On Leibniz series defined by convex functions

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Abstract

It is shown that for every \( \alpha > 0 \), we have

\[
\left| \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k^\alpha} \right| = \frac{1}{2(n + \theta_n)^\alpha}
\]

for some strictly decreasing sequence \((\theta_n)_{n \geq 1}\) such that

\[
\frac{1}{2} < \theta_n < \frac{1}{2} \left( 1 + \frac{1}{2n + 1} \right)^{\frac{\alpha + 1}{\alpha}}.
\]

hence with \( \lim_{n \to \infty} \theta_n = \frac{1}{2} \). This is only a particular case of more general new results on Leibniz series defined by convex functions.

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1. Introduction

Let \( f : [1, \infty[ \to ]0, \infty[ \) be a continuous convex function, with \( \lim_{n \to \infty} f(n) = 0 \). We will show that
for some unique sequence \((\theta_n)_{n \geq 1} \subset [0, 1]\). Under reasonable assumptions this sequence converges to some \(\theta \in \left[\frac{1}{3}, 1\right]\). In this case, among all expressions \(\frac{f(n+1)}{f(n)}\), the best asymptotic approximation for \(n\)th remainder’s absolute value is obtained for \(\lambda = \theta\). As we shall see, for all alternating generalized harmonic series

\[
\sum_{n \geq 1} \frac{(-1)^{n-1}}{n^\alpha} \quad (\alpha > 0),
\]

the sequence \((\theta_n)_{n \geq 1}\) is strictly decreasing, with \(\lim_{n \to \infty} \theta_n = \frac{1}{3}\). This “half integer” optimality is strongly related to slow convergence \((\lim_{n \to \infty} \frac{f(n+1)}{f(n)} = 1)\) of the series. If the ratio test limit is less than 1, then \(\frac{1}{3}\) is no longer optimal.

Let us recall that approximations for partial sums in terms of \(n + \frac{1}{2}\) were used in [1] for the harmonic series (slowly divergent!), and in a hidden form in [6]. In the latter, for the alternating harmonic series (slowly convergent!), \(n\)th remainder’s absolute value is expressed as

\[
\left| \sum_{k=n+1}^{\infty} (-1)^{k-1} f(k) \right| = \frac{1}{2n + x_n}.
\]

The main result from [6] states that the sequence \((x_n)_{n \geq 1}\) is strictly decreasing and provides good estimates for its convergence to 1. If we write this series as \(\sum_{n \geq 1} (-1)^{n-1} f(n)\) for \(f(x) = \frac{1}{x}\), then

\[
\frac{1}{2n + x_n} = \frac{1}{2} f\left(n + \frac{x_n}{2}\right).
\]

Thus the theorem from [6] actually has a half integer approximation nature.

Our main results (Theorems 3, 4, 7, and 10) are in the spirit of [5,6] and hold in particular for \(f(x) = \frac{1}{x^\alpha}\), with \(\alpha > 0\), hence for all alternating generalized harmonic series. For \(\alpha = 1\) we recover the results from [6].

2. Existence and convergence of \((\theta_n)_{n \geq 1}\)

Let \(f : [1, \infty[ \to [0, \infty[\) be a continuous convex function, with \(\lim_{n \to \infty} f(n) = 0\). As \(f\) must be strictly decreasing, \(\sum_{n \geq 1} (-1)^{n-1} f(n)\) is a Leibniz series, and hence converges. For every \(n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}\), set

\[
S_n := \sum_{k=1}^{n} (-1)^{k-1} f(k), \quad S := \lim_{n \to \infty} S_n,
\]

\[
\rho_n := |S - S_n| = (-1)^n (S - S_n) = \sum_{k=n+1}^{\infty} (-1)^{k-n-1} f(k).
\]
Proposition 1. There exists a unique sequence \((\theta_n)_{n \geq 1} \subset [0, 1]\), such that
\[
|S - S_n| = \frac{f(n + \theta_n)}{2} \quad \text{for every } n \in \mathbb{N}^*.
\] (2)

This sequence satisfies the recurrence relation
\[
f(n - 1 + \theta_{n-1}) + f(n + \theta_n) = 2f(n) \quad \text{for every } n \geq 2.
\] (3)

If \(f\) is strictly convex or differentiable, then \((\theta_n)_{n \geq 1} \subset [0, 1]\).

Proof. For every \(\alpha \geq 0\), let us define the sequence
\[
(X_n(\alpha))_{n \geq 1} \subset \mathbb{R}, \quad X_n(\alpha) := \rho_n - \frac{f(n + \alpha)}{2}.
\]

We clearly have \(\lim_{n \to \infty} X_n(\alpha) = 0\). By the convexity of \(f\), we deduce that
\[
X_{n+1}(0) - X_n(0) = \frac{f(n + 1) + f(n - 1)}{2} - f(n) \geq 0,
\] (4)
\[
X_{n+1}(1) - X_n(1) = f(n + 1) - \frac{f(n + 2) + f(n)}{2} \leq 0,
\] (5)
hence that \((X_{2n}(0))_{n \geq 1}\) and \((X_{2n+1}(0))_{n \geq 1}\) are increasing, and that \((X_{2n}(1))_{n \geq 1}\) and \((X_{2n+1}(1))_{n \geq 1}\) are decreasing. It follows that \(X_n(0) \leq 0 \leq X_n(1)\) for every \(n \in \mathbb{N}^*\).

We thus get the existence of a unique sequence \((\theta_n)_{n \geq 1} \subset [0, 1]\) satisfying (2), since \(f\) is continuous and strictly decreasing. The recurrence relation (3) is immediate, since \(\rho_n - \rho_{n-1} = f(n)\) for every \(n \geq 2\).

Now assume \(f\) to be strictly convex or differentiable, but \(X_m(0) = 0\) for some \(m \in \mathbb{N}^*\), that is, \(X_{m+2k}(0) = X_{m+2k+2}(0)\) for every \(k \in \mathbb{N}\). It follows that \(f|_{[m+2k,m+2k+2]}\) is an affine function\(^1\) (equality in (4)) for every \(k \in \mathbb{N}\). Thus, \(f|_{[m,\infty)}\) must be differentiable, since it is not strictly convex. We deduce that \(f|_{[m,\infty)}\) is affine, which is absurd, because \(f > 0\) and \(\lim_{n \to \infty} f(n) = 0\). Hence \(X_n(0) < 0\) for every \(n \in \mathbb{N}^*\). The proof of the inequality \(X_n(1) > 0\) is similar. We conclude that \((\theta_n)_{n \geq 1} \subset [0, 1]\).

Equality (2) can be regarded as a variant of Calabrese’s result, which asserts that if \(\sum_{n \geq 1} (-1)^{n-1}a_n\) is a Leibniz series and if \((a_n - a_{n+1})_{n \geq 1}\) is strictly decreasing, then \(\rho_n < a_n/2\) for every \(n \in \mathbb{N}^*\). This follows by (2), since there exists a strictly convex function \(f\) satisfying \(f|_{[0,\infty)} = (a_n)_{n \geq 1}\). We thus get the inequalities \(a_{n+1}/2 < \rho_n < a_n/2\) for \(n \in \mathbb{N}^*\), that is, Theorem 1.2 from [3].

Remark 2. We have \(\theta_n + \theta_{n-1} \geq 1\) for every \(n \geq 2\), hence \(\limsup_{n \to \infty} \theta_n \geq \frac{1}{2}\).

Indeed, by (3) and the convexity of \(f\) it follows that
\[
f(n) \geq f\left(n + \frac{\theta_n + \theta_{n-1} - 1}{2}\right),
\]
hence that \(\theta_n + \theta_{n-1} \geq 1\) for every \(n \geq 2\), since \(f\) is strictly decreasing.

\(^1\) That is, has the form \(x \mapsto \lambda x + \mu\) for some \(\lambda, \mu \in \mathbb{R}\).
Our next two results provide convergence tests for the sequence \((\theta_n)_{n \geq 1}\), as well as the value of its limit. Let us define

\[
\Lambda : [0, 1] \to \mathbb{R}, \quad \Lambda(x) = \begin{cases} 
1, & \text{if } x = 0, \\
\frac{\ln(2x/(x+1))}{\ln x}, & \text{if } x \in ]0, 1[, \\
\frac{1}{2}, & \text{if } x = 1.
\end{cases}
\]

It is easy to check that \(\Lambda\) is continuous and \(\frac{1}{2} \leq \Lambda \leq 1\). Hence \(\Lambda([0, 1]) = [\frac{1}{2}, 1]\).

**Theorem 3.** Assume that \(\lim_{x \to \infty} \frac{f(x + t)}{f(x)}\) exists\(^2\) for every \(t \in [0, 1]\), and that \(a := \lim_{n \to \infty} \frac{f(n+1)}{f(n)} < 1\). Then

\[
\lim_{n \to \infty} \theta_n = \Lambda(a).
\]

**Proof.** Let us first observe that

\[
\omega(t) := \lim_{x \to \infty} \frac{f(x + t)}{f(x)} \in [0, 1]
\]
exists for every \(t \geq 0\) (we can obtain it as a finite product of limits as in our statement), and that \(\omega : [0, \infty[ \to [0, 1]\) is decreasing, since so is \(f\). It is easily seen that \(\omega(t + s) = \omega(t)\omega(s)\) for all \(t, s \geq 0\). It follows that \(\omega(t) = a^t\) for every \(t > 0\), where \(a = \omega(1)\). According to the hypothesis, we have \(a \in [0, 1]\). To prove (6) we need to analyze two cases.

**Case 1.** If \(a \in ]0, 1[\), fix \(\theta \in ]0, 1[\). As \(f\) is convex, we have

\[
|2\rho_n - f(n + \theta)| = |f(n + \theta) - f(n + \theta)| \geq (f(n + 1) - f(n + 2))|\theta_n - \theta|,
\]
and consequently

\[
|\theta_n - \theta| \leq \left| 2 \frac{\rho_n}{f(n + 1)} - \frac{f(n + \theta)}{f(n + 1)} \right| \cdot \frac{1}{1 - \frac{f(n+2)}{f(n+1)}} \quad \text{for every } n \in \mathbb{N}^*.
\]

Since

\[
\lim_{n \to \infty} \frac{f(n + \theta)}{f(n + 1)} = a^{\theta - 1}.
\]

we shall next prove by applying Cesàro–Stolz theorem (see [4, p. 317]), that

\[
\lim_{n \to \infty} \frac{\rho_n}{f(n + 1)} = \frac{a^{\theta - 1}}{2}
\]

for suitable \(\theta\). We have

\[
\lim_{n \to \infty} \frac{\rho_{n+1} - \rho_{n-1}}{f(n + 2) - f(n)} = \lim_{n \to \infty} \frac{f(n+1)}{f(n)} - 1 = \frac{a - 1}{a^2 - 1} = \frac{1}{a + 1}.
\]

\(^2\) This holds if \(f\) is log-convex, that is, \(\ln(f)\) is a convex function.
Hence
\[
\lim_{n \to \infty} \frac{\rho_{2n}}{f(2n + 1)} = \lim_{n \to \infty} \frac{\rho_{2n}}{f(2n + 1)} = \lim_{n \to \infty} \frac{\rho_{2n-1}}{f(2n)} = \frac{1}{a + 1}.
\]
Since this limit equals \(a^{\theta - 1/2}\) for \(\theta = \Lambda(a)\), (7) yields (6).

Case 2. If \(a = 0\), fix \(\alpha \in [0, 1]\). As \(\lim_{n \to \infty} f(n+1) / f(n+\alpha) = 0\), there exists \(n_0 \in \mathbb{N}^*\), such that for every \(n \geq n_0\) we have \(f(n+1) / f(n+\alpha) < \frac{1}{4}\), hence
\[
f(n + \theta_n) = 2 \rho_n < 2 f(n + 1) < f(n + \alpha),
\]
that is, \(\alpha < \theta_n \leq 1\). We conclude that \(\lim_{n \to \infty} \theta_n = 1 = \Lambda(0)\). □

**Theorem 4.** If \(f\) is differentiable, and if \(f'\) is a concave function, then
\[
f(n + 1) - \frac{f(n + \frac{3}{2})}{2} < \left|S - S_n\right| < \frac{f(n + \frac{1}{2})}{2},
\]
(8)
\[
\frac{1}{2} < \theta_n < \frac{1}{2} \frac{f'(n + \frac{3}{2})}{f'(n + 1)},
\]
(9)
for every \(n \in \mathbb{N}^*\). In particular, if \(\lim_{n \to \infty} f'(n+1) / f(n) = 1\), then \(\lim_{n \to \infty} \theta_n = \frac{1}{2}\).

**Proof.** Let us define the function
\[
g : \left[3, \infty\right[ \to [0, \infty[, \quad g(x) = \frac{f(x + \frac{1}{2}) + f(x - \frac{1}{2})}{2} - f(x).
\]
As \(f'\) is concave, we have \(g' \leq 0\), and so \(g\) is decreasing and \(\lim_{x \to \infty} g(x) = 0\). Let us show that \(g > 0\). On the contrary, suppose that \(g(y) = 0\) for some \(y \in [\frac{3}{2}, \infty[\). Since \(f\) is convex and \(g|_{[\frac{3}{2}, \infty[}\) is a concave function, the restriction \(f|_{[\frac{3}{2}, \infty[}\) must be an affine function. This is absurd, because \(f > 0\) and \(\lim_{n \to \infty} f(n) = 0\). We thus conclude that \(g > 0\). An easy computation shows that
\[
\tilde{\rho}_n := \sum_{k=n+1}^{\infty} (-1)^{k-n-1} g(k) = \frac{f(n + \frac{1}{2})}{2} - \rho_n \quad \text{for every } n \in \mathbb{N}^*.
\]
As \(g > 0\) is decreasing, known facts on Leibniz series yield the inequalities \(0 < \tilde{\rho}_n < g(n + 1)\), which prove (8).

To show (9), fix \(n \in \mathbb{N}^*\). By (8) and (2) we get \(f(n + \theta_n) < f(n + \frac{1}{2})\), which forces \(\theta_n > \frac{1}{2}\). Since \(f\) is convex, we have
\[
2g(n + 1) > 2 \tilde{\rho}_n = f\left(n + \frac{1}{2}\right) - f(n + \theta_n) \geq -f'(n + 1)\left(\theta_n - \frac{1}{2}\right).
\]
As \(f'\) is concave, the function \(h : [1, \infty[ \to \mathbb{R}, h(x) = f(x + \frac{1}{2}) - f(x)\) is concave too, since \(h'\) is decreasing. Therefore
\[
2g(n + 1) = h(n + 1) - h\left(n + \frac{1}{2}\right) \leq h'\left(n + \frac{1}{2}\right) = \frac{f'(n + 1) - f'(n + \frac{1}{2})}{2}.
\]
We thus get
\[
\theta_n - \frac{1}{2} < \frac{f'(n + 1) - f'(n + \frac{1}{2})}{-2f'(n + 1)} = \frac{f'(n + \frac{1}{2})}{2f'(n + 1)} - \frac{1}{2},
\]
which completes the proof of (9). We also have
\[
\theta_n < \frac{f'(n + \frac{1}{2})}{2f'(n + 1)} < \frac{f'(n)}{2f'(n + 1)}.
\]

As Example 8 will show, all numbers from \([\frac{1}{2}, 1]\) are potential limits of the sequence \((\theta_n)_{n \geq 1}\).

3. Monotony of \((\theta_n)_{n \geq 1}\)

The sequence \((\theta_n)_{n \geq 1}\) need not be monotone in general.

Example 5. Let \(a := \frac{1}{2m^2 - 1} \in \left[\frac{1}{2}, 3\right]\). Then
\[
f : [1, \infty[ \to \mathbb{R}, \quad f(x) = \begin{cases} \frac{2a-x}{a^2}, & x \in [1, a], \\ \frac{1}{x}, & x \in [a, \infty[,
\end{cases}
\]
is a convex differentiable function. We have \(\lim_{n \to \infty} \theta_n = \frac{1}{2}\), but \(\theta_1 < \frac{1}{2} < \theta_2\). Therefore \((\theta_n)_{n \geq 1}\) is not monotone.

It is easy to check that \(f\) is a convex differentiable function. By Theorem 4, we deduce that \(\lim_{n \to \infty} \theta_n = \frac{1}{2}\), since \(f'\mid_{[3, \infty[}\) is concave and \((\theta_n)_{n \geq 3}\) depends only on the restriction \(f\mid_{[3, \infty[}\). As \(f(n) = \frac{1}{n}\) for every \(n \geq 3\), we have
\[
\rho_2 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k + 2} = \ln 2 - \frac{1}{2} = \frac{f(a)}{2},
\]
and so \(\theta_2 = a - 2 > \frac{1}{2}\). By (3) and the definition of \(f\mid_{[1, a]}\), we get at once \(\theta_1 = 3 - a < \frac{1}{2}\).

As we shall see, properties of \((\theta_n)_{n \geq 1}\) are related to those of the unique implicit function \(\Theta : [2, \infty[ \to [\frac{1}{2}, 1]\) satisfying
\[
f(x - 1 + \Theta(x)) + f(x + \Theta(x)) = 2f(x) \quad \text{for every } x \in [2, \infty).
\]

Let us observe that \(\Theta\) exists, is well-defined, and unique, since for
\[
u : [2, \infty[ \times [0, \infty[ \to \mathbb{R}, \quad u(x, y) = f(x + y - 1) + f(x + y) - 2f(x),
\]
the partial function \(u(x, \cdot)\) is strictly decreasing and \(u(x, \frac{1}{2}) \geq 0 > u(x, 1)\) for every\(^3\) \(x \in [2, \infty[\).

\(^3\) \(\Theta\) may be defined on \([\frac{1}{2}, \infty[\).
Proposition 6. For each \( n \geq 2 \) we have the inequalities
\[
(\Theta(n) - \theta_n)(\Theta(n) - \theta_{n-1}) \leq 0, \quad |\Theta(n) - \theta_{n-1}| \leq |\Theta(n) - \theta_n|.
\]
the equivalences
\[
\theta_n < \theta_{n-1} \iff \theta_n < \Theta(n) \iff \theta_{n-1} > \Theta(n) \iff \rho_n > \frac{f(n + \Theta(n))}{2},
\]
(12)
as well as the equivalences for reversed inequalities, and for equalities.

Proof. Fix \( n \in \mathbb{N}^* \), \( n \geq 2 \). By (3) and (10), we deduce that
\[
f(n + \Theta(n)) - f(n + \theta_n) = f(n - 1 + \theta_{n-1}) - f(n - 1 + \Theta(n)).
\]
(13)
Since \( f \) is strictly decreasing, we must have \((\Theta(n) - \theta_n)(\Theta(n) - \theta_{n-1}) \leq 0\). As \( f \) is convex and \( \Theta(n), \theta_n, \theta_{n-1} \in [0, 1]\), for the lateral derivatives of \( f \) at \( n \) we have \( f'_-(n) \leq f'_+(n) < 0 \) and
\[
|f'_-(n)| \cdot |\Theta(n) - \theta_{n-1}| \leq |f(n - 1 + \theta_{n-1}) - f(n - 1 + \Theta(n))| = |f(n + \Theta(n)) - f(n + \theta_n)| \leq |f'_+(n)| \cdot |\Theta(n) - \theta_n|,
\]

hence \( |\Theta(n) - \theta_{n-1}| \leq |\Theta(n) - \theta_n| \). The proof of the equivalences is straightforward and uses (13) together with the strict monotony of \( f \). \( \square \)

Theorem 7. If
\[
\psi : [2, \infty[ \to \mathbb{R}, \quad \psi(x) = f^{-1}\left(\frac{f(x - 1) + f(x)}{2}\right),
\]
is a concave function, then the sequence \( (\theta_n)_{n \geq 1} \) is decreasing and
\[
\Theta(n + 1) \leq \theta_n \leq \Theta(n) \quad \text{for every } n \geq 2.
\]
(14)
If \( \psi \) is convex, then \( (\theta_n)_{n \geq 1} \) is increasing and reversed inequalities hold in (14). Strict concavity or convexity of \( \psi \) yields strict monotony of the sequence, as well as strict inequalities into the corresponding estimates.

Proof. We shall assume that \( \psi \) is strictly concave. The reasoning is similar in all other cases. The proof will be divided into 3 steps.

Step 1. We first prove that \( \Theta \) is strictly convex. Let us observe that \( \psi \) is continuous and strictly increasing, and that \( \psi(x) > x - 1 \) for every \( x \in [2, \infty] \). Consequently, \( \psi : [2, \infty[ \to [\psi(2), \infty[ \) is invertible and \( \psi^{-1} : [\psi(2), \infty[ \to [2, \infty[ \) is strictly convex. But \( \psi(x + \Theta(x)) = x \), and so \( \Theta(x) = \psi^{-1}(x) - x \) for every \( x \in [2, \infty[ \). We conclude that \( \Theta \) is strictly convex.

\( ^4 \) Here \( f^{-1} \) denotes the inverse of \( f : [1, \infty[ \to [0, f(1)] \).
**Step 2.** We next show that $y_n := 2\rho_n - f(n + \Theta(n)) > 0$ for every $n \geq 2$. Let us observe that $\Theta$ is strictly decreasing, since it is a bounded strictly convex function. An easy computation using (10) and (13) shows that

$$y_{n+1} - y_{n-1} = z_{n-1} - z_n$$

for every $n \geq 3$, (15)

where $z_n := f(n + \Theta(n)) - f(n + \Theta(n + 1))$ for $n \geq 2$. As $f$ is convex, we have

$$f'_+(n) \leq f'_+(n + \Theta(n + 1)) \leq \frac{z_n}{\Theta(n) - \Theta(n + 1)} \leq f'_-(n + \Theta(n)) \leq f'_+(n + 1)$$

for $n \geq 2$, which leads to

$$y_{n+1} - y_{n-1} \leq f'_+(n)[\Theta(n - 1) - \Theta(n)] - f'_+(n)[\Theta(n) - \Theta(n + 1)]$$

$$= f'_+(n)[\Theta(n - 1) + \Theta(n + 1) - 2\Theta(n)] < 0$$

for every $n \geq 3$.

It follows that the subsequences $(y_{2n})_{n\geq 1}$ and $(y_{2n+1})_{n\geq 1}$ are strictly decreasing. Since $\lim_{n \to \infty} y_n = 0$, we conclude that $y_n > 0$ for every $n \geq 2$.

**Step 3.** Applying (12) finally shows that

$$\theta_n < \Theta(n) < \theta_{n-1}$$

for every $n \geq 2$, (16)

thus completing the proof. □

**Example 8.**

(a) For $f(x) = \frac{1}{x^\alpha}$ ($\alpha > 1$), we have $\lim_{n \to \infty} \theta_n = \frac{1}{2} = \Lambda(1)$ and (16) holds. In particular, $(\theta_n)_{n\geq 1}$ is strictly decreasing.

(b) For $f(x) = ax$ ($a \in [0, 1]$), we have $\lim_{n \to \infty} \theta_n = \Lambda(a)$, and $(\theta_n)_{n\geq 1}$ is constant.

(c) For $f(x) = e^{-x^2}$, we have $\lim_{n \to \infty} \theta_n = 1 = \Lambda(0)$.

**Proof.** (a) The conclusion follows by Theorems 4 and 7, since $f'$ is concave,

$$\lim_{n \to \infty} \frac{f'(n + 1)}{f'(n)} = 1,$$

and on $[2, \infty]$ we have

$$\psi(x) = \frac{2^{1/\alpha}x(x - 1)}{[x^\alpha + (x - 1)^\alpha]^{1/\alpha}}, \quad \psi''(x) = -\frac{2^{1/\alpha}(\alpha + 1)(x(x - 1))^{\alpha-1}}{[x^\alpha + (x - 1)^\alpha]^{1/\alpha+2}} < 0.$$

(b) A direct computation shows that

$$f(n + \theta_n) = 2\rho_n = \frac{2\alpha^{n+1}}{\alpha + 1} = a^{\alpha + \Lambda(a)} = f(n + \Lambda(a)),$$

and so $\theta_n = \Lambda(a)$ for every $n \in \mathbb{N}^*$, since $f$ is injective.

(c) For every $t \in [0, 1]$, we have $\lim_{t \to \infty} \frac{f(t+x)}{f(t)} = 0 < 1$, and consequently $\lim_{n \to \infty} \theta_n = \Lambda(0) = 1$, by Theorem 3. □

The first part of our previous example generalizes results obtained in [6] for $\alpha = 1$. Indeed, for $f(x) = \frac{1}{x}$ we have
\[ \psi(x) = \frac{2x(x-1)}{2x-1}, \]
\[ \Theta(x) = \psi^{-1}(x) - x = \frac{1 - x + \sqrt{x^2 + 1}}{2} = \frac{1}{2} \left( 1 + \frac{1}{x + \sqrt{x^2 + 1}} \right). \]
\[ \frac{1}{2} \left( 1 + \frac{1}{n+1 + \sqrt{(n+1)^2 + 1}} \right) < \theta_n < \frac{1}{2} \left( 1 + \frac{1}{n + \sqrt{n^2 + 1}} \right) \]
\[ < \theta_{n-1} \text{ for every } n \geq 2. \]

4. An iterative method

Let us observe that for every \( n \in \mathbb{N}^* \), the expression
\[ f(n+\frac{1}{2}) - \rho_n = (-1)^{n+1} \sum_{k=n+1}^{\infty} (-1)^{k+1} \left( \frac{f(k+\frac{1}{2}) + f(k-\frac{1}{2})}{2} - f(k) \right) \]
is related to the \( n \)th remainder of an alternating series associated to a function \( g : [\frac{1}{2}, \infty] \rightarrow [0, \infty] \) vanishing at infinity (\( \lim_{x \to \infty} g(x) = 0 \)). Under suitable assumptions, inequalities (8) can be applied to this new series, and we may repeat this argument again. This reasoning justifies our following construction.

For every \( a \in \mathbb{R} \), let \( F_a \) denote the real vector space consisting of all continuous functions \( h : [a, \infty] \rightarrow \mathbb{R} \). Let us consider the linear operator
\[ T_a : F_a \rightarrow F_{a+\frac{1}{2}}, \quad T_a h(x) = \frac{h(x+\frac{1}{2}) + h(x-\frac{1}{2})}{2} - h(x). \]
Set \( F := \bigcup_{a \in \mathbb{R}} F_a \) and define \( T : F \rightarrow F \), such that \( T|_{F_a} = T_a \) for every \( a \in \mathbb{R} \). The result of \( T(T h) \) will be written as \( T^2 h \), and so on. The needed properties of \( T \) are collected in the following lemma.

**Lemma 9.** Let \( h \in F_a \).

(a) For all \( m, n \in \mathbb{N} \) with \( m > n \geq a - \frac{1}{2} \), we have
\[ \sum_{r=n+1}^{m} (-1)^r T h(r) = \frac{(-1)^m h(m+\frac{1}{2})}{2} - \frac{(-1)^n h(n+\frac{1}{2})}{2} - \sum_{r=n+1}^{m} (-1)^r h(r). \]

(b) If \( h \) vanishes at infinity, then so does \( T h \).
(c) If \( h \) is continuously differentiable, then so is \( T h \) and \( (T h)' = T (h') \).
(d) If \( h \) is strictly convex, then \( T h > 0 \).
(e) If \( h \) is twice differentiable, then for every \( x \geq a + \frac{1}{2} \), there exists \( \xi \in ]-\frac{1}{2}, \frac{1}{2}[ \), such that
\[ T h(x) = \frac{h''(x+\xi)}{8}. \]
Proof. Properties (b)–(d) are obvious, and (a) follows by a trivial computation. To prove (e), let us observe that for every \( x \geq a + \frac{1}{2} \), a second order Taylor expansion of \([0, \frac{1}{2}] \ni t \mapsto h(x + t) + h(x - t) \in \mathbb{R}\) at 0 shows that

\[
T h(x) = \frac{h''(x + \eta) + h''(x - \eta)}{16}
\]

for some \( \eta \in ]0, \frac{1}{2}[ \). As \( h'' \) has the intermediate value property, we must have

\[
\frac{h''(x + \eta) + h''(x - \eta)}{2} = h''(x + \xi)
\]

for some \( \xi \in [-\eta, \eta] \subset ]-\frac{1}{2}, \frac{1}{2}[ \). \( \square \)

**Theorem 10.** Assume \( f \) to be \( 2p + 3 \) times continuously differentiable (\( p \in \mathbb{N} \)), with \( f^{(2p+3)} \neq 0 \). Set

\[
\sigma_p := \frac{1}{2} \sum_{k=0}^{p} (-1)^k k^p f = \frac{f - T f + T^2 f - \cdots + (-1)^p T^p f}{2} \in \mathcal{F}_{p+\frac{1}{2}}.
\]

Then for every \( n \geq \frac{p+1}{2} \) we have

\[
0 < (-1)^{p+1} \left[ S_n - S_n + (-1)^p \sigma_p \left( n + \frac{1}{2} \right) \right] < T^{p+1} f (n+1)
\]

(17)

\[
< \frac{f^{(2p+2)}(n - \frac{p-1}{2})}{8(p+1)}.
\]

(18)

**Proof.** Fix \( n \in \mathbb{N}^* \), \( n \geq \frac{p+1}{2} \). The proof will be divided into 3 steps.

**Step 1.** We first prove the equality

\[
(-1)^p \left[ \rho_n - \sigma_p \left( n + \frac{1}{2} \right) \right] + \frac{T^p f (n + \frac{1}{2})}{2} = \sum_{r=n+1}^{\infty} (-1)^{-r-n-1} T^p f (r).
\]

(19)

We can assume that \( p \in \mathbb{N}^* \), since (19) clearly holds for \( p = 0 \). By Lemma 9(b), we deduce that \( \lim_{r \to \infty} T^k f (x) = 0 \) for every \( k \in \mathbb{N} \). Applying Lemma 9(a) for \( T^k f \) shows that the series \( \sum_{r=n+1}^{\infty} (-1)^{-r-n-1} T^p f (r) \) converges for every \( k \leq p \), since it does so for \( k = 0 \) \((T^0 f = f)\). Also by Lemma 9(a) we get for \( k \leq p - 1 \) the identities

\[
\sum_{r=n+1}^{\infty} (-1)^{r} T^{k+1} f (r) = \frac{(-1)^{n+1} T^k f (n + \frac{1}{2})}{2} = \sum_{r=n+1}^{\infty} (-1)^r T^k f (r).
\]

Summation of the above equalities multiplied by \(-1\)^{k+1} leads to

\[
\sum_{r=n+1}^{\infty} (-1)^{r} f (r) + \frac{(-1)^{n+1}}{2} \sum_{k=0}^{p} (-1)^k T^k f (n + \frac{1}{2}) = (-1)^p \sum_{r=n+1}^{\infty} (-1)^r T^p f (r),
\]

which actually is (19) multiplied by \(-1\)^{n+p+1}. 

**Step 2.** We next show the inequalities

\[
T^p f(n + 1) - \frac{T^p f(n + \frac{3}{2})}{2} < \sum_{r=n+1}^{\infty} (-1)^{r-n-1} T^p f(r) < \frac{T^p f(n + \frac{1}{2})}{2}. \tag{20}
\]

Let us observe that \(T^p f \in \mathcal{F}_{1+p/2}\) is \(2p + 3 \geq 3\) times differentiable, and that \((T^p f)\)' is concave, since according to Lemma 9(c, e), for every \(x \geq 1 + \frac{p}{2}\) we have

\[(T^p f)''(x) = T^p(f''(x)) = f(2p+3)(x+\xi) < 0\]

for some \(\xi \in \left] -\frac{p}{2}, \frac{p}{2}\right[\). As \((T^p f)''\) is strictly decreasing, there exists

\[\lambda := \lim_{x \to \infty} (T^p f)''(x).\]

Using twice l'Hôpital's rule shows that

\[\frac{\lambda}{2} = \lim_{x \to \infty} \frac{T^p f(x)}{x^2} = 0.\]

It follows that \((T^p f)'' > 0\). We conclude that \(T^p f\) is convex, with \((T^p f)\)' concave, and consequently it can be extended to a function \(g : [1, \infty[ \to [0, \infty[\) keeping these properties (e.g.,

\[g(x) = T^p f \left(1 + \frac{p}{2}\right) + (T^p f)' \left(1 + \frac{p}{2}\right) \left(x - 1 - \frac{p}{2}\right)\]

\[+ \frac{1}{2} (T^p f)'' \left(1 + \frac{p}{2}\right) \left(x - 1 - \frac{p}{2}\right)^2\]

for \(x \in [1, 1 + \frac{p}{2}]\). Applying (8) for the Leibniz series \(\sum_{s \geq 1} (-1)^{s-1} g(s + n - 1)\) and \(s = 1\) now gives

\[g(n + 1) - \frac{g(n + \frac{1}{2})}{2} < \sum_{s=2}^{\infty} (-1)^{s} g(s + n - 1) < \frac{g(n + \frac{1}{2})}{2},\]

which yields (20), since \(g\big|_{1+\frac{p}{2}, \infty} = T^p f\) and \(n + \frac{1}{2} \geq 1 + \frac{p}{2}\).

**Step 3.** We finally prove (17) and (18). Inequalities (17) are just a combination of (19) and (20). Thus it remains to show (18). As

\[T^{p+1} f(n + 1) = \frac{f^{(2p+2)}(n + 1 + \eta)}{8^{p+1}}\]

for some \(\eta \in \left] -\frac{p+1}{2}, \frac{p+1}{2}\right[\) and \(f^{(2p+2)}\) is strictly decreasing, (18) follows. □

Let us note that (18) provides an a priori error estimate; for fixed \(\varepsilon > 0\), it can be used to find suitable \(p, n\). The following example shows that the error made by using \(S_n - (-1)^n\sigma_p(n + \frac{1}{2})\) as an approximation for \(S\) may be surprisingly small even for small values of \(n\) and \(p\).
Example 11. We shall apply Theorem 10 for \( f : [1, \infty] \rightarrow [0, \infty], \ f(x) = \frac{1}{x^2} \) and \( p = 1 \).

Some easy computations show that

\[
\sigma_1(x) = \frac{f(x) - Tf(x)}{2} = \frac{2x^2 - 1}{x(4x^2 - 1)}, \quad \sigma_1\left( n + \frac{1}{2} \right) = \frac{(2n + 1)^2 - 2}{4n(n+1)(2n+1)},
\]

\[
\epsilon(n) := T^2 f(n + 1) = \frac{3}{2n(n+1)(n+2)(2n+1)(2n+3)}.
\]

By Theorem 10, we have

\[
0 < (-1)^{n+1}\left[ S - S_n + (-1)^n \sigma_1\left( n + \frac{1}{2} \right) \right] < \epsilon(n) < \frac{3}{8n^3} \quad \text{for every } n \in \mathbb{N}^*.
\]

Let us note that \( \epsilon(2) = \frac{1}{560} < 0.2 \cdot 10^{-2}, \epsilon(4) = \frac{1}{7920} < 0.2 \cdot 10^{-3}, \) and \( \epsilon(13) = \frac{1}{1425060} < 0.8 \cdot 10^{-6}. \)

For estimates of \( \rho_n \) in terms of first order differences of the restrictions \( f|_{\mathbb{N}^*} \) and \( f'|_{\mathbb{N}^*} \) of a 4 times continuously differentiable function \( f \), we refer the reader to [2]. An interesting discussion on alternating series, involving the Euler transformation and integral representations, can be found in [3].

References