JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 78 (1997) 103–124

On modified Gregory rules based on a generalised mixed interpolation formula

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Received 9 January 1996

Abstract

A mixed interpolation function in its generalised form is used to derive the generalised modified Gregory formulae. These formulae are expressed in the form of the classical rules along with two correction terms. The error terms are briefly discussed. The newly derived quadrature formulae are tested with certain numerical examples, which shows the efficiency of the generalised modified rules over classical Gregory rules, as well as the modified rules based on the mixed trigonometric interpolation. The importance of the error terms are also discussed.

Keywords: Mixed interpolation; Gregory rules; Euler Maclaurin sum formula; Finite difference formula

AMS classification: 65B15, 65D32

1. Introduction

The Gregory quadrature rules have been studied and used extensively, for the purpose of approximate evaluation of definite integrals. There exist several methods of derivation of the Gregory rules, in the literature (see [10–12]). The most commonly used method is to start with the well-known Euler–Maclaurin formula, which is given by

$$\int_0^{Nh} f(x)\,dx = h[\tfrac{1}{2}f(0) + f(h) + f(2h) + \cdots + f(\overline{N-1}h) + \tfrac{1}{2}f(Nh)] - \sum_{j=1}^{\infty} h^{2j}\frac{b_{2j}}{(2j)!}(f^{(2j-1)}(Nh) - f^{(2j-1)}(0)) \tag{1.1}$$

where b_{2j} are the Bernoulli numbers and it is assumed that the function $f(x)$ is infinitely differentiable, in the interval under consideration. If the function is not known explicitly, but its values at

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0377-0427/97/$17.00
PII S0377-0427(96)00135-5

some $q+1$ equidistant points are known, then $f(x)$ is approximated by a polynomial, say $P_q(x)$, and then integrated. By differentiating $P_q(x)$, the derivatives of the function $f(x)$ can be approximated by what is called as "finite difference formulae". Thus, in this case, the infinite sum on the right-hand-side of (1.1) reduces to a finite sum, consisting of q terms, and, an error term E gets added. Now, obviously, to calculate $f^{(2j-1)}(Nh)$ one employs a backward numerical differentiation formula and to approximate $f^{(2j-1)}(0)$, one employs a forward differentiation formula. Thus, replacing the odd-order derivatives, in (1.1) by the corresponding finite difference formula, which are obtained by differentiating either the qth order backward Newton's interpolation formula (based on the nodes $Nh, (N-1)h, \ldots, (N-q)h$), or the qth order forward Newton's interpolation formula (based on the nodes $0, h, \ldots, qh$), one arrives at the relation

$$\int_0^{Nh} f(x)\,\mathrm{d}x = h[\tfrac{1}{2}f(0) + f(h) + f(2h) + \cdots + f(\overline{N-1}h) + \tfrac{1}{2}f(Nh)]$$
$$+ h\sum_{p=1}^{q} c_{p+1}^{*}[\nabla^p f(Nh) + (-1)^p \Delta_h^p f(0)] + E_*^{\mathrm{T}}$$
$$:= Q_{N+1}^{\mathrm{TGrq}}[f] + E_*^{\mathrm{T}}, \tag{1.2}$$

where c_{p+1}^{*} are calculated by using the formula

$$c_{p+1}^{*} = \frac{1}{p!}\int_{-1}^{0} x(x+1)\cdots(x+p-1)\,\mathrm{d}x, \quad p = 0, 1, \ldots . \tag{1.3}$$

The Euler–Maclaurin formula (1.2) contains, as a basic part, the composite trapezoidal rule. As a deduction from (1.1), the Euler–Maclaurin type formula consisting of the composite Simpson's rule can be obtained, by taking into consideration, that for N even, the composite Simpson rule is a linear combination of two composite trapezoidal rules, with step sizes h and $2h$, respectively. Thus, by writing

$$\int_0^{Nh} f(x)\,\mathrm{d}x = \frac{4}{3}\int_0^{Nh} f(x)\,\mathrm{d}x - \frac{1}{3}\int_0^{Nh} f(x)\,\mathrm{d}x, \tag{1.4}$$

and then applying the Euler–Maclaurin formula (1.1) to the two integrals on the right-hand side of (1.4), with step sizes h and $2h$, respectively, we obtain that,

$$\int_0^{Nh} f(x)\,\mathrm{d}x = \frac{h}{3}[f(0) + 4f(h) + 2f(2h) + \cdots + 4f(\overline{N-1}h) + f(Nh)]$$
$$+ \frac{4}{3}\sum_{j=2}^{\infty} h^{2j}\frac{b_{2j}}{(2j)!}[4^{j-1}-1](f^{(2j-1)}(Nh) - f^{(2j-1)}(0)) + E_*^{\mathrm{S}}$$
$$= \mathrm{CS}[0, Nh, h] + \frac{4}{3}\sum_{j=2}^{\infty} h^{2j}\frac{b_{2j}}{(2j)!}[4^{j-1}-1](f^{(2j-1)}(Nh) - f^{(2j-1)}(0)) + E_*^{\mathrm{S}}. \tag{1.5}$$

Again, by replacing the odd-order derivatives by the corresponding finite difference formulae, we arrive at the relation

$$\int_0^{Nh} f(x)\,\mathrm{d}x = \mathrm{CS}[0,Nh,h] + h\sum_{p=1}^{q} d^*_{p+1}[\nabla^p f(Nh) + (-1)^p \Delta_h^p f(0)] + E_*^{\mathrm{T}}$$

$$:= Q_{N+1}^{\mathrm{SGrq}}[f] + E_*^{\mathrm{S}}, \tag{1.6}$$

where the first few coefficients d^* are given to be

$$d_2^* = 0, \quad d_3^* = 0, \quad d_4^* = -\frac{1}{180}, \quad d_5^* = -\frac{1}{120}, \quad \ldots. \tag{1.7}$$

In the present work, we have constructed the modified Gregory rules, which are based on the theory of the generalised mixed interpolation, derived recently (see [6]) which are further generalisations of the results, based on the mixed trigonometric interpolation (see [2, 3]), derived in [4, 5]. We have followed the technique as explained above, to derive these modified rules. In other words, we have first considered the formula as given by the relation (1.1) and then we have replaced the odd-order derivatives by the derivatives of the appropriate generalised mixed interpolation functions. More precisely, we have approximated the odd-order derivatives by the corresponding "generalised finite difference formulae".

We have denoted the generalised rules associated with the trapezoidal rule so obtained by $Q_{N+1}^{\mathrm{GTGrq}}[f,k_b,k_f]$, and that associated with the Simpsons's rule is denoted by $Q_{N-1}^{\mathrm{GSGrq}}[f,k_b,k_f]$. Here q stands for the order of the mixed interpolation function, which depends on two free parameters k_b and k_f. We have also shown how the errors can be minimised, by choosing k_b and k_f appropriately, in order to obtain more accurate results. Some numerical examples have been taken up to illustrate the generalised Gregory rules.

In Section 2, we have derived the generalised Gregory rules associated with the trapezoidal rule. In Section 3, we have briefly discussed the derivation of the generalised Gregory rules associated with the Simpson's rule. In Section 4, the error terms in the generalised Gregory rules have been analysed, associated with trapezoidal as well Simpson's quadrature. The various choices of k_b and k_f are also discussed. In Section 5, some numerical examples are studied and the choices of k_b and k_f are explained, for the particular examples considered. The tables given at the end show the efficiency of the generalised Gregory rules, derived in the previous sections.

2. Derivation of the generalised Gregory rules associated with trapezoidal rule

In this section a few generalised Gregory-type rules for approximating the integral

$$\int_0^{Nh} f(x)\,\mathrm{d}x, \tag{2.1}$$

are derived, based on the generalised mixed interpolation function, which has the following form (see [6]):

$$\tilde{f}_q(x) = aU_1(kx) + bU_2(kx) + \sum_{i=0}^{q-2} c_i x^i, \tag{2.2}$$

where k is a free parameter. The functions $U_1(kx)$ and $U_2(kx)$ are assumed to be linearly independent, and they are chosen such that, they form a solution set of a second order, linear ODE. This choice is based on the oscillation theory of ODEs (see [8, 9]). It has been proved in [6], that this function $\tilde{f}_q(x)$ can be written in the Newtonian forward form, based on the nodes $0, h, \ldots, qh$, as given by

$$\tilde{f}_{q,0,f}(x) = \sum_{p=0}^{q} \binom{s}{p} \Delta_h^p f(0) - k_f^2 \tilde{\phi}_{q,0,f}(x; k_f) \Delta_h^{q-1} f(0) - k_f^2 \left[\tilde{\phi}_{q+1,0,f}(x; k_f) - \frac{\tilde{\phi}_{q,0,f}(x; k_f)}{\tilde{D}_{q+1,0,f}(\theta_f)} \tilde{D}^{1,1}_{q,0,f}(\theta_f) \right] \Delta_h^q f(0), \tag{2.3}$$

It can also be verified that the mixed interpolation function based on the nodes $(N-q)h, \ldots, (N-1)h, Nh$, in the backward form, (with $k = k_b$), is given by the formula:

$$\tilde{f}_{q,Nh,b}(x) = \sum_{p=0}^{q} (-1)^p \binom{-s'}{p} \nabla_h^p f(Nh) - k_b^2 \tilde{\phi}_{q,Nh,b}(x; k_b) \nabla_h^{q-1} f(Nh) - k_b^2 \left[\tilde{\phi}_{q-1,Nh,b}(x; k_b) - \frac{\tilde{\phi}_{q,Nh,b}(x; k_b)}{\tilde{D}_{q+1,Nh,b}(N\theta_b)} \tilde{D}^{1,1}_{q,Nh,b}(N\theta_b) \right] \nabla_h^q f(Nh). \tag{2.4}$$

The following notations are used for our discussion: We define

$$s = \frac{x}{h} \quad \text{and} \quad s' = \frac{x - Nh}{h},$$

$$\theta_f = k_f h \quad \text{and} \quad \theta_b = k_b h,$$

$$\tilde{\phi}_{q,0,f}(x; k_f) = \frac{1}{k^2 \tilde{D}_{q,0,f}(\theta_f)} \left[\left(\sum_{p=0}^{q-1} \binom{s}{p} \Delta_{\theta_f}^p U_1(0) - U_1(k_f x) \right) \Delta_{\theta_f}^q U_2(0) - \left(\sum_{p=0}^{q-1} \binom{s}{p} \Delta_{\theta_f}^p U_2(0) - U_2(k_f x) \right) \Delta_{\theta_f}^q U_1(0) \right], \tag{2.5}$$

$$\tilde{\phi}_{q,Nh,b}(x; k_b) = \frac{1}{k^2 \tilde{D}_{q,Nh,b}(N\theta_b)} \left[\left(\sum_{p=0}^{q-1} (-1)^p \binom{-s'}{p} \nabla_{\theta_b}^p U_1(N\theta_b) - U_1(k_b x) \right) \times \nabla_{\theta_b}^q U_2(N\theta_b) - \left(\sum_{p=0}^{q-1} (-1)^p \binom{-s'}{p} \nabla_{\theta_b}^p U_2(N\theta_b) - U_2(k_b x) \right) \times \nabla_{\theta_b}^q U_1(N\theta_b) \right], \tag{2.6}$$

$$\tilde{D}_{q,0,f}(\theta_f) = \Delta_{\theta_f}^q U_2(0) \Delta_{\theta_f}^{q-1} U_1(0) - \Delta_{\theta_f}^{q-1} U_2(0) \Delta_{\theta_f}^q U_1(0), \tag{2.7}$$

$$\tilde{D}_{q,Nh,b}(N\theta_b) = \nabla^q_{\theta_b} U_2(N\theta_b)\, \nabla^{q-1}_{\theta_b}\, U_1(N\theta_b) - \nabla^{q-1}_{\theta_b} U_2(N\theta_b)\, \nabla^q_{\theta_b}\, U_1(N\theta_b), \tag{2.8}$$

$$\tilde{D}^{r,s}_{q,0,f}(\theta_f) = \Delta^{q+r}_{\theta_f} U_2(0) \Delta^{q-s}_{\theta_f} U_1(0) - \Delta^{q+r}_{\theta_f} U_1(0) \Delta^{q-s}_{\theta_f} U_2(0), \tag{2.9}$$

$$\tilde{D}^{r,s}_{q,Nh,b}(N\theta_b) = \nabla^{q+r}_{\theta_b} U_2(N\theta_b)\, \nabla^{q-s}_{\theta_b}\, U_1(N\theta_b) - \nabla^{q+r}_{\theta_b} U_1(N\theta_b)\, \nabla^{q-s}_{\theta_b}\, U_2(N\theta_b). \tag{2.10}$$

We observe that

$$\tilde{D}^{0,1}_{q,0,f}(\theta_f) = \tilde{D}_{q,0,f}(\theta_f),$$

$$\tilde{D}^{0,1}_{q,Nh,b}(N\theta_b) = \tilde{D}_{q,Nh,b}(N\theta_b).$$

It is also observed that in the expressions (2.3) and (2.4), the first term consists of the classical polynomial interpolation, based on the equidistant nodes $0, h, \ldots, qh$ and $(N-q)h, \ldots, (N-1)h, Nh$, respectively, to which two correction terms are added. The subscripts b and f indicate, as in [4], that the formula is either in the backward form or in the forward form. It has also been discussed in [6] that the error committed, in the approximation of $f(x)$ by the formula (2.3) or (2.4), is as given by

$$\begin{aligned}\tilde{E}_{q,0,f}(x) &:= f(x) - \tilde{f}_{q,0,f}(x)\\ &= h^{q-1}\tilde{\phi}_{q,0,f}(x;k_f)\tilde{L}_{q,f} f(\xi_f),\\ &:= h^{q-1}\tilde{\phi}_{q,0,f}(x;k_f)[q-1,q,q+1,\xi_f],\end{aligned} \tag{2.11}$$

for $\xi_f \in (0, qh)$, for a chosen k_f, and

$$\begin{aligned}\tilde{E}_{q,Nh,b}(x) &:= f(x) - \tilde{f}_{q,Nh,b}(x)\\ &= h^{q-1}\tilde{\phi}_{q,Nh,b}(x;k_b)\tilde{L}_{q,b} f(\xi_b),\\ &:= h^{q-1}\tilde{\phi}_{q,Nh,b}(x;k_b)[q-1,q,q+1,\xi_b],\end{aligned} \tag{2.12}$$

for $\xi_b \in ((N-q)h, Nh)$, for a chosen k_b, where

$$\tilde{L}_{q,i} = \left(\frac{\overline{U}_q(k_i x)}{\overline{U}_{q+1}(k_i x)} \frac{\mathrm{d}^2}{\mathrm{d}x^2} - k_i \frac{\overline{U}'_q(k_i x)}{\overline{U}_{q+1}(k_i x)} \frac{\mathrm{d}}{\mathrm{d}x} + k_i^2 \right) \frac{\mathrm{d}^{q-1}}{\mathrm{d}x^{q-1}}, \quad \text{for } i = b, f, \tag{2.13}$$

with

$$\overline{U}_n(kx) = U_2^{(n)}(kx) U_1^{(n-1)}(kx) - U_1^{(n)}(kx) U_2^{(n-1)}(kx). \tag{2.14}$$

It is to be noted that, ξ_b and ξ_f depend on x.

We now describe the derivation of the modified rule, in some detail for the case $q = 2$. In a similar way the modified rules for $q = 3, 4, \ldots$ can be derived.

For $q = 2$, on using the expressions given by (2.5)–(2.10), the relations in (2.3) and (2.4) reduce to

$$\begin{aligned}\tilde{f}_{2,Nh,b}(x) &= f(Nh) + s' \nabla_h f(Nh) + \frac{s'(s'+1)}{2} \nabla_h^2 f(Nh) - \left[\frac{\tilde{D}^{0,2}_{2,Nh,b}(N\theta_b)}{\tilde{D}_{2,Nh,b}(N\theta_b)} + \frac{x-Nh}{h}\right.\\ &\quad \left. + \frac{[U_2(k_b x)\nabla^2_{\theta_b} U_1(N\theta_b) - U_1(k_b x)\nabla^2_{\theta_b} U_2(N\theta_b)]}{\tilde{D}_{2,Nh,b}(N\theta_b)}\right] \nabla_h f(Nh) - \left[\frac{\tilde{D}^{0,3}_{3,Nh,b}(N\theta_b)}{\tilde{D}_{3,Nh,b}(N\theta_b)}\right.\\ &\quad + \left(\frac{x-Nh}{h}\right)\frac{\tilde{D}^{0,2}_{3,Nh,b}(N\theta_b)}{\tilde{D}_{3,Nh,b}(N\theta_b)} + \left(\frac{x-Nh}{h}\right)\left(\frac{x-Nh+h}{h}\right) + \frac{1}{\tilde{D}_{3,Nh,b}(N\theta_b)}\\ &\quad \left. \times [U_2(k_b x)\nabla^3_{\theta_b} U_1(N\theta_b) - U_1(k_b x)\nabla^3_{\theta_b} U_2(N\theta_b)]\right] \nabla_h^2 f(Nh)\\ &\quad + \frac{\tilde{D}^{1,1}_{2,Nh,b}(N\theta_b)}{\tilde{D}_{3,Nh,b}(N\theta_b)}\left[\frac{\tilde{D}^{0,2}_{2,Nh,b}(N\theta_b)}{\tilde{D}_{2,Nh,b}(N\theta_b)} + \left(\frac{x-Nh}{h}\right)\right.\\ &\quad \left. + \frac{[U_2(k_b x)\nabla^2_{\theta_b} U_1(N\theta_b) - U_1(k_b x)\nabla^2_{\theta_b} U_2(N\theta_b)]}{\tilde{D}_{2,Nh,b}(N\theta_b)}\right] \nabla_h^2 f(Nh) \end{aligned} \tag{2.15}$$

and

$$\begin{aligned}\tilde{f}_{2,0,f}(x) &= f(0) + s\Delta_h f(0) + \frac{s(s-1)}{2}\Delta_h^2 f(0) - \left[\frac{\tilde{D}^{0,2}_{2,0,f}(\theta_f)}{\tilde{D}_{2,0,f}(\theta_f)} + \left(\frac{x}{h}\right)\right.\\ &\quad \left. + \frac{U_2(k_f x)\Delta^2_{\theta_f} U_1(0) - U_1(k_f x)\Delta_h \theta_f^2 U_2(0)}{\tilde{D}_{2,0,f}(\theta_f)}\right] \Delta_h f(0) - \left[\frac{\tilde{D}^{0,3}_{3,0,f}(\theta_f)}{\tilde{D}_{3,0,f}(\theta_f)}\right.\\ &\quad + \left(\frac{x}{h}\right)\frac{\tilde{D}^{0,2}_{3,0,f}(\theta_f)}{\tilde{D}_{3,0,f}(\theta_f)} + \left(\frac{x}{h}\right)\left(\frac{x-h}{h}\right)\\ &\quad \left. + \frac{U_2(k_f x)\Delta^3_{\theta_f} U_1(\theta_f) - U_1(k_f x)\Delta^3_{\theta_f} U_2(\theta_f)}{\tilde{D}_{3,0,f}(\theta_f)}\right] \Delta_h^2 f(0) + \left[\frac{\tilde{D}^{0,2}_{2,0,f}(\theta_f)}{\tilde{D}_{2,0,f}(\theta_f)}\right.\\ &\quad \left. + \left(\frac{x}{h}\right) + \frac{U_2(k_f x)\Delta^2_{\theta_f} U_1(\theta_f) - U_1(k_f x)\Delta^2_{\theta_f} U_2(\theta_f)}{\tilde{D}_{2,0,f}(\theta_f)}\right] \Delta_h^2 f(0). \end{aligned} \tag{2.16}$$

Now we express the odd-order derivatives $f^{(2j-1)}(Nh)$ and $f^{(2j-1)}(0)$, in the Euler–Maclaurin formula, by using the formulae

$$\begin{aligned}\tilde{f}'_{2,0,f}(0) &= \frac{1}{h}\Delta_h f(0) - \frac{1}{2h}\Delta_h^2 f(0) - \left[\frac{1}{h} + W_{2,0,1}(\theta_f)\right] \Delta_h f(0)\\ &\quad - \left[\frac{1}{h}\frac{\tilde{D}^{0,2}_{3,0,f}(\theta_f)}{\tilde{D}_{3,0,f}(\theta_f)} - \frac{1}{2h} + W_{3,0,1}(\theta_f)\right] \Delta_h^2 f(0) + \frac{\tilde{D}^{1,1}_{2,0,f}(\theta_f)}{\tilde{D}_{3,0,f}(\theta_f)}\end{aligned}$$

$$\times \left[\frac{1}{h} + W_{2,0,1}(\theta_f)\right] \Delta_h^2 f(0), \tag{2.17}$$

$$\tilde{f}_{2,0,f}^{2j-1}(0) = -W_{2,0,2j-1}(\theta_f)\Delta_h f(0) - W_{3,0,2j-1}(\theta_f)\Delta_h^2 f(0)$$
$$+ \frac{\tilde{D}_{2,0,f}^{1,1}(\theta_f)}{\tilde{D}_{3,0,f}(\theta_f)} W_{2,0,2j-1}(\theta_f)\Delta_h^2 f(0), \quad j \geqslant 2, \tag{2.18}$$

$$\tilde{f}'_{2,Nh,b} = \frac{1}{h} \nabla_h f(Nh) + \frac{1}{2h} \nabla_h^2 f(Nh) - \left[\frac{1}{h} + W_{2,Nh,1}(N\theta_b)\right] \nabla_h f(Nh)$$
$$- \left[\frac{1}{h}\frac{\tilde{D}_{3,Nh,f}^{0,2}(N\theta_b)}{\tilde{D}_{3,Nh,f}(N\theta_b)} + \frac{1}{2h} + W_{3,Nh,1}(N\theta_b)\right] \nabla_h^2 f(Nh) + \frac{\tilde{D}_{2,Nh,b}^{1,1}(N\theta_b)}{\tilde{D}_{3,Nh,b}(N\theta_b)}$$
$$\times \left[\frac{1}{h} + W_{2,Nh,1}(N\theta_b)\right] \nabla_h^2 f(Nh), \tag{2.19}$$

and

$$\tilde{f}_{2,Nh,b}^{2j-1}(Nh) = -W_{2,Nh,2j-1}(N\theta_b) \nabla_h f(Nh) - W_{3,Nh,2j-1}(N\theta_b) \nabla_h^2 f(Nh)$$
$$+ \frac{\tilde{D}_{2,Nh,b}^{1,1}(N\theta_b)}{\tilde{D}_{3,Nh,b}(N\theta_b)} W_{2,Nh,2j-1}(N\theta_b) \nabla_h^2 f(Nh), \quad j \geqslant 2, \tag{2.20}$$

wherein the following notations have been introduced for convenience:

$$W_{q,Nh,2j-1}(N\theta_b) := k_b^{2j-1} \left[\frac{U_2^{2j-1}(N\theta_b) \nabla_{\theta_b}^q U_1(N\theta_b) - U_1^{2j-1}(N\theta_b) \nabla_{\theta_b}^q U_2(N\theta_b)}{\tilde{D}_{q,Nh,b}(N\theta_b)}\right] \tag{2.21}$$

and

$$W_{q,0,2j-1}(\theta_f) := k_f^{2j-1} \left[\frac{U_2^{2j-1}(0)\Delta_{\theta_f}^q U_1(0) - U_1^{2j-1}(0)\Delta_{\theta_f}^q U_2(0)}{\tilde{D}_{q,0,f}(\theta_f)}\right]. \tag{2.22}$$

It is to be noted that, unlike in the polynomial interpolation case, the differences of the first- and the second-order appear in the approximate formulae of all the derivatives of the function $f(x)$. It is clear that the first two terms in the relations (2.17) and (2.19) are nothing but the polynomial approximations of the first-order derivatives, for $q = 2$. Thus, on substituting the relations (2.17)–(2.20) in the expression (1.1), we get

$$Q_{N+1}^{\text{GMGr2}} := Q_{N+1}^{\text{Gr2}} + (\tilde{A}_2(N\theta_b) \nabla_h f(Nh) + \tilde{B}_2(\theta_f)\Delta_h f(0))$$
$$+ (\tilde{C}_2(N\theta_b) \nabla_h^2 f(Nh) + \tilde{D}_2(\theta_f)\Delta_h^2 f(0)), \tag{2.23}$$

where $\tilde{A}_2(N\theta_b)$, $\tilde{B}_2(\theta_f)$, $\tilde{C}_2(N\theta_b)$ and $\tilde{D}_2(\theta_f)$ are given by

$$\tilde{A}_2(N\theta_b) := \frac{h}{12}(1 + 12T_{2,Nh,b}[U_2, U_1]), \tag{2.24}$$

$$\tilde{B}_2(\theta_f) := -\frac{h}{12}(1 + 12T_{2,0,f}[U_2, U_1]), \tag{2.25}$$

$$\tilde{C}_2(N\theta_b) := \frac{h}{24}\left(1 + 24T_{3,Nh,b}[U_2, U_1] - 24\frac{\tilde{D}^{1,1}_{2,Nh,b}(N\theta_b)}{\tilde{D}_{3,Nh,b}(N\theta_b)}T_{2,Nh,b}[U_2, U_1]\right), \tag{2.26}$$

$$\tilde{D}_2(\theta_f) := \frac{h}{24}\left(1 - 24T_{3,0,f}[U_2, U_1] + 24\frac{\tilde{D}^{1,1}_{2,0,f}(\theta_f)}{\tilde{D}_{3,0,f}(\theta_f)}T_{2,0,f}[U_2, U_1]\right), \tag{2.27}$$

in which $T_{q,Nh,b}[U_2, U_1]$ and $T_{q,0,f}[U_2, U_1]$ are as defined by the relations

$$T_{q,Nh,b}[U_2, U_1] := \frac{1}{\tilde{D}_{q,Nh,b}(N\theta_b)}\left[\nabla^q_{\theta_b} U_1(N\theta_b)\sum_{j=1}^{\infty}\theta_b^{2j-1}\frac{b_{2j}}{(2j)!}U_2^{(2j-1)}(N\theta_b)\right.$$
$$\left. - \nabla^q_{\theta_b} U_2(N\theta_b)\sum_{j=1}^{\infty}\theta_b^{2j-1}\frac{b_{2j}}{(2j)!}U_1^{(2j-1)}(N\theta_b)\right] \tag{2.28}$$

$$T_{q,0,f}[U_2, U_1] := \frac{1}{\tilde{D}_{q,0,f}(\theta_f)}\left[\Delta^q_{\theta_f} U_1(0)\sum_{j=1}^{\infty}\theta_f^{2j-1}\frac{b_{2j}}{(2j)!}U_2^{(2j-1)}(0)\right.$$
$$\left. - \Delta^q_{\theta_f} U_2(0)\sum_{j=1}^{\infty}\theta_f^{2j-1}\frac{b_{2j}}{(2j)!}U_1^{(2j-1)}(0)\right]. \tag{2.29}$$

We state here that, if $p(0) \neq 0$, in the equation $y''(x) + k^2 p(kx)y(x) = 0$, then, in the limit as $k \to 0$, the expressions $\tilde{A}_2(N\theta_b)$, $\tilde{B}_2(\theta_f)$, $\tilde{C}_2(N\theta_b)$ and $\tilde{D}_2(\theta_f)$ tend to zero. In other words, in the limit of k tending to zero, the generalised Gregory rules coincide with the classical Gregory rules, based on the same set of nodal points.

Now, because of the generality in the choice of the functions $U_1(kx)$ and $U_2(kx)$, the expressions in (2.24)–(2.27) cannot be simplified any further. But in the particular case when $U_1(kx) = \cos(kx)$ and $U_2(kx) = \sin(kx)$, it can be verified that the results of [4] are retrieved. It should be carefully noted that the series in the relations (2.28) and (2.29) cannot be summed up, in a closed form, for every choice of the functions $U_1(kx)$ and $U_2(kx)$. Thus, for computational purposes, the infinite series, appearing in the formulae (2.28) and (2.29) will have to be truncated at a sufficiently large number M, whenever necessary. For the particular case when $U_1(kx) = e^{kx}\cos(kx)$ and $U_2(kx) = e^{kx}\sin(kx)$, the series

$$T_1(kx) := \sum_{j=1}^{\infty}\theta^{2j-1}\frac{b_{2j}}{(2j)!}U_1^{(2j-1)}(kx) \tag{2.30}$$

and

$$T_2(kx) := \sum_{j=1}^{\infty} \theta^{2j-1} \frac{b_{2j}}{(2j)!} U_2^{(2j-1)}(kx) \tag{2.31}$$

can be expressed in the closed form as follows:

$$T_1(kx) = \frac{e^{kx}}{2}\left[\frac{\sin(kx)\sin(\theta)+\cos(kx)\sinh(\theta)}{\cosh(\theta)-\cos(\theta)} - \frac{\cos(kx)+\sin(kx)}{\theta}\right] \tag{2.32}$$

and

$$T_2(kx) = \frac{e^{kx}}{2}\left[\frac{\sin(kx)\sinh(\theta)-\cos(kx)\sin(\theta)}{\cosh(\theta)-\cos(\theta)} + \frac{\cos(kx)-\sin(kx)}{\theta}\right], \tag{2.33}$$

where $\theta = kh$.

In an analogous way, one can derive the generalised Gregory rules for $q = 3, 4, 5, 6$ and it is observed that they all have a similar structure as follows:

$$\begin{aligned} Q_{N+1}^{\mathrm{GTGrq}} &:= Q_{N+1}^{\mathrm{TGrq}} + (\tilde{A}_q(N\theta_b)\nabla_h f(Nh) + \tilde{B}_q(\theta_f)\triangle_h f(0)) \\ &\quad + (\tilde{C}_q(N\theta_b)\nabla_h^2 f(Nh) + \tilde{D}_q(\theta_f)\triangle_h^2 f(0)). \end{aligned} \tag{2.34}$$

The expressions for $\tilde{A}_q(N\theta_b)$, $\tilde{B}_q(\theta_f)$, $\tilde{C}_q(N\theta_b)$ and $\tilde{D}_q(\theta_f)$ for $q = 3, 4, 5, 6$ are summarised as below:

$$\tilde{A}_3(N\theta_b) := \frac{h}{12}\left(\frac{\tilde{D}_{3,Nh,b}^{0,2}(N\theta_b)}{\tilde{D}_{3,Nh,b}(N\theta_b)} + \frac{1}{2} + 12T_{3,Nh,b}[U_2, U_1]\right), \tag{2.35}$$

$$\tilde{B}_3(\theta_f) := -\frac{h}{12}\left(\frac{\tilde{D}_{3,0,f}^{0,2}(\theta_f)}{\tilde{D}_{3,0,f}(\theta_f)} - \frac{1}{2} + 12T_{3,0,f}[U_2, U_1]\right), \tag{2.36}$$

$$\begin{aligned} \tilde{C}_3(N\theta_b) := -\frac{h}{12}\Bigg(&\frac{\tilde{D}_{2,Nh,b}^{0,2}(N\theta_b)}{\tilde{D}_{3,Nh,b}(N\theta_b)} - \frac{19}{60} - 12T_{4,Nh,b}[U_2, U_1] \\ &+ 12\frac{\tilde{D}_{3,Nh,b}^{1,1}(N\theta_b)}{\tilde{D}_{4,Nh,b}(N\theta_b)} T_{3,Nh,b}[U_2, U_1]\Bigg), \end{aligned} \tag{2.37}$$

$$\tilde{D}_3(\theta_f) := -\frac{h}{12}\left(-\frac{\tilde{D}_{2,0,f}^{0,2}(\theta_f)}{\tilde{D}_{3,0,f}(\theta_f)} + \frac{19}{60} + 12T_{4,0,f}[U_2, U_1] - 12\frac{\tilde{D}_{3,0,f}^{1,1}(\theta_f)}{\tilde{D}_{4,0,f}(\theta_f)} T_{3,0,f}[U_2, U_1]\right), \tag{2.38}$$

$$\tilde{A}_4(N\theta_b) := \frac{h}{12}\left(\frac{\tilde{D}_{4,Nh,b}^{0,3}(N\theta_b)}{\tilde{D}_{4,Nh,b}(N\theta_b)} + \frac{1}{2}\frac{\tilde{D}_{4,Nh,b}^{0,2}(N\theta_b)}{\tilde{D}_{4,Nh,b}(N\theta_b)} + \frac{19}{60} + 12T_{4,Nh,b}[U_2, U_1]\right), \tag{2.39}$$

$$\tilde{B}_4(\theta_f) := -\frac{h}{12}\left(\frac{\tilde{D}_{4,0,f}^{0,3}(\theta_f)}{\tilde{D}_{4,0,f}(\theta_f)} - \frac{1}{2}\frac{\tilde{D}_{4,0,f}^{0,2}(\theta_f)}{\tilde{D}_{4,0,f}(\theta_f)} + \frac{19}{60} + 12T_{4,0,f}[U_2, U_1]\right), \tag{2.40}$$

$$\tilde{C}_4(N\theta_b) := -\frac{h}{12}\left(\frac{\tilde{D}^{0,2}_{3,Nh,b}(N\theta_b)}{\tilde{D}_{4,Nh,b}(N\theta_b)} + \frac{1}{2}\frac{\tilde{D}_{3,Nh,b}(N\theta_b)}{\tilde{D}_{4,Nh,b}(N\theta_b)} - \frac{27}{120} - 12T_{5,Nh,b}[U_2,U_1] + 12\frac{\tilde{D}^{1,1}_{4,Nh,b}(N\theta_b)}{\tilde{D}_{5,Nh,b}(N\theta_b)}T_{4,Nh,b}[U_2,U_1]\right), \tag{2.41}$$

$$\tilde{D}_4(\theta_f) := \frac{h}{12}\left(\frac{\tilde{D}^{0,2}_{3,0,f}(\theta_f)}{\tilde{D}_{4,0,f}(\theta_f)} - \frac{1}{2}\frac{\tilde{D}_{3,0,f}(\theta_f)}{\tilde{D}_{4,0,f}(\theta_f)} + \frac{27}{120} - 12T_{5,0,f}[U_2,U_1] + 12\frac{\tilde{D}^{1,1}_{4,0,f}(\theta_f)}{\tilde{D}_{5,0,f}(\theta_f)}T_{4,0,f}[U_2,U_1]\right), \tag{2.42}$$

$$\tilde{A}_5(N\theta_b) := \frac{h}{12}\left(\frac{\tilde{D}^{0,4}_{5,Nh,b}(N\theta_b)}{\tilde{D}_{5,Nh,b}(N\theta_b)} + \frac{1}{2}\frac{\tilde{D}^{0,3}_{5,Nh,b}(N\theta_b)}{\tilde{D}_{5,Nh,b}(N\theta_b)} + \frac{19}{60}\frac{\tilde{D}^{0,2}_{5,Nh,b}(N\theta_b)}{\tilde{D}_{5,Nh,b}(N\theta_b)} + \frac{27}{120} + 12T_{5,Nh,b}[U_2,U_1]\right), \tag{2.43}$$

$$\tilde{B}_5(\theta_f) := -\frac{h}{12}\left(\frac{\tilde{D}^{0,4}_{5,0,f}(\theta_f)}{\tilde{D}_{5,0,f}(\theta_f)} - \frac{1}{2}\frac{\tilde{D}^{0,3}_{5,0,f}(\theta_f)}{\tilde{D}_{5,0,f}(\theta_f)} + \frac{19}{60}\frac{\tilde{D}^{0,2}_{5,0,f}(\theta_f)}{\tilde{D}_{5,0,f}(\theta_f)} - \frac{27}{120} + 12T_{5,0,f}[U_2,U_1]\right), \tag{2.44}$$

$$\tilde{C}_5(N\theta_b) := -\frac{h}{12}\left(\frac{\tilde{D}^{0,3}_{4,Nh,b}(N\theta_b)}{\tilde{D}_{5,Nh,b}(N\theta_b)} + \frac{1}{2}\frac{\tilde{D}^{0,2}_{4,Nh,b}(N\theta_b)}{\tilde{D}_{5,Nh,b}(N\theta_b)} + \frac{19}{60}\frac{\tilde{D}_{4,Nh,b}(N\theta_b)}{\tilde{D}_{5,Nh,b}(N\theta_b)} - \frac{863}{5040} - 12T_{6,Nh,b}[U_2,U_1] + 12\frac{\tilde{D}^{1,1}_{5,Nh,b}(N\theta_b)}{\tilde{D}_{6,Nh,b}(N\theta_b)}T_{5,Nh,b}[U_2,U_1]\right), \tag{2.45}$$

$$\tilde{D}_5(\theta_f) := \frac{h}{12}\left(\frac{\tilde{D}^{0,3}_{4,0,f}(\theta_f)}{\tilde{D}_{5,0,f}(\theta_f)} - \frac{1}{2}\frac{\tilde{D}^{0,2}_{4,0,f}(\theta_f)}{\tilde{D}_{5,0,f}(\theta_f)} + \frac{19}{60}\frac{\tilde{D}_{4,0,f}(\theta_f)}{\tilde{D}_{5,0,f}(\theta_f)} - \frac{863}{5040} - 12T_{6,0,f}[U_2,U_1] + 12\frac{\tilde{D}^{1,1}_{5,0,f}(\theta_f)}{\tilde{D}_{6,0,f}(\theta_f)}T_{5,0,f}[U_2,U_1]\right), \tag{2.46}$$

$$\tilde{A}_6(N\theta_b) := \frac{h}{12}\left(\frac{\tilde{D}^{0,5}_{6,Nh,b}(N\theta_b)}{\tilde{D}_{6,Nh,b}(N\theta_b)} + \frac{1}{2}\frac{\tilde{D}^{0,4}_{6,Nh,b}(N\theta_b)}{\tilde{D}_{6,Nh,b}(N\theta_b)} + \frac{19}{60}\frac{\tilde{D}^{0,3}_{6,Nh,b}(N\theta_b)}{\tilde{D}_{6,Nh,b}(N\theta_b)} + \frac{9}{40}\right.$$

$$\times \frac{\tilde{D}^{0,2}_{6,Nh,b}(N\theta_b)}{\tilde{D}_{6,Nh,b}(N\theta_b)} + \frac{863}{5040} + 12T_{6,Nh,b}[U_2, U_1]\Bigg), \tag{2.47}$$

$$\tilde{B}_6(\theta_f) := -\frac{h}{12}\left(\frac{\tilde{D}^{0,5}_{6,0,f}(\theta_f)}{\tilde{D}_{6,0,f}(\theta_f)} - \frac{1}{2}\frac{\tilde{D}^{0,4}_{6,0,f}(\theta_f)}{\tilde{D}_{6,0,f}(\theta_f)} + \frac{19}{60}\frac{\tilde{D}^{0,3}_{6,0,f}(\theta_f)}{\tilde{D}_{6,0,f}(\theta_f)} - \frac{9}{40}\right.$$
$$\left.\times \frac{\tilde{D}^{0,2}_{6,0,f}(\theta_f)}{\tilde{D}_{6,0,f}(\theta_f)} + \frac{863}{5040} + 12T_{6,0,f}[U_2, U_1]\right), \tag{2.48}$$

$$\tilde{C}_6(N\theta_b) := -\frac{h}{12}\left(\frac{\tilde{D}^{0,4}_{5,Nh,b}(N\theta_b)}{\tilde{D}_{6,Nh,b}(N\theta_b)} + \frac{1}{2}\frac{\tilde{D}^{0,3}_{5,Nh,b}(N\theta_b)}{\tilde{D}_{6,Nh,b}(N\theta_b)} + \frac{19}{60}\frac{\tilde{D}^{0,2}_{5,Nh,b}(N\theta_b)}{\tilde{D}_{6,Nh,b}(N\theta_b)} + \frac{9}{40}\right.$$
$$\times \frac{\tilde{D}_{5,Nh,b}(N\theta_b)}{\tilde{D}_{6,Nh,b}(N\theta_b)} - \frac{275}{2016} - 12T_{7,Nh,b}[U_2, U_1] + 12T_{6,Nh,b}[U_2, U_1]$$
$$\left.\times \frac{\tilde{D}^{1,1}_{6,Nh,b}(N\theta_b)}{\tilde{D}_{7,Nh,b}(N\theta_b)}\right) \tag{2.49}$$

and

$$\tilde{D}_6(\theta_f) := -\frac{h}{12}\left(-\frac{\tilde{D}^{0,4}_{5,0,f}(\theta_f)}{\tilde{D}_{6,0,f}(\theta_f)} + \frac{1}{2}\frac{\tilde{D}^{0,3}_{5,0,f}(\theta_f)}{\tilde{D}_{6,0,f}(\theta_f)} - \frac{19}{60}\frac{\tilde{D}^{0,2}_{5,0,f}(\theta_f)}{\tilde{D}_{6,0,f}(\theta_f)} + \frac{9}{40}\right.$$
$$\left.\times \frac{\tilde{D}_{5,0,f}(\theta_f)}{\tilde{D}_{6,0,f}(\theta_f)} - \frac{275}{2016} + 12T_{7,0,f}[U_2, U_1] - 12\frac{\tilde{D}^{1,1}_{6,0,f}(\theta_f)}{\tilde{D}_{7,0,f}(\theta_f)}T_{6,0,f}[U_2, U_1]\right). \tag{2.50}$$

In the particular case, when $U_1(kx) = \cos(kx)$ and $U_2(kx) = \sin(kx)$, the relations in (2.35) and (2.50) reduce to the corresponding results of [4].

In the next section we briefly discuss the generalised Gregory rules involving Simpson's quadrature.

3. Generalised Gregory rules associated with the Simpson quadrature

In this section, we briefly discuss the generalised Gregory-type rules for approximating the integral as given by (2.1). The derivation is similar to that given in Section 2, and we have just given the final results of the derivation. The results of this section are the generalisations of the results derived in [5]. We have followed the same lines as in [5]. To start with, we have considered the formula as given by (1.5), in which we have replaced the derivatives $f^{(2j-1)}(Nh)$ and $f^{(2j-1)}(0)$ $(j = 2, 3, \ldots)$ by $f^{(2j-1)}_{q,Nh,b}(Nh)$ and $f^{(2j-1)}_{q,0,f}(0)$ – the derivatives of the mixed interpolation function, tabulated at the points Nh and 0, respectively. We have given below the formulae, for the cases when $q = 1, 2, 3, 4$. We have ultimately found that the Gregory rules associated with the Simpson's quadrature have the following structure:

$$Q_{N-1}^{\mathrm{GSGrq}} := Q_{N+1}^{\mathrm{SGrq}} + (\tilde{E}_q(N\theta_b)\nabla_h f(Nh) + \tilde{F}_q(\theta_f)\Delta_h f(0)) + (\tilde{G}_q(N\theta_b)\nabla_h^2 f(Nh) + \tilde{H}_q(\theta_f)\Delta_h^2 f(0)). \tag{3.1}$$

We define the two functions

$$S_{q,Nh,b}[U_2,U_1] := \frac{1}{\tilde{D}_{q,Nh,b}(N\theta_b)}[\nabla_{\theta_b}^q U_1(N\theta_b)S_2(k_bNh) - \nabla_{\theta_b}^q U_2(N\theta_b)S_1(k_bNh)] \tag{3.2}$$

and

$$S_{q,0,f}[U_2,U_1] := \frac{1}{\tilde{D}_{q,0,f}(\theta_f)}[\Delta_{\theta_f}^q U_1(0)S_2(0) - \Delta_{\theta_f}^q U_2(0)S_1(0)], \tag{3.3}$$

where we have defined $S_1(kx)$ and $S_2(kx)$ by

$$S_1(kx) := \sum_{j=2}^{\infty} \theta^{2j-1}\frac{b_{2j}}{(2j)!}(4^{j-1}-1)U_1^{(2j-1)}(kx) \tag{3.4}$$

$$S_2(kx) := \sum_{j=2}^{\infty} \theta^{2j-1}\frac{b_{2j}}{(2j)!}(4^{j-1}-1)U_2^{(2j-1)}(kx). \tag{3.5}$$

Thus, for $q = 1,2,3,4$, the expressions for $\tilde{E}_q(N\theta_b)$, $\tilde{F}_q(\theta_f)$, $\tilde{G}_q(N\theta_b)$ and $\tilde{H}_q(\theta_f)$ are derived to be the following:

$$\tilde{E}_1(N\theta_b) := -\frac{4h}{3}S_{1,Nh,b}[U_2,U_1], \tag{3.6}$$

$$\tilde{F}_1(\theta_f) := \frac{4h}{3}S_{1,0,f}[U_2,U_1], \tag{3.7}$$

$$\tilde{G}_1(N\theta_b) := \frac{4h}{3}\left(-S_{2,Nh,b}[U_2,U_1] + \frac{\tilde{D}_{1,Nh,b}^{1,1}(N\theta_b)}{\tilde{D}_{2,Nh,b}(N\theta_b)}S_{1,Nh,b}[U_2,U_1]\right), \tag{3.8}$$

$$\tilde{H}_1(\theta_f) := \frac{4h}{3}\left(S_{2,0,f}[U_2,U_1] - \frac{\tilde{D}_{1,0,f}^{1,1}(\theta_f)}{\tilde{D}_{2,0,f}(\theta_f)}S_{1,0,f}[U_2,U_1]\right), \tag{3.9}$$

$$\tilde{E}_2(N\theta_b) := -\frac{4h}{3}S_{2,Nh,b}[U_2,U_1], \tag{3.10}$$

$$\tilde{F}_2\theta_f) := \frac{4h}{3}S_{2,0,f}[U_2,U_1], \tag{3.11}$$

$$\tilde{G}_2(N\theta_b) := \frac{4h}{3}\left(-S_{3,Nh,b}[U_2,U_1] + \frac{\tilde{D}_{2,Nh,b}^{1,1}(N\theta_b)}{\tilde{D}_{3,Nh,b}(N\theta_b)}S_{2,Nh,b}[U_2,U_1]\right), \tag{3.12}$$

$$\tilde{H}_2(\theta_f) := \frac{4h}{3}\left(S_{3,0,f}[U_2,U_1] - \frac{\tilde{D}_{2,0,f}^{1,1}(\theta_f)}{\tilde{D}_{3,0,f}(\theta_f)}S_{2,0,f}[U_2,U_1]\right), \tag{3.13}$$

$$\tilde{E}_3(N\theta_b) := -\frac{4h}{3} S_{3,Nh,b}[U_2, U_1], \tag{3.14}$$

$$\tilde{F}_3(\theta_f) := \frac{4h}{3} S_{3,0,f}[U_2, U_1], \tag{3.15}$$

$$\tilde{G}_3(N\theta_b) := \frac{4h}{3}\left(\frac{1}{240} - S_{4,Nh,b}[U_2, U_1] + \frac{\tilde{D}^{1,1}_{3,Nh,b}(N\theta_b)}{\tilde{D}_{4,Nh,b}(N\theta_b)} S_{3,Nh,b}[U_2, U_1]\right), \tag{3.16}$$

$$\tilde{H}_3(\theta_f) := \frac{4h}{3}\left(-\frac{1}{240} + S_{4,0,f}[U_2, U_1] - \frac{\tilde{D}^{1,1}_{3,0,f}(\theta_f)}{\tilde{D}_{4,0,f}(\theta_f)} S_{3,0,f}[U_2, U_1]\right), \tag{3.17}$$

$$\tilde{E}_4(N\theta_b) := \frac{4h}{3}\left(\frac{1}{240} - S_{4,Nh,b}[U_2, U_1]\right), \tag{3.18}$$

$$\tilde{F}_4(\theta_f) := \frac{4h}{3}\left(-\frac{1}{240} + S_{4,0,f}[U_2, U_1]\right), \tag{3.19}$$

$$\tilde{G}_4(N\theta_b) := \frac{4h}{3}\left(\frac{1}{160} - S_{5,Nh,b}[U_2, U_1] + \frac{\tilde{D}^{1,1}_{4,Nh,b}(N\theta_b)}{\tilde{D}_{5,Nh,b}(N\theta_b)} S_{4,Nh,b}[U_2, U_1]\right) \tag{3.20}$$

and

$$\tilde{H}_4(\theta_f) := \frac{4h}{3}\left(\frac{1}{160} + S_{5,0,f}[U_2, U_1] - \frac{\tilde{D}^{1,1}_{4,0,f}(\theta_f)}{\tilde{D}_{5,0,f}(\theta_f)} S_{4,0,f}[U_2, U_1]\right). \tag{3.21}$$

As mentioned in Section 2, the infinite series in the relations (3.4) and (3.5) cannot be summed up in a closed form, for all pair of functions $U_1(kx)$ and $U_2(kx)$. But with the particular choice of $U_1(kx) = e^{kx}\cos(kx)$ and $U_2(kx) = e^{kx}\sin(kx)$, these series have a closed-form expression as given below (see [5] for the case when $U_1(kx) = \cos(kx)$ and $U_2(kx) = \sin(kx)$):

$$\begin{aligned} S_1(kx) := {} & \frac{3e^{kx}}{8}\left[\frac{\sin(kx) + \cos(kx)}{\theta}\right] \\ & + \frac{e^{kx}}{8} \times \left[\frac{\sin(kx)\sin(2\theta) + \sinh(2\theta)\cos(kx) + 4\sin(kx)\sin(\theta)\cosh(\theta) + 4\cos(kx)\sinh(\theta)\cos(\theta)}{\cos^2(\theta) - \cosh^2(\theta)}\right] \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} S_2(kx) := {} & \frac{3e^{kx}}{8}\left[\frac{\sin(kx) - \cos(kx)}{\theta}\right] \\ & + \frac{e^{kx}}{8} \times \left[\frac{\cos(kx)\sin(2\theta) - \sinh(2\theta)\sin(kx) + 4\cos(kx)\sin(\theta)\cosh(\theta) - 4\sin(kx)\sinh(\theta)\cos(\theta)}{\cosh^2(\theta) - \cos^2(\theta)}\right]. \end{aligned} \tag{3.23}$$

It is confirmed that the relations (3.6)–(3.21), reduce to the corresponding results of [5], when $U_1(kx) = \cos(kx)$ and $U_2(kx) = \sin(kx)$. In the next section, we have given the description of the error terms, for the Gregory rules associated with both trapezoidal and Simpson's quadrature.

4. The error analysis

In this section we give a brief discussion of the error term associated with the generalised Gregory rules. Because of the generality observed in the choice of the functions $U_1(kx)$ and $U_2(kx)$, it has been not possible to give a complete discription of the total error. However, for all practical purposes, we are just able to give the leading order terms of the total truncation error, which have been utilised for the computaional purposes.

We have followed the same procedure as given in [4]. The error term related to (2.23), for the case $q = 2$ and (2.34), for $q = 3,4,5,6$ can be expressed as

$$\begin{aligned}
E_*^{\mathrm{GT}} &:= \int_0^{Nh} f(x)\,\mathrm{d}x - Q_{N+1}^{\mathrm{GTGrq}}[f, k_b, k_f] \\
&= -\sum_{j=1}^{\infty} h^{2j} \frac{b_{2j}}{(2j)!}[\tilde{E}_{q,Nh,b}^{(2j-1)}(Nh) - \tilde{E}_{q,0,f}^{(2j-1)}(0)] \\
&= -\sum_{j=1}^{\infty} h^{2j} \frac{b_{2j}}{(2j)!}\left[\sum_{i=0}^{2j-1} \binom{2j-1}{i} (D_x^i[q-1,q,q+1,\xi_b])\Big|_{x=Nh} \tilde{\phi}_{q,Nh,b}^{(2j-i-1)}(Nh;k_b)\right] \\
&\quad - \left[\sum_{i=0}^{2j-1} \binom{2j-1}{i} (D_x^i[q-1,q,q+1,\xi_f])\Big|_{x=0} \tilde{\phi}_{q,0,f}^{(2j-i-1)}(0;k_f)\right],
\end{aligned} \tag{4.1}$$

where D_x denotes the differentiation operator with respect to x. Since we have that

$$\tilde{\phi}_{q,Nh,b}(Nh;k_b) = 0 = \tilde{\phi}_{q,0,f}(0;k_b),$$

the leading order term in E_*^{GT} is proportional to

$$h^{q+1}[\tilde{\phi}_{q,Nh,b}(Nh;k_b)\,[q+1,q,q-1,\xi_b]|_{x=Nh} - \tilde{\phi}_{q,0,f}(0;k_f)\,[q+1,q,q-1,\xi_f]|_{x=0}],$$

or

$$h^{q+1}[\tilde{\phi}'_{q,Nh,b}(Nh;k_b)[q+1,q,q-1,\eta_b] - \tilde{\phi}'_{q,0,f}(0;k_f)[q+1,q,q-1,\eta_f]], \tag{4.2}$$

for $(N-q)h < \eta_b < Nh$ and $0 < \eta_f < qh$. It is clear that if we choose k_b and k_f such that

$$\tilde{L}_{q,b} f(\eta_b) = 0 = \tilde{L}_{q,f} f(\eta_f), \tag{4.3}$$

then the leading order term in the error formula can be minimised. For computational purposes, as in [4], we choose η_b and η_f to be the midpoint of the respective intervals under consideration.

At this stage we mention that the expression in (4.1) cannot be simplified any further as in [4], but with the choice of $U_1(kx) = \cos(kx)$ and $U_2(kx) = \sin(kx)$, the error term has a closed-form expression, when $k_b = k_f = k$, (see [4]).

Now we consider the error analysis of the Gregory rules associated with the Simpson's quadrature. The error term associated with the formula (3.1), is expressible as

$$
\begin{aligned}
E_*^{\mathrm{GS}} &:= \int_0^{Nh} f(x)\,\mathrm{d}x - Q_{N-1}^{\mathrm{GSGrq}}[f, k_b, k_f] \\
&= \frac{4}{3}\sum_{j=2}^{\infty} h^{2j+q-1}\frac{b_{2j}}{(2j)!}[4^{j-1}-1] \\
&\quad \times \left[\sum_{i=0}^{2j-1} \binom{2j-1}{i} (D_x^i[q-1,q,q+1,\xi_b])\Big|_{x=Nh} \tilde{\phi}_{q,Nh,b}^{(2j-i-1)}(Nh;k_b) \right] \\
&\quad - \left[\sum_{i=0}^{2j-1} \binom{2j-1}{i} (D_x^i[q-1,q,q+1,\xi_f])\Big|_{x=0} \tilde{\phi}_{q,0,f}^{(2j-i-1)}(0;k_f) \right],
\end{aligned}
\tag{4.4}
$$

with $(N-q)h < \xi_b(x) < Nh$, and $0 < \xi_f(x) < qh$. We consider the leading order term in the above series, which is obtained by putting $j = 2$, in the relation (4.4), and is given by

$$
\begin{aligned}
h^{q+3} \Bigg[&\sum_{i=0}^{3} \binom{3}{i} \tilde{\phi}_{q,Nh,b}^{(3-i)}(Nh;k_b)(D_x^i[q+1,q,q-1,\xi_b])\Big|_{x=Nh} \\
&- \sum_{i=0}^{3} \binom{3}{i} \tilde{\phi}_{q,0,f}^{(3-i)}(0;k_f)\,(D_x^i[q+1,q,q-1,\xi_f])\Big|_{x=0} \Bigg].
\end{aligned}
\tag{4.5}
$$

Expanding the above expression in powers of h, we obtain the lowest degree term in h. We have observed that, for $q = 2,3,4$

$$
\begin{aligned}
\tilde{\phi}_{q,Nh,b}(Nh;k_b) &= \tilde{\phi}_{q,0,f}(0;k_f) = 0, \\
\tilde{\phi}'_{q,Nh,b}(Nh;k_b) &= \tilde{\phi}'_{q,0,f}(0;k_f) = \mathrm{O}(h), \\
\tilde{\phi}''_{q,Nh,b}(Nh;k_b) &= \tilde{\phi}''_{q,0,f}(0;k_f) = \mathrm{O}(1), \\
\tilde{\phi}'''_{q,Nh,b}(Nh;k_b) &= \tilde{\phi}'''_{q,0,f}(0;k_f) = \mathrm{O}(1/h).
\end{aligned}
\tag{4.6}
$$

Thus, the lowest degree term is obtained from the expression (4.5), by putting $i = 0$. Therefore, by choosing k_b and k_f such that

$$
\tilde{L}_{q,b}f(\eta_b) = 0 = \tilde{L}_{q,f}f(\eta_f),
\tag{4.7}
$$

for $(N-q)h < \eta_b < Nh$ and $0 < \eta_f < qh$, this contribution to the error vanishes. Again we mention here that, for all practical purposes we choose η_b and η_f to be the midpoint of the respective intervals under consideration. As a consequence of (4.7), it is clear that the leading order term in E_*^{GS} does not vanish completely.

For the case, when $q = 1$, the above method fails, since we have that

$$\begin{aligned}
\tilde{\phi}_{1,Nh,b}(Nh;k_b) &= \tilde{\phi}_{1,0,f}(0;k_f) = 0,\\
\tilde{\phi}'_{1,Nh,b}(Nh;k_b) &= \tilde{\phi}'_{1,0,f}(0;k_f) = \mathrm{O}(h),\\
\tilde{\phi}''_{1,Nh,b}(Nh;k_b) &= \tilde{\phi}''_{1,0,f}(0;k_f) = \mathrm{O}(1),\\
\tilde{\phi}'''_{1,Nh,b}(Nh;k_b) &= \tilde{\phi}'''_{1,0,f}(0;k_f) = \mathrm{O}(1).
\end{aligned} \tag{4.8}$$

Thus, from the relation (4.5), the lowest degree term is derived to be

$$\begin{aligned}
&[k^4[0,2,\xi_b(x)]|_{x=Nh}\, p'(0) + 3k^3(D_x[0,2,\xi_b(x)])|_{x=Nh}\, p(0)]\\
&\quad -[k^4[0,2,\xi_f(x)]|_{x=0}\, p'(0) + 3k^3(D_x[0,2,\xi_f(x)])|_{x=0}\, p(0)].
\end{aligned} \tag{4.9}$$

Since, the derivatives of the unknown functions $\xi_b(x)$ and $\xi_f(x)$ appear in the above relation, the leading order term for the case $q = 1$, cannot be minimised directly, as explained above, for the cases of $q = 2, 3, 4$.

The analysis is based on the assumption that the integrand is completely known. We recall that the derivation of the generalised Gregory rules associated with the Simpson's quadrature consists of approximating the function $f(x)$, and thereby generating the so called "finite difference formulae", which approximates the derivatives of the function $f(x)$, appearing in the formulae (1.5). But, now for the case $q = 1$, we first compute the values of $f'(x)$ at the nodal points $0, h, \ldots, Nh$, and then approximate $f'(x)$ by the mixed interpolation function of the form (2.2). In this case, we denote the backward and forward mixed interpolation functions by $\tilde{f}'_{q,Nh,b}(x)$ and $\tilde{f}'_{q,0,f}(x)$, respectively. We remark that, the relations as given by (2.3) and (2.4) remain unaltered, except that f gets replaced by f'. It is straightforward that the finite difference formulae are generated by differentiating the functions $\tilde{f}'_{q,Nh,b}(x)$ and $\tilde{f}'_{q,0,f}(x)$, with respect to x. For example, $\tilde{f}'''_{q,Nh,b}(x)$ is approximated by differentiating the function $\tilde{f}'_{q,Nh,b}(x)$ twice. With these adoptions, we see that the leading term in the error is proportional to

$$\begin{aligned}
h^4 \Bigg[&\sum_{i=0}^{2} \binom{2}{i} \tilde{\phi}^{(2-i)}_{1,Nh,b}(Nh;k_b)(D_x^i[2,0,\xi_b]) \Bigg|_{x=Nh}\\
&- \sum_{i=0}^{3} \binom{2}{i} \tilde{\phi}^{(2-i)}_{1,0,f}(0;k_f)(D_x^i[2,0,\xi_f]) \Bigg|_{x=0} \Bigg].
\end{aligned} \tag{4.10}$$

The lowest degree term is obtained from the above expression by putting $i = 0$. Therefore, by choosing k_b and k_f such that

$$\tilde{L}_{1,b} f(\eta_b) = 0 = \tilde{L}_{1,f} f(\eta_f), \tag{4.11}$$

for $(N-1)h < \eta_b < Nh$ and $0 < \eta_f < qh$, this contribution to the error vanishes.

It must be emphasised that the utility of the error term for the case $q = 1$ is time consuming computationally speaking. Nevertheless, for the sake of completeness, we have just explained the procedure, without providing any numerical support.

5. Numerical computations and examples

In this section, we have worked with a few examples which shows the efficiency of the generalised Gregory formulae over the classical Gregory formulae and also the modified Gregory rules based on the mixed trigonometric interpolation. We have also shown the utility of the error terms, along with the choice of the values of k_b and k_f.

Example 1. As a first example, we have considered the integral

$$\int_0^4 xe^x \cos(x)\,dx = \frac{e^4}{2}(4\cos(4) + 3\sin(4)) = -133.35548923665\cdots . \tag{5.1}$$

We have chosen the N-values arbitrarily. In this example, we have worked with the pair of functions $U_1(kx) = e^{kx}\cos(kx)$ and $U_2(kx) = e^{kx}\sin(kx)$. Table 1 gives the results of the generalised Gregory rules associated with the trapezium rule and Table 2 gives the generalised results associated with Simpson's quadrature. In the case of trapezium quadrature we have fixed the orders of the mixed interpolation function to be $q = 2,3,4,5,6$, and in the case of Simpson's rule we have chosen $q = 2,3,4$. According to the relation (4.2), which gives the leading term in the error, it is clear that unknown constants η_b and η_f lie in the intervals $(4-qh, 4)$ and $(0, qh)$, respectively. For this example, it is clear that $h = \frac{4}{N}$. For all numerical purposes, we have chosen η_b and η_f, to be the mid-point of the interval. The values of k_b and k_f are calculated such that the leading order term of the error vanishes. Thus, for the choice of the functions $U_1(kx) = e^{kx}\cos(kx)$ and $U_2(kx) = e^{kx}\sin(kx)$, the relations in (4.3) and (4.7) reduce to

$$\begin{aligned} &2k_b^2 f^{(q-1)}\left(4 - \frac{qh}{2}\right) - 2k_b f^{(q)}\left(4 - \frac{qh}{2}\right) + f^{(q+1)}\left(4 - \frac{qh}{2}\right) = 0,\\ &2k_f^2 f^{(q-1)}\left(\frac{qh}{2}\right) - 2k_f f^{(q)}\left(\frac{qh}{2}\right) + f^{(q+1)}\left(\frac{qh}{2}\right) = 0. \end{aligned} \tag{5.2}$$

In this example, we have observed that the pattern of the k_b- and k_f-values are as follows:
(i) both are positive,
(ii) one positive and one negative,
(iii) both are complex.
In the case (i), we have chosen one of them (arbitrarily, in almost all the cases the maximum of the two has been chosen), whilst in the second case, without any confusion, we have chosen the positive value; and in the third case we have invariably chosen one of the two complex roots and finally we have considered only the real part of the end result. (In this context, we would like to mention that, in the third case we obtain the same result, for both the values of k.) (See Tables 1 and 2).

Example 2. As a second example, we have considered the integral

$$\int_0^\pi e^x \cos(5x)\,dx = \frac{-1 - e^\pi}{26} = -0.928488178183\cdots . \tag{5.3}$$

For this example, we have worked with the functions $U_1(kx) = \mathrm{Ai}(-kx-1)$ and $U_2(kx) = \mathrm{Bi}(-kx-1)$, wherein, $\mathrm{Ai}(-kx)$ and $\mathrm{Bi}(-kx)$ are the two "Airy functions", (see [1]). Tables 3 and 4 give

Table 1

N	q	Polynomial	Trigonometric	Generalised
	2	−133.575634399252	−133.101941464717	−133.345840788377
	3	−133.972335706554	−133.504551287938	−133.338690249638
8	4	−133.186084966294	−133.489596019218	−133.346422068165
	5	133.301287989719	−133.391514103942	−133.356752961650
	6	−133.362219550861	−133.372653727066	−133.351019824106
	2	−133.288637723186	−133.339901433135	−133.355096229911
	3	−133.338501145666	−133.360894325828	−133.342331078137
16	4	−133.352334822654	−133.356101256662	−133.355562234551
	5	−133.355379138510	−133.355636088244	−133.355499807800
	6	−133.355681344535	−133.355536900178	−133.355490478382
	2	−133.350646448965	−133.354603715701	−133.355474669857
	3	−133.354880916847	−133.355545038578	−133.355457398299
32	4	−133.355442748906	−133.355493675563	−133.355490862975
	5	−133.355491649876	−133.355489918320	−133.355489266163
	6	−133.355490756627	−133.355489400528	−133.355489239744
	2	−133.355164377112	−133.355443552538	−133.355488735027
	3	−133.355469125587	−133.355489999935	−133.355489090807
64	4	−133.355488578920	−133.355489270110	−133.355489255209
	5	−133.355489274779	−133.355489239822	−133.355489236766
	6	−133.355489244650	−133.355489237137	−133.355489236668
	2	−133.355354313884	−133.355472064288	−133.355489068875
	3	−133.355482579062	−133.355489432220	−133.355489205114
80	4	−133.355489069073	−133.355489243696	−133.355489242126
	5	−133.355489245591	−133.355489237184	−133.355489236679
	6	−133.355489238076	−133.355489236729	−133.355489236668
	2	−133.355423567981	−133.355481588329	−133.355489168293
	3	−133.355486543408	−133.355489301200	−133.355489227460
96	4	−133.355489181698	−133.355489238825	−133.355489236946
	5	−133.355489239333	−133.355489236837	−133.355489236666
	6	−133.355489237000	−133.355489236673	−133.355489236651
	2	−133.355457104400	−133.355485832629	−133.355489208620
	3	−133.355488139029	−133.355489258251	−133.355489233904
115	4	−133.355489218403	−133.355489237492	−133.355489236749
	5	−133.355489237458	−133.355489236629	−133.355489236655
	6	−133.355489236742	−133.355489236662	−133.355489236659

respectively the results associated with the generalised Gregory trapezium rules and generalised Gregory Simpson's rule. The choice of k_b and k_f is not as simple as explained in the previous example. The reason will be clear as we proceed further. But as usual, the unknown constants η_b and η_f are fixed to be the middle points of the intervals $(\pi - qh, \pi)$ and $(0, \pi)$ respectively. In this example too, we have worked with different, arbitrarily chosen N-values and $h = \pi/N$. We mention here that, though the computations take more time as compared to the classical Gregory rules and the

Table 2

N	q	Polynomial	Trigonometric	Generalised
	2	−133.347837485634	−133.349258330867	−133.355338689535
20	3	−133.352716644345	−133.356613248564	−133.355395320707
	4	−133.354973247304	−133.355608258302	−133.355512855223
	2	−133.355009675612	−133.355144485979	−133.355484011020
40	3	−133.355396407596	−133.355502789015	−133.355485062119
	4	−133.355481866523	−133.355490135992	−133.355489440901
	2	−133.355394461311	−133.355428764298	−133.355488522839
60	3	−133.355476760131	−133.355490354917	−133.355489006236
	4	−133.355488625172	−133.355489288446	−133.355489245534
	2	−133.355459243945	−133.355472066513	−133.355489064070
80	3	−133.355486247140	−133.355489430258	−133.355489205020
	4	−133.355489131590	−133.355489243445	−133.355489237742
	2	−133.355476950659	−133.355482863161	−133.355489179456
100	3	−133.355488251527	−133.355489286623	−133.355489229663
	4	−133.355489209761	−133.355489238053	−133.355489236873
	2	−133.355483311440	−133.355486427448	−133.355489213502
120	3	−133.355488839299	−133.355489253224	−133.355489234585
	4	−133.355489227802	−133.355489237039	−133.355489236731
	2	−133.355486038289	−133.355487840681	−133.355489225880
140	3	−133.355489052338	−133.355489253224	−133.355489235909
	4	−133.355489233191	−133.355489236785	−133.355489236715
	2	−133.355487361801	−133.355488478617	−133.355489231132
160	3	−133.355489141937	−133.355489239571	−133.355489236345
	4	−133.355489235118	−133.355489236707	−133.355489236651

Gregory-type rules based on the mixed trigonometric interpolation, but the generalised Gregory-type rules, (based on the Airy functions) provide greater accuracy in the results.

The expressions for the equations $\tilde{L}_q f(x) = 0$, for $q = 2, 3, 4, 5, 6$, respectively are given below:

$$(kx+1)f'''(x) - kf''(x) + k^2(kx+1)^2 f'(x) = 0, \tag{5.4}$$

$$(kx+1)^2 f^{iv}(x) - 2k(kx+1)f'''(x) + k^2((kx+1)^2+2)f''(x) = 0, \tag{5.5}$$

$$((kx+1)^3+2)f^{v}(x) - 3k(kx+1)^2 f^{iv}(x) + k^2((kx+1)^4 + 4(kx+1))f'''(x) = 0, \tag{5.6}$$

$$\begin{aligned}&((kx+1)^4 + 4(kx+1))f^{vi}(x) - 4k((kx+1)+1)f^{v}(x)\\&\quad + k^2((kx+1)^5 + 20(kx+1)^2)f^{iv}(x) = 0,\end{aligned} \tag{5.7}$$

$$\begin{aligned}&((kx+1)^5 + 20(kx+1)^2)f^{vii}(x) - 5k((kx+1)^4 + 8(kx+1))f^{vi}\\&\quad + ((kx+1)^6 + 40(kx+1)^3)f^{v}(x) = 0,\end{aligned} \tag{5.8}$$

Table 3

N	q	Polynomial	Trigonometric	Generalised
	2	−0.834330310305	−0.918081760257	−0.921285865958
	3	−0.901362909947	−0.903309324027	−0.908496792131
16	4	−0.947372004996	−0.938769705593	−0.933258864144
	5	−0.958280690967	−0.935625632401	−0.931147736045
	6	−0.944800769068	−0.933026176733	−0.930415870416
	2	−0.889388624393	−0.924620854269	−0.925813690577
	3	−0.923581216475	−0.648020285171	−0.932644697731
20	4	−0.937224415205	−0.930086763709	−0.929088719233
	5	−0.935597850458	−0.929243164136	−0.929096012583
	6	−0.929843092718	−0.931115197934	−0.929283929373
	2	−0.922739158446	−0.927983180605	−0.928078666410
	3	−0.928784875609	−0.928783519888	−0.928688656330
32	4	−0.929305086034	−0.928546475088	−0.928507439006
	5	−0.928701993657	−0.928536999339	−0.928519833852
	6	−0.928431319469	−0.928472825720	−0.928486907141
	2	−0.926203510085	−0.928293329220	−0.928335567267
	3	−0.928681351615	−0.928549105426	−0.928527328169
40	4	−0.928720116790	−0.928502257669	−0.928492063694
	5	−0.928522959124	−0.928506877738	−0.928498213112
	6	−0.928473007369	−0.928486296798	−0.928487713123
	2	−0.928161649796	−0.928458262883	−0.928469431095
	3	−0.928520886465	−0.928491041713	−0.928489890626
64	4	−0.928502889133	−0.928488465421	−0.928488283805
	5	−0.928488701157	−0.928487628499	−0.928488002585
	6	−0.928487648990	−0.928488162197	−0.928488172639
	2	−0.928358108914	−0.928745680490	−0.928481330486
	3	−0.928500473412	−0.928488887182	−0.928488592537
80	4	−0.928492054749	−0.928488265937	−0.928488191155
	5	−0.928488223906	−0.928488123269	−0.928488164751
	6	−0.928488081490	−0.928488175832	−0.928488177442
	2	−0.928480569905	−0.928487896982	−0.928487896982
	3	−0.928488660161	−0.928488188317	−0.928488183847
160	4	−0.928488237898	−0.928488179115	−0.928488177791
	5	−0.928488177458	−0.928488178060	−0.928488178205
	6	−0.928488177761	−0.928488178179	−0.928488178182

where k stands for either k_b or k_f and x may take the value either $(\pi - qh/2)$ or $qh/2$. For a fixed x, it is clear that relations (5.4)–(5.8) give rise to polynomials in k, of certain degree. Now it is clear that the choice of k_b and k_f values is not easy. But however, we are able to choose an appropriate k_b and k_f, which has improved the previous formulae. In this example, the values of k occur in the following pattern, depending upon the degree of the polynomial.

(i) one positive, rest are complex,

(ii) one positive, one negative, rest are complex,

Table 4

N	q	Polynomial	Trigonometric	Generalised
	2	−0.927231921636	−0.927821752796	−0.928041143144
30	3	−0.928870044837	−0.928962659923	−0.928821785165
	4	−0.929262901069	−0.928576431403	−0.928498665475
	2	−0.928099551572	−0.928293332303	−0.928377591596
40	3	−0.928621202421	−0.928548371559	−0.928527340385
	4	−0.928638431387	−0.928502135840	−0.928481545779
	2	−0.928412629676	−0.928450786286	−0.928463053524
60	3	−0.928511428663	−0.928492450620	−0.928490780062
	4	−0.928501936231	−0.928488451788	−0.928488443446
	2	−0.928464406962	−0.928756026201	−0.928481313627
80	3	−0.928494378435	−0.928488882769	−0.928488593513
	4	−0.928490636807	−0.928488264917	−0.928488208506
	2	−0.928486624099	−0.928487876713	−0.928487879532
158	3	−0.928488417849	−0.928488189087	−0.928488184311
	4	−0.928488218942	−0.928488179193	−0.928488178151
	2	−0.928487018875	−0.928487961280	−0.928487966241
170	3	−0.928488346044	−0.928488185176	−0.928488182100
	4	−0.928488204394	−0.928488178800	−0.928488178132
	2	−0.928487255972	−0.928488010883	−0.928488016155
180	3	−0.928488305225	−0.928488183128	−0.928488180946
	4	−0.928488196752	−0.928488178603	−0.928488178131
	2	−0.928487573283	−0.928488075016	−0.928488079583
200	3	−0.928488254110	−0.928488180796	−0.928488179637
	4	−0.928488188020	−0.928488178389	−0.928488178140

(iii) one positive, some negative, rest are complex,
(iv) two positive, rest are complex; and so on.
In cases (i)–(iii) we have chosen the positive value for both k_b and k_f. In the case of (iv), we have chosen the maximum of the two positive values. (See Tables 3 and 4).

We mention here that, in both the examples the values of k_b and k_f are obtained in double precision, by using the MATHEMATICA package.

6. Conclusions

We have derived the generalised Gregory rules, which are based on the mixed interpolation theory. The interpolation function is of the form $U_1(kx)+U_2(kx)+\sum_{i=0}^{n-2} c_i x^i$, which clearly is a combination of a polynomial of certain degree and two other linearly independent functions $U_1(kx)$ and $U_2(kx)$. We have made the choice of these functions based on the oscillation theory of ODEs. The generalised Gregory rules are derived by considering the well-known Euler Maclaurin formula, in which the odd-order derivatives have been replaced by the corresponding finite difference formulae. We have derived the generalised Gregory rules associated with both the composite trapezium rule, as well as the composite Simpson's rule.

We have given the error anaysis in brief, for both the classes of Gregory quadrature rules. We have also discussed how to choose the appropriate k_b and k_f's, which helps in controlling the error. We have worked out a few numerical examples, which shows the efficiency of the generalised Gregory rules over the other known rules. In these examples, we have worked with two different pairs of the functions, $U_1(kx)$ and $U_2(kx)$, and in both the cases we have obtained better results.

The results, derived in this paper, will be utilised in future, to solve some Fredholm integral equations of the second kind.

Acknowledgements

Hamsapriye thanks the CSIR, New Delhi, India, for providing the financial assistance, as a research student of Indian Institute of Science, Bangalore.

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