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A Dirichlet inhomogeneous boundary value problem for a generalized Ginzburg–Landau equation

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Abstract

The Ginzburg–Landau equation has been used as a simplified mathematical model for various pattern formation systems in mechanics, physics and chemistry. In this paper, we study a generalized complex Ginzburg–Landau equation in two spatial dimension with a fifth order nonlinear term and cubic terms containing spatial derivatives. We prove a global existence and uniqueness theorem.

Keywords: Generalized 2D Ginzburg–Landau equation; Dirichlet inhomogeneous boundary value problem; Global existence

1. Introduction

There has been a large amount of literature regarding complex Ginzburg–Landau equations (GLE) and their generalized versions (GGLE). For example, Ghidaglia and Heron [1] and Doer-

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ing et al. [2] studied the finite-dimensional global attractor and related dynamical issues for the following GLE in one or two spatial dimensions:

$$u_t = (1 + i\nu)\Delta u - (1 + i\mu)|u|^2 u + au, \tag{1.1}$$

where $a > 0$, and ν, μ are given real numbers. Levermore and Oliver [3] studied GLE as a model problem. Bu [4] obtained the global existence of the Cauchy problem of the following 2D GLE

$$u_t = (\nu + i\alpha)\Delta u - (\mu + i\beta)|u|^{2q} u + \gamma u \tag{1.2}$$

with $q = 1$ and $q = 2$ under the conditions $\alpha\beta > 0$ or $|\beta| \leqslant \sqrt{5}/2$. Bartuccelli [5], Doering [6], Bartuccelli [7] and others have investigated turbulence, weak or strong solutions and length scales for Eq. (1.2) in higher dimensions. A. Mielke and G. Schneider [8–10] obtained sharper results for (1.2) on bounded and unbounded domains.

The following 1D generalized Ginzburg–Landau equation was originally derived by Doelman [11]:

$$u_t = \alpha_0 u + \alpha_1 u_{xx} + \alpha_2 |u|^2 u + \alpha_3 |u|^2 u_x + \alpha_4 u^2 \bar{u}_x - \alpha_5 |u|^4 u, \tag{1.3}$$

where $\alpha_0 > 0$, $\alpha_j = a_j + ib_j$, $j = 1, \ldots, 5$, $a_1 > 0$, $a_5 > 0$. Duan, Gao et al. [12–17] investigated a wide range of issues such as global existence of solutions, finite-dimensional global attractor, Gevery regularity of solutions, exponential attractor, number of determining nodes and inertial forms. Duan and Holmes [18] have also obtained global existence for Cauchy problem of Eq. (1.3) under the condition of $4a_1a_5 > (b_3 - b_4)^2$. Guo and Wang [19] considered the following 2D GGLE

$$u_t = \rho u + (1 + i\nu)\Delta u - (1 + i\mu)|u|^{2\sigma} u + \alpha\lambda_1 \cdot \nabla\big(|u|^2 u\big) + \beta(\lambda_2 \cdot \nabla u)|u|^2, \tag{1.4}$$

where $\rho > 0$, α, β, ν, μ are real numbers, and λ_1, λ_2 are real constant vectors. Existence of finite-dimensional global attractor of Eq. (1.4) with periodic boundary conditions was proved assuming that there exists a positive number $\delta > 0$ so that

$$\frac{1}{\sqrt{1 + (\frac{\mu - \nu\delta^2}{1+\delta^2})^2} - 1} \geqslant \sigma \geqslant 3. \tag{1.5}$$

In many papers mentioned above, appropriate boundary conditions are assumed for the existence of finite-dimensional global attractors, determining nodes and inertial forms.

Since the following 2D generalized Ginzburg–Landau equation

$$u_t = \alpha_0 u + \alpha_1 \Delta u + \alpha_2 |u|^2 u_x + \alpha_3 |u|^2 u_y + \alpha_4 u^2 \bar{u}_x + \alpha_5 u^2 \bar{u}_y - \alpha_6 |u|^{2\sigma} u \tag{1.6}$$

is more closely related to the equation derived by Doelman [11] ($\sigma = 2$), it can also be regarded as the perturbation of nonlinear derivative Schrödinger equation. When $\alpha_0 > 0$, $\alpha_j = a_j + ib_j$, $j = 1, \ldots, 6$, $a_1 > 0$, $a_6 > 0$, $\sigma > 0$, Gao and Duan [20] considered the initial value problem for the GLE in a more generalized version with the suitable initial condition $u(x, y, 0) = u_0(x, y)$, $(x, y) \in R^2$. For periodic boundary value problem of (1.6) with $\sigma = 2$, global existence was obtained by Gao and Kwek [21].

Nonetheless, there are very few results related to inhomogeneous boundary conditions. For (1.3) with Dirichlet or Neumann boundary conditions, Gao and Bu [22–24] proved a unique weak solution that exists for all time in one dimension. They also obtained global existence theorem and its inviscid limit for a Dirichlet inhomogeneous boundary value problem for the classical GLE in $n > 1$ dimensions under certain conditions [25].

In this paper, we investigate a generalized GLE (1.6) in two spatial dimensions under appropriate initial condition and inhomogeneous Dirichlet boundary condition and prove sufficient conditions for global existence.

2. A priori estimates and global existence

Let $\|\cdot\|$ denote the norm of $L^2(\Omega)$. For any $T>0$, let C and C_j be some constants depending on T and initial/boundary data, the parameters in equation, T and initial value and boundary value. We study the following initial–boundary value problem in $\Omega\in R^2$:

$$\begin{aligned}
&u_t=\alpha_0 u+(1+i\nu)\Delta u+|u|^2\vec\lambda_1\cdot\nabla u+u^2\vec\lambda_2\cdot\nabla\bar u-(1+i\mu)|u|^4u,\\
&u(x,0)=h(x),\quad x\in\Omega,\\
&u(x,t)=Q(x,t),\quad x\in\partial\Omega,
\end{aligned}\tag{2.1}$$

where $\vec\lambda_1=\alpha_2\vec i+\alpha_3\vec j$ and $\vec\lambda_2=\alpha_4\vec i+\alpha_5\vec j$ are complex vectors.

We write $P=\nabla u|_{\partial\Omega}$, $\eta=\sum_j\partial_j\xi_j=\nabla\cdot\xi$, and $\vec n=(n_1,n_2)$ as standard unit outer normal vector. Since $\partial\Omega$ is smooth, there exists a smooth function $\xi=(\xi_1,\xi_2)$ independent of t from R^2 to R^2 such that

$$\xi|_{\partial\Omega}=(n_1,n_2)=\vec n.$$

Let u be a smooth solution to the inhomogeneous boundary problem for the Ginzburg–Landau equation (2.1). The following identities are available.

First,

$$\begin{aligned}
\partial_t\int_\Omega|u|^2\,dx&=2\operatorname{Re}\int_\Omega\bar u u_t\,dx\\
&=2\operatorname{Re}\int_\Omega\bar u\big[\alpha_0u+(1+i\nu)\Delta u+|u|^2\vec\lambda_1\cdot\nabla u\\
&\quad+u^2\vec\lambda_2\cdot\nabla\bar u-(1+i\mu)|u|^4u\big]\,dx\\
&=2\alpha_0\|u\|^2-2\|u\|_6^6+2\operatorname{Re}\int_\Omega(1+i\nu)\Delta u\bar u\,dx\\
&\quad+2\operatorname{Re}\int_\Omega|u|^2\bar u\vec\lambda_1\cdot\nabla u\,dx+2\operatorname{Re}\int_\Omega|u|^2u\vec\lambda_2\cdot\nabla\bar u\,dx\\
&=2\alpha_0\|u\|^2-2\|u\|_6^6+2\operatorname{Re}(1+i\nu)\int_{\partial\Omega}(\vec n\cdot P)\bar Q\,ds-2\|\nabla u\|^2\\
&\quad+2\operatorname{Re}\int_\Omega|u|^2\bar u\vec\lambda_1\cdot\nabla u\,dx+2\operatorname{Re}\int_\Omega|u|^2u\vec\lambda_2\cdot\nabla\bar u\,dx.
\end{aligned}\tag{2.2}$$

Second,

$$\partial_t\int_\Omega|\nabla u|^2\,dx=2\operatorname{Re}\int_\Omega\nabla u\nabla\bar u_t\,dx=2\operatorname{Re}\int_{\partial\Omega}(\vec n\cdot P)\bar Q_t\,ds-2\operatorname{Re}\int_\Omega\Delta u\bar u_t\,dx$$

$$
\begin{aligned}
&= 2\,\mathrm{Re}\int_{\partial\Omega}(\vec n\cdot P)\bar Q_t\,ds \\
&\quad - 2\,\mathrm{Re}\int_{\Omega}\Delta u\big[\alpha_0\bar u+(1-i\nu)\Delta\bar u+|u|^2\overline{\vec\lambda_1}\cdot\nabla\bar u+\bar u^2\overline{\vec\lambda_2}\cdot\nabla u \\
&\quad - (1-i\mu)|u|^4\bar u\big]\,dx \\
&= 2\,\mathrm{Re}\int_{\partial\Omega}(\vec n\cdot P)\bar Q_t\,ds - 2\alpha_0\,\mathrm{Re}\int_{\Omega}\Delta u\bar u\,dx \\
&\quad - 2\,\mathrm{Re}(1-i\nu)\int_{\Omega}|\Delta u|^2\,dx - 2\,\mathrm{Re}\int_{\Omega}|u|^2\Delta u\overline{\vec\lambda_1}\cdot\nabla\bar u\,dx \\
&\quad - 2\,\mathrm{Re}\int_{\Omega}\Delta u\bar u^2\overline{\vec\lambda_2}\cdot\nabla u\,dx + 2\,\mathrm{Re}(1-i\mu)\int_{\Omega}|u|^4\bar u\Delta u\,dx \\
&= 2\,\mathrm{Re}\int_{\partial\Omega}(\vec n\cdot P)\bar Q_t\,ds + 2\alpha_0\|\nabla u\|^2 \\
&\quad - 2\alpha_0\,\mathrm{Re}\int_{\partial\Omega}(\vec n\cdot P)\bar Q\,ds - 2\|\Delta u\|^2 + 2\,\mathrm{Re}(1-i\mu)\int_{\Omega}|u|^4\bar u\Delta u\,dx \\
&\quad - 2\,\mathrm{Re}\int_{\Omega}|u|^2\Delta u\overline{\vec\lambda_1}\cdot\nabla\bar u\,dx - 2\,\mathrm{Re}\int_{\Omega}\Delta u\overline{\vec\lambda_2}\cdot\bar u^2\nabla u\,dx. \qquad (2.3)
\end{aligned}
$$

Third,

$$
\begin{aligned}
\partial_t\int_{\Omega}|u|^6\,dx &= 6\,\mathrm{Re}\int_{\Omega}|u|^4\bar u u_t\,dx \\
&= 6\,\mathrm{Re}\int_{\Omega}|u|^4\bar u\big[\alpha_0 u+(1+i\nu)\Delta u+|u|^2\vec\lambda_1\cdot\nabla u+u^2\vec\lambda_2\cdot\nabla\bar u \\
&\quad - (1+i\mu)|u|^4u\big]\,dx \\
&= 6\alpha_0\|u\|_6^6 + 6\,\mathrm{Re}(1+i\nu)\int_{\Omega}|u|^4\bar u\Delta u\,dx \\
&\quad + 6\,\mathrm{Re}\int_{\Omega}|u|^6\bar u\vec\lambda_1\cdot\nabla u\,dx + 6\,\mathrm{Re}\int_{\Omega}|u|^6u\vec\lambda_2\cdot\nabla\bar u\,dx - 6\|u\|_{10}^{10}. \qquad (2.4)
\end{aligned}
$$

Fourth,

$$
\begin{aligned}
&\partial_t\int_{\Omega}u(\xi\cdot\nabla\bar u)\,dx - \int_{\partial\Omega}Q\bar Q_t\,ds + \int_{\Omega}u\bar u_t\eta\,dx \\
&\quad = 2\alpha_0 i\,\mathrm{Im}\int_{\Omega}(\xi\cdot\nabla\bar u)u\,dx + 2i\,\mathrm{Im}\int_{\Omega}(\xi\cdot\nabla\bar u)\Delta u\,dx + 2i\,\mathrm{Im}\int_{\Omega}(\xi\cdot\nabla\bar u)|u|^2\vec\lambda_1\cdot\nabla u\,dx
\end{aligned}
$$

$$+ 2i \operatorname{Im} \int_\Omega (\xi \cdot \nabla \bar{u}) u^2 \vec{\lambda}_2 \cdot \nabla \bar{u}\, dx - 2i \operatorname{Im} \int_\Omega |u|^4 u (\xi \cdot \nabla \bar{u})\, dx$$
$$+ 2i\nu \int_{\partial\Omega} |\vec{n} \cdot P|^2\, ds - 2i\nu \sum_{m,j} \operatorname{Re} \int_\Omega (\xi_j)_m u_m \bar{u}_j\, dx - i\nu \int_{\partial\Omega} |P|^2\, ds$$
$$+ i\nu \int_\Omega \eta |\nabla u|^2\, dx - \frac{1}{3} i\mu \int_{\partial\Omega} |Q|^6\, ds + \frac{1}{3} i\mu \int_\Omega \eta |u|^6\, dx. \tag{2.5}$$

To prove the above identities, we first differentiate $|u|^2$ in t and substitute u_t by Eq. (2.1). An integration by parts over Ω yields (2.2). Similarly, (2.3) and (2.4) can be obtained fairly easily. To prove (2.5), we write

$$\xi_j (u_t \bar{u}_j - \bar{u}_t u_j) = \partial_t (u \xi_j \bar{u}_j) - \partial_j (u \bar{u}_t \xi_j) + u \bar{u}_t \partial_j \xi_j. \tag{2.6}$$

Integrating (2.6) over Ω we obtain

$$\int_\Omega \xi_j (u_t \bar{u}_j - \bar{u}_t u_j)\, dx = \partial_t \int_\Omega u \xi_j \bar{u}_j\, dx - \int_{\partial\Omega} {n_j}^2 Q \bar{Q}_t\, ds + \int_\Omega u \bar{u}_t \partial_j \xi_j\, dx. \tag{2.7}$$

On the other hand,

$$\begin{aligned}
\xi_j (u_t \bar{u}_j - \bar{u}_t u_j) &= 2i \xi_j \operatorname{Im} u_t \bar{u}_j \\
&= 2i \xi_j \operatorname{Im} \bar{u}_j \big[\alpha_0 u + (1 + i\nu) \Delta u + |u|^2 \vec{\lambda}_1 \cdot \nabla u + u^2 \nabla \vec{\lambda}_2 \cdot \bar{u} \\
&\quad - (1 + i\mu) |u|^4 u\big] \\
&= 2\alpha_0 i \xi_j \operatorname{Im} \bar{u}_j u + 2i \xi_j \operatorname{Im} \Delta u \bar{u}_j + 2i\nu \xi_j \operatorname{Re} \Delta u \bar{u}_j \\
&\quad + 2i \xi_j \operatorname{Im} |u|^2 \vec{\lambda}_1 \cdot \nabla u \bar{u}_j + 2i \xi_j \operatorname{Im} u^2 \vec{\lambda}_2 \cdot \nabla \bar{u} \bar{u}_j - 2i \xi_j \operatorname{Im} |u|^4 u \bar{u}_j \\
&\quad - 2i\mu \xi_j \operatorname{Re} |u|^4 u \bar{u}_j.
\end{aligned} \tag{2.8}$$

Terms in (2.8) are evaluated separately as follows:

$$\begin{aligned}
2i\nu \xi_j \operatorname{Re} \Delta u \bar{u}_j &= 2i \xi_j \nu \operatorname{Re} \sum_m \big[(u_m \bar{u}_j)_m - u_m \bar{u}_{jm}\big] \\
&= 2i\nu \operatorname{Re}\big[(\xi_j u_m \bar{u}_j)_m - (\xi_j)_m u_m \bar{u}_j\big] - \nu i \sum_m \big(\partial_j \big(\xi_j |u_m|^2\big) - (\partial_j \xi_j) |u_m|^2\big) \\
- 2i\mu \xi_j \operatorname{Re} |u|^4 u \bar{u}_j &= -\frac{1}{3} i\mu \xi_j \partial_j |u|^6 \\
&= -\frac{1}{3} i\mu \big(\partial_j \big(\xi_j |u|^6\big) - (\partial_j \xi_j) |u|^6\big) \\
&= -\frac{1}{3} i\mu \partial_j \big(\xi_j |u|^6\big) + \frac{1}{3} i\mu (\partial_j \xi_j) |u|^6.
\end{aligned} \tag{2.9}$$

Substituting (2.8) and (2.9) in (2.7) we get

$$\begin{aligned}
\xi_j (u_t \bar{u}_j - \bar{u}_t u_j) &= 2\alpha_0 i \xi_j \operatorname{Im} \bar{u}_j u + 2i \xi_j \operatorname{Im} \Delta u \bar{u}_j + 2i \xi_j \operatorname{Im} |u|^2 \vec{\lambda}_1 \cdot \nabla u \bar{u}_j \\
&\quad + 2i \xi_j \operatorname{Im} u^2 \vec{\lambda}_2 \cdot \nabla \bar{u} \bar{u}_j - 2i \xi_j \operatorname{Im} |u|^4 u \bar{u}_j \\
&\quad + 2i\nu \operatorname{Re}\big[(\xi_j u_m \bar{u}_j)_m - (\xi_j)_m u_m \bar{u}_j\big]
\end{aligned}$$

$$- \nu i \sum_m \big(\partial_j\big(\xi_j|u_m|^2\big) - (\partial_j\xi_j)|u_m|^2\big) - \frac{1}{3} i\mu\partial_j\big(\xi_j|u|^6\big)$$
$$+ \frac{1}{3} i\mu(\partial_j\xi_j)|u|^6. \tag{2.10}$$

Integrating (2.10) over Ω and adding $j = 1, 2$ we obtain

$$\int_\Omega \xi_j(u_t\bar{u}_j - \bar{u}_t u_j)\,dx = 2\alpha_0 i \operatorname{Im}\int_\Omega (\xi\cdot\nabla\bar{u})u\,dx$$
$$+ +2i \operatorname{Im}\int_\Omega (\xi\cdot\nabla\bar{u})\Delta u\,dx 2i \operatorname{Im}\int_\Omega (\xi\cdot\nabla\bar{u})|u|^2\vec{\lambda}_1\cdot\nabla u\,dx$$
$$+ 2i \operatorname{Im}\int_\Omega (\xi\cdot\nabla\bar{u})u^2\vec{\lambda}_2\cdot\nabla\bar{u}\,dx - 2i \operatorname{Im}\int_\Omega |u|^4 u(\xi\cdot\nabla\bar{u})\,dx$$
$$+ 2i\nu\int_{\partial\Omega} |\vec{n}\cdot P|^2\,ds - 2i\nu\sum_{m,j}\operatorname{Re}\int_\Omega (\xi_j)_m u_m\bar{u}_j\,dx$$
$$- i\nu\int_{\partial\Omega} |P|^2\,ds + i\nu\int_\Omega \eta|\nabla u|^2\,dx - \frac{1}{3}i\mu\int_{\partial\Omega} |Q|^6\,ds$$
$$+ \frac{1}{3}i\mu\int_\Omega \eta|u|^6\,dx. \tag{2.11}$$

Now we combine (2.7) and (2.11) for $j = 1, 2$ to get

$$\partial_t\int_\Omega u(\xi\cdot\nabla\bar{u})\,dx - \int_{\partial\Omega} Q\bar{Q}_t\,ds + \int_\Omega u\bar{u}_t\eta\,dx$$
$$= 2\alpha_0 i \operatorname{Im}\int_\Omega (\xi\cdot\nabla\bar{u})u\,dx + 2i \operatorname{Im}\int_\Omega (\xi\cdot\nabla\bar{u})\Delta u\,dx + 2i \operatorname{Im}\int_\Omega (\xi\cdot\nabla\bar{u})|u|^2\vec{\lambda}_1\cdot\nabla u\,dx$$
$$+ 2i \operatorname{Im}\int_\Omega (\xi\cdot\nabla\bar{u})u^2\vec{\lambda}_2\cdot\nabla\bar{u}\,dx - 2i \operatorname{Im}\int_\Omega |u|^4 u(\xi\cdot\nabla\bar{u})\,dx$$
$$+ 2i\nu\int_{\partial\Omega} |\vec{n}\cdot P|^2\,ds - 2i\nu\sum_{m,j}\operatorname{Re}\int_\Omega (\xi_j)_m u_m\bar{u}_j\,dx - i\nu\int_{\partial\Omega} |P|^2\,ds$$
$$+ i\nu\int_\Omega \eta|\nabla u|^2\,dx - \frac{1}{3}i\mu\int_{\partial\Omega} |Q|^6\,ds + \frac{1}{3}i\mu\int_\Omega \eta|u|^6\,dx. \tag{2.12}$$

This completes the proof of (2.5).

To establish a bound for u in H^1 space, we multiply (2.1) by $\eta\bar{u}$ and integrate over Ω to obtain

$$0 = \int_\Omega \eta\bar{u}\big[u_t - \alpha_0 u - (1+i\nu)\Delta u - |u|^2\vec{\lambda}_1\cdot\nabla u - u^2\vec{\lambda}_2\cdot\nabla\bar{u} + (1+i\mu)|u|^4 u\big]\,dx$$
$$= \int_\Omega \eta\bar{u}u_t\,dx - \alpha_0\eta\|u\|^2 - (1+i\nu)\int_\Omega \eta\bar{u}\Delta u\,dx - \int_\Omega \eta|u|^2\vec{\lambda}_1\cdot\nabla u\bar{u}\,dx$$

$$-\int_\Omega \eta|u|^2u\vec{\lambda}_2\cdot\nabla\bar{u}\,dx+(1+i\mu)\int_\Omega \eta|u|^6\,dx. \tag{2.13}$$

Therefore,

$$\begin{aligned}\int_\Omega \eta u\bar{u}_t\,dx &= \alpha_0\int_\Omega \eta|u|^2\,dx-(1-i\mu)\int_\Omega \eta|u|^6\,dx+(1-i\nu)\int_{\partial\Omega}\eta(\vec{n}\cdot\bar{P})Q\,ds\\ &\quad-(1-i\nu)\int_\Omega\big((\nabla\eta\cdot\nabla\bar{u})u+\eta|\nabla u|^2\big)\,dx\\ &\quad+\int_\Omega \eta|u|^2\overline{\vec{\lambda}_1}\cdot\nabla\bar{u}u\,dx+\int_\Omega \eta|u|^2\bar{u}\overline{\vec{\lambda}_2}\cdot\nabla u\,dx. \end{aligned}\tag{2.14}$$

Substituting (2.14) in (2.13) and using (2.5) we find

$$\begin{aligned}&\partial_t\int_\Omega u(\xi\cdot\nabla\bar{u})\,dx-\int_{\partial\Omega}Q\bar{Q}_t\,ds+\alpha_0\int_\Omega\eta|u|^2\,dx-(1-i\mu)\int_\Omega\eta|u|^6\,dx\\ &\quad+(1-i\nu)\int_{\partial\Omega}\eta(\vec{n}\cdot\bar{P})Q\,ds-(1-i\nu)\int_\Omega\big((\nabla\eta\cdot\nabla\bar{u})u+\eta|\nabla u|^2\big)\,dx\\ &\quad+\int_\Omega\eta|u|^2u\overline{\vec{\lambda}_1}\cdot\nabla\bar{u}\,dx+\int_\Omega\eta|u|^2\bar{u}\overline{\vec{\lambda}_2}\cdot\nabla u\,dx\\ &=2\alpha_0 i\,\mathrm{Im}\int_\Omega(\xi\cdot\nabla\bar{u})u\,dx+2i\,\mathrm{Im}\int_\Omega(\xi\cdot\nabla\bar{u})\Delta u\,dx+2i\,\mathrm{Im}\int_\Omega(\xi\cdot\nabla\bar{u})|u|^2\vec{\lambda}_1\cdot\nabla u\,dx\\ &\quad+2i\,\mathrm{Im}\,\vec{\lambda}_2\cdot\int_\Omega u^2\nabla\bar{u}(\xi\cdot\nabla\bar{u})\,dx-2i\,\mathrm{Im}\int_\Omega|u|^4u(\xi\cdot\nabla\bar{u})\,dx\\ &\quad+2i\nu\int_{\partial\Omega}|\vec{n}\cdot P|^2\,ds-2i\nu\sum\mathrm{Re}\int_\Omega(\xi_j)_m u_m\bar{u}_j\,dx-i\nu\int_{\partial\Omega}|P|^2\,ds\\ &\quad+i\nu\int_\Omega\eta|\nabla u|^2\,dx-\frac{1}{3}i\mu\int_{\partial\Omega}|Q|^6\,ds+\frac{1}{3}i\mu\int_\Omega\eta|u|^6\,dx. \end{aligned}\tag{2.15}$$

The above identity can be rearranged as follows:

$$\begin{aligned}&i\int_{\partial\Omega}\Big(\nu|P|^2-2\nu|\vec{n}\cdot P|^2+\frac{1}{3}\mu|Q|^6+iQ\bar{Q}_t-(\nu+i)\eta(\vec{n}\cdot\bar{P})Q\Big)\,ds\\ &=-\partial_t\int_\Omega u(\xi\cdot\nabla\bar{u})\,dx-\alpha_0\int_\Omega\eta|u|^2\,dx+(1-i\mu)\int_\Omega\eta|u|^6\,dx\\ &\quad+(1-i\nu)\int_\Omega\big((\nabla\eta\cdot\nabla\bar{u})u+\eta|\nabla u|^2\big)\,dx+2\alpha_0 i\,\mathrm{Im}\int_\Omega(\xi\cdot\nabla\bar{u})u\,dx\\ &\quad+2i\,\mathrm{Im}\int_\Omega(\xi\cdot\nabla\bar{u})\Delta u\,dx+2i\,\mathrm{Im}\int_\Omega(\xi\cdot\nabla\bar{u})|u|^2\vec{\lambda}_1\cdot\nabla u\,dx\end{aligned}$$

$$+2i\,\mathrm{Im}\int_\Omega u^2\vec{\lambda}_2\cdot\nabla\bar{u}(\xi\cdot\nabla\bar{u})\,dx-2i\,\mathrm{Im}\int_\Omega |u|^4u(\xi\cdot\nabla\bar{u})\,dx$$
$$-2i\nu\sum_{m,j}\mathrm{Re}\int_\Omega(\xi_j)_m u_m\bar{u}_j\,dx+i\nu\int_\Omega\eta|\nabla u|^2\,dx+\frac{1}{3}i\mu\int_\Omega\eta|u|^6\,dx$$
$$-\int_\Omega\eta|u|^2u\overline{\vec{\lambda}_1}\cdot\nabla\bar{u}\,dx-\int_\Omega\eta|u|^2\bar{u}\overline{\vec{\lambda}_2}\cdot\nabla u\,dx. \tag{2.16}$$

Our next goal is to obtain a bound on the integral of $|\vec{n}\cdot P|^2$. Let $\tilde{Q}$ be any C^4 function on $\Omega\times[0,\infty)$ with compact support in x such that (see [26])

$$(1+i\nu)\Delta\tilde{Q}=Q_t-\alpha_0Q-(a_1+ib_1)|Q|^2\nabla Q$$
$$-(a_2+ib_2)Q^2\nabla\bar{Q}+(1+i\mu)|Q|^4Q\quad\text{on }\partial\Omega, \tag{2.17}$$

and

$$\tilde{Q}=Q\quad\text{on }\partial\Omega. \tag{2.18}$$

In fact, any finite number of derivatives can be specified on $\partial\Omega$. At each point we can write

$$|P|^2=|\vec{n}\cdot P|^2+|A\cdot P|^2=|\vec{n}\cdot P|^2+|A\cdot\nabla\tilde{Q}|^2 \tag{2.19}$$

where $A\cdot P$ denotes the tangential component of P and

$$\nu|P|^2-2\nu|\vec{n}\cdot P|^2=-\nu|\vec{n}\cdot P|^2+\nu|A\cdot\nabla\tilde{Q}|^2. \tag{2.20}$$

By substituting the above identities in (2.16) we obtain

$$\nu\int_{\partial\Omega}|\vec{n}\cdot P|^2\,ds=\int_{\partial\Omega}\left[\nu|A\cdot\nabla\tilde{Q}|^2+\frac{1}{3}\mu|Q|^6+iQ\bar{Q}_t-(\nu+i)\eta(\vec{n}\cdot\bar{P})Q\right]ds$$
$$-i\partial_t\int_\Omega u(\xi\cdot\nabla\bar{u})\,dx-i\alpha_0\int_\Omega\eta|u|^2\,dx+(\mu+i)\int_\Omega\eta|u|^6\,dx$$
$$+(\nu+i)\int_\Omega\left[(\nabla\eta\cdot\nabla\bar{u})u+\eta|\nabla u|^2\right]dx-2\alpha_0\,\mathrm{Im}\int_\Omega(\xi\cdot\nabla\bar{u})u\,dx$$
$$-2\,\mathrm{Im}\int_\Omega(\xi\cdot\nabla\bar{u})\Delta u\,dx-2\,\mathrm{Im}\int_\Omega(\xi\cdot\nabla\bar{u})|u|^2\vec{\lambda}_1\cdot\nabla u\,dx$$
$$-2\,\mathrm{Im}\int_\Omega u^2\vec{\lambda}_2\cdot\nabla\bar{u}(\xi\cdot\nabla\bar{u})\,dx+2\,\mathrm{Im}\int_\Omega|u|^4u(\xi\cdot\nabla\bar{u})\,dx$$
$$+2\nu\sum\mathrm{Re}\int_\Omega(\xi_j)_m u_m\bar{u}_j\,dx-\nu\int_\Omega\eta|\nabla u|^2\,dx-\frac{1}{3}\mu\int_\Omega\eta|u|^6\,dx$$
$$-i\int_\Omega\eta|u|^2u\overline{\vec{\lambda}_1}\cdot\nabla\bar{u}\,dx-i\int_\Omega\eta|u|^2\bar{u}\overline{\vec{\lambda}_2}\cdot\nabla u\,dx. \tag{2.21}$$

Assume that $\nu\neq0$. Integrating (2.21) in t, we have

$$\int_0^t \int_{\partial\Omega} |\vec{n}\cdot P|^2\,ds\,dT$$
$$\leqslant C + \int_0^t \int_{\partial\Omega} |A\cdot\nabla\tilde{Q}|^2\,ds\,dT + \frac{N_1\sqrt{\nu^2+1}}{|\nu|}\int_0^t \int_{\partial\Omega} \big|(\vec{n}\cdot P)\big||Q|\,ds\,dT$$
$$+ N_1\int_\Omega |u||\nabla u|\,dx + C_1\int_0^t \big(\|u\|^2+\|u\|_6^6\big)\,dT + C_2\int_0^t\int_\Omega \big(|u\nabla u|+|\nabla u|^2\big)\,dx\,dT$$
$$+ \frac{2N_1}{|\nu|}\int_0^t\int_\Omega |\nabla u||\Delta u|\,dx\,dT + \frac{4MN_1}{|\nu|}\int_0^t\int_\Omega |u|^2|\nabla u|^2\,dx\,dT$$
$$+ \frac{2N_1}{|\nu|}\int_0^t\int_\Omega |u|^5|\nabla u|\,dx\,dT + \frac{2MN_1}{|\nu|}\int_0^t\int_\Omega |u|^3|\nabla u|\,dx\,dT, \tag{2.22}$$

where

$$M = \max\{|\vec{\lambda}_1|, |\vec{\lambda}_2|\}, \qquad N_1 = \max\Big\{\max_{x\in\Omega}\big|\eta(x)\big|, \max_{x\in\Omega}\big|\xi(x)\big|, \max_{x\in\Omega}\big|\nabla\xi(x)\big|\Big\}. \tag{2.23}$$

By Cauchy inequality, we have

$$\int_0^t \int_{\partial\Omega} |\vec{n}\cdot P|^2\,ds\,dT \leqslant C_3 + C_4\big(\|u\|^2+\|\nabla u\|^2\big) + C_5\int_0^t \big(\|u\|^2+\|u\|_6^6\big)\,dT$$
$$+ C_6\int_0^t\int_\Omega |\nabla u|^2\,dx\,dT + \frac{4N_1}{|\nu|}\int_0^t\int_\Omega |\nabla u||\Delta u|\,dx\,dT$$
$$+ \frac{8MN_1}{|\nu|}\int_0^t\int_\Omega |u|^2|\nabla u|^2\,dx\,dT + \frac{2N_1}{|\nu|}\int_0^t\int_\Omega |u|^5|\nabla u|\,dx\,dT$$
$$+ \frac{2MN_1}{|\nu|}\int_0^t\int_\Omega |u|^3|\nabla u|\,dx\,dT. \tag{2.24}$$

Now we are in position to prove the following global existence theorem.

Global Existence Theorem. *Let $h \in H^2 = H^2(\Omega)$, $Q \in C^1([0,\infty)\times\partial\Omega)$, $Q(0,\cdot) = h(\cdot)$ on $\partial\Omega$. Assume that u is a solution for* (2.1) *and any one of the following criteria is true* (M *is given by* (2.23)):

(i) $0 < |\nu| < \frac{\sqrt{5}}{2}$, *$\mu$ arbitrary*, $M^2 < 3 - 2\sqrt{1+\nu^2}$;
(ii) $|\nu| = \frac{\sqrt{5}}{2}$, *$\mu$ arbitrary*, $|\alpha| < \frac{\sqrt{5}}{2}$, $M^2 < 3 - 2\sqrt{1+\alpha^2}$;
(iii) $\mu\nu > 0$, $M^2 < 1$;
(iv) $\mu\nu < 0$, $|\mu| < \frac{\sqrt{5}}{2}$, $|\nu| > \frac{\sqrt{5}}{2}$, $M^2 < 3 - 2\sqrt{1+\mu^2}$;

(v) $\mu\nu < 0$, $|\mu| > \frac{\sqrt{5}}{2}$, $|\nu| > \frac{\sqrt{5}}{2}$, $|\alpha| < \frac{\sqrt{5}}{2}$, $-(1+\mu\nu) < |\alpha||\mu-\nu|$ *and* $M^2 < (3-2\sqrt{1+\alpha^2})$.

Then u is a global solution in H^2. In the other words, for any $T > 0$ there exist constants $K_1, K_2 > 0$ depending on parameters in (2.1)*, T and initial–boundary data such that*

$$\|u\|_{H^1\cap L^6} \leqslant K_1, \quad 0 \leqslant t \leqslant T,$$
$$\|u\|_{H^2} \leqslant K_2, \quad 0 \leqslant t \leqslant T.$$

Proof. Let

$$E_\delta\big(u(t)\big) = \frac{1}{2}\|\nabla u\|^2 + \frac{\delta}{6}\|u\|_6^6. \tag{2.25}$$

Then by (2.3) and (2.4), we have

$$\begin{aligned}
\frac{dE_\delta}{dt} &= \operatorname{Re}\int_{\partial\Omega} (\vec{n}\cdot P)\bar{Q}_t\,ds - \alpha_0 \operatorname{Re}\int_{\partial\Omega} (\vec{n}\cdot P)\bar{Q}\,ds + \alpha_0\|\nabla u\|^2 - \|\Delta u\|^2 \\
&\quad + \operatorname{Re}(1+i\mu)\int_\Omega |u|^4 u\Delta\bar{u}\,dx - \operatorname{Re}\int_\Omega |u|^2\Delta u\overline{\vec{\lambda}_1}\cdot\nabla\bar{u}\,dx \\
&\quad - \operatorname{Re}\int_\Omega \bar{u}^2\Delta u\overline{\vec{\lambda}_2}\cdot\nabla u\,dx + \alpha_0\delta\|u\|_6^6 + \delta\operatorname{Re}(1+i\nu)\int_\Omega |u|^4\bar{u}\Delta u\,dx \\
&\quad + \delta\operatorname{Re}\int_\Omega |u|^6\bar{u}\vec{\lambda}_1\cdot\nabla u\,dx + \delta\operatorname{Re}\int_\Omega |u|^6 u\vec{\lambda}_2\cdot\nabla\bar{u}\,dx - \delta\|u\|_{10}^{10} \\
&\leqslant -\big(\|\Delta u\|^2 + \delta\|u\|_{10}^{10}\big) + C_7E_\delta(t) \\
&\quad + \frac{1}{2}\operatorname{Re}\int_\Omega \big(|u|^4u, \Delta u\big)N_0\big(|u|^4\bar{u}, \Delta\bar{u}\big)^T\,dx + \max\{1,\alpha_0\}\int_{\partial\Omega} |\vec{n}\cdot P|\big(|Q_t| + |Q|\big)\,ds \\
&\quad + 2M\int_\Omega |u|^2|\Delta u||\nabla u|\,dx + 2M\delta\int_\Omega |u|^7|\nabla u|\,dx,
\end{aligned} \tag{2.26}$$

where $(|u|^4\bar{u}, \Delta\bar{u})^T$ denotes the transpose of $(|u|^4u, \Delta u)$ and

$$N_0 = \begin{pmatrix} 0 & 1+\delta-i(\nu\delta-\mu) \\ 1+\delta+i(\nu\delta-\mu) & 0 \end{pmatrix}.$$

Furthermore,

$$\begin{aligned}
\frac{d}{dt}\left(E_\delta + \frac{1}{2}\|u\|^2\right) &\leqslant C_8 + C_9\left(E_\delta + \frac{1}{2}\|u\|^2\right) - \big(\|u\|_6^6 + \|\nabla u\|^2\big) - \big(\|\Delta\|^2 + \delta\|u\|_{10}^{10}\big) \\
&\quad + \frac{\epsilon}{2}\int_{\partial\Omega} |\vec{n}\cdot P|^2\,ds + 2M\int_\Omega |u|^3|\nabla u|\,dx + 2M\int_\Omega |u|^2|\Delta u||\nabla u|\,dx \\
&\quad + 2M\delta\int_\Omega |u|^7|\nabla u|\,dx + \frac{1}{2}\operatorname{Re}\int_\Omega \big(|u|^4u, \Delta u\big)N_0\big(|u|^4\bar{u}, \Delta\bar{u}\big)^T\,dx.
\end{aligned} \tag{2.27}$$

If $|\alpha| < \frac{\sqrt{5}}{2}$ then $3 - 2\sqrt{1+\alpha^2} > 0$ and

$$\begin{aligned}
&\operatorname{Re}(1+i\alpha)\int_{\Omega}|u|^4\bar{u}\Delta u\,dx\\
&\quad=\operatorname{Re}(1+i\alpha)\int_{\Omega}u^2\bar{u}^3\Delta u\,dx\\
&\quad=\operatorname{Re}(1+i\alpha)\int_{\partial\Omega}(\vec{n}\cdot P)|Q|^4\bar{Q}\,ds-2\operatorname{Re}(1+i\alpha)\int_{\Omega}(\nabla u)^2|u|^2\bar{u}^2\,dx-3\int_{\Omega}|\nabla u|^2|u|^4\,dx\\
&\quad\leqslant\operatorname{Re}(1+i\alpha)\int_{\partial\Omega}(\vec{n}\cdot P)|Q|^4\bar{Q}\,ds-\big(3-2\sqrt{1+\alpha^2}\big)\int_{\Omega}|\nabla u|^2|u|^4\,dx.
\end{aligned}\tag{2.28}$$

This implies that

$$\begin{aligned}
\frac{d}{dt}\left(E_\delta+\frac{1}{2}\|u\|^2\right)&\leqslant C_{10}+C_{11}\left(E_\delta+\frac{1}{2}\|u\|^2\right)-\big(\|u\|_6^6+\|\nabla u\|^2\big)+\epsilon\int_{\partial\Omega}|\vec{n}\cdot P|^2\,ds\\
&\quad-(1-k)\big(\|\Delta u\|^2+\delta\|u\|_{10}^{10}\big)-\bar{\eta}\big(3-2\sqrt{1+\alpha^2}\big)\int_{\Omega}|u|^4|\nabla u|^2\,dx\\
&\quad+2M\int_{\Omega}|u|^2|\Delta u||\nabla u|\,dx+2M\delta\int_{\Omega}|u|^7|\nabla u|\,dx\\
&\quad+\frac{1}{2}\operatorname{Re}\int_{\Omega}\big(|u|^4u,\Delta u\big)N\big(|u|^4\bar{u},\Delta\bar{u}\big)^T\,dx
\end{aligned}\tag{2.29}$$

where

$$N=\begin{pmatrix}-2\delta k & 1+\delta-\bar{\eta}-i(\nu\delta-\mu-\alpha\bar{\eta})\\ 1+\delta-\bar{\eta}+i(\nu\delta-\mu-\alpha\bar{\eta}) & -2k\end{pmatrix}$$

and $\bar{\eta}$ is a positive constant to be chosen. In addition, k is a constant between 0 and 1 and C_{10} is a constant depending on Q and C_8. Now we integrate (2.29) in t and substitute (2.24) to get

$$\begin{aligned}
E_\delta+\frac{1}{2}\|u\|^2&\leqslant C_{11}\int_0^t\left(E_\delta+\frac{1}{2}\|u\|^2\right)dT+C_{12}-\int_0^t\big(\|u\|_6^6+\|\nabla u\|^2\big)\,dT\\
&\quad-(1-k)\int_0^t\big(\|\Delta u\|^2+\delta\|u\|_{10}^{10}\big)\,dT\\
&\quad-\bar{\eta}\big(3-2\sqrt{1+\alpha^2}\big)\int_0^t\int_{\Omega}|u|^4|\nabla u|^2\,dx\,dT\\
&\quad+2M\int_0^t\int_{\Omega}|u|^2|\Delta u||\nabla u|\,dx\,dT+2M\delta\int_0^t\int_{\Omega}|u|^7|\nabla u|\,dx\,dT\\
&\quad+\frac{1}{2}\operatorname{Re}\int_0^t\int_{\Omega}\big(|u|^4u,\Delta u\big)N\big(|u|^4\bar{u},\Delta\bar{u}\big)^T\,dx\,dT
\end{aligned}$$

$$+ C_4\epsilon\big(\|u\|^2+\|\nabla u\|^2\big)+C_5\epsilon\int_0^t\big(\|u\|^2+\|u\|_6^6\big)\,dT$$
$$+ C_6\epsilon\int_0^t\int_\Omega |\nabla u|^2\,dx\,dT+\frac{4N_1}{|\nu|}\epsilon\int_0^t\int_\Omega |\nabla u||\Delta u|\,dx\,dT$$
$$+\frac{8MN_1}{|\nu|}\epsilon\int_0^t\int_\Omega |u|^2|\nabla u|^2\,dx\,dT$$
$$+\frac{2N_1}{|\nu|}\epsilon\int_0^t\int_\Omega |u|^5|\nabla u|\,dx\,dT+\frac{2MN_1}{|\nu|}\epsilon\int_0^t\int_\Omega |u|^3|\nabla u|\,dx\,dT. \tag{2.30}$$

By choosing ϵ such that $C_4\epsilon<\frac{1}{2}$ and using some elementary inequalities, we have

$$\left(\frac{1}{2}-C_4\epsilon\right)\left(E_\delta+\frac{1}{2}\|u\|^2\right)$$
$$\leqslant C_{13}+C_{14}\int_0^t\left(E_\delta+\frac{1}{2}\|u\|^2\right)dT+C_{15}\int_0^t\big(\|u\|_6^6+\|\nabla u\|^2\big)\,dT$$
$$-(1-k-\epsilon)\int_0^t\big(\|\Delta u\|^2+\delta\|u\|_{10}^{10}\big)\,dT-\big[\bar\eta\big(3-2\sqrt{1+\alpha^2}\big)-\epsilon\big]\int_0^t\int_\Omega |u|^4|\nabla u|^2\,dx$$
$$+2M\int_0^t\int_\Omega |u|^2|\Delta u||\nabla u|\,dx\,dT+2M\delta\int_0^t\int_\Omega |u|^7|\nabla u|\,dx\,dT$$
$$+\frac{1}{2}\operatorname{Re}\int_0^t\int_\Omega \big(|u|^4u,\Delta u\big)N\big(|u|^4\bar u,\Delta\bar u\big)^T\,dx\,dT$$
$$\leqslant C_{13}+C_{14}\int_0^t\left(E_\delta+\frac{1}{2}\|u\|^2\right)dT+C_{15}\int_0^t\big(\|u\|_6^6+\|\nabla u\|^2\big)\,dT$$
$$-(1-k-\epsilon)\int_0^t\big(\|\Delta u\|^2+\delta\|u\|_{10}^{10}\big)\,dT$$
$$-\big[\bar\eta\big(3-2\sqrt{1+\alpha^2}\big)-\epsilon\big]\int_0^t\int_\Omega |u|^4|\nabla u|^2\,dx\,dT$$
$$+\frac{M^2}{1-k-\epsilon}\int_0^t\int_\Omega |u|^4|\nabla u|^2\,dx\,dT+(1-k-\epsilon)\int_0^t\|\Delta u\|^2\,dT$$

$$
\begin{aligned}
&+ (1-k-\epsilon)\delta \int_0^t \|u\|_{10}^{10}\,dT + \frac{M^2\delta}{1-k-\epsilon}\int_0^t\int_\Omega |u|^4|\nabla u|^2\,dx\,dT \\
&+ \frac{1}{2}\operatorname{Re}\int_0^t\int_\Omega \big(|u|^4u,\Delta u\big)N\big(|u|^4\bar u,\Delta\bar u\big)^T\,dx\,dT \\
\leqslant{}& C_{13} + C_{16}\int_0^t\left(E_\delta + \frac{1}{2}\|u\|^2\right)dT + \frac{1}{2}\operatorname{Re}\int_0^t\int_\Omega \big(|u|^4u,\Delta u\big)N\big(|u|^4\bar u,\Delta\bar u\big)^T\,dx\,dT \\
&+ \left[\frac{M^2(1+\delta)}{1-k-\epsilon} + \epsilon - \bar\eta\big(3-2\sqrt{1+\alpha^2}\big)\right]\int_0^t\int_\Omega |u|^4|\nabla u|^2\,dx\,dT. \qquad (2.31)
\end{aligned}
$$

Assumptions on μ, ν and α in the global existence theorem imply that N is negative semidefinite [9,10,24]. Similar to discussions in [21,24], we write $\bar\eta = 1+\delta$ and see that

$$\frac{M^2(1+\delta)}{1-k-\epsilon} + \epsilon - \bar\eta\big(3-2\sqrt{1+\alpha^2}\big) \leqslant 0, \qquad (2.32)$$

$$M^2(1+\delta) < \bar\eta\big(3-2\sqrt{1+\alpha^2}\big), \qquad (2.33)$$

for any $0<\epsilon<1-k$. Therefore,

$$E_\delta + \frac{1}{2}\|u\|^2 + \epsilon_1\int_0^t \|\Delta u\|^2\,dT \leqslant M_1 + M_2\int_0^t\left(E_\delta + \frac{1}{2}\|u\|^2\right)dT, \qquad (2.34)$$

where $\epsilon_1 = 1-k-\epsilon$. By Gronwall's lemma, $u \in H^1 \cap L^6$ for any $t \geqslant 0$, and $u \in L^2([0,T],H^2)$ for any $T>0$.

To study the local existence, we first homogenize the boundary condition by setting up $v(x,t) = u(x,t) - \tilde Q(x,t)$, where $\tilde Q(x,t)$ is defined in (2.17) and (2.18). This procedure leads to

$$v_t = (1+i\nu)\Delta v + f(\tilde Q, v). \qquad (2.35)$$

Since $f(\cdot): H^2 \to H^1$ is locally Lipschitz continuous, (2.1) has a unique solution $u \in C^1((0,T),H^1) \cap C([0,T),H^2)$ for some $T>0$ [27,28].

In order to show that $\|\Delta u\|_2$ is bounded, we rewrite (2.35) as

$$
\begin{aligned}
v_t ={}& \alpha_0 v + (1+i\nu)\Delta v + |v|^2\vec\lambda_1\cdot\nabla v + v^2\vec\lambda_2\cdot\nabla\bar v - (1+i\mu)|v|^4 v \\
&+ f_1(\tilde Q) + f_2(\tilde Q, v) \qquad (2.36)
\end{aligned}
$$

and

$$v|_{\partial\Omega} = 0 \qquad (2.37)$$

where

$$
\begin{aligned}
f_1(\tilde Q) ={}& (1+i\nu)\Delta\tilde Q - \tilde Q_t + |\tilde Q|^2\vec\lambda_1\cdot\nabla\tilde Q + \tilde Q^2\vec\lambda_2\cdot\nabla\bar{\tilde Q} - (1+i\mu)|\tilde Q|^4\tilde Q, \\
f_2(\tilde Q, v) ={}& \vec\lambda_1\cdot\big[2\operatorname{Re}(\tilde Q\bar v)\nabla v + |\tilde Q|^2\nabla v + |v|^2\nabla\tilde Q + 2\operatorname{Re}(\tilde Q\bar v)\nabla\tilde Q\big] \\
&+ \vec\lambda_2\cdot\big[v^2\nabla\bar{\tilde Q} + 2v\tilde Q\nabla\bar v + 2v\tilde Q\nabla\bar{\tilde Q} + \tilde Q^2\nabla\bar v\big]
\end{aligned}
$$

$$
\begin{aligned}
&- (1+i\mu)\big[|v|^4\tilde{Q} + 4\,\mathrm{Re}(\tilde{Q}\bar{v})^2 v + 4\,\mathrm{Re}(\tilde{Q}\bar{v})^2\tilde{Q} + 4|v|^2 v\,\mathrm{Re}(\tilde{Q}\bar{v}) \\
&+ 4|v|^2\tilde{Q}\,\mathrm{Re}(\tilde{Q}\bar{v})\big] \\
&- (1+i\mu)\big[2|v|^2 v|\tilde{Q}|^2 + 2|v|^2\tilde{Q}|\tilde{Q}|^2 + 2|\tilde{Q}|^2|v|^2 + |\tilde{Q}|^4 v\big].
\end{aligned}
$$

Since $\|u\|_{H^1}$ is bounded, $\tilde{Q} \in C^3(\Omega \times (0,T))$, by (2.36) we see that $\|\Delta v\| - C \leqslant \|\Delta u\| \leqslant \|\Delta v\| + C$ and $C_{17}(\|v_t\| - 1) \leqslant \|\Delta v\| \leqslant C_{18}(\|v_t\| + 1)$ for positive constants C, C_{17} and C_{18} are positive.

Therefore, the only remaining term to be estimated is $\|v_t\|$. Differentiating (2.36) with respect to t, we have

$$
\begin{aligned}
v_{tt} &= \alpha_0 v_t + (1+i\nu)\Delta v_t + \big(|v|^2\vec{\lambda}_1 \cdot \nabla v\big)_t + \big(v^2\vec{\lambda}_2 \cdot \nabla\bar{v}\big)_t - (1+i\mu)\big(|v|^4 v\big)_t \\
&\quad + \big(f_1(\tilde{Q})\big)_t + \big(f_2(\tilde{Q}, v)\big)_t.
\end{aligned} \tag{2.38}
$$

We now take the real part of the L^2-inner product of (2.38) with v_t. After an integration by parts, we note that $\tilde{Q} \in C^4(\Omega \times (0,T))$ and $\|u\|_{H^1}$ is bounded so we can apply Cauchy inequality to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_t\|^2 \\
&\quad \leqslant \alpha_0\|v_t\|^2 - \|\nabla v_t\|^2 + C\Big[\int_\Omega |v|^2|\nabla v_t||v_t|\,dx + \int_\Omega |v||\nabla v||v_t|^2\,dx\Big] \\
&\qquad + C\Big[\int_\Omega |v||\nabla v_t||v_t|\,dx + \int_\Omega |v||\nabla v||v_t|\,dx + \int_\Omega |\nabla v||v_t|^2\,dx + \int_\Omega |\nabla v_t||v_t|\,dx\Big] \\
&\qquad + C\Big[\int_\Omega |v|^4|v_t|^2\,dx + \int_\Omega |v|^3|v_t|^2\,dx + \int_\Omega |v|^2|v_t|^2\,dx + \int_\Omega |v||v_t|^2\,dx\Big] \\
&\qquad + C\Big[\int_\Omega |v|^4|v_t|\,dx + \int_\Omega |v|^3|v_t|\,dx + \int_\Omega |v|^2|v_t|\,dx + \int_\Omega |v_t|^2\,dx + 1\Big]
\end{aligned} \tag{2.39}
$$

and

$$
\int_\Omega |v|^2|\nabla v_t||v_t|\,dx \leqslant \epsilon_1\|\nabla v_t\|^2 + C(\epsilon_1)\int_\Omega |v|^4|v_t|^2\,dx.
$$

Using Gagliardo–Nirenberg inequality in two dimension, we get

$$
\int_\Omega |v|^4|v_t|^2\,dx \leqslant \|v\|_{L^\infty}^4\|v_t\|^2 \leqslant C\|v\|^2\|\Delta v\|^2\|v_t\|^2 \leqslant C\big(1 + \|v_t\|^4\big).
$$

Therefore we obtain

$$
\begin{aligned}
\int_\Omega |v|^2|\nabla v_t||v_t|\,dx &\leqslant \epsilon_1\|\nabla v_t\|^2 + C\int_\Omega |v|^4|v_t|^2\,dx \\
&\leqslant \epsilon_1\|\nabla v_t\|^2 + C\big(1 + \|v_t\|^4\big),
\end{aligned} \tag{2.40}
$$

$$
\int_\Omega |v||\nabla v||v_t|^2 \leqslant \|v\|_{L^\infty}\int_\Omega |\nabla v||v_t| \leqslant \epsilon_1\|\nabla v_t\|^2 + C\big(1 + \|v_t\|^4\big) \tag{2.41}
$$

and

$$\int_{\Omega} |v||\nabla v_t||v_t|\,dx \leqslant \epsilon_1\|\nabla v_t\|^2 + \int_{\Omega} |v|^2|v_t|^2\,dx \leqslant \epsilon_1\|\nabla v_t\|^2 + C\big(1+\|v_t\|^4\big). \tag{2.42}$$

The remaining terms in (2.39) could be estimated easily. Since ϵ_1 is small enough, by (2.40)–(2.42) we get

$$\frac{d}{dt}\|v_t\|^2 \leqslant C_{19}\|v_t\|^4 + C_{20}\|v_t\|^2 + C_{21}.$$

Or equivalently,

$$\frac{d}{dt}\big(\|v_t\|^2+1\big) \leqslant C\big(\|v_t\|^2+1\big)^2. \tag{2.43}$$

By (2.43), the term $\int_0^t \|\Delta u\|^2\,d\tau$ is bounded, so is $\int_0^t \|\Delta v\|^2\,d\tau$. This implies that the term $\int_0^t \|v_t\|^2\,d\tau$ is bounded. By uniform Gronwall inequality [29] we get $\|v_t\|^2 \leqslant C(T)$, hence $\|\Delta u\|^2 \leqslant C(T)$. Therefore, the solution obtained in the above exists in H^2 globally and the proof of the Global Existence Theorem is now complete. □

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