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Nilpotent Pairs, Dual Pairs, and Sheets

Dmitri I. Panyushev

Independent University of Moscow, Bol'shoi Vlasievskii per. 11, Moscow 121002, Russia
E-mail: dmitri@panyushev.mccme.ru

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INTRODUCTION

The aim of this paper is to present some results related to the theory of nilpotent pairs in semisimple Lie algebras and to give some applications of it to dual pairs and sheets. Recently, Ginzburg introduced the concept of a *principal nilpotent pair* (=*pn*-pair) in a semisimple Lie algebra $\mathfrak{g}$ [9]. It is a double counterpart of the notion of a regular nilpotent element in $\mathfrak{g}$. A pair $\mathbf{e} = (e_1, e_2) \in \mathfrak{g} \times \mathfrak{g}$ is called *nilpotent*, if $[e_1, e_2] = 0$ and there exists a pair $\mathbf{h} = (h_1, h_2)$ of semisimple elements such that $[h_1, h_2] = 0$, $[h_i, e_j] = \delta_{ij} e_j (i, j \in \{1, 2\})$. A *pn*-pair $\mathbf{e}$ is a nilpotent pair such that the simultaneous centralizer $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$ is of dimension $\operatorname{rk} \mathfrak{g}$. By a famous theorem of Richardson [16], $\operatorname{rk} \mathfrak{g}$ is the least possible value for this dimension. Evident similarity between the "double" and "ordinary" theory is manifestly seen in the following results of [9]: $\mathfrak{z}_{\mathfrak{g}}(\mathbf{h})$ is a Cartan subalgebra; the eigenvalues of ad h_1 and h_2 are integral; both e_1 and e_2 are Richardson elements; $Z_G(\mathbf{e})$ is a connected Abelian unipotent group; $Z_G(\mathbf{e})$ acts transitively on the set of semisimple pairs satisfying the above commutator relations. Excerpts from Ginzburg's theory, which by no means exhaust [9], are presented in Section 1.

In Section 2, it is shown that a considerable part of the above-mentioned results can be extended to the nilpotent pairs with $\dim \mathfrak{z}_{\mathfrak{g}}(e_1, e_2) = \operatorname{rk} \mathfrak{g} + 1$. Such pairs are called *almost* *pn*-pairs. Although almost *pn*-pairs share many properties with *pn*-pairs, with similar proofs, some new phenomena do occur for the former. For instance, it is shown that the totality of almost *pn*-pairs breaks into two natural classes (2.5). One of the distinctions between them is that the eigenvalues of ad $h_i (i = 1, 2)$ are integral for

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the first class and non-integral for the second class. We also give a description of $Z_G(\mathbf{e})$ for both classes. It is worth noting that the very existence of almost *pn*-pairs is a purely "double" phenomenon, because the dimension of "ordinary" orbits is always even.

It is not always the case that $\{e_1, e_2\}$ can be embedded in a subalgebra $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \subset \mathfrak{g}$. The pairs admitting such an embedding are called *rectangular*. Then, as usual, the $\mathfrak{sl}_2$-machinery invented by Morozov and Dynkin in the 1940s makes life much easier. For instance, a structure result and a complete classification for rectangular *pn*-pairs is found in [7]. Some results on rectangular pairs, in particular almost principal ones, are presented in Section 3.

Section 4 concerns a relationship between nilpotent pairs and dual pairs. Given a quadruple $(\mathbf{e}, \mathbf{h})$ satisfying the commutator relations as above, it is shown that $\mathfrak{k}_1 = \mathfrak{z}_\mathfrak{g}(e_1, h_1)$ and $\mathfrak{k}_2 = \mathfrak{z}_\mathfrak{g}(e_2, h_2)$ form a dual pair in $\mathfrak{g}$ under certain constraints (see Theorem 4.3). Then using results of Section 2, we prove that these constraints are satisfied for the *pn*-pairs and almost *pn*-pairs. It is curious that, for the (almost) *pn*-pair, the corresponding dual pair is reductive if and only if $\mathbf{e}$ is rectangular. Moreover, if $\mathbf{e}$ is a rectangular *pn*-pair, then $(\mathfrak{k}_1, \mathfrak{k}_2)$ is S-irreducible in the sense of Rubenthaler [18]. Thus, the concept of an (almost) *pn*-pair provides a natural framework for constructing dual pairs, not necessarily reductive ones.

In Section 5, we describe another class of rectangular nilpotent pairs such that $(\mathfrak{k}_1, \mathfrak{k}_2)$ appears to be a dual pair. These pairs are called *semi-principal*. It is worthwhile to note that, as $\mathfrak{k}_1$ is already a centralizer, $(\mathfrak{k}_1, \mathfrak{z}_\mathfrak{g}(\mathfrak{k}_1))$ is a dual pair. So, the point is that $\mathfrak{k}_2 = \mathfrak{z}_\mathfrak{g}(\mathfrak{k}_1)$ comes up also as centralizer attached to the second member of the pair.

As a by-product of our study of semi-principal pairs, we found that the double centralizer of some $\mathfrak{sl}_2$-triples has beautiful properties. It turns out that this phenomenon, appropriately formalized, had some application to sheets. Let $\{e, \tilde{h}, f\}$ be an $\mathfrak{sl}_2$-triple. Both the triple and e are called *excellent*, if e is even and $\dim \mathfrak{z}_\mathfrak{g}(\mathfrak{z}_\mathfrak{g}(\tilde{h})) = \operatorname{rk} \mathfrak{z}_\mathfrak{g}(\mathfrak{z}_\mathfrak{g}(e, \tilde{h}, f))$. In Section 6, we show that the excellent triples enjoy the following properties: $\mathfrak{z}_\mathfrak{g}(\mathfrak{z}_\mathfrak{g}(e, \tilde{h}, f))$ is semisimple; $\mathfrak{z}_\mathfrak{g}(\mathfrak{z}_\mathfrak{g}(e))$ (resp. $\mathfrak{z}_\mathfrak{g}(\mathfrak{z}_\mathfrak{g}(\tilde{h}))$) is the centralizer of e (resp. $\tilde{h}$) in $\mathfrak{z}_\mathfrak{g}(\mathfrak{z}_\mathfrak{g}(e, \tilde{h}, f))$. Then we consider the sheet $\mathcal{S}$ associated to $\{e, \tilde{h}, f\}$. It is proven that $\mathcal{S}$ is smooth and has a section, which is an affine space, and that it is the *only* sheet containing e; see Theorem 6.6. This applies, in particular, to both members of rectangular *pn*-pairs.

In Section 7, we classify the excellent elements in the simple Lie algebras.

The ground field $\Bbbk$ is algebraically closed and of characteristic zero. Throughout, $\mathfrak{g}$ is a semisimple Lie algebra and G is its adjoint group. For any set $M \subset \mathfrak{g}$, let $\mathfrak{z}_\mathfrak{g}(M)$ (resp. $Z_G(M)$) denote the centralizer of M in $\mathfrak{g}$ (resp. in G). For $M = \{a, \dots, z\}$, we simply write $\mathfrak{z}_\mathfrak{g}\{a, \dots, z\}$ or

$Z_G\{a, \dots, z\}$. If $N \subset G$, then $Z_G(N)$ stands for the centralizer of N in G. For $x \in \mathfrak{g}$ and $s \in G$, we write $s \cdot x$ in place of $(\mathrm{Ad}\, s)x$. K^o is the identity component of an algebraic group K. If $\mathfrak{a}$ is a Lie algebra, then $\mathfrak{C}(\mathfrak{a}) \subset \mathfrak{a} \oplus \mathfrak{a}$ is the *commuting variety*, i.e., the set of all pairs of commuting elements. We write $\mathfrak{C}$ in place of $\mathfrak{C}(\mathfrak{g})$. Our general reference for nilpotent orbits is [3].

1. PRINCIPAL NILPOTENT PAIRS

We first review some basic structure results on *pn*-pairs proved in [9].

1.1. DEFINITION. (V. Ginzburg). A pair $\mathbf{e} = (e_1, e_2) \in \mathfrak{g} \times \mathfrak{g}$ is called a *principal nilpotent pair* if the following holds:

(i) $[e_1, e_2] = 0$ and $\dim \mathfrak{z}_\mathfrak{g}(\mathbf{e}) = \mathrm{rk}\, \mathfrak{g}$;

(ii) For any $(t_1, t_2) \in \Bbbk^* \times \Bbbk^*$, there exists $g = g(t_1, t_2) \in G$ such that $(t_1 e_1, t_2 e_2) = (g \cdot e_1, g \cdot e_2)$.

The first step in Ginzburg's theory is that condition (ii) is equivalent to the following one: there exists an (associated semisimple) pair $\mathbf{h} = (h_1, h_2) \in \mathfrak{g} \times \mathfrak{g}$ such that $\mathrm{ad}\, h_1$ and $\mathrm{ad}\, h_2$ have rational eigenvalues and

$$[h_1, h_2] = 0, \qquad [h_i, e_j] = \delta_{ij} e_j \, (i, j \in \{1, 2\}). \tag{1.2}$$

In particular, the pair $\mathbf{e}$ is nilpotent in the sense of the Introduction. This $\mathbf{h}$ determines the bi-grading of $\mathfrak{g}$: $\mathfrak{g}_{k_1, k_2} = \{x \in \mathfrak{g} \mid [h_j, x] = k_j x,\ j = 1, 2\}$ and the induced grading of $\mathfrak{z}_\mathfrak{g}(\mathbf{e})$.

1.3. THEOREM (see [9, 1.2]). (1) *$\mathfrak{z}_\mathfrak{g}(\mathbf{h})$ is a Cartan subalgebra of $\mathfrak{g}$*;

(2) *the eigenvalues of* $\mathrm{ad}\, h_1$, $\mathrm{ad}\, h_2$, *in $\mathfrak{g}$ are integral*;

(3) $\mathfrak{z}_\mathfrak{g}(\mathbf{e}) = \bigoplus_{i, j \in \mathbb{Z}_{\geq 0}, (i, j) \neq (0, 0)} \mathfrak{z}_\mathfrak{g}(\mathbf{e})_{i, j}$, *i.e., $\mathfrak{z}_g(\mathbf{e})$ is graded by the "positive quadrant" without origin*;

(4) $\mathbf{h}$ *is determined uniquely up to conjugacy by $Z_G(\mathbf{e})^o$ (that is, the set of associated semisimple pairs forms a single $Z_G(\mathbf{e})^o$-orbit)*.

Because of the last property it is natural to work with a (fixed) quadruple $(\mathbf{e}, \mathbf{h})$ rather than with the pair $\mathbf{e}$. Denoting $\mathfrak{l}_i := \mathfrak{z}_\mathfrak{g}(h_i) (i = 1, 2)$, we get $e_1, h_1 \in \mathfrak{l}_2$ and $e_2, h_2 \in \mathfrak{l}_1$. Having the $\mathbb{Z}^2$-grading of $\mathfrak{g}$ determined by $\mathbf{h}$, one immediately sees 2 natural parabolic subalgebras containing $\mathfrak{l}_1$ and $\mathfrak{l}_2$: $\mathfrak{p}_1 := \bigoplus_{k_1 \geq 0} \mathfrak{g}_{k_1, k_2} = \mathfrak{g}_{\geq 0, *}$ (the right half-plane) and $\mathfrak{p}_2 := \bigoplus_{k_2 \geq 0} \mathfrak{g}_{k_1, k_2} = \mathfrak{g}_{*, \geq 0}$ (the upper half plane). Then $\mathfrak{l}_i$ is a Levi subalgebra of $\mathfrak{p}_i$ and e_i lies in the nilpotent radical $(\mathfrak{p}_i)^{nil}$ of $\mathfrak{p}_i$. The main structure result is:

1.4. THEOREM (see [9, Sect. 3]). *If* $\mathbf{e}$ *is a pn-pair, then*

(i) *e_i is a Richardson element in $(\mathfrak{p}_i)^{nil}$ (equivalently, $\mathfrak{p}_i$ is a polarization of e_i), $i = 1, 2$*;

(ii) *e_1 (resp. e_2) is a regular nilpotent element in $\mathfrak{l}_2$ (resp. $\mathfrak{l}_1$)*.

That the theory of pn-pairs has a rich content follows already from the description of such pairs in $\mathfrak{sl}_N$; see [9, 5.6]. In particular, the following holds: given a nilpotent element $e \in \mathfrak{sl}_N$, there exists e' such that (e, e') is a pn-pair. The partition corresponding to e' is conjugate to that for e. An explicit description of this pair is given in terms of the corresponding Young diagram. This shows $\mathfrak{g}$ may contain many pn-pairs. Nevertheless, the following fundamental result is true:

1.5. THEOREM (see [9, 3.9]). *The number of G-orbits of principal nilpotent pairs in $\mathfrak{g}$ is finite.*

Therefore the pn-pairs in simple Lie algebras can effectively be classified. The classification is obtained in [7] for the exceptional simple Lie algebras and in [8] for the classical ones. It may happen that $\mathfrak{g}$ contains no non-trivial pn-pairs at all; see, e.g., $\mathbf{C}_2$, $\mathbf{B}_3$, or $\mathbf{G}_2$.

2. ALMOST PRINCIPAL NILPOTENT PAIRS

In this section we show that a large portion of the theory in the first half of [9] can be extended to a more general setting. Our motivation partly came from studying dual pairs associated with nilpotent pairs; see Section 4. Although some of our proofs are adapted from Ginzburg's, interesting new phenomena do occur in our setting.

2.1. DEFINITION. A pair $\mathbf{e} = (e_1, e_2) \in \mathfrak{g} \times \mathfrak{g}$ is called an *almost principal nilpotent pair* if the following holds:

(i) $[e_1, e_2] = 0$ and $\dim \mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) = \mathrm{rk}\, \mathfrak{g} + 1$;

(ii) there exists a pair of semisimple elements $\mathbf{h} = (h_1, h_2) \in \mathfrak{g} \times \mathfrak{g}$ such that $[h_1, h_2] = 0$ and $[h_i, e_j] = \delta_{ij} e_j$ $(i, j \in \{1, 2\})$.

Each pair $\mathbf{h}$ satisfying condition (ii) is called an *associated semisimple* pair. As in Section 1, we shall consider the bi-grading $\mathfrak{g} = \bigoplus \mathfrak{g}_{i,j}$ determined by $\mathbf{h}$. For any subspace $M \subset \mathfrak{g}$, one may define 3 filtrations:

- e_1-filtration: $M(i, *) = \{x \in M \mid (\mathrm{ad}\, e_1)^{i+1} x = 0\}$, $i \geq 0$;
- e_2-filtration: $M(*, j) = \{x \in M \mid (\mathrm{ad}\, e_2)^{j+1} x = 0\}$, $j \geq 0$;
- the $\mathbf{e}$-filtration: Consider any $\mathbb{Z}$-linear function $u : \mathbb{Z}^2 \to \mathbb{Z}$ such that $u(1,0) > 0$, $u(0,1) > 0$, and the values $u(i,j)$ are different for all (i, j) such that $\mathfrak{g}_{i,j} \neq 0$. Given $i, j \geq 0$ we then set $M(i,j) = \{x \in M \mid (\mathrm{ad}\, e_1)^i (\mathrm{ad}\, e_2)^j x = 0$ For all (k, l) such that $u(k,l) > u(i,j)\}$.

Following an idea of Brylinski, define the corresponding limits:

$\lim_{e_1} M = \sum_{i \in \mathbb{Z}_{\geq 0}} (\mathrm{ad}\, e_1)^i M(i, *) \subset \mathfrak{g}$,

$\lim_{e_2} M = \sum_{j\in\mathbb{Z}_{\geq 0}} (\operatorname{ad} e_2)^j M(*, j) \subset \mathfrak{g}$,

$\lim_{\mathbf{e}} M = \sum_{i,\, j\in\mathbb{Z}_{\geq 0}} (\operatorname{ad} e_1)^i (\operatorname{ad} e_2)^j M(i, j) \subset \mathfrak{g}$.

2.2. THEOREM. *Let* $\mathbf{e}$ *be an almost pn-pair and* $\mathbf{h}$ *an associated semisimple pair. Then*

(i) $\mathfrak{z}_{\mathfrak{g}}(\mathbf{h})$ *is a Cartan subalgebra of* $\mathfrak{g}$;

(ii) $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) \cap \mathfrak{z}_{\mathfrak{g}}(\mathbf{h}) = 0$ *or, equivalently,* $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{0,\,0} = 0$.

Proof. We use an algebraized version of arguments in [9, Sect. 1].

(i) Consider the $\mathbf{e}$-filtration for $\mathfrak{t} := \mathfrak{z}_{\mathfrak{g}}(\mathbf{h})$. Since $\mathfrak{t} = \mathfrak{g}_{0,\,0}$ and $(\operatorname{ad} e_1)^i (\operatorname{ad} e_2)^j\ \mathfrak{t}(i, j) \subset \mathfrak{g}_{i,\, j}$, the sum in the definition is actually direct. Obviously, $\lim_{\mathbf{e}} \mathfrak{t} \subset \bigoplus_{i,\, j\in\mathbb{Z}_{\geq 0}} \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{i,\, j}$. It follows from the definition of the $\mathbf{e}$-filtration that $\dim(\lim_{\mathbf{e}} \mathfrak{t}) = \dim \mathfrak{t}$. Thus,

$$\operatorname{rk} \mathfrak{g} \leq \dim \mathfrak{t} \leq \dim \left(\bigoplus_{i,\, j\in\mathbb{Z}_{\geq 0}} \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{i,\, j} \right) \leq \dim \mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) = \operatorname{rk} \mathfrak{g} + 1.$$

Since $\mathfrak{t}$ is a Levi subalgebra, $\dim \mathfrak{t} - \operatorname{rk} \mathfrak{g}$ is even. Hence $\mathfrak{t}$ must be a Cartan subalgebra.

(ii) Assume that h is a nonzero element in $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) \cap \mathfrak{t}$. Then e_1, e_2 lie in the Levi subalgebra $\mathfrak{l} := \mathfrak{z}_{\mathfrak{g}}(h)$. By [16], $\mathfrak{E}(\mathfrak{l})$ is irreducible and the pairs of semisimple elements are dense in $\mathfrak{E}(\mathfrak{l})$. it follows that $\dim \mathfrak{z}_{\mathfrak{l}}(x, y) \geq \operatorname{rk} \mathfrak{l}$ for any pair $(x, y) \in \mathfrak{E}(\mathfrak{l})$. Thus,

$$\operatorname{rk} \mathfrak{g} = \operatorname{rk} \mathfrak{l} \leq \dim \mathfrak{z}_{\mathfrak{l}}(e_1, e_2) \leq \dim \mathfrak{z}_{\mathfrak{g}}(e_1, e_2) = \operatorname{rk} \mathfrak{g} + 1.$$

Associated with $\mathfrak{l}$, there is a decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{l} \oplus \mathfrak{n}_-$, where $[\mathfrak{l}, n_\pm] = n_\pm$. It follows that $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) = \mathfrak{z}_{\mathfrak{n}_-}(\mathbf{e}) \oplus \mathfrak{z}_{\mathfrak{l}}(\mathbf{e}) \oplus \mathfrak{z}_{\mathfrak{n}_+}(\mathbf{e})$ and $\dim \mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) = 2 \dim \mathfrak{z}_{\mathfrak{n}_+}(\mathbf{e}) + \dim \mathfrak{z}\mathfrak{l}(\mathbf{e})$. Obviously, the first summand is positive and we obtain $\dim \mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) \geq \operatorname{rk} \mathfrak{g} + 2$. This contradiction proves the claim (ii). ∎

2.3. COROLLARY. *We have* $\lim_{\mathbf{e}} \mathfrak{t} = \bigoplus_{i,\, j\in\mathbb{Z}_{\geq 0},\, (i,j)\neq(0,0)} \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{i,\, j}$. *In particular,* $(\operatorname{ad} e_1)^i (\operatorname{ad} e_2)^j \mathfrak{t}(i, j) = \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{i,\, j}$ *for all* $i, j \in \mathbb{Z}_{\geq 0}$.

Proof. It is already proved that the inclusion $\subset$ holds. Since $\mathfrak{t}$ is Cartan and the pair $\mathbf{e}$ is not principal, it follows from [9, 1.13] that $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) \neq \bigoplus_{i,\, j\in\mathbb{Z}_{\geq 0},(i,\, j)\neq(0,0)} \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{i,\, j}$. Then the assertion follows for dimension reason. ∎

Unlike the case of *pn*-pairs (see (1.3)), the eigenvalues of $\operatorname{ad} h_1$ and $\operatorname{ad} h_2$ are not necessarily integral and $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$ is not necessarily graded by “positive quadrant.” As we shall see in (2.5), these two conditions form a dichotomy in case of almost *pn*-pairs.

2.4. THEOREM. *Let* $\mathbf{e}$ *be an* (*almost*) *pn-pair with an associated semisimple pair* $\mathbf{h}$. *Put* $\mathfrak{l}_i = \mathfrak{z}_\mathfrak{g}(h_i)$ *and let* $\mathfrak{c}_i$ *denote the centre of* $\mathfrak{l}_i$ $(i = 1, 2)$. *Then*

(1) e_1 *is a regular nilpotent element in* $\mathfrak{l}_2$ *and* e_2 *is a regular nilpotent element in* $\mathfrak{l}_1$;

(2) $\lim_{e1}\mathfrak{t} = \mathfrak{z}_\mathfrak{g}(e_1, h_2)$ *and* $\lim_{e2}\mathfrak{t} = \mathfrak{z}_\mathfrak{g}(e_2, h_1)$;

(3) $\dim\mathfrak{z}_\mathfrak{g}(e_1, h_1, e_2) = \dim\mathfrak{z}_\mathfrak{g}(e_1, h_1, h_2)$ *and* $\dim\mathfrak{z}_\mathfrak{g}(e_2, h_2, e_1) = \dim\mathfrak{z}_\mathfrak{g}(e_2, h_2, h_1)$;

(4) $\mathfrak{z}_\mathfrak{g}(e_1, h_1, h_2) = \mathfrak{c}_2$ *and* $\mathfrak{z}_\mathfrak{g}(e_2, h_1, h_2) = \mathfrak{c}_1$.

Proof. By symmetry, it suffices to prove the first half of each item. The proof applies to both *pn*- and almost *pn*-pairs.

(1) and (2) These proofs are essentially the same as in [9]. Consider the e_1-limit, $\lim_{e1}\mathfrak{t} = \sum_{i\ge 0}(\operatorname{ad} e_1)^i\mathfrak{t}(i, *)$, which lies in $\mathfrak{z}_\mathfrak{g}(e_1, h_2)$. Since different summands have different weights relative to $\operatorname{ad} h_1$, the sum is direct and therefore $\operatorname{rk}\mathfrak{g} \le \dim\mathfrak{z}_\mathfrak{g}(e_1, h_2)$. The space $\mathfrak{z}_\mathfrak{g}(e_1, h_2)$ possesses the e_2-filtration and $\lim_{e_2}\mathfrak{z}_\mathfrak{g}(e_1, h_2) \subset \mathfrak{z}_\mathfrak{g}(\mathbf{e})$. For a similar reason, $\dim(\lim_{e_2}\mathfrak{z}_\mathfrak{g}(e_1, h_2)) = \mathfrak{z}_\mathfrak{g}(e_1, h_2)$ and hence $\dim\mathfrak{z}_\mathfrak{g}(e_1, h_2) \le \operatorname{rk}\mathfrak{g} + 1$. As in the proof of (2.2)(i), one may conclude by making use of the parity argument: e_1 lies in the reductive Lie algebra $\mathfrak{l}_2$ and therefore $\dim\mathfrak{z}_\mathfrak{g}(e_1, h_2) = \dim\mathfrak{z}_{\mathfrak{l}_2}(e_1)$ must have the same parity as $\operatorname{rk}\mathfrak{g} = \operatorname{rk}\mathfrak{l}_2$.

(3) Applying the formula in (2.3) with $i = 0$ gives

$$\bigoplus_j(\operatorname{ad} e_2)^j\mathfrak{t}(0, j) = \bigoplus_j\mathfrak{z}_\mathfrak{g}(\mathbf{e})_{0,j} = \mathfrak{z}_\mathfrak{g}(e_1, e_2, h_1).$$

Obviously, the dimension of the left-hand side is $\sum_j \dim(\mathfrak{t}(0, j)/\mathfrak{t}(0, j-1)) = \dim\mathfrak{t}(0, *)$. Since $\mathfrak{t}(0, *) = \mathfrak{z}_\mathfrak{g}(e_1, h_1, h_2)$, we are done.

(4) Since $e_1, h_1 \in \mathfrak{z}_\mathfrak{g}(h_2) = \mathfrak{l}_2$, we have $\mathfrak{z}_\mathfrak{g}(e_1, h_1, h_2) \supset \mathfrak{c}_2$. By either (1.4) (ii) or (2.4)(1), e_1 is a regular nilpotent element in $\mathfrak{l}_2$. Therefore $\mathfrak{z}_{\mathfrak{l}_2}(e_1) = \mathfrak{z}_\mathfrak{g}(e_1, h_2) = \mathfrak{c}_2 \oplus \mathfrak{n}$, where $\mathfrak{n} \subset [\mathfrak{l}_2, \mathfrak{l}_2]$ consists of nilpotent elements. Finally, $\mathfrak{z}_\mathfrak{g}(e_1, h_1, h_2) \subset \mathfrak{z}_\mathfrak{g}(h_1, h_2)$ and therefore $\mathfrak{z}_\mathfrak{g}(e_1, h_1, h_2)$ consists of semisimple elements, whence $\mathfrak{z}_\mathfrak{g}(e_1, h_1, h_2) = \mathfrak{c}_2$. ∎

By (2.3), $\mathfrak{z}_+ := \bigoplus_{i,j\in\mathbb{Z}_{\ge 0},(i,j)\ne(0,0)}\mathfrak{z}_\mathfrak{g}(\mathbf{e})_{i,j}$ is of codimension one in $\mathfrak{z}_\mathfrak{g}(\mathbf{e})$, if $\mathbf{e}$ is an almost *pn*-pair. Hence there is an "extra" vector x in some $\mathfrak{g}_{p,q}$ such that $\mathfrak{z}_\mathfrak{g}(\mathbf{e}) = \mathfrak{z}_+ \oplus \langle x\rangle$. We already know that $(p, q) \notin (\mathbb{Z}_{\ge o})^2$. It also follows from Theorem 2.4(2) that the eigenvalues of $\operatorname{ad} h_1$ (resp. $\operatorname{ad} h_2$) in $\mathfrak{z}_\mathfrak{g}(e_1, h_2)$ (resp. $\mathfrak{z}_\mathfrak{g}(e_2, h_1)$) are nonnegative integers. Therefore $x \notin \mathfrak{l}_1 (i = 1, 2)$. That is, $pq \ne 0$.

2.5. THEOREM. (1) *There are* 2 *mutually exclusive possibilities for* p, q. *Either*

$$(\mathbb{Z}) \qquad p, q \in \mathbb{Z} \text{ and } pq < 0, \text{ or}$$
$$(non\text{-}\mathbb{Z}) \quad p, q \in \tfrac{1}{2}\mathbb{Z}\setminus\mathbb{Z} \text{ and } p, q > 0.$$

(2) *In both cases,* $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$ *is nilpotent and contains no semisimple elements. Moreover,* $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$ *is Abelian in the non-*$\mathbb{Z}$ *case.*

Proof. (1) For ($\mathbb{Z}$), suppose that $p, q \in \mathbb{Z}$, i.e., all the eigenvalues of $\mathbf{h}$ in $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$ are integral. We need to prove here that the case $p < 0$, $q < 0$ is impossible. Assume not and $p_0 = -p > 0$, $q_0 = -q > 0$. A standard calculation with the Killing form on $\mathfrak{g}$ shows that $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{p,q} \neq 0$ if and only if $\mathfrak{g}_{p_0,q_0} \not\subset \mathrm{Im}\,(\mathrm{ad}\, e_1) + \mathrm{Im}\,(\mathrm{ad}\, e_2)$. By definition, put $\mathcal{D} = \mathfrak{g}_{p_0,q_0} \backslash (\mathrm{Im}\,(\mathrm{ad}\, e_1) + \mathrm{Im}\,(\mathrm{ad}\, e_2))$. For each $y \in \mathcal{D}$, consider the finite set $I_y = \{(k, l) \in (\mathbb{Z}_{\geq 0})^2 \mid (\mathrm{ad}\, e_1)^k (\mathrm{ad}\, e_2)^l y \neq 0\}$, with the lexicographic ordering. This means $(k, l) \prec (k', l') \Leftrightarrow k < k'$ or $k = k'$ and $l < l'$. Denote by $m(I_y)$ the unique maximal element in I_y. Let $y^* \in \mathcal{D}$ be an element such that $(k_0, l_0) := m(I_{y^*}) \preceq m(I_z)$ for all $z \in \mathcal{D}$. Then $(\mathrm{ad}\, e_1)^{k_0} (\mathrm{ad}\, e_2)^{l_0} y^*$ is a nonzero element in $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) \cap \mathfrak{g}_{p_0+k_0,\, q_0+l_0}$. By (2.3), there is $t \in \mathfrak{t}(p_0 + k_0, q_0 + l_0)$ such that $(\mathrm{ad}\, e_1)^{p_0+k_0} (\mathrm{ad}\, e_2)^{q_0+l_0} t = (\mathrm{ad}\, e_1)^{k_0} (\mathrm{ad}\, e_2)^{l_0} y^*$. Then $(\mathrm{ad}\, e_1)^{k_0} (\mathrm{ad}\, e_2)^{l_0} (y^* - (\mathrm{ad}\, e_1)^{p_0} (\mathrm{ad}\, e_2)^{q_0} t) = 0$. Since $p_0 > 0, q_0 > 0$, we have $z^* = y^* - (\mathrm{ad}\, e_1)^{p_0} (\mathrm{ad}\, e_2)^{q_0} t$ is nonzero and belongs to $\mathcal{D}$. However, $I_{z^*} \subset I_{y^*} \backslash \{(k_0, l_0)\}$. Therefore $m(I_{z^*}) < m(I_{y^*})$, which contradicts the choice of y^*. Thus, the case $p < 0, q < 0$ is impossible.

For (non-$\mathbb{Z}$), suppose $(p, q) \notin \mathbb{Z} \oplus \mathbb{Z}$. Consider the set $\mathcal{J} = \{(k, l) \mid \mathfrak{g}_{k,l} \neq 0$ and $(k, l) \notin \mathbb{Z} \oplus \mathbb{Z}\}$. Because $\langle x \rangle$ is the unique "non-integral" homogeneous subspace of $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$, $\mathcal{J}$ lies in the single coset space $(p, q) + (\mathbb{Z} \oplus \mathbb{Z})$ and has a unique "north-east" corner. Obviously, (p, q) is this corner. Since $\dim \mathfrak{g}_{m,n} = \dim \mathfrak{g}_{-m,-n}$ for all (m, n), this corner must lie in the positive quadrant. The condition $(-p, -q) \in (p, q) + (\mathbb{Z} \oplus \mathbb{Z})$ implies $p, q \in \frac{1}{2}\mathbb{Z}$. It remains to demonstrate that *both* p, q must be fractional. Assume not, and $p \in \mathbb{Z}$, while q is fractional. Consider a "path inside of $\mathfrak{g}$" connecting the points $(-p, -q)$ and (p, q): Starting from a nonzero element in $\mathfrak{g}_{-p,-q}$, we may always apply either $\mathrm{ad}\, e_1$ or $\mathrm{ad}\, e_2$ until we arrive at $\alpha x \in \mathfrak{g}_{p,q} (\alpha \neq 0)$. Since p is integral, we must intersect somewhere the vertical axis. This means $\mathrm{ad}\, h_2$ has a fractional eigenvalue in $\mathfrak{l}_1$. It then follows from nilpotency of $\mathrm{ad}\, e_2$ that $\mathrm{ad}\, h_2$ has a fractional eigenvalue in $\mathfrak{z}_{\mathfrak{l}_1}(e_2)$ as well. However, this contradicts (2.4) (2).

(2) The pairs (k, l) such that $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{k,l} \neq 0$ are said to be *bi-weights* of $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$. In either case, the bi-weights lie in an open half-plane of $\mathbb{Q} \oplus \mathbb{Q}$, hence the assertion. In the non-$\mathbb{Z}$ case, (p, q) is the unique non-integral bi-weight. Since $(0, 0)$ is not a bi-weight (see Theorem 2.2 (ii)), this implies $[\mathfrak{z}_+, x] = 0$. it is also easily seen that $\mathfrak{z}_+ = \lim_{\mathbf{e}} \mathfrak{t}$ is Abelian. ∎

2.6. Corollary. *If* $\mathbf{h}$ *is any associated semisimple pair then the eigenvalues of* $\mathrm{ad}\, h_1$, $\mathrm{ad}\, h_2$ *in* $\mathfrak{g}$ *are at least half integers.*

An almost pn-pair is said to be either of *$\mathbb{Z}$-type* or *non-$\mathbb{Z}$-type* according to the two possibilities in Theorem 2.5(1). It will be proved below that all associated semisimple pairs are $Z_G(\mathbf{e})^o$-conjugate. Therefore the type does not depend on the choice of $\mathbf{h}$.

2.7. COROLLARY. *Let* $\mathbf{e}$ *be an almost pn-pair of non-$\mathbb{Z}$-type. Then there is an inner involution* $\theta \in Aut\,\mathfrak{g}$ *such that* $\mathfrak{g}^\theta$ *is semisimple and* $\mathbf{e}$ *is a pn-pair in* $\mathfrak{g}^\theta$.

Proof. Define $\theta \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ by

$$\theta|_{\mathfrak{g}_{i,j}} = \begin{cases} \mathrm{id} & \text{if } \ i, j \in \mathbb{Z} \\ -\mathrm{id} & \text{if } \ i, j \in \mathbb{Z} + \frac{1}{2}. \end{cases}$$

It is an inner automorphism of $\mathfrak{g}$. Then $e_1, e_2 \in \mathfrak{g}^\theta$, $\mathrm{rk}\,\mathfrak{g}^\theta = \mathrm{rk}\,\mathfrak{g}$, and $\dim \mathfrak{z}_{\mathfrak{g}^\theta}(\mathbf{e}) = \mathrm{rk}\,\mathfrak{g}^\theta$. As $\mathfrak{z}_{\mathfrak{g}^\theta}(\mathbf{e})$ contains no semisimple elements, $\mathfrak{g}^\theta$ is semisimple. ∎

It is worth noting that the two cases in Theorem 2.5 really occur:

2.8. EXAMPLE. Take $\mathfrak{g} = \mathfrak{sp}_4$. Let $\alpha = \varepsilon_1 - \varepsilon_2$ and $\beta = 2\varepsilon_2$ be the usual simple roots. Denote by e_μ a nonzero root vector corresponding to μ. Then $(e_{2\alpha+\beta}, e_\beta)$ is an almost pn-pair of $\mathbb{Z}$-type and $(e_{\alpha+\beta}, e_{2\alpha+\beta})$ is an almost pn-pair of non-$\mathbb{Z}$-type. In both cases, $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) = \langle e_{2\alpha+\beta}, e_{\alpha+\beta}, e_\beta \rangle$, but associated semisimple pairs are essentially different.

As in Section 1, define the parabolic subalgebras $\mathfrak{p}_1$ and $\mathfrak{p}_2$. Unlike the pn-case, e_i is not necessarily a Richardson element in the nilpotent radical $(\mathfrak{p}_i)^{nil}$ of $\mathfrak{p}_i$. The precise statement is as follows.

2.9. THEOREM. *Let* $\mathbf{e}$ *be an almost pn-pair.*

(i) *Suppose* $\mathbf{e}$ *is of non-$\mathbb{Z}$-type. Then neither of the* e_i*'s is Richardson in* $(\mathfrak{p}_i)^{nil}$;

(ii) *Suppose* $\mathbf{e}$ *is of $\mathbb{Z}$-type with, say,* $q > 0$ *and* $p < 0$. *Then* e_2 *is Richardson in* $(\mathfrak{p}_2)^{nil}$, *while* e_1 *is not Richardson in* $(\mathfrak{p}_1)^{nil}$.

Proof. (i) Let (k_0, l_0) be the minimal element in $((p, q) + (\mathbb{Z} \oplus \mathbb{Z})) \cap (\mathbb{Q}_{>0} \oplus \mathbb{Q}_{>0})$ with respect to the lexicographic ordering. Then $\mathfrak{g}_{k_0,l_0} \subset (\mathfrak{p}_i)^{nil}$, while $\mathfrak{g}_{k_0,l_0} \not\subset [\mathfrak{p}_i, e_i] (i = 1, 2)$. It is not hard to prove that $(k_0, l_0) = (1/2, 1/2)$, but we do not need this.

(ii) Now the eigenvalues of $\mathrm{ad}\,\mathbf{h}$ are integral and the bi-weights of $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$ lie in the upper half-plane. The same argument as in [9, 1.12] shows that $\mathrm{ad}\,e_2 : \mathfrak{g}_{\alpha,\beta} \to \mathfrak{g}_{\alpha,\beta+1}$ is injective for all α and $\beta < 0$. (Otherwise we would find an element $0 \neq y \in \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{\nu,\beta}$ with $\nu \geq \alpha$, $\beta < 0$.) Then, by duality, ad e_2 is surjective for $\beta \geq 0$. In particular, $[\mathfrak{p}_2, e_2] = [\mathfrak{g}_{*,\geq 0}, e_2] = \mathfrak{g}_{*,\geq 1} = (\mathfrak{p}_2)^{nil}$.

On the other hand, ad $e_1 : \mathfrak{g}_{p,q} \to \mathfrak{g}_{p+1,q}$ is not injective. Hence ad $e_1 : \mathfrak{g}_{-p-1,-q} \to \mathfrak{g}_{-p,-q}$ is not surjective, i.e., $[\mathfrak{p}_1, e_1] = [\mathfrak{g}_{\geq 0,*}, e_1] \neq \mathfrak{g}_{\geq 1,*} = (\mathfrak{p}_1)^{nil}$. ∎

Recall the notion, due to Lusztig and Spaltenstein, of a *special* nilpotent orbit. Let $\mathcal{N}/G$ be the set of all nilponent orbits in $\mathfrak{g}$. The closure ordering "$\mathcal{O}_1 \preccurlyeq \mathcal{O}_2 \Leftrightarrow \mathcal{O}_1 \subset \overline{\mathcal{O}_2}$" makes $\mathcal{N}/G$ a finite poset. In [19, Chap. III], Spaltenstein studied a duality in $(\mathcal{N}/G, \preccurlyeq)$. He proved that there exists an order-reversing mapping $d : \mathcal{N}/G \to \mathcal{N}/G$ such that

(a) $\mathcal{O} \preccurlyeq d^2(\mathcal{O})$ for all $\mathcal{O} \in \mathcal{N}/G$;

(b) For any Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$, d takes the G-orbit through the regular nilpotent elements in $\mathfrak{l}$ to the Richardson orbit associated to $\mathfrak{l}$.

Such a mapping can uniquely be determined, in a purely combinatorial way, for the classical Lie algebras and for $\mathbf{E}_7$. In the remaining cases, a natural choice among finitely many possibilities can be done. Then one of the definitions of specialness is that $(\mathcal{N}/G)_s := d(\mathcal{N}/G)$ is just the set of special orbits. An important feature of $(\mathcal{N}/G)_s$ is that $d|_{(\mathcal{N}/G)_s}$ is an order-reversing involution. In case of $\mathfrak{sl}_n$, this is the usual conjugation on the set of all partitions of n. With these results at hand, an immediate consequence of the previous theorem is:

2.10. Proposition. *Let* $\mathbf{e}$ *be an almost pn-pair of* $\mathbb{Z}$*-type, as in* (2.9)(ii). *Then* Ge_1 *is not special.*

Proof. In view of Theorem 2.4(1), assertion 2.9(ii) can be restated as $d(Ge_1) = Ge_2$ and $d(Ge_2) \neq Ge_1$. Assume now that Ge_1 is special, i.e., $Ge_1 = d(\mathcal{O})$ for some $\mathcal{O} \in \mathcal{N}/G$. Then $d^2(\mathcal{O}) = Ge_2$ and $Ge_1 = d(\mathcal{O}) = d^3(\mathcal{O}) = d(Ge_2)$, a contradiction! ∎

2.11. Corollary. *There are no almost pn-pairs in* $\mathfrak{sl}_n$.

Proof. (1) By Corollary 2.7, any almost *pn*-pair of non-$\mathbb{Z}$-type yields an inner involution θ such that $\mathfrak{g}^\theta$ is semisimple. But $\mathfrak{sl}_n$ has no such involutions.

(2) Since all nilpotent orbits in $\mathfrak{sl}_n$ are Richardson and hence special, there are no almost *pn*-pairs of $\mathbb{Z}$-type as well. ∎

The following easy result is needed in the proof of (2.13).

2.12. Lemma. *Let* h_1, h_2 *be two commuting semisimple elements. Let* $\mathfrak{n} \subset \mathfrak{g}$ *be a subspace such that* $[h_i, \mathfrak{n}] \subset \mathfrak{n}$ $(i = 1, 2)$ *and* $\mathfrak{n} \cap \mathfrak{z}_\mathfrak{g}(h_1, h_2) = \{0\}$. *Then* $\dim\{(n_1, n_2) \subset \mathfrak{n} \oplus \mathfrak{n} \mid [h_1, n_2] = [h_2, n_1]\} = \dim \mathfrak{n}$.

2.13. Theorem. *Let* $\mathbf{e}$ *be an almost pn-pair. Let* $\mathbf{h}$ *and* $\mathbf{h}' = (h'_1, h'_2)$ *be two associated semisimple pairs. Then there exists* $u \in Z_G(\mathbf{e})^o$ *such that* $u \cdot h_i = h'_i (i = 1, 2)$.

Proof. Let $\mathcal{W}$ be the set of all associated semisimple pairs. Obviously, $Z_G(\mathbf{e})^o \cdot \mathbf{h} \subset \mathcal{W}$. If follows from (2.5) that $Z_G(\mathbf{e})^o$ is unipotent and therefore $Z_G(\mathbf{e})^o \cdot \mathbf{h}$ is closed in $\mathfrak{g} \oplus \mathfrak{g}$. Since $Z_G(\mathbf{e}) \cap Z_G(\mathbf{h})$ is finite, $\dim Z_G(\mathbf{e})^o \cdot \mathbf{h} = \operatorname{rk}\mathfrak{g} + 1$. On the other hand, $h_i' - h_i \in \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})(i = 1, 2)$. Therefore $\mathcal{W} \subset (h_1 + \mathfrak{z}_{\mathfrak{g}}(\mathbf{e}), h_2 + \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})) \cap \mathfrak{E} =: \widetilde{\mathcal{W}}$. Recall that $\mathfrak{E}$ is the commuting variety. Thus, the assertion is equivalent to that $\widetilde{\mathcal{W}}$ is irreducible and of dimension $\operatorname{rk}\mathfrak{g} + 1$. By our previous analysis, $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) = \mathfrak{z}_+ \oplus \langle x \rangle$, where $\mathfrak{z}_+$ is Abelian and $x \in \mathfrak{g}_{p,q}$. In both cases in Theorem 2.5, one has $[x, \mathfrak{z}_+] \subset \mathfrak{z}_+$. Let $(h_1 + n_1 + \nu x, h_2 + n_2 + \tau x) \in \widetilde{\mathcal{W}}$, where $n_i \in \mathfrak{z}_+$ and $\nu, \tau \in \Bbbk$. The x-coordinate of the commutator is equal to $\tau p - \nu q$. Hence $(\nu, \tau) = c(p, q)$ for some $c \in \Bbbk$. Vanishing of the $\mathfrak{z}_+$-component yields the equation

$$[h_1, n_2] - [h_2, n_1] + c(q[n_1, x] - p[n_2, x]) = 0.$$

For a fixed c, it is a system of linear equations for n_1, n_2. Consider the family of linear mappings

$$\nu_c : \mathfrak{z}_+ \oplus \mathfrak{z}_+ \to \mathfrak{z}_+,\ (n_1, n_2) \mapsto [h_1, n_2] - [h_2, n_1] + c(q[n_1, x] - p[n_2, x]).$$

Then $\widetilde{\mathcal{W}} = \bigsqcup_{c\in\Bbbk}(\mathbf{h} + \operatorname{Ker}\nu_c + c(px, qx))$. By Lemma 2.12, $\operatorname{Ker}\nu_0 \simeq \{(n_1, n_2) \mid [h_1, n_2] = [h_2, n_1]\}$ is of dimension $\dim\mathfrak{z}_+ = \operatorname{rk}\mathfrak{g}$. That is, ν_0 is onto. It follows that $\dim\operatorname{Ker}\nu_c = \operatorname{rk}\mathfrak{g}$ for all but finitely many $c \in \Bbbk\setminus\{0\}$. Therefore $\widetilde{\mathcal{W}}$ has a unique irreducible component passing through $\mathbf{h}$ and $\dim_{\mathbf{h}}\widetilde{\mathcal{W}} = \operatorname{rk}\mathfrak{g} + 1$. Let T be the (2-dimensional) subtorus of $Z_G(\mathbf{h})$ corresponding to $\langle h_1, h_2\rangle$. Clearly, $\widetilde{\mathcal{W}}$ is T-stable. By Theorem 2.5, the bi-weights of $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$ lie in an open half-space in $\mathbb{Q} \oplus \mathbb{Q}$. Therefore there exists a 1-parameter subgroup of T which contracts everything in the affine subspace $(h_1 + \mathfrak{z}_{\mathfrak{g}}(\mathbf{e}), h_2 + \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})) \subset \mathfrak{g} \oplus \mathfrak{g}$ to $\mathbf{h}$. Hence $\widetilde{\mathcal{W}}$ is a cone with vertex $\mathbf{h}$. Thus, $\widetilde{\mathcal{W}}$ is irreducible and of dimension $\operatorname{rk}\mathfrak{g} + 1$. ∎

While $Z_G(\mathbf{e})$ is always connected in case of *pn*-pairs (see [9, 3.6]), connectedness in the almost principal case depends on the type.

2.14. Proposition. *Let* $\mathbf{e}$ *be an almost pn-pair. Then*

(1) $Z_G(\mathbf{e})$ *is connected, if* $\mathbf{e}$ *is of* $\mathbb{Z}$*-type*;

(2) $Z_G(\mathbf{e})$ *is disconnected, if* $\mathbf{e}$ *is of non-*$\mathbb{Z}$*-type*.

Proof. From Theorem 2.5, it follows that $Z_G(\mathbf{e})$ is a semi-direct product of the unipotent group $Z_G(\mathbf{e})^o$ and a finite group F.

(1) Take an arbitrary $s \in F$. It is a semisimple element of finite order. Since $s \cdot \mathbf{h}$ is an associated semisimple pair for $\mathbf{e}$, it follows from (2.13) that $s \cdot \mathbf{h} = u \cdot \mathbf{h}$ for some $u \in Z_G(\mathbf{e})^o$. Hence $t := s^{-1}u \in Z_G(\mathbf{h}) = T$. By Theorem 2.9(ii), one may assume that e_2 is Richardson in $(\mathfrak{p}_2)^{nil}$. Since $t \cdot e_1 = e_1$ and e_1 is regular nilpotent in $\mathfrak{l}_2$ (see Theorem 2.4), t is in the

centre of $Z_G(h_2) =: L_2$. Because $\mathfrak{l}_2$ and e_2 generate the parabolic subalgebra $\mathfrak{p}_2$ and $t \cdot e_2 = e_2$, we get $t \cdot z = z$ for any $z \in \mathfrak{p}_2$. This clearly implies that t is in the centre of G. Since G is adjoint, we obtain $s = u = 1 \in G$.

(2) By Corollary 2.7, $Z_G(\mathbf{e})$ contains a semisimple element of order two. ∎

2.15. EXAMPLE. The following demonstrates that the notion of an almost *pn*-pair is not vacuous. Let $\mathfrak{g} = \mathfrak{sp}_{4n} = \mathfrak{sp}(\mathbb{V})$ and let $v_1, \ldots, v_{4n}$ be a basis of $\mathbb{V}$ such that the $\mathfrak{g}$-invariant skew-symmetric form is $B(z, y) = z_1 y_{4n} + \cdots + z_{2n} y_{2n+1} - z_{2n+1} y_{2n} - \cdots - z_{4n} y_1$. Define the operators $e_1, e_2 \in \mathfrak{sp}(\mathbb{V})$ by the formulas

$$e_1(v_j) = v_{j-2}\ (j \geq 2n+1), \qquad e_1(v_j) = -v_{j-2}\ (3 \leq j \leq 2n);$$

$$e_2(v_{2j}) = v_{2j-3}\ (j \geq n+1), \qquad e_2(v_{2j}) = -v_{2j-3}\ (2 \leq j \leq n).$$

If $e_i(v_j)$ is not specified, this means it is equal to zero. The orbit $G \cdot e_1$ (resp. $G \cdot e_2$) corresponds to the partition $(2n, 2n)$ (resp. $(2, \ldots, 2, 1, 1)$). Then $[e_1, e_2] = 0$ and $\mathfrak{z}_{\mathfrak{g}}(e_1, e_2) = \langle e_1, e_1^3, \ldots, e_1^{2n-1}, e_2, e_1^2 e_2, \ldots, e_1^{2n-2} e_2, x \rangle$, where x is the operator taking v_{4n-1} to v_2. Hence $\mathfrak{z}_{\mathfrak{g}}(e_1, e_2)$ is Abelian and its dimension is $2n+1$. An associated semisimple pair consists of $h_1 = \mathrm{diag}\,(t_1, \ldots, t_{4n})$, where $t_{2i} = n + 1 - i$, $t_{2i-1} = n - i$ $(i = 1, \ldots, 2n)$, and $h_2 = \mathrm{diag}\,(1/2, -1/2, 1/2, -1/2, \ldots)$. The bi-weights of $\mathfrak{z}_{\mathfrak{g}}(e_1, e_2)$ are

$$(1, 0), (3, 0), \ldots, (2n-1, 0), (0, 1), (2, 1), \ldots, (2n-2, 1), (2n, -1),$$

where the ordering corresponds to that of basis vectors. Therefore these almost *pn*-pairs are of $\mathbb{Z}$-type. Note that for $n = 1$ we obtain one of the pairs given in (2.8).

Remarks. (1) It is true that the number of G-orbits of almost *pn*-pairs is finite (cf. Theorem 1.5). This follows from the fact that the *pn*-pairs are wonderful in the sense of [15].

(2) All known examples of almost *pn*-pairs occur in $\mathbf{B}_m, \mathbf{C}_m, \mathbf{G}_2$. It can also be shown that there are no almost *pn*-pairs in $\mathbf{F}_4$ and $\mathbf{E}_n$, $n = 6, 7, 8$.

3. RECTANGULAR NILPOTENT PAIRS

Simple instances of *pn*-pairs show that in general $h_n \notin \mathrm{Im}\,(\mathrm{ad}\, e_i)$, hence the pair $\{e_i, h_i\}$ cannot be included in a simple 3-dimensional subalgebra; see Example 4.6(1). However, the theory becomes much simpler, if this can be done. This motivates the following:

3.1. DEFINITION. A pair of nilpotent elements (e_1, e_2) is called *rectangular* whenever there exists an $\mathfrak{sl}_2$-triple, containing e_1, that commutes with e_2.

Recall that an $\mathfrak{sl}_2$-triple $\{e, \tilde{h}, f\}$ satisfies the commutator relations $[\tilde{h}, e] = 2e, [\tilde{h}, f] = -2f, [e, f] = \tilde{h}$. The famous Dynkin–Kostant theory describes conjugacy classes of $\mathfrak{sl}_2$-triples and the structure of $\mathfrak{z}_\mathfrak{g}(e)$ through the use of $\{e, \tilde{h}, f\}$. (See either the original papers [4, 12] or a modern presentation in [3, Chap. 4].) Here are some results of this theory together with related notions. The semisimple element $\tilde{h}$ is called a *characteristic* of e. Given e and $\tilde{h}$, the third member of $\mathfrak{sl}_2$-triple is uniquely determined and $\mathfrak{z}_\mathfrak{g}(e, \tilde{h}) = \mathfrak{z}_\mathfrak{g}(e, \tilde{h}, f)$. Let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ be the $\mathbb{Z}$-grading defined by $\operatorname{ad}\tilde{h}$. Note that $\mathfrak{g}(0)$ is nothing but $\mathfrak{z}_\mathfrak{g}(\tilde{h})$. Then $\mathfrak{z}_\mathfrak{g}(e) = \bigoplus_{i\ge 0} \mathfrak{z}_\mathfrak{g}(e)_i$ and $\mathfrak{z}_\mathfrak{g}(e)_0$ is a maximal reductive subalgebra in $\mathfrak{z}_\mathfrak{g}(e)$. Moreover, $\mathfrak{z}_\mathfrak{g}(e)_0 = \mathfrak{z}_\mathfrak{g}(e, \tilde{h}, f)$. Putting $\mathfrak{z}_\mathfrak{g}(e)_{odd} = \bigoplus_{i\ odd} \mathfrak{z}_\mathfrak{g}(e)_i$ and likewise for "even," we have

$$\mathfrak{z}_\mathfrak{g}(e)_{even} \simeq \mathfrak{g}(0) \qquad \text{and} \qquad \mathfrak{z}_\mathfrak{g}(e)_{odd} \simeq \mathfrak{g}(1) \text{ as } \mathfrak{z}_\mathfrak{g}(e)_0\text{-module.} \tag{3.2}$$

The element e is called *even* whenever all the eigenvalues of $\operatorname{ad}\tilde{h}$ are even. Obviously, e is even if and only if $\mathfrak{g}(1) = 0$ if and only if $\dim \mathfrak{z}_\mathfrak{g}(e) = \mathfrak{z}_\mathfrak{g}(\tilde{h})$. Then the weighted Dynkin diagram of e contains only numbers 0 and 2. An $\mathfrak{sl}_2$-triple containing regular elements is called *principal*; e is regular if and only if it is even and $\mathfrak{z}_\mathfrak{g}(\tilde{h})$ is a Cartan subalgebra. Since all $\mathfrak{sl}_2$-triples containing e are $Z_G(e)^o$-conjugate, the above properties have intrinsic nature.

3.3. LEMMA. (1) *The following conditions are equivalent for a pair* $\mathbf{e}$ *of nilpotent elements*:

(i) (e_1, e_2) *is rectangular*;

(ii) *there exist commuting* $\mathfrak{sl}_2$*-triples* $\{e_1, \tilde{h}_1, f_1\}$ *and* $\{e_2, \tilde{h}_2, f_2\}$.

(2) *If* $\mathbf{e}$ *is rectangular and* $\mathfrak{g} = \oplus\mathfrak{g}_{ij}$ *is the* $\mathbb{Z}^2$*-grading defined by* $(\tilde{h}_1, \tilde{h}_2)$, *then* $\mathfrak{z}_\mathfrak{g}(\mathbf{e})$ *is graded by "positive quadrant."*

(3) *If* $\mathbf{e}$ *is a rectangular* (*almost*) *pn-pair, then we may assume that* $\mathbf{h} = (\tilde{h}_1/2, \tilde{h}_2/2)$.

Proof. (1) Suppose an $\mathfrak{sl}_2$-triple $\{e_1, \tilde{h}_1, f_2\}$ commutes with e_2. Then we may choose an $\mathfrak{sl}_2$-triple containing e_2 inside of the reductive algebra $\mathfrak{z}_\mathfrak{g}(e_1, \tilde{h}_1, f_1)$.

(2) This readily follows from the Dynkin–Kostant theory.

(3) In this case $\tilde{h}_1/2, \tilde{h}_2/2$ satisfy commutator relations (1.2). From (1.3)(4) and (2.13), we then conclude that $(\tilde{h}_1/2, \tilde{h}_2/2)$ is $Z_G(\mathbf{e})^o$-conjugate to $\mathbf{h}$.

Obviously, any rectangular pair is nilpotent in the sense of the Introduction. Because one may use the $\mathfrak{sl}_2$-machinery in the rectangular case, it seems likely that any reasonable question concerning rectangular pairs has an immediate answer. For instance, the following is proved in [7, Theorem 7.1]:

3.4. THEOREM. *Let* $\{e, \tilde{h}, f\}$ *be an* $\mathfrak{sl}_2$*-triple. Then* e *is a member of a rectangular* pn*-pair if and only if* e *is even and a (any) regular nilpotent element in* $\mathfrak{z}_g(e, \tilde{h}, f)$ *is regular in* $\mathfrak{z}_{\mathfrak{g}}(\tilde{h})$ *as well.*

It is not hard to find a similar statement in the almost principal case:

3.5. THEOREM. *Let* $\{e, \tilde{h}, f\}$ *be an* $\mathfrak{sl}_2$*-triple. Then* e *is a member of a rectangular almost* pn*-pair if and only if the following holds*:

(1) *a (any) regular nilpotent element in* $\mathfrak{k} := \mathfrak{z}_g(e, \tilde{h}, f)$ *is also regular in* $\mathfrak{z}_{\mathfrak{g}}(\tilde{h})$;

(2) e *is* not *even (i.e.,* $\mathfrak{g}(1) \neq 0$*) and* $\dim \mathfrak{z}_{\mathfrak{g}(1)}(e') = 1$, *if* $e' \in \mathfrak{k}$ *is regular nilpotent. Under these hypotheses, if* $e' \in \mathfrak{k}$ *is regular nilpotent, then* (e, e') *is an almost* pn*-pair.*

Proof. The proof is much the same as for the previous assertion. Take a nilpotent element $e' \in \mathfrak{k} \subset \mathfrak{g}_0$. It then follows from (3.2) that

$$\dim \mathfrak{z}_{\mathfrak{g}}(e, e') = \dim \mathfrak{z}_{\mathfrak{g}(0)}(e') + \dim \mathfrak{z}_{\mathfrak{g}(1)}(e').$$

Suppose $\dim \mathfrak{z}_{\mathfrak{g}}(e, e') = \operatorname{rk} \mathfrak{g} + 1$. Since $\dim \mathfrak{z}_{\mathfrak{g}(0)}(e') - \operatorname{rk} \mathfrak{g}(0)$ is nonnegative and even, we must have $\dim \mathfrak{z}_{\mathfrak{g}(0)}(e') = \operatorname{rk} \mathfrak{g}(0) = \operatorname{rk} \mathfrak{g}$ and hence $\dim \mathfrak{z}_{\mathfrak{g}(1)}(e') = 1$. Thus e' is regular in $\mathfrak{g}(0)$ and hence in $\mathfrak{k}$. This argument can reversed.

Note that any rectangular almost pn-pair is necessarily of non-$\mathbb{Z}$-type and that condition 2 can be restated as follows: $\mathfrak{g}(1)$ is a simple $\langle e', \tilde{h}', f' \rangle$-module, if $\{e', \tilde{h}', f'\} \subset \mathfrak{k}$ is a principal $\mathfrak{sl}_2$-triple.

3.6. EXAMPLE. Let $\mathfrak{g} = \mathfrak{sp}_{2n}$. For $0 < k < n$, consider the symmetric subalgebra $\mathfrak{sp}_{2k} \oplus \mathfrak{sp}_{2n-2k} \subset \mathfrak{sp}_{2n}$. Let e_1 (resp. e_2) be a regular nilpotent element in $\mathfrak{sp}_{2k}$ (resp. $\mathfrak{sp}_{2n-2k}$). Then (e_1, e_2) is a rectangular almost pn-pair.

4. DUAL PAIRS ASSOCIATED WITH NILPOTENT PAIRS

Let $\mathfrak{a}, \mathfrak{a}' \subset \mathfrak{g}$ be two subalgebras. Following Howe, we say that $\mathfrak{a}$ and $\mathfrak{a}'$ form a *dual pair*, if $\mathfrak{a}' = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ and vice versa. A *reductive dual pair* is a dual pair $(\mathfrak{a}, \mathfrak{a}')$ such that each of $\mathfrak{a}, \mathfrak{a}'$ is reductive. It is clear how to define a dual pair of groups. In the group setting the problem is however more subtle, because of connectedness questions. A classification of reductive dual pairs in reductive Lie algebras was obtained by Rubenthaler, see [18]. In the spirit of Dynkin, he introduced the notion of an "S-irreducible" dual pair and described all such pairs in the simple Lie algebras. The general classification is then reduced to that for S-irreducible pairs.

4.1. DEFINITION. A dual pair $(\mathfrak{a}, \mathfrak{a}')$ is called *S-irreducible*, if $\mathfrak{a} + \mathfrak{a}'$ is an S-subalgebra in the sense of Dynkin; i.e., it is not contained in a proper regular[1] subalgebra of $\mathfrak{g}$.

Let $\mathbf{e} \in \mathfrak{E}$ be a nilpotent pair and $\mathbf{h}$ a semisimple pair satisfying Eq. (1.2). Then the quadruple $(\mathbf{e}, \mathbf{h})$ is said to be *quasi-commutative*. By definition, put $\mathfrak{k}_i = \mathfrak{z}_\mathfrak{g}(e_1, h_i)$, $i = 1, 2$. Our aim is to demonstrate a sufficient condition for $(\mathfrak{k}_1, \mathfrak{k}_2)$ to be a dual pair. Note that $e_2, h_2 \in \mathfrak{k}_1$ and $e_1, h_1 \in \mathfrak{k}_2$. Consider the bi-grading of $\mathfrak{g}$ determined by $\mathbf{h}$: $\mathfrak{g} = \bigoplus_{i,j} \mathfrak{g}_{i,j}$, where (i, j) runs over a finite subset of $\Bbbk \times \Bbbk$ including (0,0), (1,0), and (0,1). The restriction of this bi-grading to either $\mathfrak{k}_1$ or $\mathfrak{k}_2$ gives ordinary gradings $\mathfrak{k}_1 = \bigoplus_j (\mathfrak{k}_1)_j$ and $\mathfrak{k}_2 = \bigoplus_i (\mathfrak{k}_2)_i$, where $(\mathfrak{k}_1)_j \subset \mathfrak{g}_{0,j}$ and $(\mathfrak{k}_2)_i \subset \mathfrak{g}_{i,0}$.

4.2. PROPOSITION. *Let (e_1, e_2, h_1, h_2) be a quasi-commutative quadruple. Suppose* $\dim \mathfrak{z}_\mathfrak{g}(e_1, h_1, e_2) = \dim \mathfrak{z}_\mathfrak{g}(e_1, h_1, h_2)$. *Then*

(i) *the grading of $\mathfrak{k}_1$ is actually a $\mathbb{Z}$-grading, i.e., the eigenvalues of* $\operatorname{ad} h_2$ *on $\mathfrak{k}_1$ are integral. Furthermore, the centralizer $\mathfrak{z}_{\mathfrak{k}_1}(e_2) = \mathfrak{z}_\mathfrak{g}(e_1, h_1, e_2)$ is nonnegatively graded*;

(ii) $(\operatorname{ad} e_2)_j : (\mathfrak{k}_1)_j \to (\mathfrak{k}_1)_{j+1}$ *is onto for* $j \geq 0$.

(*Of course, this has the symmetric analogue, where indices* 1 *and* 2 *are interchanged*.)

Proof. (i) The space $\mathfrak{z}_\mathfrak{g}(e_1, h_1, h_2) = \mathfrak{z}_{\mathfrak{k}_1}(h_2) = (\mathfrak{k}_1)_0$ possesses the e_2-filtration and $\lim_{e_2} \mathfrak{z}_{\mathfrak{k}_1}(h_2) \subset \mathfrak{z}_{\mathfrak{k}_1}(e_2) = \mathfrak{z}_\mathfrak{g}(e_1, h_1, e_2)$. It follows from the definition of e_2-limit that $\lim_{e_2} \mathfrak{z}_{\mathfrak{k}_1}(h_2) \subset \bigoplus_{j \in \mathbb{Z}_{\geq 0}} (\mathfrak{k}_1)_j$. Furthermore, $\dim(\lim_{e_2} \mathfrak{z}_{\mathfrak{k}_1}(h_2)) = \dim \mathfrak{z}_{\mathfrak{k}_1}(h_2)$. Under our assumption, this means that $\lim_{e_2} \mathfrak{z}_\mathfrak{g}(e_1, h_1, h_2) = \mathfrak{z}_\mathfrak{g}(e_1, h_1, e_2)$ and the eigenvalues of $\operatorname{ad} h_2$ on $\mathfrak{z}_\mathfrak{g}(e_1, h_1, e_2)$ are nonnegative integers. Assume that $(\mathfrak{k}_1)_j \neq 0$ for some $j \in \Bbbk \setminus \mathbb{Z}$. Since $(\mathfrak{k}_1)_j$ is killed by some power of $\operatorname{ad} e_2$, we have $j + c$ is an eigenvalue of $\operatorname{ad} h_2$ on $\mathfrak{z}_\mathfrak{g}(e_1, h_1, e_2)$ for some $c \in \mathbb{Z}_{\geq 0}$, which is impossible. Thus, all the eigenvalues of $\operatorname{ad} h_2$ on $\mathfrak{k}_1$ must be integral.

(ii) Set $(\mathfrak{k}_1)_{\geq j} = \bigoplus_{i \geq j} (\mathfrak{k}_1)_i$ and consider the linear map $(\operatorname{ad} e_2)_{\geq 0} : (\mathfrak{k}_1)_{\geq 0} \to (\mathfrak{k}_1)_{\geq 1}$. By part (i), we have $\operatorname{Ker} (\operatorname{ad} e_2)_{\geq 0} = \mathfrak{z}_{\mathfrak{k}_1}(e_2)$. That is, dimension of the kernel is $\dim \mathfrak{z}_\mathfrak{g}(e_1, h_1, h_2) = \dim(\mathfrak{k}_1)_0$. Thus, $(\operatorname{ad} e_2)_{\geq 0}$ must be onto. ∎

4.3. THEOREM. *Suppose a quasi-commutative quadruple* (e_1, e_2, h_2, h_2) *satisfies the conditions*

(1) $[\mathfrak{z}_\mathfrak{g}(e_1, h_1, h_2), \mathfrak{z}_\mathfrak{g}(e_1, h_1, h_2)] = 0$,

(2) $\dim \mathfrak{z}_\mathfrak{g}(e_1, h_1, e_2) = \dim \mathfrak{z}_\mathfrak{g}(e_1, h_1, h_2)$,

[1]A subalgebra of $\mathfrak{g}$ is called *regular* whenever its normalizer contains a Cartan subalgebra.

(3) $\dim \mathfrak{z}_{\mathfrak{g}}(e_2, h_2, e_1) = \dim \mathfrak{z}_{\mathfrak{g}}(e_2, h_2, h_1)$.

Then $(\mathfrak{k}_1, \mathfrak{k}_2)$ *is a dual pair in* $\mathfrak{g}$.

Proof. Since $e_1, h_1 \in \mathfrak{k}_2$ and $e_2, h_2 \in \mathfrak{k}_1$, we have $\mathfrak{k}_1 \supset \mathfrak{z}_{\mathfrak{g}}(\mathfrak{k}_2)$ and $\mathfrak{k}_2 \supset \mathfrak{z}_{\mathfrak{g}}(\mathfrak{k}_1)$. That is, the property of being a dual pair is equivalent to that $[\mathfrak{k}_1, \mathfrak{k}_2] = 0$.

We first prove that $(\mathfrak{k}_2)_{\ge 0}$ commutes with $(\mathfrak{k}_1)_{\ge 0}$. Condition (1) says that $(\mathfrak{k}_1)_0$ commutes with $(\mathfrak{k}_2)_0$. Therefore the subalgebras generated by $\{(\mathfrak{k}_1)_0, e_2\}$ and $\{(\mathfrak{k}_2)_0, e_1\}$ commute. By (4.2)(ii), the subalgebra generated by $(\mathfrak{k}_1)_0$ and e_2 is $(\mathfrak{k}_1)_{\ge 0}$. Under condition (3), the same applies to $\mathfrak{k}_2$ in place of $\mathfrak{k}_1$. That is, the subalgebra generated by $(\mathfrak{k}_2)_0$ and e_1 is $(\mathfrak{k}_2)_{\ge 0}$.

Consider the set $\mathfrak{M} = \{[x, y] \mid x \in \mathfrak{k}_1, y \in \mathfrak{k}_2\}$. It is immediate that $\mathfrak{M}$ is ad e_i- and ad h_i-stable ($i = 1, 2$). Assume that $\mathfrak{M} \neq \{0\}$, that is, $[x, y] \neq 0$ for some $x \in (\mathfrak{k}_1)_j$ and $y \in (\mathfrak{k}_2)_i$. Successively applying ad e_1 and ad e_2 to $[x, y]$, we eventually obtain a nonzero commutator $[x', y']$ with $x' \in (\mathfrak{k}_1)_{j'}$ and $y' \in (\mathfrak{k}_2)_{i'}$ such that $x' \in \mathfrak{z}_{\mathfrak{k}_1}(e_2)$ and $y' \in \mathfrak{z}_{\mathfrak{k}_2}(e_1)$. It then follows from (4.2)(i) that $i' \ge 0$ and $j' \ge 0$. Thus, $x' \in (\mathfrak{k}_1)_{\ge 0}$, $y' \in (\mathfrak{k}_2)_{\ge 0}$ and one must have $[x', y'] = 0$. This contradiction proves that $\mathfrak{M} = \{0\}$. ∎

Given $\mathbf{e}$, it may *a priori* happen that there are several non-equivalent choices of $\mathbf{h}$ such that $\mathbf{h}$ satisfies Eq. (1.2) and the hypotheses in (4.3). Fortunately, this question does not arise for (almost) *pn*-pairs. We may even give a more precise statement in these cases. Set $K_i := Z_G(e_i, h_i)$, $i = 1, 2$. These groups are not necessarily connected, but Lie $K_i = \mathfrak{k}_i$.

4.4. THEOREM. *Suppose* $\mathbf{e}$ *is either a pn-pair or an almost pn-pair and* $\mathbf{h}$ *is an associated semisimple pair. Then*

(1) $(\mathfrak{k}_1, \mathfrak{k}_2)$ *is a dual pair. The centre of* $\mathfrak{k}_i$ $(i = 1, 2)$ *is trivial*;

(2) *This dual pair is reductive if and only if the pair* $\mathbf{e}$ *is rectangular*;

(3) $K_2 = Z_G(K_1^o)$ *and* $K_1 = Z_G(K_2^o)$;

(4) *If* $\mathbf{e}$ *is a rectangular pn-pair, then* $(\mathfrak{k}_1, \mathfrak{k}_2)$ *is S-irreducible*.

Proof. (1) Since $\mathfrak{z}_{\mathfrak{g}}(\mathbf{h})$ is Abelian in both cases, hypothesis (1) in (4.3) is satisfied. By Theorem 2.4(3), the other hypotheses are satisfied, too. The centre of $\mathfrak{k}_i$ is equal to $\mathfrak{k}_1 \cap \mathfrak{k}_2 = \mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) \cap \mathfrak{z}_{\mathfrak{g}}(\mathbf{h}) = \{0\}$.

(2) Clearly, $\mathfrak{k}_1$ is reductive if and only if $\mathfrak{k}_2$ is reductive. If $\mathfrak{k}_2$ is reductive, the it contains a suitable $\mathfrak{sl}_2$-triple together with e_1. The opposite implication follows from Lemma 3.3(3).

(3) By symmetry, it suffices to prove the first equality. Since $e_2, h_2 \in \mathfrak{k}_1$, we have

$$K_2 = Z_G(e_2, h_2) \supset Z_G(\mathfrak{k}_1) = Z_G(K_1^o).$$

In the proof of the opposite inclusion we use the relation $Z_G(\mathfrak{k}_1) \supset K_2^o$ proved in the first part. Let $s \in K_2$ be an arbitrary element. One has to prove that $s \cdot x = x$ for all $x \in \mathfrak{k}_1$. By (2.4)(4), $(\mathfrak{k})_0$ is just the centre of $\mathfrak{l}_2$. Because K_2 lies in the *connected* group $L_2 := Z_G(h_2)$, it commutes with $(\mathfrak{k}_1)_0$. By the very definition, K_2 commutes with e_2. Thus, it commutes with $(\mathfrak{k}_1)_{\ge 0}$. It then follows from (4.2)(i) that $s \cdot x = x$ for $x \in \mathfrak{z}_{\mathfrak{k}_1}(e_2)$. Consider $\mathcal{Y} := \{y \in \mathfrak{k}_1 \mid s \cdot y \neq y\}$. Suppose $\mathcal{Y} \neq \emptyset$. Choose an element $y_0 \in \mathcal{Y}$ which is killed by the *least* possible power, say p, of $\operatorname{ad} e_2$. That is, $(\operatorname{ad} e_2)^p y_0 \neq 0$ and $(\operatorname{ad} e_2)^{p+1} y_0 = 0$. Since $\mathcal{Y} \cap \mathfrak{z}_{\mathfrak{k}_1}(e_2) = \emptyset$, we have $p \ge 1$. Then $(\operatorname{ad} e_2)^p y_0 \in \mathfrak{z}_{\mathfrak{k}_1}(e_2) \subset (\mathfrak{k}_1)_{\ge 0}$ and hence $(\operatorname{ad} e_2)^p y_0 = s \cdot ((\operatorname{ad} e_2)^p y_0) = (\operatorname{ad} e_2)^p (s \cdot y_0)$. In other words, $(\operatorname{ad} e_2)^p (s \cdot y_0 - y_0) = 0$. It follows that $y_1 := s \cdot y_0 - y_0 \notin \mathcal{Y}$ and $s \cdot y_1 = y_1$. Therefore $s^n.y_0 = y_0 + ny_1$ for all $n \in \mathbb{N}$. However, we have $s^n \in K_2^0 \subset Z_G(\mathfrak{k}_1)$ for some $n > 0$ and therefore y_1 must be zero. This contradiction proves that $\mathcal{Y} = \emptyset$.

(4) It follows from parts (1) and (2) that $\mathfrak{k}_1 + \mathfrak{k}_2$ is semisimple. Assume that $\mathfrak{k}_1 + \mathfrak{k}_2 \subset \mathfrak{g}^{(1)}$, where $\mathfrak{g}^{(1)}$ is a proper regular subalgebra of $\mathfrak{g}$. Then there exists a maximal semisimple subalgebra $\mathfrak{g}^{(2)} \subset \mathfrak{g}^{(1)}$ such that $\mathfrak{k}_1 + \mathfrak{k}_2 \subset \mathfrak{g}^{(2)}$. This $\mathfrak{g}^{(2)}$ is a regular subalgebra of $\mathfrak{g}$, too. According to the description of *maximal* regular semisimple subalgebras of $\mathfrak{g}$, $\mathfrak{g}^{(2)}$ is contained in the fixed-point subalgebra of some element $s \in G$ of prime order ($s \neq 1$). Then $s \in Z_G(\mathfrak{k}_1 + \mathfrak{k}_2)$. However, $Z_G(\mathfrak{k}_1 + \mathfrak{k}_2) = Z_G(\mathbf{e}) \cap Z_G(\mathbf{h}) = \{1\}$, since $Z_G(\mathbf{e})$ is connected and unipotent [9, 3.6]. ∎

4.5. Corollary. *If K_1 and K_2 are connected, then (K_1, K_2) is a dual pair of groups in G.*

Observe that the properties of (almost) *pn*-pairs were not used in full strength in the above proofs. This suggests that the notion of an (almost) *pn*-pair could be weakened so that the conclusion of Theorem 4.3 remained valid. A possible generalization in the rectangular case is discussed in the next section.

Remarks. (1) Arguing as in the proof of part (4) and using Proposition 2.14(1), one proves that if $\mathbf{e}$ is either a *pn*-pair or an almost *pn*-pair of $\mathbb{Z}$-type, then $\mathfrak{k}_1 + \mathfrak{k}_2$ is not contained in a proper *reductive* regular subalgebra of $\mathfrak{g}$. However, $\mathfrak{k}_1 + \mathfrak{k}_2$ may lie in a proper parabolic subalgebra for a non-rectangular *pn*-pair $\mathbf{e}$; see Example 4.6(1).

(2) In the rectangular case, K_i is a maximal reductive subgroup of $Z_G(e_i)$ and therefore $K_i/K_i^o \simeq Z_G(e_i)/Z_G(e_i)^o$. This group is known for all nilpotent orbits. The description is due to Springer and Steinberg [20] for the classical Lie algebras and due to Alekseevskii [1] for the exceptional ones.

4.6. EXAMPLES. We give several illustrations to Theorem 4.4.

(1) The simplest non-rectangular pn-pair occurs in $\mathfrak{g} = \mathfrak{sl}_3$. Let

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\mathfrak{z}_{\mathfrak{g}}(e_1, e_2) = \langle e_1, e_2 \rangle, h_1 = \mathrm{diag}(2/3, -1/3, -1/3)$, and $h_2 = \mathrm{diag}(-1/3, 2/3, -1/3)$, whence $\mathfrak{k}_1 = \langle e_2, h_2 \rangle$ and $\mathfrak{k}_2 = \langle e_1, h_1 \rangle$. It is clearly visible in this case that, for instance, $\tilde{h}_1 = \mathrm{diag}(1, 0, -1)$ and $\tilde{h}_1 \neq 2h_1$.

(2) In (2.15), a series of almost pn-pairs in $\mathfrak{sp}_{4n}$ is described. In that case $\mathfrak{k}_1 = \langle e_2, h_2 \rangle$, while $\dim \mathfrak{k}_2 = 2n^2 - n + 1$. The Levi decomposition of $\mathfrak{k}_2$ is as follows: $\mathfrak{k}_2^{red} \simeq \mathfrak{so}_{2n-1} \oplus \Bbbk$; $\mathfrak{l}_2^{nil}$ is Abelian and affords the simplest representation of $\mathfrak{so}_{2n-1}$.

(3) The rectangular pn-pairs in simple Lie algebras were classified in [7]. For instance, there are 4 such pairs in $\mathbf{E}_7$ and 1 pair in either of $\mathbf{F}_4, \mathbf{E}_6, \mathbf{E}_8$. The corresponding S-irreducible reductive dual pairs are

$$(\mathbf{G}_2, \mathbf{A}_1) \text{ in } \mathbf{F}_4; \qquad (\mathbf{G}_2, \mathbf{A}_2) \text{ in } \mathbf{E}_6; \qquad (\mathbf{G}_2, \mathbf{F}_4) \text{ in } \mathbf{E}_8;$$
$$(\mathbf{G}_2, \mathbf{A}_1), (\mathbf{G}_2, \mathbf{C}_3), (\mathbf{F}_4, \mathbf{A}_1), (\mathbf{A}_1, \mathbf{A}_1) \text{ in } \mathbf{E}_7;$$

By [1], the groups $Z_G(e_i)$ are connected (in the adjoint group!) for all nilpotent orbits occurring in this situation. It then follows from Corollary 4.5 that the above two lines represent also the dual pairs of connected groups in the respective adjoint group G.

Remark. A classification of *reductive* dual pairs in the Lie algebraic setting was obtained by Rubenthaler. However, the "tableau récapitulatif" in [18, p. 70] contains several inaccuracies. Below we use Rubenthaler's notation. Each time an orthogonal Lie algebra $o(m)$ occurs as a factor, one has either to require that $m \neq 2$, or to replace the given dual pair by a correct one. This refers to the following possibilities in that table:

$$\mathbf{B}_n : 2(n - kp) + 1 - p = 2; \quad \mathbf{C}_n\ 2) : p = 2; \quad \mathbf{D}_n\ 1) : 2n - 2kp - p = 2.$$

For instance, if $p = 2$ for $\mathbf{C}_n$, then the dual pair must be $(\mathfrak{gl}(k+1), o(2))$, not $(\mathfrak{sp}(k+1), o(2))$. However, unlike the case $p \neq 2$, this dual pair is not S-irreducible.

It is also interesting to observe that Rubenthaler's "diagrammes en dualité" correspond exactly to the dual pairs arising from the rectangular pn-pairs.

5. SEMI-PRINCIPAL PAIRS

We shall say that a subalgebra $a \subset \mathfrak{g}$ is *reflexive* whenever $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}))$. This is tantamount to saying that $(\mathfrak{a}, \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}))$ is a dual pair. Obviously, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ is reflexive for *any* algebra $a \subset \mathfrak{g}$. In particular, the centralizer of any $\mathfrak{sl}_2$-triple is reflexive. It is however interesting to find out those $\mathfrak{sl}_2$-triples whose double centralizer has some natural description, e.g., is again the centralizer of an $\mathfrak{sl}_2$-triple. For instance, in the dual pair associated to a rectangular *pn*-pair, both algebras $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are the centralizers of $\mathfrak{sl}_2$-triples. Moreover, $\mathfrak{k}_1$ can be described as the centralizer in $\mathfrak{g}$ of a principal $\mathfrak{sl}_2$-triple in $\mathfrak{k}_2$. This fact and the criterion given in (3.4) provide some motivation for the following definition. Recall that a nilpotent element e in a reductive Lie algebra $\mathfrak{l}$ is called *distinguished* whenever any semisimple element of $\mathfrak{z}_{\mathfrak{l}}(e)$ lies in the center of $\mathfrak{l}$.

5.1. DEFINITION. A pair of nilpotent elements $(e_1, e_2) \in \mathfrak{g} \times \mathfrak{g}$ is called *semi-principal rectangular* (=*spr*-pair), if the following holds:

(i) there exist commuting $\mathfrak{sl}_2$-triples $\{e_1, \tilde{h}_1, f_1\}$ and $\{e_2, \tilde{h}_2, f_2\}$ (rectangularity);

(ii) e_1 is distinguished in $\mathfrak{z}_{\mathfrak{g}}(\tilde{h}_2) =: \mathfrak{l}_2$;

(iii) e_2 is even in $\mathfrak{z}_{\mathfrak{g}}(e_1, \tilde{h}_1) =: \mathfrak{k}_1$.

Define the subalgebras $\mathfrak{k}_i, \mathfrak{l}_i, \mathfrak{c}_i$, and the subgroups $K_i (i = 1, 2)$ as above, with $\tilde{h}_i$ in place of h_i. The meaning of condition (ii) is that e_1 should be a distinguished element in $\mathfrak{k}_2$ which remains distinguished as an element of $\mathfrak{l}_2$. Note that (e_2, e_1) need not be an *spr*-pair and e_2 need not be even in $\mathfrak{g}$. But if e_2 is even in $\mathfrak{g}$, it is also even in $\mathfrak{k}_1$. It follows from Theorem 2.4 that each rectangular (almost) *pn*-pair is an *spr*-pair.

5.2. THEOREM. *Let (e_1, e_2) be an spr-pair. Then*

(i) $\mathfrak{z}_g(e_1, \tilde{h}_1, \tilde{h}_2) = \mathfrak{c}_2$;

(ii) $\mathfrak{c}_2$ *is a Cartan subalgebra and* e_2 *is a regular nilpotent element in* $\mathfrak{k}_1$;

(iii) $(\mathfrak{k}_1, \mathfrak{k}_2)$ *is a reductive dual pair in* $\mathfrak{g}$.

Proof. (i) The argument is close to that in Theorem 2.4(4). By definition, $\mathfrak{z}_{\mathfrak{g}}(e_1, \tilde{h}_2) = \mathfrak{z}_{\mathfrak{l}_2}(e_1) = \mathfrak{c}_2 \oplus \mathfrak{n}$, where $\mathfrak{n} \subset [\mathfrak{l}_2, \mathfrak{l}_2]$ consists of nilpotent elements. As $e_1, \tilde{h}_1 \in \mathfrak{l}_2$, we have $\mathfrak{z}_{\mathfrak{g}}(e_1, \tilde{h}_1, \tilde{h}_2) \supset \mathfrak{c}_2$. Thus, $\mathfrak{c}_2 \subset \mathfrak{z}_{\mathfrak{g}}(e_1, \tilde{h}_1, \tilde{h}_2) \subset \mathfrak{z}_{\mathfrak{g}}(e_1, \tilde{h}_2) = \mathfrak{c}_2 \oplus \mathfrak{n}$. In the rectangular case, $\mathfrak{k}_1 = \mathfrak{z}_{\mathfrak{g}}(e_1, \tilde{h}_1, f_1)$ is reductive. Hence $\mathfrak{z}_{\mathfrak{g}}(e_1, \tilde{h}_1, \tilde{h}_2) = \mathfrak{z}_{\mathfrak{k}_1}(\tilde{h}_2)$ is reductive, too. This clearly forces that $\mathfrak{z}_{\mathfrak{k}_1}(\tilde{h}_2) = \mathfrak{c}_2$.

(ii) Since $\tilde{h}_2$ is semisimple, the previous equality means $\mathfrak{c}_2$ is a Cartan subalgebra in $\mathfrak{k}_1$. Because e_2 is assumed to be even in $\mathfrak{k}_1$, the $\mathfrak{sl}_2$-triple $\{e_2, \tilde{h}_2, f_2\}$ is principal in $\mathfrak{k}_1$.

(iii) As in the proof of (4.3), it is enough to prove that $[\mathfrak{k}_1, \mathfrak{k}_2] = 0$. It follows from (ii) that $\mathfrak{k}_1$ is generated by $\mathfrak{c}_2$, e_2, and f_2 as a Lie algebra. Since $\mathfrak{k}_2 \subset \mathfrak{l}_2$ and $\mathfrak{c}_2$ is the centre of $\mathfrak{l}_2$, we see that $\mathfrak{k}_2$ commutes with $\mathfrak{f}_2$, e_2, and f_2, as required. ∎

A procedure of searching *spr*-pairs is as follows. Let $\{e_2, \tilde{h}_2, f_2\}$ be an $\mathfrak{sl}_2$-triple. First, one has to explicitly determine $\mathfrak{k}_2, \mathfrak{l}_2$, and the embedding $\mathfrak{k}_2 \hookrightarrow \mathfrak{l}_2$. The next step is to find a distinguished element $e_1 \in \mathfrak{k}_2$ which remains distinguished in $\mathfrak{l}_2$. In the case e_2 being even in $\mathfrak{g}$, this is enough. Otherwise, one needs to check that e_2 is even in $\mathfrak{k}_1$. The first candidate for e_1 is a regular nilpotent element in $\mathfrak{k}_2$. However, it can happen that regular nilpotent elements in $\mathfrak{k}_2$ fail to be distinguished in $\mathfrak{l}_2$, while elements of a smaller orbit in $\mathfrak{k}_2$ satisfy our requirements. Furthermore, it can happen that there are several such orbits in $\mathfrak{k}_2$. This means that we may find *spr*-pairs (e', e_2) and (e'', e_2) such that e' and e'' lie in different G-orbits in $\mathfrak{g}$; see, e.g., Example 3 below. Nevertheless, Theorem 5.2(iii) guarantees that reductive parts of the centralizers of e' and e'' will coincide—they are just equal to $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{k}_2)$.

5.3. Examples. We refer to [3, Chaps. 4 and 8] for standard facts on weighted Dynkin diagrams and labelling of nilpotent orbits.

(1) Let $\mathcal{O}_2$ be the nilpotent orbit in $\mathfrak{g} = \mathbf{E}_7$, labelled by $2\mathbf{A}_2$. The weighted Dynkin diagram of $\mathcal{O}_2$ is

$$\left(\begin{array}{c} 0\text{–}2\text{–}0\text{–}\underset{\displaystyle 0}{\overset{}{0}}\text{–}0\text{–}0 \end{array} \right).$$

Therefore $\mathfrak{l}_2 \simeq \mathfrak{so}_{10} \oplus \Bbbk$ and one finds in [5] that $\mathfrak{k}_2 \simeq \mathbf{G}_2 \oplus \mathfrak{sl}_2$. The embedding $\mathfrak{k}_2 \hookrightarrow [\mathfrak{l}_2, \mathfrak{l}_2]$ is as follows: $[\mathfrak{l}_2, \mathfrak{l}_2]$ has the tautological 12-dimensional module $\mathbb{V}_{12} = \mathbb{V}_{10} \oplus \mathbb{V}_2$. Then $\mathbb{V}_{10}|_{\mathfrak{k}_2}$ = (7-dim repr. $\mathbf{G}_2$) $\oplus$ ad $\mathfrak{sl}_2$ and $\mathbb{V}_2|_{\mathfrak{k}_2}$=(2-dim repr. $\mathfrak{sl}_2$). Let e_1 be a regular nilpotent element in $\mathfrak{k}_2$. The above description of embedding shows that e_1 is distinguished as an element of $\mathfrak{l}_2$. More precisely, $e_1 = e' + e''$, where $e' \in \mathfrak{so}_{10}$ corresponds to the partition (7, 3) and $e'' \in \mathfrak{sl}_2$ is regular. (The distinguished nilpotent orbits in $\mathfrak{so}_N$ correspond bijectively to the partitions of N into distinct odd parts.) Since e_2 is even in $\mathfrak{g}$, it is also even in $\mathfrak{k}_1$. Hence a dual pair comes up and it remains to realize what $\mathfrak{k}_1$ is. The orbit of e' in $\mathfrak{so}_{10}$ is subregular and is labelled by $\mathbf{D}_5(a_1)$. Therefore the label of $\mathcal{O}_1 = G \cdot e_1$ is $\mathbf{D}_5(a_1) + \mathbf{A}_1$. Now, one finds in the list of weighted Dynkin diagrams for $\mathbf{E}_7$ that the diagram corresponding to $\mathcal{O}_1$ is

$$\left(\begin{array}{c} 0\text{–}0\text{–}2\text{–}\underset{\displaystyle 0}{\overset{}{0}}\text{–}0\text{–}2 \end{array} \right).$$

Hence $\mathfrak{l}_1 \simeq \mathfrak{sl}_4 \oplus \mathfrak{sl}_3 \oplus \Bbbk^2$ and, by [5], $\mathfrak{k}_1 \simeq \mathfrak{sl}_2$. Thus, the dual pair is $(\mathbf{A}_1, \mathbf{G}_2 \oplus \mathbf{A}_1)$. By [1], the groups $Z_G(e_i)$ $(i = 1, 2)$ are connected here. The connected groups $K_i (i = 1, 2)$ form therefore a dual pair of groups.

(2) The members of *spr*-pairs are not necessarily even. Let $\mathcal{O}_2$ be the nilpotent orbit in $\mathfrak{g} = \mathbf{E}_7$, labelled by $2\mathbf{A}_1$. Its weighted Dynkin diagram is

$$\left(\begin{array}{c}0\text{–}1\text{–}0\text{–}\overset{}{0}\text{–}0\text{–}0\\ 0\end{array}\right).$$

Here $\mathfrak{l}_2 \simeq \mathfrak{so}_{10} \oplus \mathfrak{sl}_2 \oplus \Bbbk$ and $\mathfrak{k}_2 \simeq \mathfrak{so}_9 \oplus \mathfrak{sl}_2$ with the obvious embedding. Therefore a regular nilpotent element $e_1 \in \mathfrak{k}_2$ is also regular in $\mathfrak{l}_2$. The label of $G \cdot e_1$ is $\mathbf{D}_5 + \mathbf{A}_1$ and the weighted Dynkin diagram is

$$\left(\begin{array}{c}0\text{–}1\text{–}1\text{–}0\text{–}1\text{–}2\\ 1\end{array}\right).$$

Then one finds $\mathfrak{k}_1 \simeq \mathfrak{sl}_2$. Hence e_2 is certainly even in $\mathfrak{k}_1$ and we obtain an *spr*-pair. The corresponding reductive dual pair is $(\mathbf{B}_4 + \mathbf{A}_1, \mathbf{A}_1)$. It is not S-irreducible, since it is contained in the semisimple subalgebra of maximal rank $\mathbf{D}_6 + \mathbf{A}_1 \subset \mathbf{E}_7$.

(3) Let $\mathfrak{g} = \mathfrak{sp}_{2N}$ and let $G \cdot e_2$ be the orbit corresponding to the partition

$$(\underbrace{m, \dots, m}_{2n}, \underbrace{1, \dots, 1}_{2l}),$$

where m is odd, $m \neq 1$, and $N = nm + l$. Since the parts have the same parity, e_2 is even. Making use of the weighted Dynkin diagram, one finds that $[\mathfrak{l}_2, \mathfrak{l}_2] = (\mathfrak{sl}_{2n})^{(m-1)/2} \oplus \mathfrak{sp}_{2(n+l)}$ and $\mathfrak{k}_2 = \mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2l}$. The embedding $\mathfrak{k}_2 \hookrightarrow \mathfrak{l}_2$ is determined by the maps $\nu_1 : \mathfrak{k}_2 \to \mathfrak{sp}_{2(n+l)}$ and $\nu_2 : \mathfrak{k}_2 \to (\mathfrak{sl}_{2n})^{(m-1)/2}$. Here ν_1 is the direct sum of matrices and ν_2 corresponds to the diagonal embedding $\mathfrak{sp}_{2n} \hookrightarrow \mathfrak{sl}_{2n} \xrightarrow{\Delta} (\mathfrak{sl}_{2n})^{(m-1)/2}$. Let us realize which elements $e_1 = e' + e''(e' \in \mathfrak{sp}_{2n}, e'' \in \mathfrak{sp}_{2l})$ remain distinguished in $\mathfrak{l}_2$. Since the only distinguished elements in $\mathfrak{sl}_N$ are the regular ones, e' must be regular in $\mathfrak{sp}_{2n}$. This already guarantees us that $\nu_2(e_1)$ is regular in $(\mathfrak{sl}_{2n})^{(m-1)/2}$. The orbits of distinguished elements in $\mathfrak{sp}_{2N}$ correspond bijectively to the partitions of $2N$ into even unequal parts. Since e' is already chosen, the partition of $\nu_1(e_1)$ has a part equal to $2n$. Thus, e'' must be a distinguished element in $\mathfrak{sp}_{2l}$ whose partition contains *no* parts equal to $2n$. For instance, one may take e'' to be regular whenever $n \neq l$. In case $n = l$, it is easy to see that a required partition exists if and only if $n \notin \{1, 2\}$. Thus, *spr*-pairs come up if and only if $(n, l) \notin \{(1, 1), (2, 2)\}$ and the choice of e_1 is not unique in general. The partition of e_1 can be either of

$$(\underbrace{2n, \dots, 2n}_{m}, 2l_1, \dots, 2l_t),$$

where $\sum_i l_i = l$, $l_i \neq l_j$, and $l_i \neq n$. For all such choices, $\mathfrak{k}_1$ is equal to $\mathfrak{so}_m$ and we obtain the dual pair $(\mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2l}, \mathfrak{so}_m)$. By [18], these algebras form a dual pair even if $(n, l) = (1, 1)$ or $(2, 2)$. But, for these "bad" values $\mathfrak{so}_m$ has no interpretation as the centralizer in $\mathfrak{g}$ of an $\mathfrak{sl}_2$-triple in $\mathfrak{k}_2$. Observe also that one obtains a rectangular *pn*-pair, if $l = 0$.

6. EXCELLENT ELEMENTS AND EXCELLENT SHEETS

For an *spr*-pair (e_1, e_2), Theorem 5.2 says that e_2 is a regular nilpotent element in $\mathfrak{k}_1 = \mathfrak{z}_\mathfrak{g}(\mathfrak{k}_2) = \mathfrak{z}_\mathfrak{g}(\mathfrak{z}_g(e_2, \tilde{h}_2, f_2))$ and that $\mathfrak{c}_2 = \mathfrak{z}_\mathfrak{g}(\mathfrak{z}_\mathfrak{g}(\tilde{h}_2))$ is the centralizer of $\tilde{h}_2$ in $\mathfrak{k}_1$. That is, the functor of taking the double centralizer, applied to $\{e_2, \tilde{h}_2, f_2\}$, has nice properties. Our goal in this section is to further investigate and give applications of such phenomenon.

6.1. Definition. A nilpotent element $e \in \mathfrak{g}$ is called *excellent*, if it is even and $\dim \mathfrak{z}_\mathfrak{g}(\mathfrak{z}_\mathfrak{g}(\tilde{h})) = \operatorname{rk} \mathfrak{z}_\mathfrak{g}(\mathfrak{z}_\mathfrak{g}(e, \tilde{h}, f))$ for a (any) $\mathfrak{sl}_2$-triple $\{e, \tilde{h}, f\}$ containing e. The same terminology applies to the $\mathfrak{sl}_2$-triple itself.

Set $\mathfrak{k} = \mathfrak{z}_\mathfrak{g}(e, \tilde{h}, f)$, $K = Z_G(e, \tilde{h}, f)$, $\mathfrak{k}^\vee = \mathfrak{z}_\mathfrak{g}(\mathfrak{k})$, and $\mathfrak{l} = \mathfrak{z}_\mathfrak{g}(\tilde{h})$. Then $\mathfrak{c} := \mathfrak{z}_\mathfrak{g}(\mathfrak{z}_\mathfrak{g}(\tilde{h}))$ is the center of $\mathfrak{l}$ and $(\mathfrak{k}, \mathfrak{k}^\vee)$ is a dual pair. We shall write $\mathfrak{z}_\mathfrak{g}^2(\cdot)$ in place of $\mathfrak{z}_\mathfrak{g}(\mathfrak{z}_\mathfrak{g}(\cdot))$.

Examples. (1) If e is distinguished in $\mathfrak{g}$, then $\mathfrak{k} = 0$ and $\mathfrak{k}^\vee = \mathfrak{g}$. But $\mathfrak{c}$ is a Cartan subalgebra if and only if e is regular. Hence an excellent distinguished element is regular.

(2) If (e_1, e_2) is an *spr*-pair, then e_2 satisfies the second condition of Definition 6.1. But the converse is not true. If $(n, l) = (1, 1)$ or $(2, 2)$ in Example 5.3(3), then e_2 is excellent, whereas it cannot be included in an *spr*-pair.

6.2. Theorem. *Let e satisfy the second condition in Definition* 6.1. *Then*

(1) $\mathfrak{c}$ *is a Cartan subalgebra and* $\{e, \tilde{h}, f\}$ *is a principal* $\mathfrak{sl}_2$*-triple in* $\mathfrak{k}^\vee$;

(2) $K = Z_G(\mathfrak{k}^\vee)$.

Proof. (1) Since $\tilde{h} \in \{e, \tilde{h}, f\}$, taking the double centralizer gives $\mathfrak{c} \subset \mathfrak{k}^\vee$, whence $\mathfrak{c}$ is Cartan in $\mathfrak{k}^\vee$. Next, $\mathfrak{z}_{\mathfrak{k}^\vee}(\tilde{h}) = \mathfrak{z}_\mathfrak{g}(\tilde{h}) \cap \mathfrak{k}^\vee = \mathfrak{z}_\mathfrak{g}(\mathfrak{c}) \cap \mathfrak{k}^\vee = \mathfrak{z}_{\mathfrak{k}^\vee}(\mathfrak{c}) = \mathfrak{c}$, which means $\tilde{h}$ is regular in $\mathfrak{k}^\vee$. The centralizer of $\{e, \tilde{h}, f\}$ in $\mathfrak{k}^\vee$ is equal to $\mathfrak{k} \cap \mathfrak{k}^\vee$, the centre of $\mathfrak{k}^\vee$. That is, e is distinguished in $\mathfrak{k}^\vee$. Since any distinguished element is even (see, e.g., [3, Chap. 8]), the assertion follows.

(2) As $\{e, \tilde{h}, f\} \subset \mathfrak{k}^\vee$, we obtain $K \supset Z_G(\mathfrak{k}^\vee)$. In view of part (1), $\mathfrak{k}^\vee$ is generated by e, $\mathfrak{c}$, and f as a Lie algebra. By definition, K centralizes e and f; and K centralizes $\mathfrak{c}$, because $\mathfrak{c}$ is the center of $\mathfrak{l}$ and K is contained in the *connected* group $L = Z_G(\tilde{h})$. Hence $K \subset Z_G(\mathfrak{k}^\vee)$. ∎

Recall from Section 2 the notions of e-filtration and e-limit, which apply to any nilpotent element and any linear subspace of $\mathfrak{g}$.

6.3. LEMMA. *Let $e \in \mathfrak{g}$ be an arbitrary nilpotent element. Then*

(1) $[\lim_e \mathfrak{z}^2_{\mathfrak{g}}(\tilde{h}), \lim_e \mathfrak{z}_{\mathfrak{g}}(\tilde{h})] = 0$;

(2) *If e is even, then* $\lim_e \mathfrak{z}^2_{\mathfrak{g}}(\tilde{h}) \subset \mathfrak{z}^2_{\mathfrak{g}}(e)$ *and* $\dim \mathfrak{z}^2_{\mathfrak{g}}(\tilde{h}) \le \dim \mathfrak{z}^2_{\mathfrak{g}}(e)$.

Proof. Note first that $\mathfrak{z}^2_{\mathfrak{g}}(x)$ is the center of $\mathfrak{z}_{\mathfrak{g}}(x)$ for any $x \in \mathfrak{g}$.

(1) By definition, the linear space $\lim_e M$ is generated by all elements of the form $(\operatorname{ad} e)^i x (x \in M)$ that lie in $\mathfrak{z}_{\mathfrak{g}}(e)$. Let $x \in \mathfrak{z}^2_{\mathfrak{g}}(\tilde{h})$ and $y \in \mathfrak{z}_{\mathfrak{g}}(\tilde{h})$. If $0 \ne (\operatorname{ad} e)^i x \in \mathfrak{z}_{\mathfrak{g}}(e)$ and $0 \ne (\operatorname{ad} e)^j y \in \mathfrak{z}_{\mathfrak{g}}(e)$, then $[(\operatorname{ad} e)^i x, (\operatorname{ad} e)^j y] = (i^! j^! /(i+j)!)(\operatorname{ad} e)^{i+j}[x, y] = 0$.

(2) If e is even, then $\dim \mathfrak{z}_{\mathfrak{g}}(\tilde{h}) = \dim \mathfrak{z}_{\mathfrak{g}}(e)$. Since $\dim(\lim_e \mathfrak{z}_{\mathfrak{g}}(\tilde{h})) = \dim \mathfrak{z}_{\mathfrak{g}}(\tilde{h})$, we conclude that $\lim_e \mathfrak{z}_{\mathfrak{g}}(\tilde{h}) = \mathfrak{z}_{\mathfrak{g}}(e)$. Hence $\lim_e \mathfrak{z}^2_{\mathfrak{g}}(\tilde{h}) \subset \mathfrak{z}^2_{\mathfrak{g}}(e)$, by the first claim. ∎

6.4. THEOREM. *Let e be excellent. Then*

(1) $\mathfrak{k}$ *and* $\mathfrak{k}^\vee$ *are semisimple*;

(2) $\mathfrak{z}^2_{\mathfrak{g}}(e)$ *is the centralizer of e in* $\mathfrak{k}^\vee$.

Proof. (1) Let $\mathfrak{z}$ be the centre of $\mathfrak{k}^\vee$. Then $\mathfrak{z} \subset \mathfrak{c}$ and therefore $[\mathfrak{z}, \mathfrak{l}] = 0$. In case e is even, $\mathfrak{g}$ is generated by $e, \mathfrak{l}$ and f as a Lie algebra. Hence $\mathfrak{z}$ is in the center of $\mathfrak{g}$, i.e., $\mathfrak{z} = 0$.

(2) Since $\mathfrak{z}^2_{\mathfrak{g}}(e) \subset \mathfrak{z}_{\mathfrak{k}^\vee}(e)$ and $\dim \mathfrak{z}_{\mathfrak{k}^\vee}(e) = \dim \mathfrak{z}_{\mathfrak{k}^\vee}(\tilde{h})$, the assertion follows from the previous lemma. ∎

EXAMPLE. The properties in Lemma 6.3(ii) and Theorem 6.4 need not hold, if e is not even. Let e be a nilpotent element in $\mathfrak{sl}_5$ whose weighted Dynkin diagram is (1–1–1–1). Since $\tilde{h}$ is regular semisimple, $\mathfrak{z}^2_{\mathfrak{g}}(\tilde{h})$ is a Cartan subalgebra and hence the second condition in Definition 6.1 is satisfied. But here $\mathfrak{k}$ is a 1-dimensional toral subalgebra and $\dim \mathfrak{z}^2_{\mathfrak{g}}(e) = 2 < \operatorname{rk} \mathfrak{k}^\vee = 4$.

We may express beautiful properties possessed by the excellent elements (or excellent $\mathfrak{sl}_2$-triples) in the following form. If $\{e, \tilde{h}, f\}$ is excellent, then:

$\dim \mathfrak{z}^2_{\mathfrak{g}}(e) = \dim \mathfrak{z}^2_{\mathfrak{g}}(\tilde{h})$;

$\mathfrak{z}^2_{\mathfrak{g}}(e)$ is the centralizer of e in $\mathfrak{z}^2_{\mathfrak{g}}(e, \tilde{h}, f)$;

$\mathfrak{z}^2_{\mathfrak{g}}(\tilde{h})$ is the centralizer of $\tilde{h}$ in $\mathfrak{z}^2_{\mathfrak{g}}(e, \tilde{h}, f)$.

(6.5) *Sheets.* Now, we show that the excellent elements provide an excellent framework for constructing sections of sheets. A *sheet* in $\mathfrak{g}$ is an irreducible component of the set of points whose G-orbits have a fixed dimension. The unique open sheet consists of the regular elements in $\mathfrak{g}$. This sheet has been thoroughly studied in [13]. The general theory of sheets was started in [2]. We refer to that paper for basic results of the theory. Each sheet is locally closed and contains a unique nilponent G-orbit. However, a nilpotent orbit may lie in several sheets. We shall only deal with *Dixmier sheets*, i.e., sheets containing semisimple elements. These are described as follows. For $Z \subset \mathfrak{g}$, we set $Z^{reg} = \{x \in Z \mid \dim G \cdot x \geq \dim G \cdot y \text{ for all } y \in Z\}$. Let $\mathfrak{l} \subset \mathfrak{g}$ be a Levi subalgebra with centre $\mathfrak{c}$. Then $(\overline{G \cdot \mathfrak{c}})^{reg}$ is a Dixmier sheet and all Dixmier sheets are of this form. To any even nilpotent element, one naturally associates a Dixmier sheet. If e is even and $\tilde{h}$ is a characteristic of e, then applying the above construction to the centre of $\mathfrak{z}_{\mathfrak{g}}(\tilde{h})$, one obtains a Dixmier sheet containing e. This sheet will be denoted by $\mathcal{S}_{\tilde{h}}(e)$. In this case, one has $\dim \mathcal{S}_{\tilde{h}}(e) = \dim G \cdot e + \dim \mathfrak{c} = \dim G \cdot \tilde{h} + \dim \mathfrak{c}$. Let us say that $Y \subset \mathcal{S}_{\tilde{h}}(e)$ is a *section* if Y is irreducible, $G \cdot y \cap \mathcal{S}_{\tilde{h}}(e) = \{y\}$ for all $y \in Y$, and $G \cdot Y = \mathcal{S}_{\tilde{h}}(e)$. In addition to the notation in (6.1), let $K^\vee$ denote the *connected* group with Lie algebra $\mathfrak{k}^\vee$.

6.6. THEOREM. *Suppose e is excellent and $(e, \tilde{h}, f)$ is an $\mathfrak{sl}_2$-triple. Then*

(1) *$\mathcal{S}_{\tilde{h}}(e)$ is smooth;*

(2) *$\mathcal{S}_{\tilde{h}}(e) \cap (e + \mathfrak{z}_{\mathfrak{g}}(f)) = e + \mathfrak{z}_{\mathfrak{k}^\vee}(f)$;*

(3) *$e + \mathfrak{z}_{\mathfrak{k}^\vee}(f)$ is a section of $\mathcal{S}_{\tilde{h}}(e)$;*

(4) *$\mathcal{S}_{\tilde{h}}(e)$ is the unique sheet containing e.*

Proof. (1) Since $[\mathfrak{g}, e] \oplus \mathfrak{z}_{\mathfrak{g}}(f) = \mathfrak{g}$, the affine space $e + \mathfrak{z}_{\mathfrak{g}}(f)$ is transversal to the orbit $G \cdot e$ at e. Consider the subspace $\mathcal{A} := e + \mathfrak{z}_{\mathfrak{k}^\vee}(f) = (e + \mathfrak{z}_{\mathfrak{g}}(f)) \cap \mathfrak{k}^\vee$. Since $\{e, \tilde{h}, f\}$ is a principal $\mathfrak{sl}_2$-triple in $\mathfrak{k}^\vee$ (see Theorem 6.2(1)), $\mathcal{A}$ is a section of the open sheet in $\mathfrak{k}^\vee$. This is a classical result of Kostant [13]. Therefore almost all elements in $\mathcal{A}$ are semisimple and $K^\vee$-conjugate to elements in $\mathfrak{c}$, the latter being both a Cartan subalgebra in $\mathfrak{k}^\vee$ and the centre of $\mathfrak{l}$. It follows that $\max_{x \in \mathcal{A}} \dim G \cdot x = \dim G - \dim \mathfrak{l} = \dim G \cdot \tilde{h}$ and $\overline{G \cdot \mathcal{A}} = \overline{G \cdot \mathfrak{c}} = \overline{\mathcal{S}_{\tilde{h}}(e)}$. Consider the 1-parameter group $\{\lambda(t) \mid t \in \mathbf{k}^*\} \subset GL(\mathfrak{g})$, where $\lambda(t) = \exp(t(\operatorname{ad} \tilde{h} - 2 \cdot \mathrm{Id}_{\mathfrak{g}}))$. It is easily seen that $\mathcal{A}$ is $\lambda(\mathbb{k}^*)$-stable and $e \in \overline{\lambda(\mathbb{k}^*)x}$ for all $x \in \mathcal{A}$, whence $\dim G \cdot e \leq \dim G \cdot x$. Because e is assumed to be even and hence $\dim G \cdot e = \dim G \cdot \tilde{h}$, all G-orbits intersecting $\mathcal{A}$ have the same dimension. Thus $\mathcal{A} \subset \mathcal{S}_{\tilde{h}}(e)$.

Our next argument relies on results of Katsylo [10]. He studied the variety $\mathcal{S} \cap (e' + \mathfrak{z}_{\mathfrak{g}}(f'))$ for an arbitrary sheet $\mathcal{S}$ containing an arbitrary nilpotent element e'. By [10, 0.1], we have

- $\mathcal{B} := \mathcal{S} \cap (e' + \mathfrak{z}_{\mathfrak{g}}(f'))$ is closed in $e' + \mathfrak{z}_{\mathfrak{g}}(f')$,
- the G-orbits in $\mathcal{S}$ intersect $\mathcal{B}$ transversally,
- $G \cdot \mathcal{B}_i = \mathcal{S}$ for any irreducible component $\mathcal{B}_i$ of $\mathcal{B}$.

Applying this to $\mathcal{B}^{\tilde{h}} := \mathcal{S}_{\tilde{h}}(e) \cap (e + \mathfrak{z}_{\mathfrak{g}}(f))$ and the irreducible components $\mathcal{B}_i^{\tilde{h}}$, we see that $\dim \mathcal{B}_i^{\tilde{h}} = \dim \mathfrak{c}$. Since $\dim \mathcal{A} = \dim \mathfrak{z}_{\mathfrak{k}^\vee}(f) = \dim \mathfrak{c}$ and $\mathcal{A} \subset \mathcal{B}^{\tilde{h}}$, we have $\mathcal{A}$ is an irreducible component of $\mathcal{B}^{\tilde{h}}$ and $\mathcal{S}_{\tilde{h}}(e) = G \cdot \mathcal{A}$. It follows from the transversality condition that the natural map $G \times \mathcal{A} \to \mathcal{S}_{\tilde{h}}(e)$ is smooth and hence $\mathcal{S}_{\tilde{h}}(e)$ is smooth, too.

(2) By [10, 0.2], the connected group K^o acts trivially on $\mathcal{B}^{\tilde{h}}$ or, equivalently, $\mathcal{B}^{\tilde{h}}$ is contained in $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{k}) = \mathfrak{k}^\vee$. Therefore

$$\mathcal{A} = (e + \mathfrak{z}_{\mathfrak{g}}(f)) \cap \mathfrak{k}^\vee \subset (e + \mathfrak{z}_{\mathfrak{g}}(f)) \cap \mathcal{S}_{\tilde{h}}(e) = \mathcal{B}^{\tilde{h}} \subset \mathfrak{k}^\vee,$$

whence $\mathcal{A} = \mathcal{B}^{\tilde{h}}$.

(3) By [10, 0.3], two points $x', x'' \in \mathcal{A}$ lie in the same G-orbit if and only if these lie in the same K/K^o-orbit. Thus, $\mathcal{A}$ is a section of $\mathcal{S}_{\tilde{h}}(e)$ if and only if K acts trivially on $\mathcal{A}$. Let x' be a generic point in $\mathcal{A}$. Then x' is a regular semisimple element in $\mathfrak{k}^\vee$ and hence $K^\vee \cdot x'$ contains a point $y \in \mathfrak{c}$. We have $Z_G(y) = Z_G(\tilde{h}) \supset K$ and $x' = s \cdot y$ for some $s \in K^\vee$. Then $Z_G(x') \supset sKs^{-1}$. By Theorem 6.2(2), the subgroups K and $K^\vee$ commute. Hence $K \subset Z_G(x')$ and we are done.

(4) Let $\mathcal{S}$ be an arbitrary sheet containing e. Arguing as in the proof of part (2), we obtain $(e + \mathfrak{z}_{\mathfrak{g}}(f)) \cap \mathcal{S} \subset \mathfrak{k}^\vee$. Therefore $(e + \mathfrak{z}_{\mathfrak{g}}(f)) \cap \mathcal{S} \subset e + \mathfrak{z}_{\mathfrak{k}^\vee}(f) \subset \mathcal{S}_{\tilde{h}}(e)$. Since $\mathcal{S} = G \cdot ((e + \mathfrak{z}_{\mathfrak{g}}(f)) \cap \mathcal{S})$ by Katsylo's result, we must have $\mathcal{S} = \mathcal{S}_{\tilde{h}}(e)$. ∎

6.7. COROLLARY. *The assertions of Theorem* 6.6 *are valid for both members of the rectangular pn-pairs.*

Proof. By Theorems 3.4 and 5.2, each member of a rectangular *pn*-pair is excellent. ∎

In view of Theorem 6.6(4), the sheet containing an excellent element is said to be *excellent*, too.

One may remember that each sheet in $\mathfrak{sl}_N$ is smooth and has a section, and each nilpotent element belongs to a unique sheet. On the other hand, it is shown by Ginzburg that each nilpotent element in $\mathfrak{sl}_N$ can be included in a *pn*-pair, see [9, 5.6] (this is no longer true for the other simple Lie algebras). It is therefore natural to suggest that something like Theorem 6.6 holds for arbitrary *pn*-pairs:

6.8. *Conjecture.* Let e be a member of a *pn*-pair. Then e belongs to a unique sheet; this sheet is smooth and has a section.

Making use of the classification of *pn*-pairs [7, 8], one can verify uniqueness of the sheet containing e in a case-by-case fashion. Indeed, the explicit description of induction on the set of nilpotent orbits in all simple Lie algebras is known; see [6, 11, 19].[2] Therefore, given a nilpotent orbit, one can say whether it belongs to a unique sheet. But the $\mathfrak{sl}_2$-framework breaks down completely in the non-rectangular case, and it is not clear how to produce a section.

7. CLASSIFICATION AND TABLES

Since the excellent orbits (or sheets) enjoy excellent properties, it is worth getting the list of them. Our classification is presented in two tables and we give the necessary details concerning our computations.

(7.1) *The Exceptional Case.* In $\mathbf{G}_2$, the only excellent orbit is the regular nilpotent one. For the non-regular excellent orbits, $\mathfrak{k}$ has to be non-trivial and semisimple. Looking through the tables in [5], one finds that the number of such even orbits in $\mathbf{F}_4, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8$ is equal to 3, 3, 15, 13, respectively. Having computed $\mathfrak{z}_\mathfrak{g}(\mathfrak{k})$ in each case, one distinguishes the excellent orbits among them. The actual number of non-regular excellent orbits is equal to 2, 2, 9, 6, respectively.

(7.2) *The Classical Case.* If $d_1, \ldots, d_m$ are all nonzero *different* parts of a partition $\mathbf{d}$ such that $d_1 > d_2 > \cdots > d_m$ and d_i occurs with multiplicity $r_i (i = 1, \ldots, m)$, then we write $\mathbf{d} = (d_1^{r_1}, \ldots, d_m^{r_m})$. For a classical simple Lie algebra, let $G(\mathbf{d})$ denote the orbit corresponding to $\mathbf{d}$. It is assumed that $\mathbf{d}$ satisfies the necessary constraints in the symplectic and orthogonal case. (If $\mathfrak{g} = \mathfrak{so}_N$ and $\mathbf{d}$ is "very even," then $G(\mathbf{d})$ can be either of the two SO_N-orbits.) It is well known (and easy to prove) that $G(\mathbf{d})$ is even if and only if the d_i's have the same party.

Given an even orbit $G(\mathbf{d})$, we describe the structure of $\mathfrak{l}, \mathfrak{k}$, and $\mathfrak{z}_\mathfrak{g}(\mathfrak{k})$ in terms of $\mathbf{d}$. The formulas for $\mathfrak{l}$ are easy and those for $\mathfrak{k}$ are found in [3, 6.1.3]. Then it is not hard to realize what $\mathfrak{z}_\mathfrak{g}(\mathfrak{k})$ is. Some accuracy is however needed while dealing with algebras $\mathfrak{so}_r$, since these are not semisimple for $r = 2$. Since $\mathfrak{k}$ and $\mathfrak{z}_\mathfrak{g}(\mathfrak{k})$ must be semisimple for the excellent elements (see Theorem 6.4(1)), we will assume that $r \neq 2$ whenever $\mathfrak{k}$ contains a summand $\mathfrak{so}_r$. With explicit formulas for $\mathfrak{l}$ and $\mathfrak{z}_\mathfrak{g}(\mathfrak{k})$, verification of the arithmetical condition $\dim \mathfrak{c} = \operatorname{rk} \mathfrak{z}_\mathfrak{g}(\mathfrak{k})$ becomes trivial.

[2]The results of Ref. [6] were announced in [19, pp. 171–177].

It is important to stress that our formulas for $\mathfrak{l}$ are only valid for *even* orbits.

(1) $\mathfrak{g} = \mathfrak{sl}_N$. Here

$$\mathfrak{l} = (\mathfrak{sl}_{r_1})^{d_1-d_2} \oplus (\mathfrak{sl}_{r_1+r_2})^{d_2-d_3} \oplus \cdots \oplus (\mathfrak{sl}_{r_1+\cdots+r_m})^{d_m} \oplus \mathbb{k}^{d_1-1},$$
$$\mathfrak{k} = \mathfrak{sl}_{r_1} \oplus \mathfrak{sl}_{r_2} \oplus \cdots \oplus \mathfrak{sl}_{r_m} \oplus \mathbb{k}^{m-1},$$

and

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{k}) = \mathfrak{sl}_{d_1} \oplus \mathfrak{sl}_{d_2} \oplus \cdots \oplus \mathfrak{sl}_{d_m} \oplus \mathbb{k}^{m-1}.$$

Thus, $\mathfrak{k}$ is semisimple if and only if $m = 1$ and then $G(\mathbf{d})$ is excellent. Actually, e is a member of a pn-pair in this case.

(2) $\mathfrak{g} = \mathfrak{sp}_{2N}$. Now we have to distinguish two possibilities.

(a) $d_1, \ldots, d_m$ are odd. Then $r_1, \ldots, r_m$ must be even. Here

$$\mathfrak{l} = (\mathfrak{sl}_{r_1})^{(d_1-d_2)/2} \oplus (\mathfrak{sl}_{r_1+r_2})^{(d_2-d_3)/2} \oplus \cdots \oplus (\mathfrak{sl}_{r_1+\cdots+r_m})^{(d_m-1)/2}$$
$$\oplus\, \mathfrak{sp}_{r_1+\cdots+r_m} \oplus \mathbb{k}^{(d_1-1)/2},$$
$$\mathfrak{k} = \mathfrak{sp}_{r_1} \oplus \mathfrak{sp}_{r_2} \oplus \cdots \oplus \mathfrak{sp}_{r_m},$$

and

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{k}) = \mathfrak{so}_{d_1} \oplus \mathfrak{so}_{d_2} \oplus \cdots \oplus \mathfrak{so}_{d_m}.$$

The arithmetical condition reads $(d_1 - 1)/2 = (d_1 - 1)/2 + \cdots + (d_m - 1)/2$, whence the excellent orbits correspond to either $m = 1$ or $m = 2$ and $d_2 = 1$. The first possibility gives us a member of a pn-pair.

(b) $d_1, \ldots, d_m$ are even. Then

$$\mathfrak{l} = (\mathfrak{sl}_{r_1})^{(d_1-d_2)/2} \oplus (\mathfrak{sl}_{r_1+r_2})^{(d_2-d_3)/2} \oplus \cdots \oplus (\mathfrak{sl}_{r_1+\cdots+r_m})^{d_m/2} \oplus \mathbb{k}^{d_1/2},$$

and

$$\mathfrak{k} = \mathfrak{so}_{r_1} \oplus \mathfrak{so}_{r_2} \oplus \cdots \oplus \mathfrak{so}_{r_m}.$$

If none of the r_i's is equal to 2, then $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{k}) = (\bigoplus_{i:\ r_i \neq 1} \mathfrak{sp}_{d_i}) \oplus \mathfrak{sp}_d$, where $d = \sum_{j:\ r_j=1} d_j$. But the rank of $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{k})$ does not depend on the number of r_i's that are equal to 1 and the condition $d_1/2 = d_1/2 + \cdots + d_m/2$ implies that $m = 1$.

(3) $\mathfrak{g} = \mathfrak{so}_N$. Here again are two possibilities.

(a) $d_1, \dots, d_m$ are even. Then $r_1, \dots, r_m$ must be even. Here

$$\mathfrak{l} = (\mathfrak{sl}_{r_1})^{(d_1-d_2)/2} \oplus (\mathfrak{sl}_{r_1+r_2})^{(d_2-d_3)/2} \oplus \cdots \oplus (\mathfrak{sl}_{r_1+\cdots+r_m})^{d_m/2} \oplus \Bbbk^{d_1/2},$$
$$\mathfrak{k} = \mathfrak{sp}_{r_1} \oplus \mathfrak{sp}_{r_2} \oplus \cdots \oplus \mathfrak{sp}_{r_m},$$

and

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{k}) = \mathfrak{sp}_{d_1} \oplus \mathfrak{sp}_{d_2} \oplus \cdots \oplus \mathfrak{sp}_{d_m}.$$

Then the equality $d_1/2 = (d_1 + \cdots + d_m)/2$ leads to $m = 1$.

(b) $d_1, \dots, d_m$ are odd. Then

$$\mathfrak{l} = (\mathfrak{sl}_{r_1})^{(d_1-d_2)/2} \oplus (\mathfrak{sl}_{r_1+r_2})^{(d_2-d_3)/2} \oplus \cdots \oplus (\mathfrak{sl}_{r_1+\cdots+r_m})^{(d_m-1)/2}$$
$$\oplus\, \mathfrak{so}_{r_1+\cdots+r_m} \oplus \Bbbk^{(d_1-1)/2},$$

and

$$\mathfrak{k} = \mathfrak{so}_{r_1} \oplus \mathfrak{so}_{r_2} \oplus \cdots \oplus \mathfrak{so}_{r_m}.$$

If none of the r_i's is equal to 2, then $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{k}) = (\bigoplus_{i:\ r_i \neq 1} \mathfrak{so}_{d_i}) \oplus \mathfrak{so}_d$, where $d = \sum_{j:\ r_j=1} d_j$. Observe that an anomaly occurs if $r_1 + \cdots + r_m = 2$, i.e., $m = 2$ and $r_1 = r_2 = 1$. Then $\dim \mathfrak{c} = (d_1 - 1)/2 + 1$. This case leads to the "excellent" partition $(d_1, 1)$, which represents the regular nilpotent orbit in $\mathfrak{so}_{d_1+1}$. Otherwise, we have $\dim \mathfrak{c} = (d_1 - 1)/2$ and $\operatorname{rk} \mathfrak{z}_{\mathfrak{g}}(\mathfrak{k}) \geq \sum_i ((d_i - 1)/2)$. Then a quick analysis leads to the following "excellent" partitions: $m = 1$ and $r_1 \neq 2$; $m = 2, d_2 = 1$, and $r_i \neq 2$ $(i = 1, 2)$.

Thus, a classification of excellent orbits is completed.

Here are some explanations to the tables. Nilpotent orbits in the exceptional (resp. classical) Lie algebras are represented by their weighted Dynkin diagrams (resp. partitions). The rightmost column gives dimension of the section of the excellent sheet. Recall that $\dim \mathcal{A} = \dim \mathfrak{z}_{\mathfrak{k}^\vee}(e) = \operatorname{rk} \mathfrak{z}_{\mathfrak{g}}(\mathfrak{k})$. In Table I, the pairs of orbits corresponding to the rectangular *pn*-pairs are placed in adjacent rows that are not separated. The "duality" between the label of $G \cdot e$ and the Cartan type of $[\mathfrak{l}, \mathfrak{l}]$ visible in each such pair is a manifestation of the properties stated in (1.4) or in (3.4). In Table II, the label of an orbit has the same meaning as for exceptional Lie algebras. It represents the (unique up to conjugation) minimal Levi subalgebra meeting this orbit. An algorithm for finding the label through the partition is found in [14, Sect. 3].

TABLE I
The Non-regular Excellent Orbits in the Exceptional Case

$\mathfrak{g}$	Diagram of $G\cdot e$	Label of $G\cdot e$	$[\mathfrak{l},\mathfrak{l}]$	$\mathfrak{k}$	$\dim \mathscr{A}$
$\mathbf{F}_4$	2–0 ⇐ 0–0	$\mathbf{A}_2$	$\mathbf{B}_3$	$\mathbf{G}_2$	1
	0–0 ⇐ 2–2	$\mathbf{B}_3$	$\widetilde{\mathbf{A}}_2$	$\mathbf{A}_1$	2
$\mathbf{E}_6$	2–0–0–0–2 0	$2\mathbf{A}_2$	$\mathbf{D}_4$	$\mathbf{G}_2$	2
	0–0–2–0–0 2	$\mathbf{D}_4$	$2\mathbf{A}_2$	$\mathbf{A}_2$	2
$\mathbf{E}_7$	2–0–0–0–0–0 0	$[3\mathbf{A}_1]''$	$\mathbf{E}_6$	$\mathbf{F}_4$	1
	0–2–0–2–2–2 0	$\mathbf{E}_6$	$[3\mathbf{A}_1]''$	$\mathbf{A}_1$	4
	0–0–0–0–0–0 2	$\mathbf{A}_2+3\mathbf{A}_1$	$\mathbf{A}_6$	$\mathbf{G}_2$	1
	0–2–0–2–0–0 0	$\mathbf{A}_6$	$\mathbf{A}_2+3\mathbf{A}_1$	$\mathbf{A}_1$	2
	0–0–0–0–2–2 0	$\mathbf{D}_4$	$[\mathbf{A}_5]''$	$\mathbf{C}_3$	2
	2–2–0–0–0–2 0	$[\mathbf{A}_5]''$	$\mathbf{D}_4$	$\mathbf{G}_2$	3
	0–0–2–0–0–0 0	$\mathbf{A}_3+\mathbf{A}_2+\mathbf{A}_1$	$\mathbf{A}_4+\mathbf{A}_2$	$\mathbf{A}_1$	1
	0–0–0–2–0–0 0	$\mathbf{A}_4+\mathbf{A}_2$	$\mathbf{A}_3+\mathbf{A}_2+\mathbf{A}_1$	$\mathbf{A}_1$	1
	0–2–0–0–0–0 0	$2\mathbf{A}_2$	$\mathbf{D}_5+\mathbf{A}_1$	$\mathbf{G}_2+\mathbf{A}_1$	1
$\mathbf{E}_8$	2–2–0–0–0–0–0 0	$\mathbf{D}_4$	$\mathbf{E}_6$	$\mathbf{F}_4$	2
	2–2–2–0–0–0–2 0	$\mathbf{E}_6$	$\mathbf{D}_4$	$\mathbf{G}_2$	4
	0–0–0–0–0–0–2 0	$2\mathbf{A}_2$	$\mathbf{D}_7$	$2\mathbf{G}_2$	1
	0–0–0–0–0–0–0 2	$\mathbf{D}_4(a_1)+\mathbf{A}_2$	$\mathbf{A}_7$	$\mathbf{A}_2$	1
	0–0–2–0–0–0–0 0	$\mathbf{A}_4+\mathbf{A}_2$	$\mathbf{D}_5+\mathbf{A}_2$	$2\mathbf{A}_1$	1
	0–0–2–0–0–0–2 0	$\mathbf{A}_6$	$\mathbf{D}_4+\mathbf{A}_2$	$2\mathbf{A}_1$	2

7.3. *Remark.* In [17], Rubenthaler introduced the notion of an *admissible* sheet and proved that each admissible sheet has a section, which is an affine space. He also gave a classification of the admissible sheets. It follows from comparing the two classifications that each excellent sheet is admissible. But the converse is not true and, furthermore, the assertions of

TABLE II
The Classical Case

$\mathfrak{g}$	Partition	Label of $G\cdot e$	$[\mathfrak{l},\mathfrak{l}]$	$\mathfrak{k}$	$\dim\mathscr{A}$
$\mathfrak{sl}_{nm}$	$(n,\dots,n)$	$m\mathbf{A}_{n-1}$	$(\mathfrak{sl}_m)^n$	$\mathfrak{sl}_m$	$n-1$
$\mathfrak{sp}_{2nm}$ $(m\neq 2)$	$(2n,\dots,2n)$	$\frac{m-1}{2}\tilde{\mathbf{A}}_{2n-1}+\mathbf{C}_n$ if m is odd; $\frac{m}{2}\widetilde{\mathbf{A}}_{2n-1}$ if m is even;	$(\mathfrak{sl}_m)^n$	$\mathfrak{so}_m$	n
$\mathfrak{sp}_{2(nm+l)}$ $(m$ is odd$)$	$(m^{2n},1^{2l})$	$n\widetilde{\mathbf{A}}_{m-1}$	$(\mathfrak{sl}_{2n})^{\frac{m-1}{2}}\oplus\mathfrak{sp}_{2(n+l)}$	$\mathfrak{sp}_{2n}\oplus\mathfrak{sp}_{2l}$	$\frac{m-1}{2}$
$\mathfrak{so}_{nm}$ $(m,n$ are even$)$	$(m,\dots,m)$	$\frac{n}{2}\mathbf{A}_{m-1}$	$(\mathfrak{sl}_n)^{m/2}$	$\mathfrak{sp}_n$	$\frac{m}{2}$
$\mathfrak{so}_{nm+l}$ $(m$ is odd$)$ $n\neq 2, l\neq 2$	$(m^n,1^l)$	$\frac{n-1}{2}\mathbf{A}_{m-1}+\mathbf{D}_{\frac{m+1}{2}}$ if l is odd; $\frac{n-1}{2}\mathbf{A}_{m-1}+\mathbf{B}_{\frac{m-1}{2}}$ if l is even;	$(\mathfrak{sl}_n)^{\frac{m-1}{2}}\oplus\mathfrak{so}_{n+l}$	$\mathfrak{so}_n\oplus\mathfrak{so}_l$	$\frac{m-1}{2}$

Theorem 6.6 do not hold for the nilpotent orbit lying in an arbitrary admissible sheet. For instance, the nilpotent orbit labelled by $\mathbf{D}_4(a_1)$ in $\mathfrak{g}=\mathbf{E}_6$ lies in an admissible sheet, while the total number of sheets containing it is equal to 3; see [6, Table 1]. It should also be noted that Rubenthaler writes nothing about smoothness of admissible sheets that our approach to the problem is less technical.

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