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A new iterative criterion for *H*-matrices: The reducible case

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It is dedicated to Professor Hans Schneider on the occasion of his 80th birthday.

Abstract

H-matrices appear in various areas of science and engineering and it is of vital importance to have an Algorithm to identify the *H*-matrix character of a certain matrix $A \in \mathbb{C}^{n,n}$. Recently, the present authors have proposed a new iterative criterion (Algorithm $\mathbb{AH}$) to completely identify the *H*-matrix property of an *irreducible* matrix. The present work extends the previous Algorithm to cover the *reducible* case as well.

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1. Introduction

The theory of *H*-matrices is very important for the numerical solution of linear systems of algebraic equations arising in various applications. E.g., in the Linear Complementarity Problem

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(LPC), in the Free Boundary Value Problems in Fluid Analysis, etc. [2]. The most common way to define an H-matrix $A \in \mathbb{C}^{n,n}$ is the following:

Definition 1.1. $A \in \mathbb{C}^{n,n}$ is an H-matrix if and only if (*iff*) there exists a positive diagonal matrix $D \in \mathbb{R}^{n,n}$ so that AD is (**row-wise**) **strictly diagonally dominant**, that is

$$|a_{ii}|d_i > \sum_{j=1,\ j\neq i}^{n} |a_{ij}|d_j, \quad i = 1(1)n. \tag{1.1}$$

For the identification of an H-matrix, many criteria have been proposed the majority of which are iterative ones (see, e.g. [6,10,9,11,12,5,1]). This is because direct criteria seem to have high computational complexities. The only iterative criterion that takes into account the sparsity of A is the one in [5], where an extension of the *compact profile* technique of [8] was developed and can also be used in the present case.

The new Algorithm in this paper extends Algorithm $\mathbb{AH}$ in [1] to cover the *reducible* case as well, since the latter was constructed to deal with *irreducible* matrices **only**.

In Section 2, basic notation, terminology and statements are presented. In Section 3, Algorithm $\mathbb{AH}$ is illustrated and some explanations on it are given. In Section 4, use of combinatorial matrix theory allows us to solve the problem of the general $p \times p$ block *reducible* case. In Section 5, we propose a new Jacobi type iterative criterion for identifying H-matrices (Algorithm $\mathbb{AH}\mathbf{2}$). Contrary to Algorithm $\mathbb{AH}$ the convergence of the new Algorithm is guaranteed for all *irreducible* and *reducible* matrices. Finally, in Section 6, we present a number of numerical examples worked out with both Algorithms $\mathbb{AH}$ and $\mathbb{AH}\mathbf{2}$.

2. Preliminaries and background material

For our analysis some definitions are recalled and a number of useful statements are given. Most of them can be found in [2,7,16,19].

Definition 2.1 [2]. A matrix $A \in \mathbb{R}^{n,n}$ is called an M-**matrix** if it can be written as $A = sI - B$, where $B \geqslant 0$ and $\rho(B) < s$, with $\rho(.)$ denoting spectral radius. (*Note*: In Definition 2.1 and in this context an M-matrix is always nonsingular.)

Lemma 2.1 [16]. *Let $A \in \mathbb{R}^{n,n}$ be an irreducible M-matrix, then its inverse exists and is a strictly positive matrix, that is $A^{-1} > 0$.*

Lemma 2.2. *If $A \in \mathbb{R}^{n,n}$ is an M-matrix so is PAP^{T}, where P is a permutation matrix.*

Lemma 2.3 [16]. *Any principal submatrix of an M-matrix $A \in \mathbb{R}^{n,n}$ is also an M-matrix.*

Definition 2.2 [2]. The **comparison** matrix of a matrix $A \in \mathbb{C}^{n,n}$ is the matrix $\mathscr{M}(A)$ with elements

$$m_{ij} = \begin{cases} |a_{ii}|, & \text{if } i = j = 1(1)n, \\ -|a_{ij}|, & \text{if } i, j = 1(1)n,\ i \neq j. \end{cases}$$

Lemma 2.4 [16]. *A matrix $A \in \mathbb{C}^{n,n}$ is an H-matrix iff its comparison matrix is an M-matrix.*

Lemma 2.5. *A matrix $A \in \mathbb{C}^{n,n}$ is an H-matrix iff the Jacobi iteration matrix associated with its comparison matrix is convergent.*

Lemma 2.6. *A matrix $A \in \mathbb{C}^{n,n}$ is* **not** *an H-matrix iff there exists at least one principal submatrix of A that is* **not** *an H-matix.*

Lemma 2.7. *Let $A \in \mathbb{C}^{n,n}$, with $a_{ii} \neq 0$, $i = 1(1)n$, and $B = EA$, where $E = \mathrm{diag}(e_1, e_2, \ldots, e_n) \in \mathbb{C}^{n,n}$ be any nonsingular diagonal matrix. Let J_A and J_B be the Jacobi iteration matrices associated with A and B, respectively. Then J_A and J_B are identical.*

Definition 2.3 [16]. Let $A \in \mathbb{R}^{n,n}$, $A \geqslant 0$, be an irreducible matrix and k be the number of eigenvalues of A of modulus equal to its spectral radius $\rho(A)$. If $k = 1$, then A is **primitive**. If $k > 1$, then A is **cyclic of index** k.

Definition 2.4 [2]. **Index** of a given matrix $A \in \mathbb{C}^{n,n}$, denoted by **index**(A), is the smallest nonnegative integer such that **rank**$(A^k) =$ **rank**(A^{k+1}).

Lemma 2.8 [2]. *If $A \in \mathbb{R}^{n,n}$, $A \geqslant 0$, is an irreducible matrix with positive trace, $\sum_{i=1}^{n} a_{ii} > 0$, then A is primitive.*

Theorem 2.1 [7]. *If $A \in \mathbb{C}^{n,n}$ and if $\lambda, \mu \in \sigma(A)$, with $\lambda \neq \mu$, then any left eigenvector of A corresponding to μ is orthogonal to any right eigenvector of A corresponding to λ.*

Theorem 2.2 [16]. *Let $A \in \mathbb{R}^{n,n}$, $A \geqslant 0$, be an irreducible matrix. Then, its spectral radius $\rho(A)$ is a simple (positive) eigenvalue of A (the Perron root) and a positive eigenvector (the Perron vector) is associated with it.*

Theorem 2.3 [16]. *For any given irreducible matrix $A \in \mathbb{R}^{n,n}$, $A \geqslant 0$, let P^* be the* **hyperoctant** *of vectors $x > 0$. Then, for any $x \in P^*$, either*

$$\min_{i=1(1)n} \left\{ \frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right\} < \rho(A) < \max_{i=1(1)n} \left\{ \frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right\},$$

or

$$\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} = \rho(A), \ i = 1(1)n.$$

Algorithm $\mathbb{AH}$ [1] is based on a modification of the well-known **Power Method** (see, e.g. [18] and more specifically [4] and [7]), applied to a nonnegative, irreducible and primitive $n \times n$ matrix. The *Power Method* Theorem and the one on which Algorithm $\mathbb{AH}$ is based are stated below.

Theorem 2.4 (The Power Method). *Let $A \in \mathbb{C}^{n,n}$, with its eigenvalues satisfying*

$$|\lambda_1| > |\lambda_j|, \quad j = 2(1)n.$$

Define

$$x^{(k)} = Ax^{(k-1)}, \quad k = 1, 2, 3, \ldots, \text{ for any } x^{(0)} \in \mathbb{C}^n \backslash \{0\}. \tag{2.1}$$

Assume that $x^{(0)}$ has a nonzero component along the eigenvector corresponding to λ_1. Then

$$\lambda_1 = \lim_{k \to \infty} \frac{(Ax^{(k)})_i}{x_i^{(k)}} \quad \text{for } x_i^{(k)} \neq 0, \ i = 1(1)n. \tag{2.2}$$

Theorem 2.5 [1]. *For any given irreducible and primitive matrix $A \in \mathbb{R}^{n,n},\ A \geqslant 0$, let $\lambda_1 = \rho(A)$ and let $A = SJS^{-1}$ be its Jordan canonical form, with $J = \mathrm{diag}(J_1, J_2, \ldots, J_p)$, $J_i \in \mathbb{C}^{n_i,n_i}$, $i = 1(1)p$, $\sum_{i=1}^{p} n_i = n$, and with $S = [s_1\, s_2\, s_3\, \ldots\, s_n]$ being the matrix of the principal vectors of A. Then, any $x \in P^*$, analyzed along the principal vectors $s_i,\ i = 1(1)n$, has a positive component along the Perron vector s_1 corresponding to the Perron root λ_1.*

3. Algorithm $\mathbb{AH}$ and main statements

The new algorithm (Algorithm $\mathbb{AH}$**2**) we are to propose is an extension of Algorithm $\mathbb{AH}$ and as is proved converges also in a finite number of iterations. The latter Algorithm is illustrated below after some definitions are given.

For both Algorithms the following matrices are needed. A sequence of positive diagonal matrices $D^{(k)}$, that will be defined in the Algorithm, and $A^{(k)}$.

$$D^{(k)}, \quad k = 0, 1, 2, \ldots, \quad D^{(0)} = I, \tag{3.1}$$

$$A^{(k)} = (D^{(k-1)})^{-1} A^{(k-1)} D^{(k-1)}, \quad k = 1, 2, 3, \ldots, \quad A^{(0)} = (\mathrm{diag}(A))^{-1} A, \tag{3.2}$$

assuming $a_{ii} \neq 0,\ i = 1(1)n$. From (3.1) and (3.2), it is readily seen that

$$a_{ii}^{(k)} = 1, \quad i = 1(1)n, \quad k = 0, 1, 2, \ldots \tag{3.3}$$

Algorithm $\mathbb{AH}$

INPUT: An irreducible matrix $A := [a_{ij}] \in \mathbb{C}^{n,n}$.

OUTPUT: $D = D^{(0)} D^{(1)} \cdots D^{(k)} \in \mathfrak{D}_{D^{-1}A} \equiv \mathfrak{D}_A$ or $\notin \mathfrak{D}_A$ [3] if A **is** or is **not** an H-matrix, respectively.

1. If $a_{ii} = 0$ for some $i \in \{1, 2, \ldots, n\}$, "$A$ is **not** an H-matrix", STOP; Otherwise
2. Set $D = I$, $A^{(0)} = (\mathrm{diag}(A))^{-1} A$, $D^{(0)} = I$, $k = 1$
3. Compute $D = DD^{(k-1)}$, $A^{(k)} = (D^{(k-1)})^{-1} A^{(k-1)} D^{(k-1)} = [a_{ij}^{(k)}]$
4. Compute $s_i^{(k)} = \sum_{j=1,\ j\neq i}^{n} |a_{ij}^{(k)}|,\ i = 1(1)n$, $s^{(k)} = \min_{i=1(1)n} s_i^{(k)},\ S^{(k)} = \max_{i=1(1)n} s_i^{(k)}$
5. If $s^{(k)} > 1$, "A is **not** an H-matrix", STOP; Otherwise
6. If $S^{(k)} < 1$, "A **is** an H-matrix", STOP; Otherwise
7. If $S^{(k)} = s^{(k)}$, "$\mathscr{M}(A)$ is **singular**", STOP; Otherwise
8. Set $d = [d_i]$, where
$$d_i = \frac{1 + s_i^{(k)}}{1 + S^{(k)}}, \quad i = 1(1)n$$
9. Set $D^{(k)} = \mathrm{diag}(d),\ k = k + 1$; Go to Step 3. END

For Algorithm $\mathbb{AH}$ the following two statements were proved in [1]:

Theorem 3.1. *Let $A \in \mathbb{C}^{n,n}$ be an irreducible matrix. Then, Algorithm $\mathbb{AH}$ always terminates in a finite number of iterations (except, maybe, when $\det(\mathscr{M}(A)) = 0$).*

[3] $\mathfrak{D}_A$ denotes the class of all positive diagonal matrices D so that AD is strictly diagonally dominant.

Theorem 3.2. *Let $A \in \mathbb{C}^{n,n}$ be any irreducible matrix. If Algorithm $\mathbb{AH}$ terminates in a finite number of iterations, then its output is correct.*

If $A \in \mathbb{C}^{n,n}$ is irreducible, with $a_{ii} \neq 0,\ i = 1(1)n$, and we set as in Algorithm $\mathbb{AH}$

$$A^{(k)} = \left(\operatorname{diag}(d_1^{(k-1)}, d_2^{(k-1)}, \ldots, d_n^{(k-1)})\right)^{-1} \times A^{(k-1)} \operatorname{diag}(d_1^{(k-1)}, d_2^{(k-1)}, \ldots, d_n^{(k-1)}), \text{ with } d^{(0)} = e, \tag{3.4}$$

$$|A^{(k)}| = I + B^{(k)}, \quad k = 0, 1, 2, \ldots \tag{3.5}$$

where $e \in \mathbb{R}^n$ is the vector of ones and $|X|$ denotes the matrix whose elements are the moduli of the corresponding elements of X. Note that $B^{(0)}$ is the Jacobi matrix associated with the comparison matrix of A, $J_{\mathscr{M}(A)}$. If in the Algorithm we allow $k \to \infty$ then in the proofs of Theorems 3.1 and 3.2 it was also proved in [1], among others, that

Corollary 3.1. *Under the assumptions and notations so far the Perron vector d of $|A^{(0)}|$ (and $B^{(0)}$) is given by*

$$d = \left(\lim_{k\to\infty}\left(\prod_{i=1}^{k} D^{(i)}\right)\right) e. \tag{3.6}$$

Corollary 3.2. *Under the assumptions and notations so far there hold*

$$\lim_{k\to\infty} |A^{(k)}|\, e = \rho(|A^{(0)}|)e \quad \text{and} \quad \lim_{k\to\infty} a_{ij}^{(k)} = \frac{d_j}{d_i}\, a_{ij}^{(0)}. \tag{3.7}$$

Algorithm $\mathbb{AH}$ was designed to work for *irreducible* matrices. However, we had observed that it worked perfectly well for certain classes of *reducible* matrices. This motivated the investigation of the effect of the application of Algorithm $\mathbb{AH}$ to reducible matrices a little further. So, we were led to extend it and create Algorithm $\mathbb{AH}\mathbf{2}$ which is shown to converge for both *irreducible* and *reducible* matrices and terminates in a finite number of iterations.

4. The general reducible case

To study the general $p \times p$ block *reducible* matrices we introduce some more theoretical material. Although some of it holds for general $n \times n$ complex matrices we will restrict to nonnegative matrices. Most of the basic material is taken from the works of Rothblum [13], Schneider [14], Bru and Neumann [3] and also from the book of Berman and Plemmons [2].

Lemma 4.1. *Let $A \geqslant 0,\ A \in \mathbb{R}^{n,n}$ be a reducible matrix. Then there exists a permutation matrix P such that A can be reduced to a block triangular form*

$$PAP^{\mathrm{T}} = \begin{bmatrix} A_{11} & A_{12} & \cdots & \cdots & A_{1,p-1} & A_{1p} \\ & A_{22} & \cdots & \cdots & A_{2,p-1} & A_{2p} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \vdots & \vdots \\ & & & & A_{p-1,p-1} & A_{p-1,p} \\ & & & & & A_{pp} \end{bmatrix}, \tag{4.1}$$

where each block $A_{ii} \in \mathbb{R}^{n_i,n_i}$, $i = 1(1)p$, $\sum_{i=1}^{p} n_i = n$, *is either irreducible or a* 1×1 *nullmatrix. This form is known as the* **Frobenius normal form**.

Note : To be in agreement with the main body of Algorithm $\mathbb{AH}$ it will be assumed that in (4.1) $a_{ii} \neq 0$, $i = 1(1)n$, and that a_{ii} are normalized so that $a_{ii} = 1$, $i = 1(1)n$.

Definition 4.1. Let $A \geqslant 0$, $A \in \mathbb{R}^{n,n}$. We define the **(directed) graph** of A, $G(A)$, to be the graph with vertices (nodes) $1(1)n$, where an edge (arc) leads from i to j *iff* $a_{ij} \neq 0$.

Definition 4.2. Let $A \geqslant 0$, $A \in \mathbb{R}^{n,n}$. For $i, j \in \{1, \ldots, n\}$, we say that **i has access to j** if in $G(A)$ there is a path from i to j and that i and j **communicate** if i has access to j and j has access to i. (*Note*: **Communication** is an **equivalence** relation.)

Definition 4.3. The **classes** of $A \geqslant 0$ are the equivalence classes of the communication relation induced by $G(A)$. A **class** α has access to a class β if for $i \in \alpha$ and $j \in \beta$, i has access to j. A class is **initial** if it is not accessed by any other class and is **final** if it has access to no other class. A class is **basic** if $\rho(A[\alpha]) = \rho(A)$, where $A[\alpha]$ is the submatrix of A based on the indices in α, and **nonbasic** if $\rho(A[\alpha]) < \rho(A)$.

Remark 4.1. The blocks A_{ii}, $i = 1(1)p$, in the Frobenius normal form (4.1) of A correspond to the classes of A. From (4.1), every $A \geqslant 0$ has at least one basic class and one final class. The class that corresponds to A_{ii}, $i = 1(1)p$, is basic *iff* $\rho(A_{ii}) = \rho(A)$ and final *iff* $A_{ij} = 0$, $j > i$. In particular, A is irreducible *iff* it has only one (basic and final) class.

Theorem 4.1. *Let* $A \geqslant 0$, $A \in \mathbb{R}^{n,n}$. *Then to the spectral radius* $\rho(A)$ *there corresponds a positive eigenvector iff the final classes of A are exactly its basic ones. (Note: As, e.g., in the case of an irreducible matrix or of a block diagonal matrix with* $\rho(A_{ii}) = \rho(A)$, $i = 1(1)p$.*)*

Theorem 4.2. *Let* $A \geqslant 0$, $A \in \mathbb{R}^{n,n}$. *Then to the spectral radius* $\rho(A)$ *there corresponds a positive eigenvector and a positive eigenvector of* A^T *iff all the classes of A are basic and final.*

Definition 4.4. Let $A \geqslant 0$, $A \in \mathbb{R}^{n,n}$. The **degree** of A is $\nu(A) = \mathbf{index}(\rho(A)I - A)$. The nullspace $\mathcal{N}((\rho(A)I - A)^{\nu(A)})$ is called the **algebraic eigenspace** of A and its elements are called **generalized eigenvectors**.

Definition 4.5. Let $A \geqslant 0$, $A \in \mathbb{R}^{n,n}$ and $\alpha_1, \alpha_2, \ldots, \alpha_k$ be classes of A. The collection $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is a **chain** from α_1 to α_k, if a_i has access to a_{i+1}, $i = 1(1)k - 1$. The **length** of a chain is the number of basic classes it contains. A class α has access to a class β in m **steps** if m is the length of the longest chain from α to β. The **height** of a class β is the length of the longest chain of classes that terminate in β.

Theorem 4.3. *Let* $A \geqslant 0$, $A \in \mathbb{R}^{n,n}$ *have spectral radius* $\rho(A)$ *and m basic classes* $\alpha_1, \alpha_2, \ldots, \alpha_m$. *Then the algebraic eigenspace of A contains nonnegative vectors* $x^{(1)}, x^{(2)}, \ldots, x^{(m)}$, *such that the subvector* ${x_i}^{(j)} > 0$ *iff* α_i *has access to* α_j *and any such collection is a basis of the algebraic eigenspace of A. (Note: It is understood that* $x_i^{(i)} > 0$.*)*

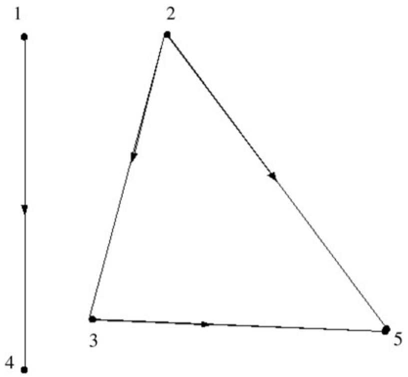

Fig. 1. Block graph of A in (4.2).

Remark 4.2. Based on Definitions 4.4, 4.5 and Theorem 4.3 it is clear that the only "genuine" eigenvectors of A, that is those nonnegative vectors $x \in \mathbb{R}^{n,n}$ for which $Ax = \rho(A)x$, correspond to basic classes of height 1.

The material presented so far suffices to develop our new Algorithm for $p \times p$ block *reducible* matrices. For this we will draw some conclusions when the Power Method Theorem 2.4 and/or Algorithm $\mathbb{AH}$ is applied to the reducible matrix $A \geqslant 0$, supposedly that A is in its Frobenius normal form (4.1), with $a_{ii} = 1,\ i = 1(1)n$.

(a) If the graph of A, $G(A)$, consists of the union of the disjoint subgraphs $g_i(A),\ i = 1(1)k,\ 1 < k \leqslant p$, then A can be written as $A = \operatorname{diag}(B_{11}, B_{22}, \ldots, B_{kk})$, where each $B_{ii},\ i = 1(1)k$, is reducible and is already in its Frobenius normal form. So, the terms *initial*, *final*, *basic*, *nonbasic*, etc. have to be redefined for each of the new major blocks $B_{ii},\ i = 1(1)k$. Therefore, the application of the Power Method Theorem 2.4 and/or of Algorithm $\mathbb{AH}$ to A is equivalent to its application to each B_{ii} separately. Obviously, if $k = p$, A is block diagonal with each block being an irreducible matrix.

Example 1. Consider the matrix

$$A = \begin{bmatrix} A_{11} & 0 & 0 & A_{14} & 0 \\ 0 & A_{22} & A_{23} & 0 & A_{25} \\ 0 & 0 & A_{33} & 0 & A_{35} \\ 0 & 0 & 0 & A_{44} & 0 \\ 0 & 0 & 0 & 0 & A_{55} \end{bmatrix}. \tag{4.2}$$

It is seen that $G(A)$ (Fig. 1) consists of two disjoint subgraphs. One has vertices the nodes $\{1, 4\}$ and the other subgraph the nodes $\{2, 3, 5\}$. So, there is an obvious block similarity permutation that puts A into the form

$$A = \operatorname{diag}(B_{11}, B_{22}),\ \text{where } B_{11} = \begin{bmatrix} A_{11} & A_{14} \\ 0 & A_{44} \end{bmatrix}, \quad B_{22} = \begin{bmatrix} A_{22} & A_{23} & A_{25} \\ 0 & A_{33} & A_{35} \\ 0 & 0 & A_{55} \end{bmatrix}. \tag{4.3}$$

Then, the Power Method Theorem (and Algorithm $\mathbb{AH}$) is applied to B_{11} and B_{22} separately.

(b) If A is of the form (4.1), with $a_{ii} = 1,\ i = 1(1)n$, and its graph, $G(A)$, does **not** consist of a union of disjoint subgraphs. Then applying a block similarity permutation on A, say QAQ^{T}, preserving the Frobenius normal form, its basic classes are put, in increasing order of their heights, into major principal blocks as final classes. If any two or more basic classes are of the same height they are put in the same major principal block in any order. If there are nonbasic classes that do not access any basic one they are put in a separate last major principal block. In this way, in each major principal block with one or more final basic classes α_r, all the nonbasic classes have access to at least one of the α_r's. If there is a last major principal block of nonbasic classes, let it be denoted by $\widetilde{A}$, its graph $G(\widetilde{A})$ is considered and depending on whether $G(\widetilde{A})$ consists of the union of disjoint subgraphs or not the procedure in (a) before or the present one is followed. The rearrangement proposed is the one in [13] and [3], apart from the last major principal block of nonbasic classes.

From the theory presented so far we can state, without any formal proof, the following proposition whose validity will be made clear by a more general example.

Theorem 4.4. *Under the assumptions of* (b) *previously, the application of the Power Method Theorem* 2.4 *and/or Algorithm* $\mathbb{AH}$ *to the new form of A makes all row sums of the major principal blocks corresponding to basic classes tend to* $\rho(A)$ *in the limit. If there is a last major principal block of nonbasic classes that do not have access to any basic one, let it be* $\widetilde{A}$, *then, by following the previous described rules and depending on* $G(\widetilde{A})$, *the application of Algorithm* $\mathbb{AH}$ *makes the row sums of* $\widetilde{A}$ *tend to limits that are strictly less than* $\rho(A)$.

Having raised some basic issues in (a) and (b) above, we present and treat a more general example, where all the previously raised issues will be discussed and made clear.

Example 2

$$\begin{bmatrix}
A_{11} & 0 & 0 & 0 & A_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A_{22} & 0 & A_{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A_{33} & A_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A_{44} & 0 & A_{46} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_{55} & A_{56} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A_{66} & A_{67} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{77} & A_{78} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{88} & A_{89} & A_{8,10} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{99} & 0 & A_{9,11} & A_{9,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{10,10} & 0 & 0 & A_{10,13} & A_{10,14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{11,11} & 0 & A_{11,13} & A_{11,14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{12,12} & A_{12,13} & A_{12,14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{13,13} & A_{13,14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{14,14}
\end{bmatrix}. \tag{4.4}$$

Suppose that A in (4.4) is already in its normalized Frobenius normal form (4.1). Its graph $G(A)$ (Fig. 2), which does not consist of the union of disjoint subgraphs, contains fourteen classes $(\alpha_1, \alpha_2, \ldots, \alpha_{14})$ and suppose that the basic ones are $\alpha_5, \alpha_6, \alpha_8, \alpha_{10}, \alpha_{12}$. Making the previously suggested block similarity permutation, say QAQ^{T}, we have the new block matrix below and which, to simplify the notation, is denoted again by A

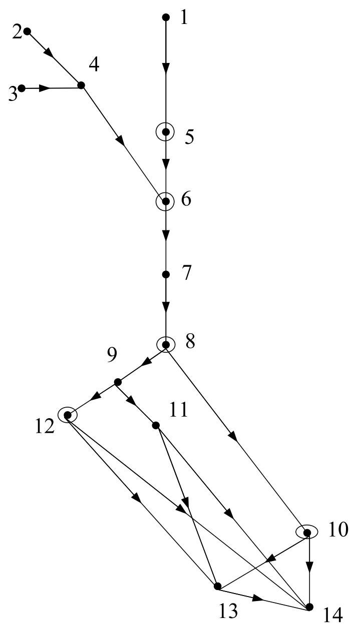

Fig. 2. Block graph of A in (4.4), where the basic classes are encircled.

$$
\left[\begin{array}{cc|cccc|cc|ccc|ccc}
A_{11} & A_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A_{55} & 0 & 0 & 0 & A_{56} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & A_{22} & 0 & A_{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A_{33} & A_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_{44} & A_{46} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A_{66} & A_{67} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & A_{77} & A_{78} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{88} & A_{8,10} & A_{89} & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{10,10} & 0 & 0 & 0 & A_{10,13} & A_{10,14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{99} & A_{9,12} & A_{9,11} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{12,12} & 0 & A_{12,13} & A_{12,14} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{11,11} & A_{11,13} & A_{11,14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{13,13} & A_{13,14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{14,14}
\end{array}\right]. \tag{4.5}
$$

As is seen, we have made a block partitioning in the new form of A so that the first four major principal blocks have in each of them the basic classes as final, while all three diagonal blocks of the fifth major principal block are nonbasic. Note that the heights of the basic blocks (classes) in the new block partitioning are $1, 2, 3, 4, 4$, as they were before, so $\operatorname{index}(\rho(A)I - A) = 4$, with the last two basic classes α_{10} and α_{12} belonging to the fourth major principal block. By virtue of Theorem 4.3, the new A has one nonnegative eigenvector and three nonnegative generalized eigenvectors as follows:

$$\begin{aligned}
x &= [x_1^{\mathrm{T}}\, x_5^{\mathrm{T}}\, 0_{n_2}^{\mathrm{T}}\, 0_{n_3}^{\mathrm{T}}\, 0_{n_4}^{\mathrm{T}}\, 0_6^{\mathrm{T}}\, 0_{n_7}^{\mathrm{T}}\, 0_{n_8}^{\mathrm{T}}\, 0_{n_{10}}^{\mathrm{T}}\, 0_{n_9}^{\mathrm{T}}\, 0_{n_{12}}^{\mathrm{T}}\, 0_{n_{11}}^{\mathrm{T}}\, 0_{n_{13}}^{\mathrm{T}}\, 0_{n_{14}}^{\mathrm{T}}]^{\mathrm{T}},\\
y &= [y_1^{\mathrm{T}}\, y_5^{\mathrm{T}}\, y_2^{\mathrm{T}}\, y_3^{\mathrm{T}}\, y_4^{\mathrm{T}}\, y_6^{\mathrm{T}}\, 0_{n_7}^{\mathrm{T}}\, 0_{n_8}^{\mathrm{T}}\, 0_{n_{10}}^{\mathrm{T}}\, 0_{n_9}^{\mathrm{T}}\, 0_{n_{12}}^{\mathrm{T}}\, 0_{n_{11}}^{\mathrm{T}}\, 0_{n_{13}}^{\mathrm{T}}\, 0_{n_{14}}^{\mathrm{T}}]^{\mathrm{T}},\\
u &= [u_1^{\mathrm{T}}\, u_5^{\mathrm{T}}\, u_2^{\mathrm{T}}\, u_3^{\mathrm{T}}\, u_4^{\mathrm{T}}\, u_6^{\mathrm{T}}\, u_7^{\mathrm{T}}\, u_8^{\mathrm{T}}\, 0_{n_{10}}^{\mathrm{T}}\, 0_{n_9}^{\mathrm{T}}\, 0_{n_{12}}^{\mathrm{T}}\, 0_{n_{11}}^{\mathrm{T}}\, 0_{n_{13}}^{\mathrm{T}}\, 0_{n_{14}}^{\mathrm{T}}]^{\mathrm{T}},\\
v &= [v_1^{\mathrm{T}}\, v_5^{\mathrm{T}}\, v_2^{\mathrm{T}}\, v_3^{\mathrm{T}}\, v_4^{\mathrm{T}}\, v_6^{\mathrm{T}}\, v_7^{\mathrm{T}}\, v_8^{\mathrm{T}}\, v_{10}^{\mathrm{T}}\, v_9^{\mathrm{T}}\, v_{12}^{\mathrm{T}}\, 0_{n_{11}}^{\mathrm{T}}\, 0_{n_{13}}^{\mathrm{T}}\, 0_{n_{14}}^{\mathrm{T}}]^{\mathrm{T}},
\end{aligned} \tag{4.6}$$

where any nonzero subvector is positive. To find the left eigenvector(s) of A we have to find the eigenvector(s) of A^{T}. The graph of A^{T} is nothing but that of Figure 2, where the arrows on it point in the opposite directions. This means that the heights of the various basic classes in A^{T} will be the complements of those in A with respect to the highest previous height increased by one ($\nu(A) + 1 = 4 + 1 = 5$). The nonbasic classes that were in the fifth major principal block will have height 0. Thus, the basic class(es) of A^{T} that have height 1, according to Remark 4.2, will give the left eigenvector(s) of A corresponding to $\rho(A)$. This will be the one associated with the classes α_{10} and α_{12}. Those associated with the basic classes $\alpha_5, \alpha_6, \alpha_8$ will give generalized eigenvectors (see Theorem 2.3.20 of [2]). By writing down analytically the 14 block equations from $A^{\mathrm{T}}v' = \rho(A)v'$ it is found out that the aforementioned left eigenvectors are

$$v' = [0_{n_1}^{\mathrm{T}}\, 0_{n_5}^{\mathrm{T}}\, 0_{n_2}^{\mathrm{T}}\, 0_{n_3}^{\mathrm{T}}\, 0_{n_4}^{\mathrm{T}}\, 0_6^{\mathrm{T}}\, 0_{n_7}^{\mathrm{T}}\, 0_{n_8}^{\mathrm{T}}\, {v'_{10}}^{\mathrm{T}}\, 0_{n_9}^{\mathrm{T}}\, {v'_{12}}^{\mathrm{T}}\, 0_{n_{11}}^{\mathrm{T}}\, {v'_{13}}^{\mathrm{T}}\, {v'_{14}}^{\mathrm{T}}]^{\mathrm{T}}, \tag{4.7}$$

where v'_{10} and v'_{12} are the *Perron vectors* of $A_{10,10}^{\mathrm{T}}$ and $A_{12,12}^{\mathrm{T}}$, respectively, and

$$\begin{aligned}
v'_{13} &= (\rho(A)I_{n_{13}} - A_{13,13}^{\mathrm{T}})^{-1}(A_{10,13}^{\mathrm{T}}v'_{10} + A_{12,13}^{\mathrm{T}}v'_{12}) > 0,\\
v'_{14} &= (\rho(A)I_{n_{14}} - A_{14,14}^{\mathrm{T}})^{-1}(A_{10,14}^{\mathrm{T}}v'_{10} + A_{12,14}^{\mathrm{T}}v'_{12} + A_{13,14}^{\mathrm{T}}v'_{13}) > 0.
\end{aligned}$$

From (4.6) and (4.7) one readily gets that

$$v'^{\mathrm{T}}x = 0, \quad v'^{\mathrm{T}}y = 0, \quad v'^{\mathrm{T}}u = 0, \quad v'^{\mathrm{T}}v > 0. \tag{4.8}$$

Suppose, without loss of generality, the new A undergoes one more similarity transformation, with permutation matrix $\widehat{Q}$, so that $\widehat{Q}A\widehat{Q}^{\mathrm{T}}$, denoted by A again, is put in its Jordan canonical form and at the same time indicates that $v \in \mathscr{N}((\rho(A)I_l - A)^4)$, where $l = n - n_{11} - n_{13} - n_{14}$. More specifically, the first l components of v, let them constitute the subvector $\widetilde{v} \in \mathbb{R}^l$, will be such that $\widetilde{v} \in \mathscr{N}((\rho(A)I_l - A[\alpha_1, \alpha_5, \alpha_3, \alpha_4, \alpha_6, \alpha_7, \alpha_8, \alpha_{10}, \alpha_9, \alpha_{12}])^4)$.[4] It is

$$A[x\, y\, u\, v\, s_5\, \ldots\, s_n] = [x\, y\, u\, v\, s_5\, \ldots\, s_n]\left[\begin{array}{cccc|c}
\rho(A) & 1 & & & \\
 & \rho(A) & 1 & & \\
 & & \rho(A) & 1 & \\
 & & & \rho(A) & \\
\hline
 & & & & S'
\end{array}\right], \tag{4.9}$$

from which one takes

$$Ax = \rho(A)x, \quad Ay = x + \rho(A)y, \quad Au = y + \rho(A)u, \quad Av = u + \rho(A)v. \tag{4.10}$$

It is reminded that the vectors x, y, u, v used in (4.9), are the ones in (4.6) premultiplied by $\widehat{Q}$. Let $d^{(0)} = e$, and suppose that $d^{(0)}$ is written as a linear combination of the generalized eigenvectors of S. It will be $d^{(0)} = \eta_1 x + \eta_2 y + \eta_3 u + \eta_4 v + \sum_{i=5}^{n} \eta_i s_i$. Forming $v'^{\mathrm{T}}d^{(0)} > 0$ and taking into account (4.8) and Theorem 2.5 we have $0 < v'^{\mathrm{T}}d^{(0)} = \eta_4 v'^{\mathrm{T}}v$, from which

[4] The second author would like to express his sincere thanks to Professor Hans Schneider [15] for making clear to him a point regarding the *index* of a nonnegative matrix in a 2 × 2 block *reducible* case.

$\eta_4 > 0$, and so $d^{(0)}$ has a positive component along the generalized eigenvector v. Using successively relations (4.10) we can obtain by induction that

$$\begin{aligned} A^k x &= \rho^k(A)x, \\ A^k y &= \binom{k}{1}\rho^{k-1}(A)x + \rho^k(A)y, \\ A^k u &= \binom{k}{2}\rho^{k-2}(A)x + \binom{k}{1}\rho^{k-1}(A)y + \rho^k(A)u, \\ A^k v &= \binom{k}{3}\rho^{k-3}(A)x + \binom{k}{2}\rho^{k-2}(A)y + \binom{k}{1}\rho^{k-1}(A)u + \rho^k(A)v. \end{aligned} \tag{4.11}$$

Therefore,

$$\begin{aligned} A^k d^{(0)} = {} & \rho^k(A)\left[\eta_1 + \binom{k}{1}\frac{\eta_2}{\rho(A)} + \binom{k}{2}\frac{\eta_3}{\rho^2(A)} + \binom{k}{3}\frac{\eta_4}{\rho^3(A)}\right]x \\ & + \rho^k(A)\left[\eta_2 + \binom{k}{1}\frac{\eta_3}{\rho(A)} + \binom{k}{2}\frac{\eta_4}{\rho^2(A)}\right]y \\ & + \rho^k(A)\left[\eta_3 + \binom{k}{1}\frac{\eta_4}{\rho(A)}\right]u + \rho^k(A)\eta_4 v + \rho^k(A)\sum_{i=5}^{n} A^k\frac{\eta_i}{\rho^k(A)}s_i. \end{aligned} \tag{4.12}$$

Forming the ratios $\frac{(A^{k+1}d^{(0)})_j}{(A^k d^{(0)})_j}$ for all j's that do not correspond to the rows of the fifth major principal block (11th, 13th and 14th block rows), then, for the Power Method Theorem 2.4 and/or Algorithm $\mathbb{AH}$, we have $\lim_{k\to\infty}\frac{(A^{k+1}d^{(0)})_j}{(A^k d^{(0)})_j} = \rho(A)$, except for the rows corresponding to the fifth major principal block. A formal proof for the aforementioned convergence in a general case, which is an "obvious" extension of the present one, is to be given elsewhere. Here, we simply note the following: For the first $n_1 + n_5$ rows, one has to divide both terms of the fractions by k^3 before one takes limits, as $k \to \infty$. For the next $n_2 + n_3 + n_4 + n_6$ rows one has to divide both terms by k^2, for the following $n_7 + n_8$ rows by k, while for the subsequent $n_{10} + n_9 + n_{12}$ rows one takes limits without any further division by a power of k. Recall that all the aforementioned rows are actually in the positions the similarity permutation by $\widehat{Q}$ has brought them. On the other hand, it is readily seen that Algorithm $\mathbb{AH}$ applies to the fifth major principal block, $\widetilde{A}$, quite independently of its application to all the previous ones. So, we have to consider $G(\widetilde{A})$, which, in the present case, does not consist of the union of disjoint subgraphs, and define the terms basic, nonbasic, initial, final, etc classes for $\widetilde{A}$ locally. If, e.g., $\rho(A[\alpha_{13}]) > \rho(A[\alpha_{11}]),\ \rho(A[\alpha_{14}])$ then by Theorem 4.4 and the previous analysis we know that for $k \to \infty$ the limiting row sums of the first and second block rows of $\widetilde{A}$ will equal $\rho(A_{13,13})$ while those of the last block row will equal $\rho(A_{14,14})$.

Before we close this section we mention in passing that Corollaries analogous to Corollaries 3.1 and 3.2 can be stated formally. In the case of *Example* 2, the analogous to Corollary 3.1 will give that

$$d = (\widehat{Q}Q)^{\mathrm{T}}\widehat{d},\ \widehat{d} = [\widehat{d}_{\hat{1}}^{\mathrm{T}}\ \widehat{d}_{\hat{2}}^{\mathrm{T}}\ \widehat{d}_{\hat{3}}^{\mathrm{T}}]^{\mathrm{T}}, \tag{4.13}$$

where

$$\widehat{d}_{\hat{1}} = \lim_{k\to\infty}\left(\prod_{i=1}^{k} D^{(i)}_{n-n_{11}-n_{13}-n_{14}}\right)e_{n-n_{11}-n_{13}-n_{14}},$$

$$\widehat{d}_{\hat{2}} = \lim_{k\to\infty} \left(\prod_{i=1}^{k} D^{(i)}_{n_{11}+n_{13}} \right) e_{n_{11}+n_{13}}, \tag{4.14}$$
$$\widehat{d}_{\hat{3}} = \lim_{k\to\infty} \left(\prod_{i=1}^{k} D^{(i)}_{n_{14}} \right) e_{n_{14}}.$$

It should be noted that $\widehat{d}_{\hat{1}}$ is the *Perron vector* of $A[\alpha_1, \alpha_5, \alpha_3, \alpha_4, \alpha_6, \alpha_7, \alpha_8, \alpha_{10}, \alpha_9, \alpha_{12}]$, $\widehat{d}_{\hat{2}}$ is the *Perron vector* of $A[\alpha_{11}, \alpha_{13}]$ and $\widehat{d}_{\hat{3}}$ is the *Perron vector* of $A[\alpha_{14}]$, where A is the original matrix in its Frobenius form given in (4.4). The analogous statement to Corollary 3.2 will give that

$$\lim_{k\to\infty} A^{(k)}e = (\widehat{Q}Q)^{\mathrm{T}} \mathrm{diag}(\rho(A)I_{n-n_{11}-n_{13}-n_{14}},$$
$$\times \rho(A_{13,13})I_{n_{11}+n_{13}}, \rho(A_{14,14})I_{n_{14}})(\widehat{Q}Q)\, e, \tag{4.15}$$
$$\lim_{k\to\infty} a^{(k)}_{ij} = \frac{d_i}{d_j} a_{ij}, \quad i, j = 1(1)n.$$

5. The new algorithm

Based on the theory, the analysis and Example 2 of the previous section, we are ready to make some observations and present our new Algorithm which will be called Algorithm $\mathbb{AH}$**2**.

(a) Suppose that $A \in \mathbb{C}^{n,n}$, $a_{ii} \neq 0$, $i = 1(1)n$, is irreducible or reducible, with its Frobenius normal form (4.1) being normalized so that $a_{ii} = 1$, $i = 1(1)n$, and that all its basic classes are final. Then application of Algorithm $\mathbb{AH}$, as $k \to \infty$, will give as a limit a similar matrix whose all block rows will have row sums equal to $\rho(A)$. This means that the new Algorithm must coincide with Algorithm $\mathbb{AH}$.

(b) Suppose that A, as before, is reducible, with its Frobenius normal form (4.1) being normalized and that **not** all its basic classes are final. Then, application of Algorithm $\mathbb{AH}$, as $k \to \infty$, will bring us to a situation similar to that of *Example* 2 of Section 4. Namely, some of the block rows of the limiting matrix will have row sums equal to $\rho(A)$ while some others will have them strictly less than $\rho(A)$. So, we have to distinguish two subcases.

(b1) Suppose that the application of Algorithm $\mathbb{AH}$ to A, as $k \to \infty$, gives that **all** limiting block rows have sums $s_i = \lim_{k\to\infty} s^{(k)}_i \geqslant 1$, $i = 1(1)n$, (the $s^{(k)}_i$'s are defined in Step 4 of Algorithm $\mathbb{AH}$) or **all** limiting block rows have sums $s_i < 1$, $i = 1(1)n$. Then Algorithm $\mathbb{AH}$ makes the correct identification for A, that is "A is **not** an H-matrix" and "A **is** an H-matrix", respectively.

(b2) Suppose that the application of Algorithm $\mathbb{AH}$ to A, as $k \to \infty$, gives that some limiting block rows have sums $s_i \geqslant 1$ while some others have sums $s_i < 1$. Then Algorithm $\mathbb{AH}$ **cannot** make any identification for A, which should have been "A is **not** an H-matrix". This is the **only** case where Algorithm $\mathbb{AH}$ needs modification in such a way as to cope with the situation just described.

To present our new Algorithm, let

$$\mathbb{N} := \{1, 2, \ldots, n\},\ \mathbb{N}^{(k)}_0 \equiv \mathbb{N}_0(A^{(k)}) := \left\{ i \in \mathbb{N} : s^{(k)}_i \geqslant 1 \right\}, \tag{5.1}$$

where $n^{(k)}_0 := n_0(A^{(k)})$ the cardinality of $\mathbb{N}^{(k)}_0$.

Algorithm $\mathbb{AH}$2

INPUT: A matrix $A := [a_{ij}] \in \mathbb{C}^{n,n}$ and a maximum number of iterations allowed ("maxit")

OUTPUT: $D = D^{(0)}D^{(1)}\cdots D^{(k)} \in \mathfrak{D}_{D^{-1}A} \equiv \mathfrak{D}_A$ or $\notin \mathfrak{D}_A$ if A **is** or is **not** an H-matrix, respectively.

1. If $a_{ii} = 0$ for some $i \in \mathbb{N}$, "A is **not** an H-matrix", STOP; Otherwise
2. Set $D = I$, $A^{(0)} = (\text{diag}(A))^{-1}A$, $D^{(0)} = I, k = 1$
3. Compute $D = DD^{(k-1)}$, $A^{(k)} = (D^{(k-1)})^{-1}A^{(k-1)}D^{(k-1)} = [a_{ij}^{(k)}]$
4. Compute $s_i^{(k)} = \sum_{j=1,\ j\neq i}^{n} |a_{ij}^{(k)}|,\ i = 1(1)n,\ s^{(k)} = \min_{i=1(1)n} s_i^{(k)},\ S^{(k)} = \max_{i=1(1)n} s_i^{(k)}$
5. If $s^{(k)} > 1$, "A is **not** an H-matrix", STOP; Otherwise
6. If $S^{(k)} < 1$, "A **is** an H-matrix", STOP; Otherwise
7. If $S^{(k)} = s^{(k)}$, "$\mathscr{M}(A)$ is **singular**", STOP; Otherwise
8. Set $d = [d_i]$, where
$$d_i = \frac{1 + s_i^{(k)}}{1 + S^{(k)}}, \quad i = 1(1)n$$
9. Set $D^{(k)} = \text{diag}(d)$, If $k <$ maxit, $k = k + 1$, Go to Step 3; Otherwise
10. Determine $\mathbb{N}_0^{(\text{iter})}$ and $n_0^{(\text{iter})}$
11. If $n_0^{(\text{iter})} = 1$, "Inconclusive, increase maxit", STOP; Otherwise
12. Compute
$$s_{i_j}^{(\text{iter})} = \sum_{l=1,\ l\neq j}^{n_0^{(\text{iter})}} |a_{i_j,i_l}^{(\text{iter})}|, \quad j = 1(1)n_0^{(\text{iter})}, \quad i_j, i_l \in \mathbb{N}_0^{(\text{iter})}$$
13. If $s_{i_j}^{(\text{iter})} \geqslant 1,\ j = 1(1)n_0^{(\text{iter})},\ i_j \in \mathbb{N}_0^{(\text{iter})}$, "$A$ is **not** an H-matrix", STOP; Otherwise
14. Update $\mathbb{N}_0^{(\text{iter})}$ (by discarding $i_j \in \mathbb{N}_0^{(\text{iter})} : s_{i_j} < 1$) and $n_0^{(\text{iter})}$; Go to Step 11. END

A couple of explanations should be given regarding the Algorithm above:

(a) In cases (a) and (b1) described in the observations preceding Algorithm $\mathbb{AH}$**2** the exit of the Algorithm from one of the Steps 5, 6 or 7 is guaranteed, since then the new Algorithm is nothing but Algorithm $\mathbb{AH}$, provided, of course, that "maxit" is big enough.

(b) In case (b2) the exhaustion of "maxit" means one of two things. Either the Algorithm converges very slowly, in which case "maxit" should be increased, or the matrix is **not** an H-matrix. To check if A is **not** an H-matrix we appeal to Lemma 2.6. So, we consider the principal submatrix of $A^{(\text{iter})}$ that consists of the $n_0^{(\text{iter})}$ rows and the corresponding columns for which the sums $s_{i_j}^{(\text{iter})}$, restricted to the submatrix in question, are $\geqslant 1$. If all these $n_0^{(\text{iter})}$ rows of the principal submatrix satisfy the same inequalities, then this submatrix is **not** an H-matrix. Hence, by Lemma 2.6, A is **not** an H-matrix. If not all the rows of this submatrix satisfy the previous inequalities then we discard the rows (and columns) whose sums $s_{i_j}^{(\text{iter})}$ are < 1, we update our information by considering a strictly smaller principal submatrix whose rows have sums $s_{i_j}^{(\text{iter})} \geqslant 1$. This procedure leads to either the conclusion that a smaller principal submatrix is **not** an H-matrix in which case **nor** is A or to a 1×1 submatrix in which case no conclusion can be drawn and so "maxit" should be increased.

We close this section by stating two theorems the proofs of which can be directly drawn from the analysis made so far and from the corresponding proofs of theorems in [1] already presented in Section 3 as Theorems 3.1 and 3.2.

Theorem 5.1. *Let* $A \in \mathbb{C}^{n,n}$ *be any given matrix. Then Algorithm*, $\mathbb{A}\mathbb{H}\mathbf{2}$ *always terminates* (*except*, *maybe*, *when* $\det(\mathscr{M}(A)) = 0$) *in a finite number of iterations.*

Theorem 5.2. *Let* $A \in \mathbb{C}^{n,n}$ *be any given matrix. If Algorithm* $\mathbb{A}H\mathbf{2}$ *terminates in a finite number of iterations*, *then its output is correct.*

6. Numerical examples

To cover all possible cases that were studied previously we have examined many examples some of which are given below.

Example 1. The irreducible matrix A of the example of [5]:

$$A = \begin{bmatrix} -1 & a_{12} & 0 & 0 & 0 \\ 0.5 & -1 & 0 & -0.6 & 0 \\ 0 & -0.1 & 1 & 0 & 0.5 \\ 0 & 0.5 & 0 & 1 & -0.5 \\ -0.2 & 0.1 & 0.3 & 0 & -1 \end{bmatrix}.$$

For $a_{12} = 1.146392$, by application of Algorithm $\mathbb{A}\mathbb{H}$ (or Algorithm $\mathbb{A}\mathbb{H}\mathbf{2}$) we have as *OUTPUT*: "$A$ is NOT an H-matrix", $s_{\min}^{(37)} = 1.00000002036218$.

Example 2. For the irreducible matrix:

$$\begin{bmatrix} 1 & 0.01 & 0.02 & 0.01 & 0.03 & 0.01 \\ 0.05 & 1 & 0.1 & 0.02 & 0.01 & 0.01 \\ 0.01 & 0.01 & 1 & 1.001 & 0.01 & 0.01 \\ 0.01 & 0.03 & 1.002 & 1 & 0.01 & 0.02 \\ 0.02 & 0.01 & 0.02 & 0.01 & 1 & 0.1 \\ 0.07 & 0.01 & 0.01 & 0.01 & 0.01 & 1 \end{bmatrix},$$

by application of Algorithm $\mathbb{A}\mathbb{H}$ we have *OUTPUT*: "A is NOT an H-matrix", $s_{\min}^{(16)} = 1.00163711553673$, whereas by application of Algorithm $\mathbb{A}\mathbb{H}\mathbf{2}$ the *OUTPUT* is the same in 8 iterations. Specifically, $A^{(8)}[3, 4]$ is not an H-matrix.

Example 3. Consider the following reducible matrix already in its Frobenius normal form with unit diagonal elements, whose basic classes are $\alpha_2,\ \alpha_3$ are not both final

$$\left[\begin{array}{cc|ccc|cc} 1 & 0.001 & 0 & 0 & 0 & 0 & 0.03 \\ 0.02 & 1 & 0 & 0 & 0 & 0.01 & 0 \\ \hline 0 & 0 & 1 & 0 & 0.1 & 0.03 & 0.01 \\ 0 & 0 & 20 & 1 & 0 & 0.05 & 0.01 \\ 0 & 0 & 0 & 4 & 1 & 0.03 & 0.02 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

It can be found out that

$$\rho(A[1]) < 1 < \rho(A[2]) = \rho(A_{33}) = \rho(A)$$

and $O_{12} = O$. For this matrix we have that *OUTPUT*: "A is NOT an H-matrix", $s_{\min}^{(6)} =$ 1.26837418775458, with Algorithm $\mathbb{AH}$. However, that A is not an H-matrix can be obtained with Algorithm $\mathbb{AH}2$ in 2 iterations. Specifically, $A^{(2)}[6, 7]$ is not an H-matrix.

Example 4. Consider the following reducible matrix already in its Frobenius normal form with unit diagonal elements:

$$\left[\begin{array}{c|cc|ccc|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0.8 & 0 & 0 \\ \hline
0 & 1 & 0.1 & 0.02 & 0.01 & 0.01 & 0 & 0 & 0.2 & 0.3 \\
0 & 0.1 & 1 & 0.1 & 0.2 & 0.03 & 0 & 0 & 0.3 & 0.2 \\ \hline
0 & 0 & 0 & 1 & 0.01 & 1.02 & 0 & 0 & 0.1 & 0.8 \\
0 & 0 & 0 & 1 & 1 & 0.1 & 0 & 0 & 0.1 & 0.2 \\
0 & 0 & 0 & 0.01 & 0.1 & 1 & 0 & 0 & 0.8 & 0.4 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 1 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.6 & 1
\end{array}\right]$$

and

$$\rho(A_{11}) = \rho(A_{44}) < \rho(A_{22}) = \rho(A_{33}) = \rho(A_{55}) = \rho(A).$$

It works only with Algorithm $\mathbb{AH}\mathbf{2}$ and it can be found out that "A is NOT an H-matrix" in 1 iteration. Specifically, $A^{(1)}$ [4, 5, 6, 9, 10] is not an H-matrix.

Example 5. Consider the matrix in (4.4) with block submatrices:

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 0.25 & 1 \end{bmatrix}, \quad A_{22} = [1], \quad A_{33} = \begin{bmatrix} 1 & 2 & 0.1 \\ 0.1 & 1 & 0.1 \\ 0.2 & 0.3 & 1 \end{bmatrix}, \quad A_{44} = \begin{bmatrix} 1 & 0.3 \\ 1.5 & 1 \end{bmatrix},$$

$$A_{55} = \begin{bmatrix} 1 & 3.6 \\ 0.4 & 1 \end{bmatrix}, \quad A_{66} = \begin{bmatrix} 1 & 7.2 \\ 0.2 & 1 \end{bmatrix}, \quad A_{77} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix},$$

$$A_{88} = \begin{bmatrix} 1 & 0.32 \\ 4.5 & 1 \end{bmatrix}, \quad A_{99} = \begin{bmatrix} 1 & 0.4 \\ 0.8 & 1 \end{bmatrix}, \quad A_{10,10} = \begin{bmatrix} 1 & 2.25 \\ 0.64 & 1 \end{bmatrix},$$

$$A_{11,11} = \begin{bmatrix} 1 & 0.1 \\ 0.2 & 1 \end{bmatrix}, \quad A_{12,12} = \begin{bmatrix} 1 & 72 \\ 0.02 & 1 \end{bmatrix}, \quad A_{13,13} = \begin{bmatrix} 1 & 0.2 \\ 6.05 & 1 \end{bmatrix},$$

$$A_{14,14} = \begin{bmatrix} 1 & 0.1 \\ 11.025 & 1 \end{bmatrix}, \quad A_{15} = \begin{bmatrix} 0.1 & 0.03 \\ 0.02 & 0.01 \end{bmatrix}, \quad A_{24} = \begin{bmatrix} 0.01 & 0.05 \end{bmatrix},$$

$$A_{34} = \begin{bmatrix} 0.002 & 0.01 \\ 0.003 & 0.004 \\ 0.05 & 0.06 \end{bmatrix}, \quad A_{46} = \begin{bmatrix} 0.07 & 0.08 \\ 0.09 & 0.01 \end{bmatrix}, \quad A_{56} = \begin{bmatrix} 0.1 & 0.12 \\ 0.12 & 0.1 \end{bmatrix},$$

$$A_{67} = \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.3 \end{bmatrix}, \quad A_{78} = \begin{bmatrix} 0.15 & 0.25 \\ 0.2 & 0.1 \end{bmatrix}, \quad A_{89} = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix},$$

$$A_{8,10} = \begin{bmatrix} 0.2 & 0.3 \\ 0.2 & 0.4 \end{bmatrix}, \quad A_{9,11} = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.1 \end{bmatrix}, \quad A_{9,12} = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.2 \end{bmatrix},$$

$$A_{10,13} = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.3 \end{bmatrix}, \quad A_{10,14} = \begin{bmatrix} 0.2 & 0.3 \\ 0.4 & 0.5 \end{bmatrix},$$
$$A_{11,13} = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.4 \end{bmatrix}, \quad A_{11,14} = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix},$$
$$A_{12,13} = \begin{bmatrix} 0.15 & 0.25 \\ 0.25 & 0.35 \end{bmatrix}, \quad A_{12,14} = \begin{bmatrix} 0.15 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}, \quad A_{13,14} = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{bmatrix}$$

and basic classes $\alpha_5, \alpha_6, \alpha_8, \alpha_{10}, \alpha_{12}$, with

$$\rho(A) > \rho(A_{13,13}) > \rho(A_{14,14}) > 1.$$

OUTPUT: "A is NOT an H-matrix", $s_{\min}^{(10)} = 1.05000000000000$, with Algorithm $\mathbb{AH}$, and the same result in 1 iteration, with Algorithm $\mathbb{AH}2$. Particularly, $A^{(1)}[15, 16]$ is not an H-matrix.

Example 6. Similar to *Example* 5 except that

$$A_{13,13} = \begin{bmatrix} 1 & 0.2 \\ 0.8 & 1 \end{bmatrix}, \quad A_{14,14} = \begin{bmatrix} 1 & 0.1 \\ 0.4 & 1 \end{bmatrix}$$

and

$$\rho(A) > 1 > \rho(A_{13,13}) > \rho(A_{11,11}), \quad \rho(A_{14,14}).$$

Obviously Algorithm $\mathbb{AH}$ does not work. On the other hand, *OUTPUT*: "A is NOT an H-matrix" in 1 iteration with Algorithm $\mathbb{AH}2$. Particularly, $A^{(1)}[15, 16]$ is not an H-matrix.

Example 7. Similar to *Example* 5 except that

$$A_{55} = \begin{bmatrix} 1 & 1.6 \\ 0.4 & 1 \end{bmatrix}, \quad A_{66} = \begin{bmatrix} 1 & 3.2 \\ 0.2 & 1 \end{bmatrix}, \quad A_{10,10} = \begin{bmatrix} 1 & 1 \\ 0.64 & 1 \end{bmatrix},$$
$$A_{13,13} = \begin{bmatrix} 1 & 0.2 \\ 0.8 & 1 \end{bmatrix}, \quad A_{14,14} = \begin{bmatrix} 1 & 0.1 \\ 0.4 & 1 \end{bmatrix}$$

and

$$1 > \rho(A) > \rho(A_{13,13}) > \rho(A_{11,11}), \quad \rho(A_{14,14}).$$

OUTPUT: "A IS an H-matrix", $s_{\max}^{(49)} = 0.28796103805572$, with either Algorithm.

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References

[1] M. Alanelli, A. Hadjidimos, A New Iterative Criterion for H-Matrices, SIAM J. Matrix Anal. Appl. 29 (2007) 160–176.

[2] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Classics in Applied Mathematics, SIAM, Philadelphia, 1994.

[3] R. Bru, M. Neumann, Nonnegative Jordan basis, Linear Multilinear Algebra 23 (1988) 95–109.

[4] D.K. Faddeev, V.N. Faddeeva, Computational Methods of Linear Algebra, W.H. Freeman, San Francisco, 1963.

[5] A. Hadjidimos, An extended compact profile iterative method criterion for sparce *H*-matrices, Linear Algebra Appl. 389 (2004) 329–345.

[6] M. Harada, M. Usui, H. Niki, An extension of the criteria for generalized diagonally dominant matrices, Int. J. Comput. Math. 60 (1996) 115–119.

[7] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, England, 1985.

[8] D.R. Kincaid, J.R. Respess, D.M. Young, R.G. Grimes, ITPACK 2C: A Fortran package for solving large sparse linear systems by adaptive accelerated iterative methods, ACM Trans. Math. Software 8 (1982) 302–322.

[9] T. Konho, H. Niki, H. Sawami, Y.-M. Gao, An iterative test for *H*-matrix, J. Comput. Appl. Math. 115 (2000) 349–355.

[10] B. Li, L. Li, M. Harada, H. Niki, M.J. Tsatsomeros, An iterative criterion for *H*-matrices, Linear Algebra Appl. 271 (1998) 179–190.

[11] L. Li, On the iterative criterion for generalized diagonally dominant matrices, SIAM J. Matrix Anal. Appl. 24 (2002) 17–24.

[12] K. Ojiro, H. Niki, M. Usui, A new criterion for *H*-matrices, J. Comput. Appl. Math. 150 (2003) 293–302.

[13] U.G. Rothblum, Algebraic eigenspaces of nonnegative matrices, Linear Algebra Appl. 12 (1975) 281–292.

[14] H. Schneider, The influence of the marked reduced graph of a nonnegative matrix on the Jordan form and related properties: a survey, Linear Algebra Appl. 84 (1986) 161–189.

[15] H. Schneider, Personal Communication, 2006.

[16] R.S. Varga, Matrix Iterative Analysis, 2nd revised and expanded ed., Springer, Berlin, 2000.

[17] R.S. Varga, Notes on Power Method for Nonnegative Matrices, Personal Communication, 2006.

[18] J.H. Wilkinson, The Algebraic Eigenvalue Problem, Clarendon Press, Oxford, 1965.

[19] D.M. Young, Iterative Solution of Large Linear Systems, Academic Press, New York, 1971.