of a solution of the mixed variational inequalities. This fact motivated Glowinski *et al.* [5] to develop another technique, which is called the auxiliary principle technique. For the recent applica-

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tions of the auxiliary principle technique, see Noor [14, 15]. In this paper, we consider a new technique, which depends on the concept of the resolvent of the related maximal monotone operator. Using the resolvent operator technique, we establish the equivalence between mixed variational inequalities, fixed-point problems, and resolvent equations. These alternative equivalent formulations are used to suggest and analyze a number of well-known iterative methods for general monotone mixed variational inequalities. As special cases, we apply a number of well-known iterative methods to the classical monotone variational inequalities.

2. PRELIMINARIES

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$, respectively. Let K be a nonempty closed convex set in H. Let φ: $H \to R \cup \{+\infty\}$ be a proper, convex, and lower semicontinuous function.

For given nonlinear operators T, g: $H \to H$, consider the problem of finding $u \in H$ such that

$$\langle Tu, g(v) - g(u) \rangle + \varphi(g(v)) - \varphi(g(u)) \geq 0, \qquad \text{for all } g(v) \in H. \tag{2.1}$$

The inequality of type (2.1) is called the general mixed variational inequality or the general variational inequality of the second kind. It can be shown that a wide class of linear and nonlinear problems arising in pure and applied sciences can be studied via the general mixed variational inequalities (2.1).

We remark that if $g \equiv I$, the identity operator, then problem (2.1) is equivalent to finding $u \in H$ such that

$$\langle Tu, v - u \rangle + \varphi(v) - \varphi(u) \geq 0, \qquad \text{for all } v \in H, \tag{2.2}$$

which are called the mixed variational inequalities. For the applications, numerical methods, and formulations, see [6, 11, 12, 16] and the references therein.

We note that if φ is the indicator function of a close convex set K in H, that is,

$$\varphi(u) \equiv I_K(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{otherwise}, \end{cases}$$

then the general mixed variational inequality (2.1) is equivalent to finding $u \in H$, $g(u) \in K$, such that

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \qquad \text{for all } g(v) \in K. \tag{2.3}$$

The inequality of type (2.3) is known as the general variational inequality, which was introduced and studied by Noor [9] in 1988. It turned out that the odd-order and nonsymmetric free, unilateral, obstacle, and equilibrium problems can be studied by the general variational inequality (2.1) (see [9, 10]).

For $g \equiv I$, the identity operator, the general variational inequality (2.3) collapses to: find $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0, \qquad \text{for all } v \in K, \tag{2.4}$$

which is called the standard variational inequality, introduced and studied by Stampacchia [18] in 1964. For the recent state of the art, see [1−18].

It is worth mentioning that the projection technique and its variant forms, including the Wiener−Hopf equations, cannot be used to suggest iterative methods for solving the (general) mixed variational inequalities of types (2.1) and (2.2) because of the presence of the nonlinear term φ. To overcome this difficulty, another technique, which is called the resolvent operator technique, mainly due to Noor [11−13], is used to suggest some iterative methods for solving problem (2.2). In this paper, we extend the resolvent operator technique for the general mixed variational inequality (2.1). For this purpose, we recall the following well-known concepts and results.

Definition 2.1 [2]. If A is a maximal monotone operator on H, then, for a constant $\rho > 0$, the resolvent operator associated with A is defined by

$$J_A(u) = (I + \rho A)^{-1}(u), \qquad \text{for all } u \in H,$$

where I is the identity operator. It is well known that a monotone operator is maximal if and only if its resolvent operator is defined everywhere. In addition, the resolvent operator is single-valued and nonexpansive, that is, for all $u, v \in H$,

$$\| J_A(u) - J_A(v) \| \leq \|u - v\|.$$

Remark 2.1. It is well-known that the subdifferential $\partial\varphi$ of a proper, convex, and lower semicontinuous function φ: $H \to R \cup \{+\infty\}$ is a maximal monotone; so we denote by

$$J_\varphi(u) = (I + \varphi\, \partial\varphi)^{-1}(u), \qquad \text{for all } u \in H,$$

the resolvent operator associated with $\partial\varphi$, which is defined everywhere on H.

LEMMA 2.1 [2]. *For a given $z \in H$, $u \in H$ satisfies the inequality*

$$\langle u - z, v - u\rangle + \rho\varphi(v) - \rho\varphi(u) \geq 0, \qquad \text{for all } v \in H,$$

if and only if

$$u = J_\varphi z,$$

where $J_\varphi = (I + \varphi\,\partial\varphi)^{-1}$ is the resolvent operator and ρ is a constant. This property of the resolvent operator J_φ plays an important part in obtaining our results.

Let $R_\varphi \equiv I - J_\varphi$, where I is the identity operator and $J_\varphi \equiv (I + \rho\,\partial\varphi)^{-1}$ is the resolvent operator. For a given nonlinear operator $T, g\colon H \to H$, consider the problem of finding $z \in H$ such that

$$Tg^{-1}J_\varphi z + \rho^{-1}R_\varphi z = 0, \tag{2.5}$$

where $\rho > 0$ is a constant and g is invertible. The equations of type (2.7) are called the general resolvent equations. If $g \equiv I$, the identity operator, then problem (2.7) reduces to: find $z \in H$ such that

$$TJ_\varphi z + \rho^{-1}R_\varphi z = 0, \tag{2.6}$$

which are the resolvent equations. For the applications, formulation, and numerical methods of the resolvent equations, see [11, 12].

We remark that if φ is the indicator function of a closed convex set K in H, then $J_\varphi \equiv P_K$, the projection of H onto K. Consequently problem (2.7) is equivalent to finding $z \in H$ such that

$$Tg^{-1}P_K z + \rho^{-1}Q_K z = 0. \tag{2.7}$$

Equations of type (2.9) are known as the general Wiener−Hopf equations, which are mainly due to Noor [9]. We would like to mention that the Wiener−Hopf equations are being used to develop some implementable and efficient iterative algorithms for solving variational inequalities and related fields. For the recent state of the art, see [14] and the references therein.

3. MAIN RESULTS

In this section, we suggest and analyze some new iterative methods for solving the general monotone mixed variational inequality (2.1). First of all, we prove that the variational inequality (2.1) is equivalent to the fixed-point problem by invoking Lemma 2.1.

LEMMA 3.1. *The function* $u \in H$ *is a solution of the mixed variational inequality* (2.1) *if and only if* $u \in H$ *satisfies the relation*

$$g(u) = J_{\varphi}[g(u) - \rho T u], \tag{3.1}$$

where $J_{\varphi} = (I + \rho\, \partial\varphi)^{-1}$ *is the resolvent and* $\rho > 0$ *is a constant.*

Proof. Let $u \in H$ be a solution of (2.1). Then

$$\langle Tu, g(v) - g(u) \rangle + \varphi(g(v)) - \varphi(g(u)) \geq 0, \qquad \text{for all } g(v) \in H,$$

which can be written as

$$\langle g(u) - (g(u) - \rho Tu), g(v) - g(u) \rangle + \rho\varphi(g(v)) - \rho\varphi(g(u)) \geq 0,$$

which is equivalent to

$$g(u) = J_{\varphi}[g(u) - \rho Tu],$$

by invoking Lemma 2.1.

Lemma 3.1 implies that the general mixed variational inequality (2.1) is equivalent to the fixed-point problem. This alternative equivalent formulation is very useful from a numerical point of view. This fixed-point formulation enables us to suggest and analyze the following iterative algorithm.

ALGORITHM 3.1. *For a given* $u_0 \in H$, *compute the approximate solution* u_{n+1} *by the iterative scheme*

$$u_{n+1} = u_n - g(u_n) + J_{\varphi}[g(u_n) - \rho T u_n], \qquad n = 0, 1, 2, \ldots .$$

For the convergence analysis of Algorithm 3.1, *see Noor* [16], *if the operators* T, g *are strongly monotone and Lipschitz continuous.*

We define the residue vector $R(u)$ by the relation

$$R(u) = g(u) - J_{\varphi}[g(u) - \rho Tu]. \tag{3.2}$$

Thus it is obvious that $u \in H$ is a solution of the general mixed variational inequality (2.1) if and only if $u \in H$ is a zero of the equation

$$R(u) = 0. \tag{3.3}$$

For a constant $\gamma \in (0, 2)$, Eq. (3.3) can be written as

$$g(u) + \rho Tu = g(u) + \rho Tu - \gamma R(u).$$

This formulation can be used to suggest the following new implicit method for solving the general mixed variational inequality (2.1).

ALGORITHM 3.2. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = g(u_n) + \rho Tu_n - \rho Tu_{n+1} - \gamma R(u_n), \qquad n = 0, 1, 2, \dots . \tag{3.4}$$

We remark that if φ is the indicator function of the closed convex set K in H, then the resolvent operator $J_\varphi = P_K$, the projection of H onto K. Consequently, relation (3.2) becomes

$$R_K(u) = g(u) - P_K[g(u) - \rho Tu], \tag{3.5}$$

and Algorithm 3.2 collapses to Algorithm 3.3 for the general variational inequalities.

ALGORITHM 3.3. *For a given $u_0 \in H$, $g(u_0) \in K$, compute u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = g(u_n) + \rho Tu_n - \rho Tu_{n+1} - \gamma R_K(u_n), \qquad n = 0, 1, 2, \dots .$$

If $g \equiv I$, the identity operator, then Algorithm 3.2 reduces to:

ALGORITHM 3.4 [12]. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = u_n + \rho Tu_n - \rho T_{u_{n+1}} - \gamma R(u_n), \qquad n = 0, 1, 2, \dots, \tag{3.6}$$

where

$$R(u_n) = u_n - J_\varphi[u_n - \rho Tu_n], \qquad n = 0, 1, 2, \dots .$$

If $g = I$, the identity operator, and $\gamma = 1$, then the iterative scheme (3.6) *can be written as*

$$u_{n+1} = (I + T)^{-1}\{J_\varphi[I - T] + T\}(u_n), \qquad n = 0, 1, 2, \dots,$$

which can be considered as the operator splitting method.

If φ is the indicator function of the closed convex set K in H, then $J_\varphi = P_K$, the projection of H onto K. Consequently Algorithm 3.4 collapses to the algorithm of He [7].

ALGORITHM 3.5 [7]. *For a given* $u_0 \in K$, *compute* u_{n+1} *by the iterative scheme*

$$u_{n+1} = u_n + \rho T u_n - \rho T u_{n+1} - \gamma [u_n - P_K[u_n - \rho T u_n]], \qquad n = 0, 1, 2, \dots .$$

It is clear that Algorithm 3.2 *can be considered an extension of the iterative methods of He* [6, 7] *and Noor* [12].

For the convergence analysis of Algorithm 3.2, we need the following result.

THEOREM 3.1. *Let* $\bar{u} \in H$ *be a solution of* (2.1). *If* $T: H \to H$ *is a g-monotone operator, then*

$$\langle g(u) - g(\bar{u}) + \rho(Tu - T\bar{u}), R(u) \rangle \geq \| R(u) \|^2, \qquad \text{for all } u \in H. \tag{3.7}$$

Proof. Since $\bar{u} \in H$ is a solution of (2.1), then

$$\langle T\bar{u}, g(v) - g(\bar{u}) \rangle + \varphi(g(v)) - \varphi(g(\bar{u})) \geq 0, \qquad \text{for all } g(v) \in H. \tag{3.8}$$

Taking $g(v) = J_\varphi[g(u) - \rho Tu]$ in (3.8), we have

$$\rho \langle T\bar{u}, J_\varphi[g(u) - \rho Tu] - g(\bar{u}) \rangle + \rho\varphi(J_\varphi[g(u) - \rho Tu]) - \rho\varphi(g(\bar{u})) \geq 0. \tag{3.9}$$

Setting $z = g(u) - \rho Tu$, $u = J_\varphi[g(u) - \rho Tu]$, $v = g(\bar{u})$ in (2.6), we obtain

$$\langle g(u) - \rho Tu - J_\varphi[g(u) - \rho Tu], J_\varphi[g(u) - \rho Tu] - g(\bar{u}) \rangle + \rho\varphi(g(\bar{u})) - \rho\varphi(J_\varphi[g(u) - \rho Tu]) \geq 0 \tag{3.10}$$

Adding (3.9) and (3.10), we have

$$\langle g(u) - \rho(Tu - T\bar{u}) - J_\varphi[g(u) - \rho Tu], J_\varphi[g(u) - \rho Tu] - g(\bar{u}) \rangle \geq 0,$$

which can be written as

$$\langle R(u) - \rho(Tu - T\bar{u}), g(u) - g(\bar{u}) - R(u) \rangle \geq 0, \tag{3.11}$$

by using (3.2).

From (3.11), it follows that

$$\begin{aligned}\langle g(u) - g(\bar{u}) + \rho(Tu - T\bar{u}), R(u)\rangle &\geq \langle R(u), R(u)\rangle + \rho\langle Tu - T\bar{u}, g(u) - g(\bar{u})\rangle \\ &\geq \langle R(u), R(u)\rangle, \qquad \text{since } T \text{ is } g\text{-monotone},\end{aligned}$$

which implies that

$$\langle g(u) - g(\bar{u}) + \rho(Tu - T\bar{u}), R(u)\rangle \geq \|R\|^2,$$

the required result.

THEOREM 3.2. *Let $\bar{u} \in H$ be the solution of* (2.1) *and u_{n+1} be the approximate solution obtained from Algorithm* 3.2. *Then*

$$\|g(u_{n+1}) - g(\bar{u}) + \rho(Tu_{n+1} - T\bar{u})\|^2 \leq \|g(u_n) - g(\bar{u}) + \rho(Tu_n - T\bar{u})\|^2 - \gamma(2-\gamma)\|R(u_n)\|^2. \quad (3.12)$$

Proof. Since $\bar{u}$ is a solution of (2.1) and u_{n+1} satisfies relation (3.4),

$$\begin{aligned}\|g(u_{n+1}) - g(\bar{u}) + \rho(Tu_{n+1} - T\bar{u})\|^2 &= \|g(u_n) - g(\bar{u}) + \rho(Tu_n - T\bar{u}) - \gamma R(u_n)\|^2 \\ &= \|g(u_n) - g(\bar{u}) + \rho(Tu_n - T\bar{u})\|^2 + \gamma^2\|R(u_n)\|^2 \\ &\quad - 2\gamma\langle g(u_n) - g(\bar{u}) + \rho(Tu_n - T\bar{u}), R(u_n)\rangle \\ &\leq \|g(u_n) - g(\bar{u}) + \rho(Tu_n - T\bar{u})\|^2 \\ &\quad - 2\gamma\|R(u_n)\|^2 + \gamma^2\|R(u_n)\|^2, \qquad \text{by using (3.7)} \\ &= \|g(u_n) - g(\bar{u}) + \rho(Tu_n - T\bar{u})\|^2 - \gamma(2-\gamma)\|R(u_n)\|^2,\end{aligned}$$

the required result.

THEOREM 3.3. *Let $g\colon H \to H$ be invertible. Then the approximate solution u_{n+1} obtained from Algorithm* 3.2 *converges to a solution $\bar{u}$ of the general variational inequality* (2.1).

Proof. Let $u \in H$ be a solution of (2.1). From (3.12), it follows that

$$\sum_{n=0}^{\infty} \gamma(2-\gamma)\|R(u_n)\|^2 \leq \|g(u_0) - g(\bar{u}) + \rho(Tu_0 - T\bar{u})\|^2,$$

and consequently,

$$\lim_{n\to\infty} R(u_n) = 0.$$

Let $\bar{u}$ be the cluster point of $\{u_n\}$, and the subsequence $\{u_{n_j}\}$ converges to $\bar{u}$. Since $R(u)$ is continuous,

$$R(\bar{u}) = \lim_{j\to\infty} R(u_{n_j}) = 0,$$

and $\bar{u}$ is the solution of the general mixed variational inequality (2.1) by invoking Lemma 3.1 and

$$\left\| g(u_{n+1}) - g(\bar{u}) + \rho\left(Tu_{n+1} - T\bar{u}^2\right\| \right.$$
$$\leq \left\| g(u_n) - g(\bar{u}) + \rho(Tu_n - T\bar{u})\right\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point and

$$\lim_{n\to\infty} g(u_n) = g(\bar{u}).$$

Since g is invertible,

$$\lim_{n\to\infty} (u_n) = \bar{u},$$

which is the solution of the general monotone mixed variational inequality.

Using Lemma 2.1, Lemma 3.1, and the technique of Noor [9], we establish the equivalence between the general mixed variational inequalities (2.1) and the resolvent equations (2.7). This equivalence is used to suggest a new iterative algorithm for solving the mixed variational inequality (2.1).

THEOREM 3.4. *The general mixed variational inequality* (2.1) *has a solution* $u \in H$, *if and only if the general resolvent equation* (2.7) *has a solution* $z \in H$, *where*

$$g(u) = J_\varphi z \tag{3.13}$$

and

$$z = g(u) - \rho Tu. \tag{3.14}$$

Proof. Let $u \in H$ be a solution of (2.1). Then by Lemmas 2.1 and 3.1 we have

$$g(u) = J_\varphi[g(u) - \rho Tu].$$

Using the fact that $R_\varphi \equiv I - J_\varphi$ and the above equation repeatedly, we obtain

$$\begin{aligned} R_\varphi[g(u) - \rho Tu] &= g(u) - \rho Tu - J_\varphi[g(u) - \rho Tu] \\ &= -\rho Tu \\ &= -\rho T g^{-1} J_\varphi[g(u) - \rho Tu]. \end{aligned}$$

This implies that

$$Tg^{-1}J_\varphi z + \rho^{-1}R_\varphi z = 0,$$

with

$$z = g(u) - \rho Tu.$$

Conversely, let $z \in H$ be a solution of (2.7). Then

$$\rho T g^{-1} J_\varphi z = -R_\varphi z = J_\varphi z - z.$$

Now by invoking Lemma 2.1 and the above relation, we have

$$\begin{aligned} 0 &\leq \langle J_\varphi z - z, g(v) - J_\varphi z\rangle + \rho\varphi(g(v)) - \rho\varphi(J_\varphi z) \\ &= \rho\left\{\langle Tg^{-1}J_\varphi z, g(v) - J_\varphi z\rangle + \varphi(g(v)) - \varphi(J_\varphi z)\right\}. \end{aligned}$$

Thus $u = g^{-1}J_\varphi z$ is a solution of the mixed variational inequalities (2.1).

From Theorem 3.1, it follows that general mixed variational inequality (2.1) and the resolvent equations (2.7) are equivalent. We use this equivalence to suggest a new iterative algorithm for solving the general monotone mixed variational inequalities (2.1).

Using the fact that $R_\varphi = I - J_\varphi$, the resolvent equations (2.7) can be written as

$$z - J_\varphi z + \rho T g^{-1} J_\varphi z = 0.$$

Thus, for a stepsize γ, we can write this as

$$g(u) = g(u) - \gamma\left\{z - J_\varphi z + \rho T g^{-1} J_\varphi z\right\}.$$

This fixed-point formulation allows us to suggest the following iterative algorithm for solving general monotone mixed variational inequalities (2.1).

ALGORITHM 3.5. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned} z_n &= g(u_n) - \rho T u_n \\ w_n &= z_n - J_\varphi z_n + \rho T g^{-1} J_\varphi z_n \\ g(u_{n+1}) &= g(u_n) - \gamma w_n, \qquad n = 0, 1, 2, \ldots . \end{aligned}$$

We note that for $g \equiv I$ (the identity operator), Algorithm 3.5 collapses to the following algorithm of Noor [13] for solving the monotone mixed variational inequalities (2.2).

ALGORITHM 3.6. *For a given $u_0 \in H$, compute the approximate solution*

$$\begin{aligned} z_n &= u_n - \rho T u_n \\ w_n &= z_n - J_\varphi z_n + \rho T J_\varphi z_n \\ u_{n+1} &= u_n - \gamma w_n, \qquad n = 0, 1, 2, \ldots . \end{aligned}$$

In brief, for a suitable and appropriate choice of the operators, T, φ, and the space H, one can obtain a number of algorithms, including those of He [6, 7], Solodov and Tseng [17], and Noor [13] for solving various classes of variational inequalities and the related optimization problems.

For the convergence analysis of Algorithm 3.5, we need the following concepts.

DEFINITION 3.1. For all $u, v \in H$, an operator $T: H \to H$ is said to be

(i) *g-monotone* if

$$\langle Tu - Tv, g(u) - g(v) \rangle \geq 0.$$

(ii) *g-Lipschitz continuous* if there exists a constant $\delta > 0$ such that

$$\langle Tu - Tv, g(u) - g(v) \rangle \leq \delta \| g(v) - g(u) \|^2.$$

Note that for $g \equiv I$, the identity operator, Definition 3.1 reduces to the standard definition of the monotonicity and Lipschitz continuity of the operator T.

THEOREM 3.5. *Let $\bar{u} \in H$ be a solution of* (2.1) *and $T: H \to H$ be a g-monotone and g-Lipschitz continuous operator with a constant* $\delta > 0$. *Then*

$$\begin{aligned} \Big\langle g(u) - g(\bar{u}), R(u) - \rho T u + \rho T g^{-1} J_\varphi [g(u) - \rho T u] \Big\rangle & \\ \geq \{1 - \rho\delta\} \| R(u) \|^2, \qquad \text{for all } u \in H. & \qquad (3.15) \end{aligned}$$

Proof. Let $\bar{u} \in H$ be a solution of (2.1). Then

$$\begin{aligned} \rho \langle T\bar{u}, g(v) - g(\bar{u}) \rangle + \rho\varphi(g(v)) - \rho\varphi(g(\bar{u})) \geq 0, & \\ \text{for all } g(v) \in H. & \qquad (3.16) \end{aligned}$$

Taking $g(v) = J_\varphi[g(u) - Tu]$ in (3.16), we have

$$\rho\left\langle T\bar{u}, J_\varphi[g(u) - \rho Tu] - g(\bar{u})\right\rangle + \rho\varphi\big(J_\varphi[g(u) - \rho Tu]\big) - \rho\varphi(g(\bar{u})) \geq 0. \tag{3.17}$$

Letting $z = g(u) - \rho Tu$, $u = J_\varphi[g(u) - \rho Tu]$, $v = g(\bar{u})$ in (2.6), we obtain

$$\left\langle J_\varphi[g(u) - \rho Tu] - g(u) + \rho Tu, g(\bar{u}) - J_\varphi[g(u) - \rho Tu]\right\rangle + \rho\varphi(g(\bar{u})) - \rho\varphi\big(J_\varphi[g(u) - \rho Tu]\big) \geq 0,$$

from which it follows that

$$\left\langle R(u) - \rho Tu, J_\varphi[g(u) - \rho Tu] - g(\bar{u})\right\rangle + \rho\varphi(g(\bar{u})) - \rho\varphi\big(J_\varphi[g(u) - \rho Tu]\big) \geq 0. \tag{3.18}$$

Since T is g-monotone, for all $u, \bar{u} \in H$,

$$\rho\left\langle Tg^{-1}J_\varphi[g(u) - \rho Tu] - T(\bar{u}), J_\varphi[g(u) - \rho Tu] - g(\bar{u})\right\rangle \geq 0. \tag{3.19}$$

Adding (3.17), (3.18), and (3.19), we have

$$\left\langle g(u) - g(\bar{u}), R(u) - \rho Tu + \rho Tg^{-1}J_\varphi[g(u) - \rho Tu]\right\rangle \geq \left\langle R(u), R(u) - \rho Tu + \rho Tg^{-1}J_\varphi[g(u) - \rho Tu]\right\rangle. \tag{3.20}$$

Since T is a g-Lipshitz continuous operator with a constant $\delta > 0$,

$$\langle Tu - Tv, g(u) - g(v)\rangle \leq \delta\|g(u) - g(v)\|^2. \tag{3.21}$$

From (3.2), (3.20), and (3.21), we obtain

$$\begin{aligned}\left\langle R(u) - \rho Tu + \rho Tg^{-1}J_\varphi[g(u) - \rho Tu], R(u)\right\rangle &= \|R(u)\|^2 - \rho\left\langle Tu - Tg^{-1}J_\varphi[g(u) - \rho Tu], R(u)\right\rangle \\ &\geq \{1 - \rho\delta\}\|R(u)\|^2.\end{aligned} \tag{3.22}$$

Combining (3.20) and (3.22), we have

$$\left\langle R(u) - \rho Tu + \rho Tg^{-1}J_\varphi[g(u) - \rho Tu], g(u) - g(\bar{u})\right\rangle \geq \{1 - \rho\delta\}\|R(u)\|^2,$$

the required result.

THEOREM 3.6. *The sequence* $\{u_n\}$ *generated by Algorithm* 3.5 *for general monotone mixed variational inequalities* (2.1) *satisfies the inequality*

$$\|g(u_{n+1}) - g(\bar{u})\|^2 \leq \|g(u_n) - g(\bar{u})\|^2 - \gamma(2 - \gamma - 2\rho\delta)\|R(u_n)\|^2, \quad \text{for all } \bar{u} \in H. \tag{3.23}$$

Proof. From (3.15) and Algorithm 3.5, we have

$$\begin{aligned} &\|g(u_{n+1}) - g(\bar{u})\|^2 \\ &\quad = \Big\| g(u_n) - g(\bar{u}) - \gamma\big\{g(u_n) - \rho T u_n - J_\varphi[g(u_n) - \rho T u_n] \\ &\qquad\qquad + \rho T g^{-1} J_\varphi[g(u_n) - \rho T u_n]\big\}\Big\|^2 \\ &\quad \leq \|g(u_n) - g(\bar{u})\|^2 - \gamma(2 - \gamma - 2\rho\delta)\|R(u_n)\|^2. \end{aligned}$$

Following the technique of Theorem 3.3, one can easily show that the approximate solution u_{n+1} obtained from Algorithm 3.5 converges to the exact solution $\bar{u} \in H$ of the general monotone mixed variational inequality (2.1).

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