Static analysis for multi-layered piezoelectric cantilevers

H.J. Xiang, Z.F. Shi *

School of Civil Engineering, Beijing Jiaotong University, Beijing 100044, PR China

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Abstract

Taking the bonding layers and electrodes into account, the multi-layered piezoelectric cantilevers are studied based on the theory of elasticity. Different from the traditional investigations based on the elementary theory of elasticity, the Airy stress function method is used in the present paper. The stress function and induction function are proposed and determined, and then the exact solutions of the static governing equations are found. The material properties and thickness of different layers may be different in the present investigations. As two special cases, the exact static solutions for both unimorph and bimorph are directly obtained by using the present general solutions. The exact solutions obtained in the present paper are compared with the numerical results and others’ investigations, and good agreements are found. In addition, the effects of the properties of both bonding layers and electrodes are discussed. Moreover, the present solution can be used for function graded piezoelectric cantilever beams when the thickness of each bonding layer is taken as zero.

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1. Introduction

As well-known, multi-layered piezoelectric structures play an important role and are widely used in the engineering. Usually these multi-layered devices can elaborate original transducers such as Dual-frequency or Barker Code transducers or 2-D array elements with relative low electrical impedance, which lead to a greater sensitivity compared with single-layer element (Desmare, 1999). If they are made of piezoelectric polymeric materials, these devices can also offer some advantages over piezoelectric ceramic transducers, including flexibility, ease of preparing large sheets and the ability to undergo large deflections without damage (Marcus, 1984). Because of these advantages and wide applications, multi-layered piezoelectric structures have attracted much attention in recent years. For example, based on the one-dimensional constitutive equations, some simple behaviors of symmetric or non-symmetric piezoelectric cantilever bimorphs were studied (Smits et al., 1991; Smits and Ballato, 1994; Brissaud et al., 2003; Brissaud, 2004). Using the same constitutive equations, impedance and admittance matrices were presented for the analysis of the beam-type piezoelectric
multi-morph devices (Ha and Kim, 2002). Based on the Bernoulli–Euler beam model, the natural frequencies, maximum displacement and resultant force of a symmetric multi-morph cantilever were obtained (Lee et al., 2005). In another investigation, a new approach for laminated plates with piezoelectric layers was proposed based on a refinement improvement of the electric potential as a function of the thickness coordinate (Fernandes and Pouget, 2002). In this study, the shearing correction of elastic displacement was accounted. For the actuator consisting of a metallic layer covered symmetrically by two transversely isotropic piezoelectric layers, the basic behaviors were analyzed (Lim and He, 2004). For an intelligent beam with single elastic layer and two piezoelectric layers, a static analysis with a voltage applying on these two piezoelectric layers was performed by using Airy stress function method (Lin et al., 2001). In addition, a modeling and optimal design method for piezoelectric micro actuators was proposed (DeVoe and Pisano, 1997).

In order to improve the reliability of the sensors or actuators, the functionally graded piezoelectric materials (FGPM) have been proposed and manufactured (Zhu and Meng, 1995). Assuming the piezoelectric coefficient $d_{31}$ as a linear function in thickness direction and keeping other material parameters as constant, the experiment study on the functionally graded piezoelectric actuator showed that with respect to a classical bimorph, the deflection of a functionally graded piezoelectric cantilever actuator is only slightly smaller, whereas the internal mechanical stress is drastically reduced (Hauke et al., 2000). Utilizing the similar assumption but based on the theory of elasticity, some exact solutions for functionally graded piezoelectric sensors and actuators are obtained (Shi, 2002; Liu and Shi, 2004; Chen and Shi, 2005; Shi, 2005). The investigation on a linearly graded flat actuator showed that there is not any stress component in the flat actuator when it is subjected to an external voltage (Liu and Shi, 2004), but it will do in a linearly graded curved actuator (Shi, 2005). Moreover, piezoelectric actuators with functionally graded properties are designed with the aim of maintaining high bending displacement and reducing the stress concentration at the middle interface that exists in standard bimorph actuators (Taya et al., 2003).

Though there is considerable number of papers dealing with multi-layered sensors or actuators, most of the investigations were following Timoshenko’s approach or based on the elementary theory of elasticity. On the other hand, the simply supported boundary conditions were taken into account in most previous investigations (Heyliger and Brooks, 1996). Moreover, it is noted that the effects of bonding layers cannot be fully ignored. So, the objective of the current research is to give a precise analysis for the multi-layered piezoelectric sensors or actuators with bonding layers and electrodes based on the theory of elasticity. All the equilibrium conditions and continuous conditions for the stress, displacement and induction as well as electric potential on the interfaces between neighbor layers are exactly satisfied. In the present investigation, a multi-layered cantilever with different material properties and thickness for different layers is studied. It should be noted that the present solution can also be used for analyzing functionally graded piezoelectric cantilever beams by assuming the thickness of each bonding layer to be zero. The organization of the rest of this paper is as follows. The basic equations for piezoelectric materials are summarized in Section 2. The exact solutions for a kind of multi-layered piezoelectric composite cantilevers are obtained in Section 3 and then applied to two special cases in Section 4. In Section 5, some numerical results and comparisons are addressed and good agreements are found.

### 2. Basic equations

For the piezoelectric actuators, the design theory was presented under the consideration of the effect of bonding layers (Marcus, 1984). This effect on a multi-layered actuator was investigated, which all layers of the actuator were connected in parallel (Shi et al., 2006). Here, we will study a multi-layered actuator which all layers are connected in series. As shown in Fig. 1, the piezoelectric layer and bonding layer are placed alternately. There are $n + 1$ elastic layers (including two electrodes and $n − 1$ bonding layers) and $n$ piezoelectric layers. Actually, the actuator will become a multi-layered pure piezoelectric actuator when the thickness of each elastic layer is zero, which is a multi-layered model for functionally graded piezoelectric beam. Between the upper and lower surfaces of the actuator there is an external electrical potential $V_0$. The thickness of the elastic layer $k$ is determined by $(h_{2k-1} - h_{2k-2})$ and the thickness of the piezoelectric layer $k$ is determined by $(h_{2k} - h_{2k-1})$ as shown in Fig. 2. The thickness of both the elastic and the piezoelectric layers may be different. In one implementation of these devices, both the bonding layers are the elastic electrodes. Referring to a
Cartesian coordinate system \((x–0–z)\) and setting \(e_{ij}, \sigma_{ij}, D_i\) and \(E_i\) as the components of strain, stress, induction and electric field, respectively, the constitutive equations for transversely isotropic elastic materials and piezoelectric materials under the condition of plane deformation can be written as follows:

\[
\begin{align*}
e_{x} & = S_{11E_k} \sigma_{x} + S_{13E_k} \sigma_{z} \\
e_{z} & = S_{13E_k} \sigma_{x} + S_{33E_k} \sigma_{z} \quad \text{(in elastic layer)} \\
\gamma_{xz} & = S_{44E_k} \tau_{xz} \\
e_{x} & = S_{11P_k} \sigma_{x} + S_{13P_k} \sigma_{z} + g_{31k} D_z \\
e_{z} & = S_{13P_k} \sigma_{x} + S_{33P_k} \sigma_{z} + g_{33k} D_z \\
\gamma_{xz} & = S_{44P_k} \tau_{xz} + g_{15k} D_x \\
E_x & = -g_{15k} \tau_{xz} + \beta_{11k} D_x \\
E_z & = -g_{31k} \sigma_{x} - g_{33k} \sigma_{z} + \beta_{33k} D_z 
\end{align*}
\]  

where \(S_{ijE_k}\) and \(S_{ijP_k}\) are the coefficient of the effective elastic compliance for elastic layers and piezoelectric layers, respectively. The coefficients of the piezoelectric and dielectric impermeability for the piezoelectric layers are denoted by \(g_{ij}\) and \(\beta_{ij}\), respectively. The subscript \(k\) is added in the coefficients to distinguish different layers. Above material coefficients \(S_{ijE_k}\), \(S_{ijP_k}\), \(g_{ij}\) and \(\beta_{ij}\) in the plane stress condition are the same as the corresponding coefficients in 3D case, but they should be changed in the plane strain case (Xiang, 2007).

The strain components for both elastic and piezoelectric materials can be expressed by means of the displacement components \((u\) and \(w)\) as

\[
\begin{align*}
e_{x} & = \frac{\partial u}{\partial x}, \quad e_{z} = \frac{\partial w}{\partial z}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} 
\end{align*}
\]

For piezoelectric materials, there is another set of geometrical equations between the electric field and the electrical potential \(\varphi\) as

\[
\begin{align*}
V_0 & = N_0 \\
M_0 & = L
\end{align*}
\]
\[ E_z = -\frac{\partial \phi}{\partial z}, \quad E_z = -\frac{\partial \psi}{\partial z} \quad (2b) \]

On the other hand, without consideration of body force the static equilibrium equations for both elastic and piezoelectric materials can be given as
\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0, \quad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} = 0 \quad (3a)
\]

Moreover, without consideration of body charge, the induction components in the piezoelectric materials should satisfy the following equation
\[
\frac{\partial D_x}{\partial x} + \frac{\partial D_z}{\partial z} = 0 \quad (3b)
\]

To ensure the displacement and electric potential, can be obtained by integrating Eq. (2), the components of strain and electric field must satisfy a set of equations as follows:
\[
\frac{\partial^2 e_x}{\partial z^2} + \frac{\partial^2 e_z}{\partial x^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z}, \quad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 0 \quad (4)
\]

Under the consideration of some detailed boundary conditions, these basic equations will be solved in the following sections.

3. Exact analysis for multi-layered piezoelectric sensors and actuators

To find the solution of the basic equations, the Airy stress function method is used in the present paper. The stress function \( \tilde{\phi} \) and the induction function \( \psi \) are introduced so that the components of stress and induction can be expressed as
\[
\begin{align*}
\sigma_x &= \frac{\partial^2 \tilde{\phi}}{\partial z^2}, \quad \sigma_z = \frac{\partial^2 \tilde{\phi}}{\partial x \partial z}, \quad \tau_{xz} = -\frac{\partial^2 \tilde{\phi}}{\partial x \partial z} \\
D_x &= \frac{\partial \tilde{\psi}}{\partial z}, \quad D_z = -\frac{\partial \tilde{\psi}}{\partial x}
\end{align*}
\quad (5)
\]

Further the stress function \( \tilde{\phi}_k \) and induction function \( \psi_k \) of layer \( k \) are assumed as:
\[
\begin{align*}
\tilde{\phi}_k &= -a_{Ek} z^2 + b_{Ek} z^2 \quad \text{(for elastic layer)} \quad (6a) \\
\tilde{\psi}_k &= -a_{pk} z^2 + b_{pk} z^2, \quad \psi_k = l_k x \quad \text{(for piezoelectric layer)} \quad (6b)
\end{align*}
\]

where \( a_{Ek}, b_{Ek}, a_{pk}, b_{pk} \) and \( l_k \) are constants to be determined. Further the components of stress and induction in layer \( k \) can be expressed as
\[
\begin{align*}
\sigma_x &= -6a_{Ek} z + 2b_{Ek} \quad \text{(in elastic layer)} \quad (7a) \\
\sigma_z &= \tau_{xz} = 0 \\
\sigma_x &= -6a_{pk} z + 2b_{pk}, \quad \sigma_z = \tau_{xz} = 0 \quad \text{(in piezoelectric layer)} \quad (7b)
\end{align*}
\]

From Eqs. (1) and (7), it is easily verified that Eq. (4) is satisfied. Further, by the use of Eqs. (1) and (2) the displacement and electrical potential in layer \( k \) can be expressed as follows:
\[
\begin{align*}
\text{u} &= -6a_{Ek} S_{11Ek} z + 2b_{Ek} S_{11Ek} z + \omega_{Ek} z + u_{Ek} \quad \text{(in elastic layer)} \quad (8a) \\
w &= -3a_{Ek} S_{13Ek} z + 2b_{Ek} S_{13Ek} z + 3a_{Ek} S_{11Ek} x^2 - \omega_{Ek} x + w_{Ek} \\
\text{u} &= -6a_{pk} S_{11pk} z + 2b_{pk} S_{11pk} z - g_{33k} l_{pk} x + \omega_{pk} z + u_{pk} \\
w &= -3a_{pk} S_{13pk} z + 2b_{pk} S_{13pk} z - g_{33k} l_{pk} x + 3a_{pk} S_{11pk} x^2 - \omega_{pk} x + w_{pk} \quad \text{(in piezoelectric layer)} \quad (8b) \\
\phi &= -3a_{pk} g_{313k} z + 2b_{pk} g_{313k} z + \beta_{33k} l_{pk} z + \phi_k
\end{align*}
\]

where \( a_{pk}, b_{pk}, a_{Ek}, b_{Ek}, l_{pk}, \omega_{pk}, \omega_{Ek}, u_{pk}, u_{Ek}, w_{pk}, w_{Ek}, \phi_k \) are constants to be determined by using geometrical and electrical boundary conditions. To find the exact solution of the multi-layered piezoelectric cantilever, the solutions expressed by Eqs. (7) and (8) should be correctly assembled by considering some continuous
conditions at the interfaces and boundary conditions. It is obvious that the following boundary conditions are automatically satisfied

\[ D_i = 0 \quad \text{at } x = 0, L \]  
\[ \tau_{xz} = 0 \quad \text{at } x = L \]  
\[ \sigma_z = 0, \quad \tau_{xz} = 0 \quad \text{at } z = 0 \quad \text{and} \quad z = h_{2n+1} \]

(9a)  
(9b)  
(9c)

Besides, the continuous conditions for the stresses at the interfaces are also automatically satisfied. The continuous conditions of induction in \( z \)-direction at the interfaces yield to

\[ l_1 = l_2 = \cdots = l_n = l \]

(10)

In addition, the displacement \((u \text{ and } w)\) should be continuous at the interfaces,

\[
\begin{cases}
    u(x, h_{i-}) = u(x, h_{i+}) & \text{for } (1 \leq i \leq 2n) \\
w(x, h_{i-}) = w(x, h_{i+})
\end{cases}
\]

(11)

which leads to the following relations:

\[
\begin{align*}
    u(x, h_{2k-1}) &= -6a_E k S_{11E}x h_{2k-1} + 2b_E k S_{11E}x + \omega_E k h_{2k-1} + u_E k \\
    &\quad -6a_P k S_{11P}x h_{2k-1} + 2b_P k S_{11P}x - g_{33k} l x + \omega_P k h_{2k-1} + u_P k \\
    u(x, h_{2k}) &= -6a_E k+1 S_{11E,k+1}x h_{2k} + 2b_E k+1 S_{11E,k+1}x + \omega_E k+1 h_{2k} + u_E k+1 \\
    &\quad -6a_P k+1 S_{11P}x h_{2k} + 2b_P k+1 S_{11P}x - g_{33k} l x + \omega_P k h_{2k} + u_P k \\
    w(x, h_{2k-1}) &= -3a_E k S_{13E} h_{2k-1} + 2b_E k S_{13E} h_{2k-1} + 3a_E k S_{11E} h_{2k-1}^2 - \omega_E k x + w_E k \\
    &\quad -3a_P k S_{13P} h_{2k-1} + 2b_P k S_{13P} h_{2k-1} - g_{33k} l h_{2k-1} + 3a_P k S_{11P} h_{2k-1}^2 - \omega_P k x + w_P k \\
    w(x, h_{2k}) &= -3a_E k+1 S_{13E,k+1} h_{2k} + 2b_E k+1 S_{13E,k+1} h_{2k} + 3a_E k+1 S_{11E,k+1} h_{2k}^2 - \omega_E k+1 x + w_E k+1 \\
    &\quad -3a_P k+1 S_{13P} h_{2k} + 2b_P k+1 S_{13P} h_{2k} - g_{33k} l h_{2k} + 3a_P k+1 S_{11P} h_{2k}^2 - \omega_P k x + w_P k
\end{align*}
\]

(12)

Besides yielding to \( \omega_E k = \omega_P k = \omega_0 \), \( u_E k = u_P k = u_0 \), the above equations are satisfied only if

\[
\begin{align*}
    a_E k S_{11E} &= a_E k S_{11E} = a_P k S_{11P} \\
    2b_E k S_{11E} &= 2b_P k S_{11P} - g_{31k} l \\
    2b_P k S_{11P} &= 2b_P k S_{11P} + (g_{31k} - g_{31i}) l \\
    -3a_E k S_{13E} h_{2k-1} + 2b_E k S_{13E} h_{2k-1} + w_E k &= -3a_P k S_{13P} h_{2k-1} + 2b_P k S_{13P} h_{2k-1} - g_{33k} l h_{2k-1} + w_P k \\
    -3a_E k+1 S_{13E,k+1} h_{2k} + 2b_E k+1 S_{13E,k+1} h_{2k} + w_E k+1 &= -3a_P k+1 S_{13P} h_{2k} + 2b_P k+1 S_{13P} h_{2k} - g_{33k} l h_{2k} + w_P k
\end{align*}
\]

(13)  
(14)  
(15)  
(16)  
(17)

The boundary conditions at the upper and lower surfaces as well as the conditions at the interfaces of the elastic layer between two neighbor piezoelectric layers for electrical potential can be expressed as

\[
\begin{align*}
    \varphi(x, 0) &= 0 \\
    \varphi(x, h_{2i-2}) &= \varphi(x, h_{2i-1}) & (i = 2, 3, \ldots, n) \\
    \varphi(x, h_{2n}) &= V_0
\end{align*}
\]

(18)

For simplicity, the following expressions are introduced

\[
\begin{align*}
    H_{Ej,k} &= \frac{h_{2k-1}^j - h_{2k-2}^j}{S_{11E}} & (k = 1, 2, \ldots, n, n + 1; \ j = 1, 2, 3) \\
    H_{Pj,k} &= \frac{h_{2k}^j - h_{2k-1}^j}{S_{11P}} & (k = 1, 2, \ldots, n; \ j = 1, 2, 3)
\end{align*}
\]

(19a)  
(19b)
From Eqs. (8b), (13)–(15) and (18), the following equation can be obtained (See Appendix A)
\[ k_{11}a_{p1} + k_{12}b_{p1} + k_{13}l = V_0 \]  \tag{20} 

in which
\[
\begin{aligned}
k_{11} &= -3 \sum_{k=1}^{n} H_{P2,k}g_{31k}S_{11P1}, \\
k_{12} &= 2 \sum_{k=1}^{n} H_{P1,k}g_{31k}S_{11P1}, \\
k_{13} &= \sum_{k=1}^{n} (\xi_{1k} - \xi_{10k})g_{31k} + \beta_{33k}(h_{2k} - h_{2k-1})
\end{aligned}
\tag{21} 

For the loading end, the Saint-Venant’s principle is considered, which leads to the following mechanical boundary conditions
\[
\begin{aligned}
\sum_{k=1}^{n} \int_{h_{2(k-1)}}^{h_{2k}} (-6a_{rk}z + 2b_{rk})dz + \sum_{k=1}^{n+1} \int_{h_{2(k-2)}}^{h_{2k-1}} (-6a_{rEk}z + 2b_{rEk})dz &= N_0 \\
\sum_{k=1}^{n} \int_{h_{2(k-1)}}^{h_{2k}} (-6a_{rk}z + 2b_{rk})dz + \sum_{k=1}^{n+1} \int_{h_{2(k-2)}}^{h_{2k-1}} (-6a_{rEk}z + 2b_{rEk})dz &= M_0
\end{aligned}
\tag{22} 

Substituting Eqs. (13)–(15) into Eq. (22), the following two equations are obtained
\[
\begin{aligned}
k_{21}a_{p1} + k_{22}b_{p1} + k_{23}l &= N_0 \\
k_{31}a_{p1} + k_{32}b_{p1} + k_{33}l &= M_0
\end{aligned}
\tag{23} 

in which
\[
\begin{aligned}
k_{21} &= -3 \left( \sum_{k=1}^{n+1} H_{E2,k} + \sum_{k=1}^{n} H_{P2,k} \right) S_{11P1}, \\
k_{22} &= 2 \left( \sum_{k=1}^{n+1} H_{E1,k} + \sum_{k=1}^{n} H_{P1,k} \right) S_{11P1} \\
k_{23} &= -\sum_{k=1}^{n+1} H_{E1,k}g_{311} + \sum_{k=1}^{n} H_{P1,k}(g_{31k} - g_{311}) \\
k_{31} &= -2 \left( \sum_{k=1}^{n+1} H_{E3,k} + \sum_{k=1}^{n} H_{P3,k} \right) S_{11P1}, \\
k_{32} &= \left( \sum_{k=1}^{n+1} H_{E2,k} + \sum_{k=1}^{n} H_{P2,k} \right) S_{11P1} \\
k_{33} &= -\frac{1}{2} \sum_{k=1}^{n+1} H_{E2,k}g_{311} + \frac{1}{2} \sum_{k=1}^{n} H_{P2,k}(g_{31k} - g_{311})
\end{aligned}
\tag{24} 

By solving Eqs. (20) and (23), the parameters \( a_{p1}, b_{p1} \) and \( l \) can be determined
\[
\begin{pmatrix}
a_{p1} \\
b_{p1} \\
l
\end{pmatrix} = 
\begin{pmatrix}
k_{11} & k_{12} & k_{13} \\
k_{21} & k_{22} & k_{23} \\
k_{31} & k_{32} & k_{33}
\end{pmatrix}^{-1}
\begin{pmatrix}
V_0 \\
N_0 \\
M_0
\end{pmatrix} = K^{-1}
\begin{pmatrix}
V_0 \\
N_0 \\
M_0
\end{pmatrix}
\tag{25} 

The parameters \( a_{rEk}, a_{rk}, b_{rEk}, b_{rk} \) can be determined by using Eqs. (13)–(15).
\[
\begin{aligned}
a_{rEk} &= S_{11P1}a_{p1}, \\
a_{rk} &= S_{11E1}a_{p1}, \\
b_{rEk} &= S_{11P1}b_{p1} - \frac{g_{311}}{2S_{11E1}} l, \\
b_{rk} &= S_{11P1}b_{p1} + \frac{g_{31k} - g_{311}}{2S_{11E1}} l.
\end{aligned}
\tag{26} \tag{27} \tag{28} \tag{29} 

To determine other parameters, the following geometrical restraint conditions are introduced
\[
u(L, h_{2l-1}) = 0, \quad w(L, h_{2l-1}) = 0, \quad \frac{\partial w(L, h_{2l-1})}{\partial x} = 0 \quad (I \in \{1, 2, 3, \ldots, n\})
\tag{30} 

which leads to
\[
\begin{align*}
-6a_{EI}S_{11EI}Lh_{2j-1} + 2b_{EI}S_{11EI}L + \omega_0 h_{2j-1} + u_0 &= 0 \\
-3a_{EI}S_{13EI}h_{2j-1}^2 + 2b_{EI}S_{13EI}h_{2j-1} + 3a_{EI}S_{11EI}L^2 - \omega_0 L + w_{EI} &= 0 \\
6a_{EI}S_{11EI}L - \omega_0 &= 0
\end{align*}
\]  
(31)

Substituting Eqs. (26)–(29) into above equation and letting \( \lambda_{EI} = S_{13EI}/S_{11EI} \), the parameters \( u_0, \omega_0 \) and \( w_{EI} \) can be expressed as
\[
\begin{align*}
u_0 &= -2S_{11PI}Lb_{P1} + g_{3311}Ll, \quad \omega_0 = 6S_{11PI}La_{P1} \\
w_{EI} &= 3(\lambda_{EI}h_{2j-1}^2 + L^2)S_{11PI}a_{P1} - 2\lambda_{EI}S_{11PI}h_{2j-1}b_{P1} + \lambda_{EI}g_{3311}h_{2j-1}l
\end{align*}
\]  
(32)

All the other unknown parameters can be found from Eqs. (16), (17) and (A.4).
\[
\begin{align*}
w_{Ek} &= w_{Ek} + 3(H_{2,k-1} - H_{2,j-1})a_{P1} + 2(H_{1,k-1} - H_{1,j-1})b_{P1} + (\tilde{H}_{1,k-1} - \tilde{H}_{1,j-1})l \\
w_{Pk} &= w_{Ek} + 3(\lambda_{Pk} - \lambda_{Ek})h_{2k-1}^2S_{11PI}a_{P1} - 2(\lambda_{Pk} - \lambda_{Ek})h_{2k-1}S_{11PI}b_{P1} - \lambda_{Ek}g_{33k}g_{33k}^T h_{2k-1}^T \\
\varphi_k &= 3S_{11PI}(\lambda_{Pk}h_{2k-1}^2 - \sum_{i=1}^{k-1} g_{331i}H_{P1,i}a_{P1} - 2S_{11PI}(\lambda_{Pk}h_{2k-1} - \sum_{i=1}^{k-1} g_{331i}H_{P1,i})b_{P1} + \tilde{H}_{1,k}l
\end{align*}
\]  
(33)

in which
\[
\begin{align*}
H_{1,k} &= \sum_{i=1}^{k-1} [\lambda_{Pl}(h_{2i} - h_{2i-1}) - \lambda_{E,i+1}h_{2i} + \lambda_{EI}h_{2i-1}]S_{11PI} \\
\tilde{H}_{1,k} &= \sum_{i=1}^{k-1} [(h_{2i} - h_{2i-1})[\lambda_{Pl}(g_{331i} - g_{331}) - g_{33k}] + \lambda_{E,i+1}h_{2i}g_{331} - \lambda_{EI}h_{2i-1}g_{331}] \\
\tilde{H}_{1,k} &= \sum_{i=1}^{k-1} [g_{331i}(g_{331i} - g_{331})H_{P1,i} + \beta_{33k}(h_{2i} - h_{2i-1})] - [\lambda_{P}g_{33k} - g_{331}]h_{2k-1} \\
H_{2,k} &= \sum_{i=1}^{k-1} [-\lambda_{Pl}(h_{2i} - h_{2i-1}) + \lambda_{E,i+1}h_{2i} - \lambda_{EI}h_{2i-1}]S_{11PI} \\
\lambda_{Ek} &= \frac{S_{11Ek}}{S_{11Ek}}, \quad \lambda_{P} = \frac{S_{11Ek}}{S_{11Ek}}, \quad \lambda_{Pk} = \frac{S_{11Ek}}{S_{11Ek}}
\end{align*}
\]  
(34)

Till now all the unknown parameters have been determined. From Eq. (8a), the tip deflection of the multilayered cantilever can be given as
\[
\delta = w(0, 0) = w_{EI}
\]  
(35)

**4. Solutions for some special cases**

These smart devices can be constructed in a variety of configurations. The simplest one is the unimorph as shown in Fig. 3, in which a single piezoelectric layer is bonded to a non-piezoelectric substrate. The next device

![Fig. 3. Piezoelectric unimorph.](image-url)
in complexity is the bimorph, a sandwich of two piezoelectric sheets with a bonding layer between them as shown in Fig. 4.

4.1. Piezoelectric unimorph

As shown in Fig. 3, the non-piezoelectric layer (substrate) serves as an electrode (Marcus, 1984). Using the solutions obtained in Section 3, the solution for this case can be obtained. Let \( n = 1 \), Eqs. (21) and (24) can be simplified as

\[
k_{11} = -3(h_2^2 - h_1^2)g_{311}, \quad k_{12} = 2(h_2 - h_1)g_{311}, \quad k_{13} = (h_2 - h_1)\beta_{311},
\]

\[
k_{21} = -3\left(\frac{h_1^2}{S_{1111}} + \frac{h_1^2 - h_2^2}{S_{1112}} + \frac{h_1^2 - h_1^2}{S_{1111}}\right)S_{1111}P_1,
\]

\[
k_{22} = 2\left(\frac{h_1}{S_{1111}} + \frac{h_1 - h_2}{S_{1112}} + \frac{h_1^2 - h_1^2}{S_{1111}}\right)S_{1111}P_1, \quad k_{23} = -(h_1 + h_1 - h_1^2)g_{311},
\]

\[
k_{31} = -2\left(\frac{h_1^2}{S_{1111}} + \frac{h_1^2 - h_1^2}{S_{1112}} + \frac{h_1^2 - h_1^2}{S_{1111}}\right)S_{1111}P_1,
\]

\[
k_{32} = -\frac{h_1^2}{S_{1112}}, \quad k_{33} = -\frac{1}{2}\left(\frac{h_1}{S_{1111}} + \frac{h_1^2 - h_1^2}{S_{1112}}\right)g_{311}
\]

From Eq. (25), the parameters \( a_{P1}, b_{P1} \) and \( l \) can be obtained. From Eqs. (26) and (28), the parameters \( a_{E1}, a_{E2}, b_{E1} \) and \( b_{E2} \) can be written as follows:

\[
\begin{align*}
a_{E1} &= \frac{S_{1111}}{S_{1111}} a_{P1}, \quad b_{E1} = \frac{S_{1111}}{S_{1111}} b_{P1} - \frac{g_{311}}{S_{1111}} l, \\
a_{E2} &= \frac{S_{1112}}{S_{1112}} a_{P1}, \quad b_{E2} = \frac{S_{1112}}{S_{1112}} b_{P1} - \frac{g_{312}}{S_{1112}} l
\end{align*}
\]

In order to determine the other parameters, some geometrical restraint conditions should be considered. For example, letting \( u(L, h_1) = w(L, h_1) = \frac{\partial w(L, h_1)}{\partial x} = 0 \), we have

\[
\begin{align*}
\omega_0 &= 6S_{1111} L a_{P1}, \\
u_0 &= -2S_{1111} L b_{P1} + g_{311} L l \\
w_{E1} &= 3(\lambda_{E1} h_1^2 + L^2)S_{1111} a_{P1} - 2\lambda_{E1} S_{1111} h_1 b_{P1} + \lambda_{E1} g_{311} h_1 l
\end{align*}
\]

From Eqs. (32) and (33), the parameters \( w_{E2}, w_{P1} \) and \( \varphi_1 \) are determined as

\[
\begin{align*}
w_{E2} &= 3(\lambda_{E2} h_2^2 a_{P1} + L^2)S_{1111} - (h_2^2 - h_1^2)S_{1311} a_{P1} \\
&\quad - 2\lambda_{E2} h_2^2 h_1 S_{1111} - (h_2 - h_1)S_{1311} b_{P1} + \lambda_{E2} S_{1311} h_2 + g_{331}(h_2 - h_1)l \\
w_{P1} &= 3(L^2 S_{1111} + h_2^2 S_{1311})a_{P1} - 2S_{1311} h_1 b_{P1} + g_{331} h_1 l \\
\varphi_1 &= 3a_{P1} g_{311} h_1^2 - 2b_{P1} g_{311} h_1 - \beta_{311} l h_1
\end{align*}
\]
4.2. Piezoelectric bimorph

Fig. 4 shows a piezoelectric bimorph structure with two outer electrodes and one bonding layer included. The total layers of this device is five, so we have \( n = 2 \). Assuming that two electrodes are made of the same materials, the following relationship is obtained

\[
S_{ijE1} = S_{ijE3} = S_{ijE}
\]

To simplify the denotation, the material coefficient \( S_{ijE2} \) of the bonding layer for this piezoelectric bimorph device is rewritten as \( S_{B} \). In addition, the thickness of both piezoelectric layers and both electrodes are the same, respectively, i.e.

\[
\begin{align*}
  h_2 - h_3 &= h_2 - h_1 = t_p \\
  h_5 - h_4 &= h_1 = t_e \\
  h_3 - h_2 &= t_p
\end{align*}
\]

It is noted that the key step to obtain the solution is to find out the parameters \( a_{p1}, b_{p1} \) and \( l \). To find out these parameters, the matrix \( K \) has to be calculated at first. Using Eqs. (21) and (24), all elements of matrix \( K \) are obtained as follows:

\[
\begin{align*}
  k_{11} &= -3t_p(2t_e + t_p)g_{311} + \frac{S_{111}}{S_{112}}(h + t_p + t_b)g_{312}, \\
  k_{12} &= 2t_p\left(g_{311} + \frac{S_{111}}{S_{112}}g_{312}\right), \quad k_{13} = t_p\left(\frac{S_{111} - S_{112}}{S_{112}}g_{312} + \beta_{332} + \beta_{311}\right), \\
  k_{21} &= -3h\left(\frac{n}{S_{112}} + \frac{2t_e}{S_{112}}\right) + \left(\frac{2t_e + t_p}{S_{112}} + \frac{h + t_p + t_b}{S_{112}}\right)t_pS_{1111}, \\
  k_{22} &= \frac{4t_pS_{111}}{S_{112}} + \frac{2t_eS_{111}}{S_{112}} + 2\left(\frac{S_{111}}{S_{112}} + 1\right)t_p, \quad k_{23} = \frac{S_{111} - S_{112}}{S_{112}}t_p - \frac{2t_eS_{111}}{S_{112}} - \frac{t_bS_{111}}{S_{112}}, \\
  k_{31} &= -2\left(\frac{T_1}{S_{112}} + \frac{h + t_p + t_b}{S_{112}}\right) + \left(\frac{2t_e + t_p}{S_{112}}\right)^2 + \frac{t_b}{S_{112}}, \\
  k_{32} &= -\frac{k_{31}}{3}, \quad k_{33} = \frac{h + t_p + t_b}{S_{112}}\left(g_{312} - g_{311}\right)t_p - h\left(\frac{t_e}{S_{112}} + \frac{t_b}{S_{112}}\right)g_{311}
\end{align*}
\]

in which

\[
\begin{align*}
  h &= 2(t_e + t_p) + t_b \\
  T_1 &= (t_e + t_p + t_b)^3 - (t_e + t_p)^3 \\
  T_2 &= (h - t_e)^3 - (t_e + t_p + t_b)^3
\end{align*}
\]

Using the same procedure as described in Section 4.1, the exact solutions of this bimorph can be obtained.

5. Numerical results and comparisons

In previous sections, the exact solutions for multi-layered sensors and actuators are obtained. In order to give a clear explanation, some numerical results and comparisons are presented in this section. For the multi-layered actuator as shown in Fig. 1, the case \( n = 10 \) is considered. That is to say that this structure consists of ten piezoelectric layers and nine bonding layers as well as two electrode layers. For simplicity, the thickness of every electrode layer, bonding layer and piezoelectric layer is taken as \( t_e, t_b \) and \( t_p \), respectively. The geometrical sizes of these different kinds of layers are taken as \( t_e = 0.01 \) mm, \( t_b = 0.01 \) mm, \( t_p = 0.12 \) mm and \( L = 16 \) mm unless pointed out. The elastic modulus \( Y \) and Poisson’s ratio \( \mu \) of both electrodes (Al) are taken as 70 GPa and 0.35, respectively. For the bonding layers made of (Gold–tin, 80 wt.% Au–20 wt.% Sn) (Rassaian et al., 1999), the elastic modulus and Poisson’s ratio are taken as 137.3 GPa and 0.3, respectively. The material coefficients \( \sigma_y \) for elastic layers used in this paper can be determined from above elastic modulus and Poisson’s ratio, for example, \( S_{11} = S_{33} = (1 - \mu^2)/Y \), \( S_{13} = -\mu(1 + \mu)/Y \), \( S_{44} = 2(1 + \mu)/Y \) for isotropic materials under the plane strain condition. For the piezoelectric layers made of PZT-5H, the material parameters are listed in Table 1 (Wang, 1983; Ruan et al., 2000).
The piezoelectric layers are polarized in the thickness direction. For comparison, the following two different microstructures are considered.

Case 1: Referring to Fig. 5a, the polling directions of piezoelectric layers 1, 3, 5, 7, 9 are the same as the direction of $+Z$ axis. While the polling directions of piezoelectric layers 2, 4, 6, 8, 10 are opposite to the direction of $+Z$ axis.

Case 2: Referring to Fig. 5b, the polling directions of piezoelectric layers 1, 2, 3, 4, 5 are the same as the direction of $+Z$ axis. While the polling directions of piezoelectric layers 6, 7, 8, 9, 10 are opposite to the direction of $+Z$ axis.

In previous analysis in Section 3, it is assumed that the piezoelectric layers are polarized along $+Z$ direction. So the sign of the piezoelectric coefficients $g_{ij}$ should be changed when the polling directions of the piezoelectric layers are opposite to the direction of $+Z$ axis (Smits et al., 1991; He et al., 2000). Considering that most of the piezoelectric actuators used in engineering are under plane strain condition, so the calculations in the present section are performed under the plane strain assumption unless it is pointed out. On the other hand, the loading condition $V_0 = 100 \text{ V}$, $N_0 = 0$ and $M_0 = 0$ is considered. That is to say that an electrical potential $V_0 = 100 \text{ V}$ is supplied only.

The deflections of the multi-layered piezoelectric actuators obtained analytically and numerically in the present paper are shown in Fig. 6. For comparison, the results obtained by DeVoe and Pisano (1997) are also plotted in the same figure and good agreements are found. Fig. 6 shows that DeVoe’s assumptions for simplification are correction. The distribution of electrical potential in a cross section of these two actuators is shown in Fig. 7. It is easily found from these two figures that the difference for the electrical potential in Case 1 and Case 2 is not obvious, but it does for the deflections of these two actuators. Figs. 8 and 9 show the stress distribution in a cross section of the actuators in Case 1 and Case 2, respectively. The analytical results agree very well with the numerical findings. These two figures also show that the stress difference between different bonding layers in Case 1 is not as larger as that in Case 2.

Keeping the thickness of both electrodes and piezoelectric layers as constant, Fig. 10 shows the relationship between the tip deflection and the thickness of bonding layers. It is found that the tip deflection of the ideal piezoelectric multi-layered actuator ($t_b = 0$) is 1.24 and 8.03 $\mu$m in Case 1 and Case 2, respectively. When the
Fig. 6. The deflection of the piezoelectric multi-morph.

Fig. 7. Electrical potential distribution in the cross section.

Fig. 8. Stress distribution in a cross section (Case 1).
thickness of each bonding layer becomes 10 μm, the tip deflection reduces to be 1.05 and 6.65 μm, respectively. Fig. 11 shows the normal stresses \( \sigma_z(h_{1+}) \) and \( \sigma_z(h_{1-}) \) changing with the thickness of bonding layers. Figs. 10 and 11 show that both the tip deflection and stress decrease almost linearly with the increase of the thickness of the bonding layers for the considered two kinds of actuators. Keeping the Poisson’s ratio as a constant, the tip deflection changing with the elastic modulus of the bonding layer \( E_b \) is plotted in Fig. 12. It is found that the tip deflection decreases almost linearly with the increase of the elastic modulus \( E_b \) for both two kinds of actuators, too. Figs. 13 and 14 show the relationship between the tip deflection and the thickness as well as the elastic modulus of electrode, respectively. It is found that the responses of the actuator with the change of electrode and the change of bonding layer are similar.

6. Conclusions

Based on the theory of elasticity, the static analytical solutions of the multi-layered piezoelectric actuators are obtained by solving a set of equations. The effect of both bonding layers and electrodes are taken into account. When the thickness of bonding layers is zero, the solutions of the ideal multi-layered piezoelectric actuators can be obtained as special cases of the present paper, which can be used for the case of functionally graded piezoelectric cantilever beams. Moreover, the present solution can be simplified as the exact solutions.
of some typical piezoelectric devices, such as unimorph and bimorph. In addition to find all the mechanical
and electrical fields such as the stress, strain, displacement, induction and electrical potential for all the pie-
zoelectric and bonding as well as electrode layers, this paper strictly proves that all the layers are under uni-
axial tension or compression stress state when the multi-layered structure is subjected to a mixed loading
condition: an axial tensile force and a couple at the free end as well as an electric potential between its upper
and lower surfaces. The present results are valid and have a good agreement with the results obtained by other
investigators. This paper also shows that the elementary theory of elasticity can be used to study the bending
behaviors of this kind of microstructures under above mentioned loading conditions and a pleased accuracy
can be obtained. In addition, due to the material coefficient and thickness of different layers may be different in
the present investigation, the present work makes it convenient to model and design different piezoelectric
actuators.
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Appendix A

Substituting Eq. (8b) into Eq. (18), it is found

\[ u_i/C_0 = 3a_{31}g_{1i}h_i/C_0 \quad \text{for } i = 2, 3, \ldots, n \]  

Then the following expressions can be obtained from Eq. (18) as

\begin{align*}
\varphi_n &= V_0 + 3a_{3n}g_{31n}h_n^2 - 2b_{3n}g_{31n}h_n - \beta_{33n}lh_n \\
\varphi_1 &= 3a_{p1}g_{311}h_1^2 - 2b_{p1}g_{311}h_1 - \beta_{331}lh_1
\end{align*}

Fig. 13. Relationship between the tip deflection and the thickness of electrode.

Fig. 14. Tip deflection changing with the elastic modulus of electrode (Poisson’s ratio \( \mu = 0.3 \)).
The following relationship can be easily found
\[
\varphi_u - \varphi_1 = \sum_{i=2}^{n} (\varphi_i - \varphi_{i-1}) \quad (A.3a)
\]
\[
\varphi_k - \varphi_1 = \sum_{i=2}^{k} (\varphi_i - \varphi_{i-1}) \quad (A.3b)
\]
Substituting Eqs. (A.1) and (A.2) into Eq. (A.3a), Eq. (20) is obtained.
Substituting Eqs. (A.1) and (A.2) into Eq. (A.3b), yields to
\[
\varphi_k = \varphi_1 + \sum_{i=2}^{k} (\varphi_i - \varphi_{i-1})
\]
\[
= \sum_{i=1}^{k-1} \left[ -3a_{pn} g_{31l} (h_{2i}^2 - h_{2i-1}^2) + 2b_{pn} g_{31l} (h_{2i} - h_{2i-1}) + \beta_{33l} (l(h_{2i} - h_{2i-1})) \right]
\]
\[
+ 3a_{pn} g_{31l} h_{2k-1}^2 - 2b_{pn} g_{31l} h_{2k-1} - \beta_{33l} h_{2k-1} \quad (A.4)
\]

References


