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Homology of $(n+1)$-types and Hopf type formulas

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Abstract

The tripleability of the category of crossed n-cubes is studied. The leading cotriple homology of these homotopy $(n+1)$-types is investigated, describing it as Hopf type formulas.

MSC: 18G10; 18G50

It is well known that there exist elegant algebraic models of connected CW-spaces whose homotopy groups are trivial in dimensions more than $n+1$, called homotopy $(n+1)$-types. These algebraic models are cat^n-groups introduced by Loday in [18], generalising the notion of crossed modules firstly given by Whitehead in [24] as a means of representing homotopy 2-types, or equivalently more combinatorial algebraic systems, crossed n-cubes, invented by Ellis and Steiner in [9]. A number of papers of the last years are dedicated to the investigation of homological properties of these objects.

In [8] Ellis and in [2] Baues introduced and investigated the (co)homology of crossed module as the (co)homology of its classifying space, neglecting its algebraic structure.

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In [16] Ladra and Grandjeán gave the first approach to an internal homology theory of crossed modules taking into account its algebraic structure.

Later, in [4] Carrasco et al. made the observation that the category of crossed modules is an algebraic category, that is, there is a tripleable "underlying" functor from the category of crossed modules to the category of sets, implying a purely algebraic construction and study of cotriple (co)homology theory.

In [10] Grandjeán et al. gave a connection of these two homology theories of crossed modules by the dimension shifting isomorphism, whilst, in [5] Casas et al. have recently generalised this result to higher pattern for cat^n-groups.

The aim of this paper is to investigate the homology of homotopy $(n+1)$*-types from a Hopf formulas point of view*, using a purely algebraic method of m-fold Čech derived functors introduced in [6] (see also [12]).

The Čech derived functors of group valued functors were introduced in [19] (see also [11,6]) as an algebraic analogue of the Čech (co)homology construction of open covers of topological spaces with coefficients in sheaves of abelian groups (see [23]). Some applications of Čech derived functors to classical group (co)homology theory, K-theory, non-abelian homology of groups and Lie algebras and Conduché-Ellis homology of precrossed modules were given in [19,20,13–15]. In [12] (see also [6]) the notion of the Čech derived functors has been generalised to that of the m-fold Čech derived functors of group valued functors (where m is a fixed non-negative integer). Based on this notion a new purely algebraic method for the investigation of higher integral group homology is given in [6] (see also [12]).

In the current paper the m-fold Čech derived functors of group valued functors from the category of crossed n-cubes is treated. In particular, we calculate the mth m-fold Čech derived functor of the certain abelianization functor $\sigma\mathfrak{Ab}$ from the category of crossed n-cubes to the category of groups, implying the expression of the cotriple homology of crossed n-cubes (cat^n-groups) as generalised Hopf type formulas. Our main result has the form:

Main Theorem (*Generalised Hopf Type Formulas*). *Let $\mathscr{M}$ be a crossed n-cube and $\mathfrak{X}$ its projective exact m-presentation. Then there is an isomorphism*

$$H_{m+1}(\mathscr{M}) \cong \frac{\bigcap_{i\in\langle m\rangle} R^i_{\langle n\rangle} \cap \prod_{B\cup C=\langle n\rangle}[\mathfrak{X}(\emptyset)_B, \mathfrak{X}(\emptyset)_C]}{\prod_{A\subseteq\langle m\rangle}(\prod_{B\cup C=\langle n\rangle}[\bigcap_{i\in A} R^i_B, \bigcap_{i\notin A} R^i_C])}, \quad m \geqslant 1,$$

where $R^i = \mathrm{Ker}(\mathfrak{X}(\emptyset) \to \mathfrak{X}(\{i\}))$ *for* $i \in \langle m\rangle$.

This result generalises the Hopf formulas for higher integral group homology [3] (see also [6]) and Hopf formula for the second CCG-homology of crossed modules [4].

We are setting up the following *Notations and Conventions*: For a non-negative integer n we denote by $\langle n\rangle$ the set of first n natural numbers $\{1, \ldots, n\}$. For any set A its cardinality is denoted by $|A|$. For a and b elements of a group, $[a, b]$ is the commutator $aba^{-1}b^{-1}$. We denote by **Gr** and **Set** the categories of groups and sets, respectively.

We begin by the examination of two equivalent algebraic models of homotopy $(n+1)$-types, cat^n-groups and crossed n-cubes [18,9], recalling some needed results and notions for our future purpose.

Recall from [18] that a cat^n-group is a group G together with $2n$ endomorphisms $s_i, t_i : G \to G$, $1 \leqslant i \leqslant n$, such that

$$\begin{array}{l} t_i s_i = s_i, \quad s_i t_i = t_i, \quad [\operatorname{Ker} s_i, \operatorname{Ker} t_i] = 1 \quad \text{for all } i, \\ s_i s_j = s_j s_i, \quad t_i t_j = t_j t_i, \quad s_i t_j = t_j s_i \quad \text{for } i \neq j. \end{array}$$

A morphism of cat^n groups $f : (G, s_i, t_i) \to (G', s'_i, t'_i)$ is a group homomorphism $f : G \to G'$ satisfying $s'_i f = f s_i$ and $t'_i f = f t_i$ for $1 \leqslant i \leqslant n$. We obtain the category of cat^n-groups denoted by $\mathbf{Cat^n}$.

Later, in [9], the higher dimensional analogues of crossed modules were introduced, called crossed n-cubes.

A *crossed n-cube of groups* is a family $\mathscr{M} = \{\mathscr{M}_A : A \subseteq \langle n \rangle\}$ of groups together with homomorphisms $\mu_i : \mathscr{M}_A \to \mathscr{M}_{A \setminus \{i\}}$ for $i \in \langle n \rangle$, $A \subseteq \langle n \rangle$ and functions $h : \mathscr{M}_A \times \mathscr{M}_B \longrightarrow \mathscr{M}_{A \cup B}$ for $A, B \subseteq \langle n \rangle$, such that if ${}^a b$ denotes $h(a, b) \cdot b$ for $a \in \mathscr{M}_A$ and $b \in \mathscr{M}_B$ with $A \subseteq B$, then for all $a, a' \in \mathscr{M}_A$, $b, b' \in \mathscr{M}_B$, $c \in \mathscr{M}_C$ and $i, j \in \langle n \rangle$, the following conditions hold:

$$\begin{array}{l} \mu_i(a) = a \text{ if } i \notin A, \\ \mu_i \mu_j(a) = \mu_j \mu_i(a), \\ \mu_i h(a, b) = h(\mu_i(a), \mu_i(b)), \\ h(a, b) = h(\mu_i(a), b) = h(a, \mu_i(b)) \quad \text{if } i \in A \cap B, \\ h(a, a') = [a, a'], \\ h(a, b) = h(b, a)^{-1}, \\ h(a, b) = 1 \quad \text{if } a = 1 \text{ or } b = 1, \\ h(aa', b) = {}^a h(a', b) h(a, b), \\ h(a, bb') = h(a, b)^b h(a, b'), \\ {}^a h(h(a^{-1}, b), c)^c h(h(c^{-1}, a), b)^b h(h(b^{-1}, c), a) = 1, \\ {}^a h(b, c) = h({}^a b, {}^a c) \quad \text{if } A \subseteq B \cap C. \end{array}$$

A morphism of crossed n-cubes, $\alpha : \mathscr{M} \to \mathscr{N}$, is a family of group homomorphisms $\{\alpha_A : \mathscr{M}_A \to \mathscr{N}_A, A \subseteq \langle n \rangle\}$ commuting with the μ_i and the h-functions. The resulted category of crossed n-cubes of groups will be denoted by $\mathbf{Crs^n}$.

According to [18] the category of cat^1-groups is equivalent to that of crossed modules and the category of cat^2-groups to that of crossed squares. One of the main result of [9] says that the categories $\mathbf{Crs^n}$ and $\mathbf{Cat^n}$ are equivalent. Namely, one has the following.

Theorem 1. *There are inverse equivalences of categories*

$$\mathbf{Crs^n} \underset{\Psi^n}{\overset{\Phi^n}{\rightleftarrows}} \mathbf{Cat^n}$$

given by

$$\Phi^n(\mathscr{M}) = \bigvee_{A \subseteq \langle n \rangle} \mathscr{M}_A \big/ \{h(a, b) = [a, b] \textit{ for all } a \in \mathscr{M}_A, b \in \mathscr{M}_B, A, B \subseteq \langle n \rangle\},$$
$$\mathscr{M} \in \mathbf{Crs^n}$$

and

$$\Psi^n(G)_A = \bigcap_{i\in A} \operatorname{Ker} s_i \cap \bigcap_{i\notin A} \operatorname{Im} s_i, \quad G \in \mathbf{Cat^n}, \quad A \subseteq \langle n\rangle.$$

Note that in this paper we mainly prefer to use crossed n-cubes instead of cat^n-groups, except those cases when using of cat^n-groups will make things easier to understand.

The following example of a crossed n-cube appears naturally from normal $(n+1)$-ad of groups.

Example 2. Let G be a group and $N_1, \ldots, N_n$ be normal subgroups of G. Let $\mathscr{M}_A = \bigcap_{i\in A} N_i$ for $A \subseteq \langle n\rangle$ (here $\mathscr{M}_\emptyset$ is understood to mean G); if $i \in \langle n\rangle$, define $\mu_i : \mathscr{M}_A \xrightarrow{i\in A} \mathscr{M}_{A\setminus\{i\}}$ to be the inclusion and given $A, B \subseteq \langle n\rangle$, let $h : \mathscr{M}_A \times \mathscr{M}_B \to \mathscr{M}_{A\cup B}$ be given by the commutator: $h(a,b) = [a,b]$ for $a \in \mathscr{M}_A$, $b \in \mathscr{M}_B$. Then $\{\mathscr{M}_A : A \subseteq \langle n\rangle, \mu_i, h\}$ is a crossed n-cube, called the *inclusion crossed n-cube* given by the *normal* $(n+1)$-ad *of groups* $(G; N_1, \ldots, N_n)$.

Now we give the notion of a crossed n-subcube, which is consistent with the categorical notion of subobject in the category $\mathbf{Crs^n}$. We say that a crossed n-cube $\mathscr{M}'$ is a *crossed n-subcube* of $\mathscr{M}$ if $\mathscr{M}'_A$ is a subgroup of $\mathscr{M}_A$, the homomorphism $\mu'_i : \mathscr{M}'_A \to \mathscr{M}'_{A\setminus\{i\}}$ and the function $h' : \mathscr{M}'_A \times \mathscr{M}'_B \longrightarrow \mathscr{M}'_{A\cup B}$ are the restrictions of $\mu_i : \mathscr{M}_A \to \mathscr{M}_{A\setminus\{i\}}$ and $h : \mathscr{M}_A \times \mathscr{M}_B \longrightarrow \mathscr{M}_{A\cup B}$ respectively for every $i \in \langle n\rangle$, $A, B \subseteq \langle n\rangle$.

Moreover, a crossed n-subcube $\mathscr{M}'$ of $\mathscr{M}$ is said to be a *normal crossed n-subcube* if $h(a,b') \in \mathscr{M}'_{A\cup B}$ and $h(a',b) \in \mathscr{M}'_{A\cup B}$ for all $a \in \mathscr{M}_A$, $b' \in \mathscr{M}'_B$, $a' \in \mathscr{M}'_A$, $b \in \mathscr{M}_B$.

Let $\alpha : \mathscr{M} \to \mathscr{N}$ be a morphism of crossed n-cubes and $\operatorname{Ker}\alpha$ denote the family $\{\operatorname{Ker}\alpha_A : A \subseteq \langle n\rangle\}$ of groups, which essentially is a normal crossed n-subcube of $\mathscr{M}$.

Now we give the example which will play an important role in the sequel.

Example 3. Let $\mathscr{N}$ be a crossed n-cube and $R^1, \ldots, R^m$ be normal crossed n-subcubes of $\mathscr{N}$. Let $A \subseteq \langle m+n\rangle$, $A_1 = A \cap \{n+1, \ldots, n+m\}$, $A_2 = A \cap \langle n\rangle$ and consider $\mathscr{M}_A = \bigcap_{j\in A_1} R^{j-n}_{A_2}$ (here $\bigcap_{j\in\emptyset} R^{j-n}_{A_2}$ is understood to mean $\mathscr{N}_{A_2}$); define $\mu_i : \mathscr{M}_A \xrightarrow{i\in A} \mathscr{M}_{A\setminus\{i\}}$ to be the inclusion $\bigcap_{j\in A_1} R^{j-n}_{A_2} \hookrightarrow \bigcap_{j\in A_1\setminus\{i\}} R^{j-n}_{A_2}$ if $i \in A_1$ and to be induced by $\mu_i : R^{j-n}_{A_2} \to R^{j-n}_{A_2\setminus\{i\}}$ if $i \in A_2$; let $h : \mathscr{M}_A \times \mathscr{M}_B \to \mathscr{M}_{A\cup B}$ be defined naturally by commutators and h-functions of the crossed n-cubes $\mathscr{N}$, $R^1, \ldots, R^m$. Then $\{\mathscr{M}_A : A \subseteq \langle n\rangle, \mu_i, h\}$ is a crossed $(m+n)$-cube, called the *crossed* $(m+n)$*-cube* of groups induced by the *normal* $(m+1)$-ad *of crossed n-cubes* $(\mathscr{N}; R^1, \ldots, R^m)$.

Note that for $n = 0$ this construction agrees with Example 2, if we assume that a crossed 0-cube is just a group.

It will be shown that the category $\mathbf{Crs^n}$ is an algebraic category (see also [5]), that is, there is a tripleable forgetful functor from $\mathbf{Crs^n}$ to $\mathbf{Set}$. In fact, we need only to construct 'free' cotriple in the category $\mathbf{Crs^n}$.

At first we construct the adjoint pair of functors $\mathbf{Crs^n} \underset{F}{\overset{U}{\rightleftarrows}} \mathbf{Gr}$.

Assume that the functor $U : \mathbf{Crs^n} \to \mathbf{Gr}$ assigns to any crossed n-cube $\mathscr{M} = \{\mathscr{M}_A : A \subseteq \langle n\rangle\}$ the direct product of groups $\mathscr{M}_A$, $A \subseteq \langle n\rangle$, i.e.,

$$U(\mathscr{M}) = \prod_{A\subseteq\langle n\rangle} \mathscr{M}_A.$$

Now, define the functor $F : \mathbf{Gr} \to \mathbf{Crs^n}$ as follows: for any group G, let $F(G)$ denote the inclusion crossed n-cube induced by the normal $(n+1)$-ad of groups $(\bigvee_{A\subseteq\langle n\rangle} G_A;$ $\operatorname{Ker} p_1, \ldots, \operatorname{Ker} p_n)$ (see Example 2), where $\bigvee_{A\subseteq\langle n\rangle} G_A$ is the sum of groups $G_A = G$, $A \subseteq \langle n\rangle$ and

$$p_i : \bigvee_{A\subseteq\langle n\rangle} G_A \longrightarrow \bigvee_{B\subseteq\langle n-1\rangle} G_B, \quad i \in \langle n\rangle,$$

are natural projections given by

$$p_i = \begin{cases} 1_G : G_A \to G_B & \text{if } A \subseteq \langle n\rangle\backslash\{i\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\delta_i : \langle n\rangle\backslash\{i\} \to \langle n-1\rangle$ is the unique monotone bijection.

Proposition 4. *The functor F is left adjoint to the functor U.*

To prove this proposition we use the following easily verified facts requiring only care over the notation. Given a crossed n-cube $\mathscr{M}=\{\mathscr{M}_A : A \subseteq \langle n\rangle\}$, for any $B \subseteq \langle n\rangle$ denote by $\mathscr{M}^B$ and $\mathscr{M}^{\overline{B}}$ the families $\{\mathscr{M}_A : A \subseteq \langle n\rangle, B \subseteq A\}$ and $\{\mathscr{M}_A : A \subseteq \langle n\rangle, B \cap A = \emptyset\}$ respectively. Then $\mathscr{M}^B$ and $\mathscr{M}^{\overline{B}}$ have the structure of crossed $(n-|B|)$-cubes (see Proposition 5 [21]).

Proof of Proposition 4. We claim that for any group G, the homomorphism

$$u = \{u_A\} : G \longrightarrow \prod_{A\subseteq\langle n\rangle} F(G)_A = UF(G),$$

where $u_A : G \to F(G)_A = \bigcap_{i\in A} \operatorname{Ker} p_i$ is given by the identity from G to G_A, is a universal arrow from G to the functor U.

Let $\mathscr{M}$ be a crossed n-cube and let $\alpha_A : G \to \mathscr{M}_A$, $A \subseteq \langle n\rangle$ be homomorphisms defining a homomorphism $\alpha : G \to \prod_{A\subseteq\langle n\rangle} \mathscr{M}_A = U(\mathscr{M})$. Then there is a commutative diagram with splitting short exact sequences of groups:

$$\begin{array}{ccccc} \operatorname{Ker} p_i & \rightarrowtail & \bigvee_{A\subseteq\langle n\rangle} G_A & \overset{p_i}{\twoheadrightarrow} & \bigvee_{B\subseteq\langle n-1\rangle} G_B \\ \widetilde{\gamma}_i \downarrow & & \gamma \downarrow & & \downarrow \gamma_i \\ \Phi^{n-1}(\mathscr{M}^{\{i\}}) & \rightarrowtail & \Phi^n(\mathscr{M}) & \twoheadrightarrow & \Phi^{n-1}(\mathscr{M}^{\overline{\{i\}}}), \end{array}$$

where Φ^* is the equivalence given in Theorem 1, γ_i is induced by $G_B \overset{\alpha_A}{\to} \mathscr{M}_A$ with $A \subseteq \langle n\rangle\backslash\{i\}$ such that $\delta_i(A) = B$, γ is induced by $G_A \overset{\alpha_A}{\to} \mathscr{M}_A$, $A \subseteq \langle n\rangle$ and $\widetilde{\gamma}_i$ is the restriction of γ. It is easy to see that the homomorphisms $\widetilde{\gamma}_i$ induce the homomorphisms $\widetilde{\gamma}_A : \bigcap_{i\in A} \operatorname{Ker} p_i \to \Phi^{n-|A|}(\mathscr{M}^A)$.

Now define the homomorphism $\widetilde{\widetilde{\gamma}}_A : \bigcap_{i\in A} \mathrm{Ker}\, p_i \to \mathscr{M}_A$, $A \subseteq \langle n \rangle$ as the composition of $\widetilde{\gamma}_A$ and $\beta_A : \Phi^{n-|A|}(\mathscr{M}^A) \to \mathscr{M}_A$ given by $\mathscr{M}_B \overset{\mu_{B\setminus A}}{\to} \mathscr{M}_A$ for $B \supseteq A$, where $\mu_{B\setminus A}$ is the composition of the homomorphisms μ_{i_j}, $j=1,\ldots,|B\setminus A|$, with any $i_j \in (B\setminus A)\setminus \cup_{k=1}^{j-1}\{i_k\}$. Finally, it is easy to verify that $\widetilde{\widetilde{\gamma}} = \{\widetilde{\widetilde{\gamma}}_A\} : F(G) \to \mathscr{M}$ is the unique morphism of crossed n-cubes with $U(\widetilde{\widetilde{\gamma}})u = \alpha$. $\square$

We denote by $U_1 : \mathbf{Gr} \to \mathbf{Set}$ the usual forgetful functor and by $F_1 : \mathbf{Set} \to \mathbf{Gr}$ its left adjoint, the free group functor. Composing these two adjunctions,

$$\mathbf{Crs^n} \underset{F}{\overset{U}{\rightleftarrows}} \mathbf{Gr} \underset{F_1}{\overset{U_1}{\rightleftarrows}} \mathbf{Set},$$

we deduce the following.

Proposition 5. *The underlying set functor* $\mathscr{U} = U_1 \circ U : \mathbf{Crs^n} \to \mathbf{Set}$ *has a left adjoint* $\mathscr{F} = F \circ F_1 : \mathbf{Set} \to \mathbf{Crs^n}$.

It is routine to verify that the category $\mathbf{Crs^n}$, $n \geqslant 2$, similarly to that of crossed modules (i.e., $n=1$) [4], has kernel pairs and coequalizers preserved and reflected by the functor $\mathscr{U}$. Then by Proposition 5 and Linton's criterion on tripleability [17] the underlying set functor $\mathscr{U} : \mathbf{Crs^n} \to \mathbf{Set}$ is tripleable.

It is well known for an algebraic category $\mathbf{C}$ the obvious inclusion functor of the category of abelian group objects $\mathfrak{Ab}\mathbf{C} \hookrightarrow \mathbf{C}$ has left adjoint $\mathfrak{Ab} : \mathbf{C} \to \mathfrak{Ab}\mathbf{C}$, called the abelianization functor, which plays a fundamental role in the description of homology of objects in the category $\mathbf{C}$. Namely, the kth homology of an object $X \in \mathbf{C}$ is defined to be $\mathscr{L}_k\mathfrak{Ab}(X)$, where $\mathscr{L}_k\mathfrak{Ab}$ denotes the kth derived functor of $\mathfrak{Ab}$ [22].

An abelian group object in $\mathbf{Crs^n}$, an abelian crossed n-cube, is a crossed n-cube whose h maps are trivial [6]. The abelianization functor

$$\mathfrak{Ab}^{(n)} : \mathbf{Crs^n} \longrightarrow \mathfrak{Ab}\mathbf{Crs^n}, \tag{1}$$

is given by:

(a) for $A \subseteq \langle n \rangle$

$$\mathfrak{Ab}^{(n)}(\mathscr{M})_A = \frac{\mathscr{M}_A}{\prod_{B\cup C=A} D_{B,C}},$$

where $D_{B,C}$ is the subgroup of $\mathscr{M}_A$ generated by the elements $h(b,c)$, $h : \mathscr{M}_B \times \mathscr{M}_C \to \mathscr{M}_{B\cup C=A}$ for all $b \in \mathscr{M}_B$, $c \in \mathscr{M}_C$,

(b) if $i \in \langle n \rangle$, the homomorphism $\widetilde{\mu}_i : \mathfrak{Ab}^{(n)}(\mathscr{M})_A \to \mathfrak{Ab}^{(n)}(\mathscr{M})_{A\setminus\{i\}}$ is induced by the homomorphism μ_i,

(c) for $A, B \subseteq \langle n \rangle$, the function $\widetilde{h} : \mathfrak{Ab}^{(n)}(\mathscr{M})_A \times \mathfrak{Ab}^{(n)}(\mathscr{M})_B \to \mathfrak{Ab}^{(n)}(\mathscr{M})_{A\cup B}$ is induced by h and therefore is trivial,

for any $\mathscr{M} = \{\mathscr{M}_A : A \subseteq \langle n \rangle, \mu_i, h\} \in \mathbf{Crs^n}$.

The equivalent abelian group object to abelian crossed n-cube in the category $\mathbf{Cat^n}$ is just a cat^n-group whose underlying group is abelian, which is called abelian cat^n-group (see also [5]). Moreover, the abelianization functor

$$\mathfrak{Ab}^{(n)} : \mathbf{Cat^n} \longrightarrow \mathfrak{Ab}\mathbf{Cat^n} \tag{2}$$

sends a cat^n-group $G = (G, s_i, t_i)$ to the abelian cat^n-group $(G/[G, G], \overline{s_i}, \overline{t_i})$, where $\overline{s_i}$ and $\overline{t_i}$ are induced by s_i and t_i.

Given a crossed module $\mathscr{M}$, the corresponding cat^1-group (G, s, t) has an internal category structure within the category $\mathbf{Gr}$. The objects are the elements of $N = \mathrm{Im}(s) = \mathrm{Im}(t)$, the morphisms are the elements of G, the source and target maps are s and t. The morphisms g and h are composable if $t(g) = s(h)$ and their composite is $h \circ g = hs(h)^{-1}g$. The nerve of this category forms the simplicial group $E(\mathscr{M})_*$ which is called the nerve of the crossed module $\mathscr{M}$ [18] (see also [21]).

Now define the functor

$$E^{(m)} : \mathbf{Crs^n} \longrightarrow \mathfrak{Simpl}\mathbf{Crs^{n-m}} \text{(the category of simplicial crossed (n-m)-cubes)}, \tag{3}$$

$1 \leqslant m \leqslant n$, as follows: given a crossed n-cube $\mathscr{M}$, consider an associated cat^n-group G, which is equivalent to a crossed $(n-m)$-cube endowed with m compatible category structures. Then by applying the nerve structure E to the m-independent directions, this construction leads naturally to an m-simplicial crossed $(n-m)$-cube. Then the simplicial crossed $(n-m)$-cube $E^{(m)}(\mathscr{M})_*$ is the diagonal of this m-simplicial crossed $(n-m)$-cube.

Note that this construction depends upon the sequence of the m-independent directions.

In Proposition 2 of [6] it is established that the abelianization of a crossed module commutes with its nerve. We provide more general result for functors (1) and (3), which plays essential role to obtain generalised Hopf formulas for the homology of crossed n-cubes.

Proposition 6. *Let $n \geqslant 0$, $m \geqslant 1$ and $\mathscr{M}$ be a crossed $(n+m)$-cube. Then there is an isomorphism of simplicial crossed n-cubes*

$$\mathfrak{Ab}^{(n)} E^{(m)}(\mathscr{M})_* \cong E^{(m)} \mathfrak{Ab}^{(n+m)}(\mathscr{M})_*,$$

where $E^{(m)}$ functors in both sides of the isomorphism are applied to the same directions.

Proof. To simplify things, according to Theorem 1, instead of the crossed $(n+m)$-cube $\mathscr{M}$ we use its equivalent object, cat^{n+m}-group, $G = (G, s_i, t_i) = \Phi^{n+m}(\mathscr{M})$. The proof will be done by induction on m.

Let $m = 1$ and $n = 0$, then the assertion reduces to Proposition 2 [6]. This case plays the key role in the whole proof.

In fact, for $m = 1$, $n \geqslant 1$ and for the cat^{n+1}-group G, let us fix some $k \in \langle n+1 \rangle$ and apply the functor $E^{(1)}$ to this 'direction'. By the definition, the simplicial cat^n-group , $E^{(1)}(\mathscr{M})_*$, is just the simplicial group $E(\Psi^1(G, s_k, t_k))_*$ endowed with n compatible category structures induced by the respective structural endomorphisms s_j, t_j $(0 \leqslant j \leqslant n+1,\ j \neq k)$ of the cat^{n+1}-group G. The fact that the abelianization of a cat^n-group is just the abelianization of underling group endowed with induced structural endomorphism and our key fact above completes the assertion in this case.

Proceeding by induction, we suppose that the assertion is true for $m-1$ and we will prove it for m.

By the construction, $E^{(m)}(\mathscr{M})_*$ is the diagonal of the bisimplicial crossed n-cube induced by applying the crossed module nerve construction $E^{(1)}$ to the simplicial crossed $(n+1)$-cube $E^{(m-1)}(\mathscr{M})_*$. Hence one has

$$E^{(m)}(\mathscr{M})_k = E^{(1)}(E^{(m-1)}(\mathscr{M})_k)_k,$$

for all $k \geqslant 0$. Using the inductive hypothesis one has isomorphisms

$$\begin{aligned}\mathfrak{Ab}^{(n)}E^{(m)}(\mathscr{M})_k &= \mathfrak{Ab}^{(n)}E^{(1)}(E^{(m-1)}(\mathscr{M})_k)_k \cong E^{(1)}\mathfrak{Ab}^{(n+1)}(E^{(m-1)}(\mathscr{M})_k)_k \\ &\cong E^{(1)}(E^{(m-1)}(\mathfrak{Ab}^{(n+m)}(\mathscr{M}))_k)_k = E^{(m)}(\mathfrak{Ab}^{(n+m)}(\mathscr{M}))_k. \qquad \square\end{aligned}$$

Now we construct the cotriple homology of crossed n-cubes (cat^n-groups). We refer to the work of Barr and Beck [1] for the background about cotriple (co)homology.

The above constructed pair of adjoint functors $\mathbf{Set} \underset{\mathscr{U}}{\overset{\mathscr{F}}{\rightleftarrows}} \mathbf{Crs^n}$ induces the cotriple $\mathbb{F} \equiv (\mathbb{F}, \delta, \tau)$ on the category $\mathbf{Crs^n}$ by the obvious way: $\mathbb{F} = \mathscr{F}\mathscr{U} : \mathbf{Crs^n} \to \mathbf{Crs^n}$, $\tau : \mathbb{F} \to 1_{\mathbf{Crs^n}}$ is the counit and $\delta = \mathscr{F}u\mathscr{U} : \mathbb{F} \to \mathbb{F}^2$, where $u : 1_{\mathbf{Set}} \to \mathscr{U}\mathscr{F}$ is the unit of the adjunction.

Using the general theory of cotriple homology due to [1], define the kth homology of a given crossed n-cube $\mathscr{M}$ as the $(k-1)$th cotriple derived functor of the abelianization functor $\mathfrak{Ab}^{(n)}$

$$\mathscr{H}_k(\mathscr{M}) = \mathscr{L}^{\mathbb{F}}_{k-1}\mathfrak{Ab}^{(n)}(\mathscr{M}), \quad k \geqslant 1.$$

Let $\mathbb{P}$ be the projective class induced by the'free' cotriple $\mathbb{F}$, namely $X \in \mathbb{P}$ if and only if there exists a morphism $\alpha : X \to \mathbb{F}(X)$ such that $\tau_X\alpha = 1_X$. It is well known that derived functors relative to the cotriple are isomorphic to the derived functors relative to the projective class induced by the cotriple [11]. Thus there is an isomorphism

$$\mathscr{L}^{\mathbb{F}}_k\mathfrak{Ab}^{(n)} \cong \mathscr{L}^{\mathbb{P}}_k\mathfrak{Ab}^{(n)}.$$

Recall also that an object P of a category $\mathbf{C}$ is projective if given a regular epimorphism $f : X \twoheadrightarrow Y$, each morphism $g : P \to Y$ can be lifted to a morphism $h : P \to X$ such that $fh = g$. We say that $\mathbf{C}$ has enough projective objects if any object X admits a projective presentation, i.e., there exists a regular epimorphism $P \to X$ with P a projective object. If $\mathbf{C}$ is a tripleable category with the adjunction $\mathbf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{C}$, then $F(X)$, $X \in \mathbf{Set}$, is a projective object and the natural morphism $FU(C) \to C$, $C \in \mathbf{C}$, is a regular epimorphism in $\mathbf{C}$, implying that $\mathbf{C}$ has enough projectives. It is also known that the projective class of all projective objects in the algebraic category $\mathbf{C}$ coincides with the projective class $\overline{\mathbb{P}}$ induced by the adjunction and regular epimorphisms are just $\overline{\mathbb{P}}$-epimorphisms.

It is easy to check that if $\mathscr{M}_*$ is a $\mathbb{F}$-cotriple resolution of a crossed n-cube $\mathscr{M}$, then $\mathscr{M}_*^{\overline{\langle n\rangle\setminus A}}$ is a projective resolution of $\mathscr{M}^{\overline{\langle n\rangle\setminus A}}$ for $A \subseteq \langle n\rangle$, $A \neq \langle n\rangle$. Hence

$$\mathscr{H}_k(\mathscr{M})_A = \mathscr{H}_k(\mathscr{M}^{\overline{\langle n\rangle\setminus A}})_{\langle|A|\rangle}, \quad k \geqslant 1.$$

Therefore the interest of our investigation is the group $\mathscr{H}_k(\mathscr{M})_{\langle n\rangle}$, which we denote by $H_k(\mathscr{M})$. If we define the functor $\sigma : \mathbf{Crs^n} \to \mathbf{Gr}$ by $\sigma(\mathscr{M}) = \mathscr{M}_{\langle n\rangle}$ for $\mathscr{M} \in \mathbf{Crs^n}$, then

$$H_k(\mathscr{M}) = \mathscr{L}^{\mathbb{F}}_{k-1}(\sigma\mathfrak{Ab}^{(n)})(\mathscr{M}), \quad k \geqslant 1.$$

Now we consider the m-fold Čech derivatives of functors from the category of crossed n-cubes to the category of groups, whilst the general situation has been dealt in [12] (see also [6]). In particular, we give an explicit computation of the m-fold Čech derived functors of the functor $\sigma\mathfrak{Ab}^{(n)} : \mathbf{Crs^n} \to \mathfrak{Ab}\mathbf{Gr}$, implying a purely algebraic approach to the homology groups of crossed n-cubes from a Hopf type formula point of view.

Let m be a natural number. The subsets of $\langle m\rangle$ are ordered by inclusion. This ordered set determines in the usual way a category $\underline{C_m}$. For every pair (A, B) of subsets with $A \subseteq B \subseteq \langle m\rangle$, there is the unique morphism $\rho^A_B : A \to B$ in $\underline{C_m}$.

An *m-cube of crossed n-cubes* is a functor $\mathfrak{X} : \underline{C_m} \to \mathbf{Crs^n}$. A *morphism between m-cubes* $\mathfrak{X}, \mathfrak{Y} : \underline{C_m} \to \mathbf{Crs^n}$ is a natural transformation $\kappa : \mathfrak{X} \to \mathfrak{Y}$.

Note that a crossed n-cube of groups gives an n-cube on forgetting structure, but there is a reversal of the role of the index A. The top corner of a crossed n-cube is $\mathscr{M}_{\langle n\rangle}$, that in an n-cube is $\mathfrak{X}(\emptyset)$.

Example 7. Let $(\mathscr{M}_*, d^0_0, \mathscr{M})$ be an augmented simplicial object in the category $\mathbf{Crs^n}$. A natural m-cube of crossed n-cubes $\mathscr{M}^{(m)} : \underline{C_m} \to \mathbf{Crs^n}$, $m \geqslant 1$ is defined by the following way:

$$\mathscr{M}^{(m)}(A) = \mathscr{M}_{m-1-|A|} \quad \text{for all } A \subseteq \langle m\rangle,$$
$$\mathscr{M}^{(m)}(\rho^A_{A\cup\{j\}}) = d^{m-1-|A|}_{k-1} \quad \text{for all } A \neq \langle m\rangle,\ \ j \notin A,$$

where $\mathscr{M}_{-1} = \mathscr{M}$, $\delta(k) = j$ and $\delta : \langle m - |A|\rangle \to \langle m\rangle \backslash A$ is the unique monotone bijection.

An m-cube of crossed n-cubes $\mathfrak{X}$ determines a normal $(m+1)$-ad of crossed n-cubes $(\mathfrak{X}(\emptyset); R^1, \ldots, R^m)$, where $R^i = \operatorname{Ker} \mathfrak{X}(\rho^{\emptyset}_{\{i\}})$, $i \in \langle m\rangle$. This $(m+1)$-ad will be called the *normal* $(m+1)$*-ad of crossed n-cubes induced by* $\mathfrak{X}$.

Given an m-cube of crossed n-cubes $\mathfrak{X}$. It is easy to see that there exists a natural morphism of crossed n-cubes $\mathfrak{X}(A) \xrightarrow{\alpha_A} \lim_{B\supset A} \mathfrak{X}(B)$ for any $A \subseteq \langle m\rangle$, $A \neq \langle m\rangle$.

Definition 8. Let $\mathscr{M}$ be a crossed n-cube. An m-cube of crossed n-cubes $\mathfrak{X}$ will be called an *m-presentation of the crossed n-cube* $\mathscr{M}$ if $\mathfrak{X}(\langle m\rangle) = \mathscr{M}$. An m-presentation $\mathfrak{X}$ of $\mathscr{M}$ is called *projective* if the crossed n-cube $\mathfrak{X}(A)$ is a projective crossed n-cube for all $A \neq \langle m\rangle$ and called *exact* if the morphism α_A is a regular epimorphism for all $A \neq \langle m\rangle$.

The following lemma is straightforward.

Lemma 9. *Let $(\mathscr{M}_*, d^0_0, \mathscr{M})$ be an augmented simplicial object in the category* $\mathbf{Crs^n}$ *and suppose that $\pi_0(\mathscr{M}_*) = \mathscr{M}$. Then $(\mathscr{M}_*, d^0_0, \mathscr{M})$ is an exact simplicial resolution of $\mathscr{M}$ if and only if the m-cube of crossed n-cubes $\mathscr{M}^{(m)}$ is an exact m-presentation of $\mathscr{M}$ for all* $m \geqslant 1$.

Given a crossed n-cube $\mathscr{M}$ and a morphism $\alpha : \mathscr{Q} \to \mathscr{M}$ of the category $\mathbf{Crs^n}$. The *Čech augmented complex* $(\check{C}(\alpha)_*, \alpha, \mathscr{M})$ for α is the following augmented simplicial crossed n-cube

$$\cdots \begin{smallmatrix}\longrightarrow\\ \vdots\\ \longrightarrow\end{smallmatrix} \quad \mathscr{Q}\times_{\mathscr{M}}\cdots\times_{\mathscr{M}}\mathscr{Q} \quad \begin{smallmatrix}\longrightarrow\\ \vdots\\ \longrightarrow\end{smallmatrix} \cdots \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} \quad \mathscr{Q}\times_{\mathscr{M}}\mathscr{Q} \quad \begin{smallmatrix}\longrightarrow\\ \longrightarrow\end{smallmatrix} \quad \mathscr{Q} \xrightarrow{\alpha} \mathscr{M},$$

thus

$$\check{C}(\alpha)_k = \underbrace{\mathscr{Q}\times_{\mathscr{M}}\cdots\times_{\mathscr{M}}\mathscr{Q}}_{(k+1)\text{-times}} \quad \text{for } k \geqslant 0.$$

Now let $\mathfrak{X}$ be an m-presentation of the crossed n-cube $\mathscr{M}$. Applying $\check{C}$ above, in the m-independent directions, this construction leads naturally to an augmented m-simplicial object in the category $\mathbf{Crs^n}$. Taking the diagonal of this augmented m-simplicial object gives the augmented simplicial crossed n-cube $(\check{C}^{(m)}(\mathfrak{X})_*, \alpha, \mathscr{M})$ called an *augmented m-fold Čech complex for* $\mathfrak{X}$, where $\alpha = \mathfrak{X}(\rho^{\emptyset}_{\langle m\rangle}) : \mathfrak{X}(\emptyset) \to \mathscr{M}$. In case $\mathfrak{X}$ is a projective exact m-presentation of $\mathscr{M}$, then $(\check{C}^{(m)}(\mathfrak{X})_*, \alpha, \mathscr{M})$ will be called an *m-fold Čech resolution of* $\mathscr{M}$.

Definition 10. Let $T : \mathbf{Crs^n} \to \mathbf{Gr}$ be a covariant functor. Define kth *m-fold Čech derived functor* $\mathscr{L}_k^{m\text{-fold}}T : \mathbf{Crs^n} \to \mathbf{Gr}$, $k \geqslant 0$, of the functor T by choosing for each $\mathscr{M}$ in $\mathbf{Crs^n}$, a projective exact m-presentation $\mathfrak{X}$ and setting

$$\mathscr{L}_k^{m\text{-fold}}T(\mathscr{M}) = \pi_k(T\check{C}^{(m)}(\mathfrak{X})_*),$$

where $(\check{C}^{(m)}(\mathfrak{X})_*, \alpha, \mathscr{M})$ is the m-fold Čech resolution of the crossed n-cube $\mathscr{M}$ for the projective exact m-presentation $\mathfrak{X}$ of $\mathscr{M}$.

Note that by [12] (see also Theorem 16 [6]) the m-fold Čech derived functors are well defined.

Lemma 11. *Let $\mathfrak{X}$ be an m-presentation of a crossed n-cube of groups. There is an isomorphism of simplicial crossed n-cubes*

$$\check{C}^{(m)}(\mathfrak{X})_* \cong E^{(m)}(\mathscr{N})_*,$$

where $\mathscr{N}$ is the crossed $(m+n)$-cube of groups given by the normal $(m+1)$-ad of crossed n-cubes $(\mathfrak{X}(\emptyset); R^1, \ldots, R^m)$ induced by $\mathfrak{X}$.

Proof. Directly follows from Lemma 19 [6]. □

The following theorem gives the calculation of the mth m-fold Čech derived functors of the functor $\sigma\mathfrak{Ab}^{(n)} : \mathbf{Crs^n} \to \mathfrak{Ab}\mathbf{Gr} \subseteq \mathbf{Gr}$.

Theorem 12. *Let $\mathscr{M}$ be a crossed n-cube and $\mathfrak{X}$ its projective exact m-presentation. Then there is an isomorphism*

$$\mathscr{L}_m^{m\text{-fold}}(\sigma\mathfrak{Ab}^{(n)})(\mathscr{M}) \cong \frac{\bigcap_{i\in\langle m\rangle} R^i_{\langle n\rangle} \cap \prod_{B\cup C=\langle n\rangle}[\mathfrak{X}(\emptyset)_B, \mathfrak{X}(\emptyset)_C]}{\prod_{A\subseteq\langle m\rangle}(\prod_{B\cup C=\langle n\rangle}[\bigcap_{i\in A} R^i_B, \bigcap_{i\notin A} R^i_C])}, \quad m \geqslant 1,$$

where $R^i = \mathrm{Ker}(\mathfrak{X}(\emptyset) \to \mathfrak{X}(\{i\}))$ *for* $i \in \langle m\rangle$.

Proof. Using Lemma 11, $\mathscr{L}_m^{m\text{-fold}}(\sigma\mathfrak{Ab}^{(n)})(\mathscr{M}) \cong \pi_m(\sigma\mathfrak{Ab}^{(n)}E^{(m)}(\mathscr{N})_*)$, where $\mathscr{N}$ is the crossed $(n+m)$-cube of groups induced by the normal $(m+1)$-ad of crossed n-cubes $(\mathfrak{X}(\emptyset); R^1, \ldots, R^m)$. Hence Proposition 6 implies an isomorphism $\mathscr{L}_m^{m\text{-fold}}(\sigma\mathfrak{Ab}^{(n)})(\mathscr{M}) \cong \pi_m(\sigma E^{(m)}\mathfrak{Ab}^{(n+m)}(\mathscr{N})_*)$. Then, by Proposition 14 [6] (see also Proposition 3.4 [18]),

$$\begin{aligned}\mathscr{L}_m^{m\text{-fold}}(\sigma\mathfrak{Ab}^{(n)})(\mathscr{M}) \cong & \bigcap_{l\in\langle m\rangle} \mathrm{Ker}(\mathfrak{Ab}^{(n+m)}(\mathscr{N})_{\langle n+m\rangle} \\ & \xrightarrow{\widetilde{\mu}_l} \mathfrak{Ab}^{(n+m)}(\mathscr{N})_{\langle n+m\rangle\setminus\{l\}}).\end{aligned} \tag{4}$$

Now we set up the inductive hypothesis. Let $m = 1$, then

$$\begin{aligned}&\mathscr{L}_1^{1\text{-fold}}(\sigma\mathfrak{Ab}^{(n)})(\mathscr{M}) \\ &\quad \cong \mathrm{Ker}\left(\frac{R^1_{\langle n\rangle}}{\prod_{A\subseteq\langle 1\rangle}(\prod_{B\cup C=\langle n\rangle}[\bigcap_{i\in A} R^i_B, \bigcap_{i\notin A} R^i_C])} \right. \\ &\qquad \left. \longrightarrow \frac{\mathfrak{X}(\emptyset)_{\langle n\rangle}}{\prod_{B\cup C=\langle n\rangle}[\mathfrak{X}(\emptyset)_B, \mathfrak{X}(\emptyset)_C]}\right) \\ &\quad = \frac{R^1_{\langle n\rangle} \cap \prod_{B\cup C=\langle n\rangle}[\mathfrak{X}(\emptyset)_B, \mathfrak{X}(\emptyset)_C]}{\prod_{A\subseteq\langle 1\rangle}(\prod_{B\cup C=\langle n\rangle}[\bigcap_{i\in A} R^i_B, \bigcap_{i\notin A} R^i_C])}.\end{aligned}$$

Proceeding by induction, we suppose that the result is true for $m-1$ and we will prove it for m.

Let us consider $l \in \langle m\rangle$ and denote by $\mathfrak{X}^{\overline{\{l\}}}$ the restriction of the functor $\mathfrak{X} : \underline{C_m} \to \mathbf{Crs^n}$ to the subcategory of $\underline{C_m}$ consisting of all subsets $A \subseteq \langle m\rangle$ with $l \notin A$. It is easy to check that $\mathfrak{X}^{\overline{\{l\}}}$ is a projective exact $(m-1)$-presentation of the crossed n-cube $\mathfrak{X}(\langle m\rangle\setminus\{l\})$ which itself is projective crossed n-cube. Since the values of m-fold Čech derived functors of any functor for an object belonging to the projective class are trivial, our inductive hypothesis implies that

$$\begin{aligned}&\mathscr{L}_{m-1}^{(m-1)\text{-fold}}(\sigma\mathfrak{Ab}^{(n)})(\mathfrak{X}(\langle m\rangle\setminus\{l\})) \\ &\quad \cong \frac{\bigcap_{i\in\langle m\rangle\setminus\{l\}} R^i_{\langle n\rangle} \cap \prod_{B\cup C=\langle n\rangle}[\mathfrak{X}(\emptyset)_B, \mathfrak{X}(\emptyset)_C]}{\prod_{A\subseteq\langle m\rangle\setminus\{l\}}(\prod_{B\cup C=\langle n\rangle}[\bigcap_{i\in A} R^i_B, \bigcap_{i\notin A} R^i_C])} = 1.\end{aligned} \tag{5}$$

Now from (4) and (5) one can easily deduce the required isomorphism. □

Finally we give the proof of our main theorem which expresses the homology of crossed n-cubes as Hopf type formulas generalising to that of recently obtained in [6] for the non-abelian derived functors of group nilization functors and the Hopf formula for the second CCG-homology of crossed modules [4].

Proof of Main Theorem. Let $(F_*, d_0^0, \mathscr{M})$ be a projective simplicial resolution of $\mathscr{M}$ in the category $\mathbf{Crs^n}$ and consider the short exact sequence of augmented simplicial groups

$$
\begin{array}{ccccccccc}
 & 1 & & 1 & & 1 & & 1 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \begin{smallmatrix}\to \\ \vdots \\ \to\end{smallmatrix} & D(F_m) & \begin{smallmatrix}\xrightarrow{\widetilde{d^m_{0,\langle n\rangle}}} \\ \vdots \\ \xrightarrow{\widetilde{d^m_{m,\langle n\rangle}}}\end{smallmatrix} \cdots \begin{smallmatrix}\to \\ \to \\ \to\end{smallmatrix} & D(F_1) & \begin{smallmatrix}\xrightarrow{\widetilde{d^1_{0,\langle n\rangle}}} \\ \xrightarrow{\widetilde{d^1_{1,\langle n\rangle}}}\end{smallmatrix} & D(F_0) & \xrightarrow{\widetilde{d^0_{0,\langle n\rangle}}} & D(\mathscr{M}) \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \begin{smallmatrix}\to \\ \vdots \\ \to\end{smallmatrix} & F_{m,\langle n\rangle} & \begin{smallmatrix}\xrightarrow{d^m_{0,\langle n\rangle}} \\ \vdots \\ \xrightarrow{d^m_{m,\langle n\rangle}}\end{smallmatrix} \cdots \begin{smallmatrix}\to \\ \to \\ \to\end{smallmatrix} & F_{1,\langle n\rangle} & \begin{smallmatrix}\xrightarrow{d^1_{0,\langle n\rangle}} \\ \xrightarrow{d^1_{1,\langle n\rangle}}\end{smallmatrix} & F_{0,\langle n\rangle} & \xrightarrow{d^0_{0,\langle n\rangle}} & \mathscr{M}_{\langle n\rangle} \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \begin{smallmatrix}\to \\ \vdots \\ \to\end{smallmatrix} & \sigma\mathfrak{Ab}^{(n)}(F_m) & \begin{smallmatrix}\to \\ \vdots \\ \to\end{smallmatrix} \cdots \begin{smallmatrix}\to \\ \to \\ \to\end{smallmatrix} & \sigma\mathfrak{Ab}^{(n)}(F_1) & \begin{smallmatrix}\to \\ \to\end{smallmatrix} & \sigma\mathfrak{Ab}^{(n)}(F_0) & \to & \sigma\mathfrak{Ab}^{(n)}(\mathscr{M}) \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & 1 & & 1 & & 1 & & 1
\end{array},
$$

where $D(\mathscr{M})$ denotes the group $\prod_{B\cup C=\langle n\rangle}[\mathscr{M}_B, \mathscr{M}_C]$ for any crossed n-cube $\mathscr{M}$.

By the induced long exact homotopy sequence, one has the isomorphisms of groups

$$\pi_m\sigma\mathfrak{Ab}^{(n)}(F_*) \cong \frac{\bigcap_{i=0}^{m-1}\operatorname{Ker}\widetilde{d^{m-1}_{i,\langle n\rangle}}}{\widetilde{d^m_{m,\langle n\rangle}}(\bigcap_{i=0}^{m-1}\operatorname{Ker}\widetilde{d^m_{i,\langle n\rangle}})}, \quad m \geqslant 1. \tag{6}$$

Since $\widetilde{d^m_{i,\langle n\rangle}}$ is the restriction of $d^m_{i,\langle n\rangle}$ to $D(F_m)$, $\operatorname{Ker}\widetilde{d^m_{i,\langle n\rangle}} = \operatorname{Ker} d^m_{i,\langle n\rangle} \cap D(F_m)$. Hence $\bigcap_{i=0}^{m-1}\operatorname{Ker}\widetilde{d^{m-1}_{i,\langle n\rangle}} = (\bigcap_{i=0}^{m-1}\operatorname{Ker} d^{m-1}_{i,\langle n\rangle}) \cap D(F_{m-1})$ and $\bigcap_{i=0}^{m-1}\operatorname{Ker}\widetilde{d^m_{i,\langle n\rangle}} = (\bigcap_{i=0}^{m-1}\operatorname{Ker} d^m_{i,\langle n\rangle}) \cap D(F_m)$.

Since the shift of simplicial object F_* is the contractible augmented simplicial object $(Dec(F_*), d_0^1, F_0)$ (see [7]), by Lemma 9 the m-cube of crossed n-cubes $Dec(F)^{(m)}$ is a projective exact m-presentation of F_0. Hence, by Theorem 12 one has

$$\begin{aligned}
&\mathscr{L}_m^{m\text{-fold}}(\sigma\mathfrak{Ab}^{(n)})(F_0) \\
&\quad\cong \frac{\bigcap_{i\in\langle m\rangle}\operatorname{Ker} d^m_{i-1,\langle n\rangle} \cap \prod_{B\cup C=\langle n\rangle}[F_{m,B}, F_{m,C}]}{\prod_{A\subseteq\langle m\rangle}(\prod_{B\cup C=\langle n\rangle}[\bigcap_{i\in A}\operatorname{Ker} d^m_{i-1,B}, \bigcap_{i\notin A}\operatorname{Ker} d^m_{i-1,C}])} = 1, \;\; m \geqslant 1,
\end{aligned}$$

implying the equality

$$\bigcap_{i\in\langle m\rangle} \operatorname{Ker} d^m_{i-1,\langle n\rangle} \bigcap \prod_{B\cup C=\langle n\rangle} [F_{m,B}, F_{m,C}]$$
$$= \prod_{A\subseteq\langle m\rangle} \left(\prod_{B\cup C=\langle n\rangle} \left[\bigcap_{i\in A} \operatorname{Ker} d^m_{i-1,B}, \bigcap_{i\notin A} \operatorname{Ker} d^m_{i-1,C} \right] \right), \quad m \geqslant 1, \tag{7}$$

Since $(F_{*,\langle n\rangle}, d^0_{0,\langle n\rangle}, \mathscr{M}_{\langle n\rangle})$ is an aspherical augmented simplicial group, $d^m_{m,\langle n\rangle}(\bigcap_{i\in\langle m\rangle} \operatorname{Ker} d^m_{i-1,\langle n\rangle}) = \bigcap_{i\in\langle m\rangle} \operatorname{Ker} d^{m-1}_{i-1,\langle n\rangle}$, $m \geqslant 1$. Using this fact and Lemma 8 [6], by (7) it is easy to see that one has an equality

$$\widetilde{d^m_{m,\langle n\rangle}} \left(\bigcap_{i=0}^{m-1} \operatorname{Ker} \widetilde{d^m_{i,\langle n\rangle}} \right)$$
$$= d^m_{m,\langle n\rangle} \left(\prod_{A\subseteq\langle m\rangle} \left(\prod_{B\cup C=\langle n\rangle} \left[\bigcap_{i\in A} \operatorname{Ker} d^m_{i-1,B}, \bigcap_{i\notin A} \operatorname{Ker} d^m_{i-1,C} \right] \right) \right)$$
$$= \prod_{A\subseteq\langle m\rangle} \left(\prod_{B\cup C=\langle n\rangle} \left[\bigcap_{i\in A} \operatorname{Ker} d^{m-1}_{i-1,B}, \bigcap_{i\notin A} \operatorname{Ker} d^{m-1}_{i-1,C} \right] \right).$$

Thus by (6) one has

$$H_{m+1}(\mathscr{M}) \cong \frac{(\bigcap_{i=0}^{m-1} \operatorname{Ker} d^{m-1}_{i,\langle n\rangle}) \cap \prod_{B\cup C=\langle n\rangle} [F_{m-1,B}, F_{m-1,C}]}{\prod_{A\subseteq\langle m\rangle} (\prod_{B\cup C=\langle n\rangle} [\bigcap_{i\in A} \operatorname{Ker} d^{m-1}_{i-1,B}, \bigcap_{i\notin A} \operatorname{Ker} d^{m-1}_{i-1,C}])}.$$

Using again Lemma 9 and Theorem 12 completes the proof. □

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