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CHOICE AND WELL-ORDERING

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It is one thing to say that the axiom of choice implies that everything may be well-ordered. It is quite another thing to say that if a set has a choice function then it has a well-ordering. Even in the classical setting the difference shows in the length of proof: using the (global) axiom of choice one may quickly apply a Zorn's lemma argument to the family of partial well-orderings on a set to obtain an entire well-ordering. The difference is clearest in the intuitionistic setting, so clear that a number of us have been searching for a counterexample. The main purpose of the following is to end that search.

Theorem. *In any topos an object has a choice map iff it has a well-ordering.*

Corollaries. *In intuitionistic set theory with atoms* (ZI, ZFI, ZFIA), *indeed, in intuitionistic type-theory, every set with a choice function may be well-ordered.*

The paper closes with a classification theorem for the well-ordered objects in Grothendieck topoi and a detailed description of the well-ordered functors on arbitrary small domains and well-ordered sheaves over arbitrary spaces. The coda is a speculation.

The proof will be presented in the topos-theoretical setting. In previous cases of intuitionistic proofs topos theory has been a convenience to those of us who like it. In this case it appears to be a practical necessity. In theory, one may extract from this proof a syntactical description of the well-ordering starting with a set, A and choice function, c. In theory, that is, there is an binary predicate on A derived from equality, membership and c, with quantification ranging over A and its first few iterated power-sets, which binary predicate may then be shown to be a well-ordering without using excluded middle. It is the author's estimation that such an 'elementary proof' will be extracted from this proof only by machine translation using machines not yet in existence. It is the author's further estimation that such a proof will be verifiable only via further machines not yet even in theoretical existence.

1. Two definitions

The existence of a choice map is equivalent to a more primitive property (primitive enough to work in any regular category). Recall first that the axiom of choice (on any category) is equivalent to the statement that all objects are projective and that an object is projective iff every entire binary relation therefrom contains a map. We say that an object is a CHOICE OBJECT if every entire relation targeted *thereto* contains a map. In a topos there is a universal entire relation targeted at an object B; namely, the universal relation from the power-object, PB, restricted to its domain, P^+B. B is choice iff this one entire relation contains a map, which map is, of course, called a choice map.

The easiest definition of the well-orderability of B is the existence of an ORDERED CHOICE FUNCTION, that is a choice map $c: P^+B \to B$ such that

$$c(B_1 \cup B_2) = c\{c(B_1), c(B_2)\}.$$

We may then, of course, define $x \leqslant y$ as $x = c\{x, y\}$ and easily verify that $\leqslant$ is a total ordering and that $c(B_1) \leqslant y$ for all $y \in B_1$.

We will sometimes use the more primitive definition: B is said to be WELL-ORDERED by a binary relation, $\leqslant$, if the relation is a total ordering such that there is a choice map, c, so that $c(B') \leqslant y$ for all $y \in B' \subset B$. It is, of course, easy to show that such a choice function is an ordered choice function. Note that it suffices for $\leqslant$ to be a partial ordering: the existence of an ordered choice map for $\leqslant$ forces it to be a total ordering.

2. Choice objects

Lemma 2.1. *If B is choice and $B \to C$ is epic, then there is a left-inverse $C \to B$.*

Proof. Because we may regard the map $f: B \to C$ as a relation, reciprocate it to obtain an entire relation $f^\circ: C \to B$, use the choiceness of B to obtain a map $g \subset f^\circ$, and easily check that $gf = 1_B$. □

Lemma 2.2. *Choice objects are closed under the formation of:*
(a) *subobjects;*
(b) *quotient-objects;*
(c) *finite products.*

Proof. (a) is immediate from the definition of choice.

(b) is immediate from Lemma 2.1: every quotient-object appears as a subobject.

(c) requires work. The empty product is, of course, easy: every entire relation targeted at **1** is already a map. For binary products let B_1 and B_2 be choice objects and $R:A \to B_1 \times B_2$ an entire relation. All maps when regarded as relations are entire and entire relations are closed under composition. Use the choiceness of B_1 to obtain a map $f_1 \subset Rp_1$. Verify that $R \cap f_1 p_1^\circ$ is entire and use the choiceness of B_2 to obtain a map $f_2 \subset (R \cap f_1 p_1^\circ)p_2$. Then the map $\langle f_1, f_2 \rangle$ is contained in R. □

It is worth knowing for the above two lemmas that they hold in the regular category setting and that the representation theorem for regular categories yields a metatheorem for the calculus of relations: any universal Horn sentence in the operations of composition, intersection and reciprocation true for relations between sets is true in any regular category.

The next lemma, which holds in the pre-topos setting, may be regarded as a sharpening of Diaconescu's theroem that the axiom of choice (which may be rephrased as saying that all objects are choice) implies excluded middle (which will be rephrased as saying that the category is boolean: all subobject-lattices are boolean algebras).

Lemma 2.3. *The following are equivalent conditions on a topos*:
(a) *choice objects are closed under finite co-product*;
(b) $\mathbf{1}+\mathbf{1}$ *is choice*;
(c) *the topos is boolean.*

Proof. (c) *implies* (a): Let B_1 and B_2 be choice and let $R:A \to B_1 + B_2$ be an entire relation. Define $A_1 \subset A$ as the domain of $Ru_1^\circ: A \to B_1$, and $A_2 \subset A$ as its complement. Let R_1 be the restriction of Ru_1° to A_1 and R_2 the restriction of Ru_2° to A_2. R_1 and R_2 are entire and the relation $R_1 + R_2$ is entire and contained in R. It now clearly suffices to use the choiceness of the B's to obtain $f_1 + f_2 \subset R_1 + R_2$.

(a) trivially implies (b).

To get from (b) to (c) recall that the fundamental lemma of topos theory says that if C is an arbitrary object in a topos $\mathbb{A}$ and $\mathbb{A}/C$ is the category whose objects are maps to C (and whose maps are commutative triangles), then $\mathbb{A}/C$ is a topos and the functor $\mathbb{A} \to \mathbb{A}/C$ that sends an object A to $A \times C \to C$ is a *representation of topoi,* that is, it preserves all relevant structure (and is sometimes called a 'logical morphism of topoi'). The existence of a choice map is preserved by representations of topoi, hence condition (b) holds in $\mathbb{A}/C$ and it suffices to show that condition (b) implies that the lattice of subobjects of $\mathbf{1}$ is boolean: the lattice of subobjects of C is isomorphic to the lattice of subobjects of the terminator in $\mathbb{A}/C$.

To that end we recall that an object B is *decidable* if the diagonal map $B \to B \times B$ has a complement. $\mathbf{1}$ is vacuously decidable. Coproducts of decidables are decidable. Subobjects of decidables are decidable. For any pair of maps $f, g:A \to B$ into a decidable object it is the case that the equalizer of f and g is complemented because the equalizer may be constructed from the pullback diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & A \arrow[d, "\langle f, g\rangle"] \\ B \arrow[r] & B \times B \end{tikzcd}$$

and the fact that pullbacks preserve unions as well as intersections.

Finally then, for any subobject $U \subset \mathbf{1}$ we form the pushout diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \mathbf{1} \arrow[d] \\ \mathbf{1} \arrow[r] & D \end{tikzcd}$$

U is the equalizer of the two maps from $\mathbf{1}$ to D. D is a quotient of $\mathbf{1}+\mathbf{1}$. Condition (b) and Lemma 2.1 imply, therefore, that $C = D$ is a subobject of a decidable and is itself decidable. □

(For the record: this lemma holds in the pre-topos setting. The above proof works except for the argument that condition (b) is preserved under slicing. For that purpose show that condition (b) is equivalent to a fourth condition, a condition on the lattices that appear as lattices of subobjects, hence a condition clearly preserved under slicing. Such a condition is:

(d) *Finite covers have partitions as refinements, that is, if an object A is the union of two subobjects A_1 and A_2, then there exist subobjects A_1' and A_2' such that:*

$$A_1' \subset A_1, \qquad A_2' \subset A_2,$$
$$A_1' \cup A_2' = A,$$
$$A_1' \cap A_2' = 0.$$

The equivalence of (b) and (d) is a simple matter of translation. Maps from A to $\mathbf{1}+\mathbf{1}$ are in natural correspondence with pairs of complemented subobjects of A. Relations from A to $\mathbf{1}+\mathbf{1}$ are in natural correspondence with pairs of subobjects, and entire relations from A to $\mathbf{1}+\mathbf{1}$ with covering pairs of subobjects. It is easily checked that an entire relation contains a map if the corresponding covering pair has a partition as a refinement.)

3. The boolean case

It would be nice if our proof were a straightforward construction that simply avoided excluded middle. Instead it is a rather curious reduction to the boolean case. For the sake of completeness we give a proof for that special case.

In a boolean topos let B be a choice object, $c: P^+B \to B$ a choice map. Define a 'successor operation' $s: PB \to PB$ by $s(B') = B' \cup \{s(B \backslash B')\}$ for proper subobjects and let the entire subobject be a fixed point of s (necessarily its only fixed point). Define $F \subset PB$ to be the smallest subobject closed with respect to arbitrary unions and invariant under the action of s.

Lemma 3.1. *F is linearly ordered by containment.*

Proof. It is useful to abstract the situation. Let P be any partially ordered subject, $s: P \to P$ a function such that

s is inflationary: $x \leq s(x)$,

s is a successor function: $x \leq y \leq s(x)$ *implies* $x = y$ *or* $y = s(x)$.

Let $F \subset P$ be the smallest subobject closed with respect to arbitrary suprema and invariant under the action of s. (The proof will not use any completeness condition on P. It needs only that F be closed with respect to any suprema that may exist. If P does not have a bottom then F is empty. The proof does use the successor condition on s, but, in fact, the result still holds without it.)

Let F' be the set of 'pinch points' in F, that is, $x \in F'$ iff $x \leq y$ or $y \leq x$ for all $y \in F$. It suffices to show that F' is closed with respect to arbitrary suprema and invariant under the action of s. In any poset the set of pinch points is closed with respect to arbitrary suprema (using the rules of classical logic). We need only show that F' is s-invariant. To that end let $x \in F'$. We wish to show that $s(x) \in F'$. Let F'' be the subobject of F' defined by $y \in F''$ iff $s(x) \leq y$ or $y \leq s(x)$. We will finish by showing that F'' is closed with respect to arbitrary suprema and invariant under the action of s. Again the suprema take care of themselves. We need the s-invariance. Let $y \in F''$. We must show that either $s(x) \leq s(y)$ or $s(y) \leq s(x)$. We are given:

$$x \leq s(y) \quad \text{or} \quad s(y) \leq x \qquad (\text{because } x \in F');$$
$$s(x) \leq y \quad \text{or} \quad y \leq s(x) \qquad (\text{because } y \in F'').$$

If either $s(y) \leq x$ or $s(x) \leq y$ we are done because s is inflationary. We may thus assume that $x \leq s(y)$ and $y \leq s(x)$. Since x is a pinch point we have that either $x \leq y$ or $y \leq x$. In the first case we have $x \leq y \leq s(x)$ hence $x = y$ or $y = s(x)$. In the second case we have $y \leq x \leq s(y)$ hence $y = x$ or $x = s(y)$. All together, then, we have $x = y$ or $s(x) = y$ or $s(y) = x$, hence $s(x) = s(y)$ or $s(x) \leq s(y)$ or $s(y) \leq s(x)$. □

We could now proceed to show that F is well-ordered (even in the general poset case), and imbed B into F (in the case at hand) by sending $x \in B$ to the union of all members of F that do not include x as an element. It is easier, however, to use the ordered choice function definition of well-orderability. Let C be the family of complements of members of F. C is linearly ordered by containment and closed with respect to arbitrary intersections. For any non-empty $B' \subset B$ let $\tilde{B}' \in C$ be the intersection of all members which contain B'. Define an ordered choice function by $\hat{c}(B') = c(\tilde{B}')$.

4. Restricting to the well-supported case

Let C be a choice object in a topos $\mathbb{A}$, $U = \mathrm{spt}(C) \subset \mathbf{1}$ its support. The slice topos $\mathbb{A}/U$ may be viewed as the full subcategory of $\mathbb{A}$ of those objects with support contained in U. C is well-supported as an object in $\mathbb{A}/U$. The inclusion functor $\mathbb{A}/U \to \mathbb{A}$ preserves the calculus of relations and hence C is a choice object in $\mathbb{A}/U$. The inclusion functor does not preserve the construction of power objects but it does preserve P^+ and it preserves the binary operation of union on P^+. It thus suffices to consider the case of well-supported choice objects.

By Lemma 2.1 any well-supported choice C object has a point $\mathbf{1} \to C$ which leads us to:

5. The topos of pointed objects

The standard category of pointed objects, $\mathbf{1}\backslash\mathbb{A}$, has maps of the form $\mathbf{1} \to B$ as objects and commutative squares as maps. It is not a topos. (The terminator and coterminator coincide in $\mathbf{1}\backslash\mathbb{A}$. Topoi in which that occurs are degenerate. They are inconsistent: they are such that true equals false.) By the TOPOS OF POINTED OBJECTS, denoted $^{\blacksquare}\mathbb{A}$, nicknamed 'dot $\mathbb{A}$', we do not mean the standard category of pointed objects. We mean the same objects, but put a further condition on the maps. In the standard case the maps 'preserve the base point'. We require them to be *strict maps* between pointed objects, which means not just commutative squares but pullback squares. Thus the objects of $^{\blacksquare}\mathbb{A}$ are maps of the form $\mathbf{1} \to B$. A map from one object to another is a *pullback* square

$$\begin{array}{ccc} \mathbf{1} & \longrightarrow & B \\ \downarrow & & \downarrow \\ \mathbf{1} & \longrightarrow & B'. \end{array}$$

Lemma 5.1. *$^{\blacksquare}\mathbb{A}$ is a topos.*

Proof. This (apparently new) way of getting a topos from a topos is a composition of special cases of two old ways. First slice by Ω to obtain $\mathbb{A}/\Omega$. The objects of $\mathbb{A}/\Omega$ may be reinterpreted as *monomorphisms* $A' \to A$, the maps as pullback squares. $^{\blacksquare}\mathbb{A}$ is a full subcategory of $\mathbb{A}/\Omega$.

Recall the construction of CLOSED SHEAVES. Let U be a subterminator in any topos $\mathbb{B}$. Define $\mathbb{B}:U$ to be the full subcategory sent to the terminator under $\mathbb{B}\to\mathbb{B}/U$. $\mathbb{B}:U$ is always a topos. (If $\mathbb{B}$ is a spatial topos and U is regarded as an open subset of the base space, then $\mathbb{B}/U$ may be regarded as the topos of sheaves over U and $\mathbb{B}:U$ is equivalent to the topos of sheaves over a closed subset of the base space, namely the complement of U.)

The universal subobject $t:\mathbf{1}\to\Omega$ is a subterminator in $\mathbb{A}/\Omega$. The functor $\mathbb{A}/\Omega\to\mathbb{A}/t$ sends an object $A'\to A$ to A'. $(\mathbb{A}/\Omega):t$ is the full subcategory of those objects such that A' is isomorphic to $\mathbf{1}$. $(\mathbb{A}/\Omega):t$ is ${}^\blacksquare A$. □

Lemma 5.2. *The forgetful functor* ${}^\blacksquare\mathbb{A}\to\mathbb{A}$ *reflects choiceness.*

This forgetful functor is the composition of the inclusion functor ${}^\blacksquare\mathbb{A}\to\mathbb{A}/\Omega$ and the standard forgetful functor $\mathbb{A}/\Omega\to\mathbb{A}$, hence this lemma is a consequence of:

Lemma 5.3. *For any subterminator, U, in any topos, $\mathbb{B}$, the inclusion functor $\mathbb{B}:U\to\mathbb{B}$ reflects choiceness.*

Lemma 5.4. *For any object, B, in any topos, $\mathbb{B}$, the forgetful functor, $\mathbb{B}/B\to\mathbb{B}$, reflects choiceness.*

Proofs. 5.3 is an immediate consequence of the fact that the inclusion functor preserves the construction of P^+. (The functor $\mathbb{B}\to\mathbb{B}/U$ preserves everything. If it sends an object A to $\mathbf{1}$, then it sends P^+A to $P^+\mathbf{1}$. But $P^+\mathbf{1}$ is always $\mathbf{1}$.)

For 5.4 note that the forgetful functor carries entire relations to entire relations. If the target of the relation is choice in $\mathbb{B}$, then the entire relation contains a map in $\mathbb{B}$. It is easy to check that any map in $\mathbb{B}$ contained in a relation coming from $\mathbb{B}/B$ also comes from $\mathbb{B}/B$. □

(The construction of ${}^\blacksquare\mathbb{A}$ may be generalized: for any object B in a topos $\mathbb{A}$, define $\underline{\mathrm{Ext}}(B)$, the topos of extensions of B, as the category whose objects are *monics* of the form $B\to C$ and whose maps are pullback squares. We may construct $\underline{\mathrm{Ext}}(B)$ by first slicing by the partial-map classifier $\tilde{B}$, then viewing $B\subset\tilde{B}$ as a subterminator in $\mathbb{A}/\tilde{B}$ and, finally, moving to its topos of closed sheaves $(\mathbb{A}/\tilde{B}):B$. We may as well do even more. For any map $f:B\to A$ define $\underline{\mathrm{Ext}}(f)$, *the topos of extensions of f* as the category whose objects are pairs of the form $B\to X\to A$ where $B\to X$ is a monic and the composition from B to A is f. The maps of $\underline{\mathrm{Ext}}(f)$ are 2 by 3 commutative diagrams such that the left hand squares are pullbacks. We may construct $\underline{\mathrm{Ext}}(f)$ by first slicing by A, viewing f as an object therein and then proceeding as above. If f is a map to the terminator, then this last construction reduces to the previous. If f is a map from the coterminator, we obtain just an ordinary slice topos. If f is the identity map of B, the objects of $\underline{\mathrm{Ext}}(f)$ are what the topologists have called the extracts of B ('extract' taken as the converse of 'retract').

6. The construction

Let C be a choice object in a topos $\mathbb{A}$. Following Section 4 we may assume that C is well-supported and choose a point $\mathbf{1} \to C$. (It will be the bottom point of the well-ordering.) We may view C as an object in ${}^{\blacksquare}\mathbb{A}$. By 5.2 it is still a choice object. It need not be well-supported as an object in ${}^{\blacksquare}\mathbb{A}$. Its support is a pointed subobject Ω' of Ω. Ω' appears as a quotient object of C and Lemma 2.2 says that it is a choice object in $\mathbb{A}$.

Lemma 6.1. *If Ω' is a pointed subobject of Ω that is choice, then $({}^{\blacksquare}\mathbb{A})/\Omega'$ is a boolean topos.*

Proof. Define the 'wedge', W, via the pushout diagram

$$\begin{tikzcd} \mathbf{1} \arrow[r] \arrow[d] & \Omega' \arrow[d] \\ \Omega' \arrow[r] & W. \end{tikzcd}$$

W may be embedded in $\Omega' \times \Omega'$ and Lemma 2.2 says that W is choice. Using the diagonal map of the pushout view W as an object in $({}^{\blacksquare}\mathbb{A})/\Omega'$ where it may be directly verified to be the double coproduct of the terminator. Lemma 5.2 says, therefore, that $\mathbf{1}+\mathbf{1}$ is choice in $({}^{\blacksquare}\mathbb{A})/\Omega'$. Lemma 2.3 implies that $({}^{\blacksquare}\mathbb{A})/\Omega'$ is a boolean topos. $\square$

Lemma 5.2 says that C is still a choice object in the boolean topos $({}^{\blacksquare}\mathbb{A})/\Omega'$ and Section 3 says that it is well ordered in $({}^{\blacksquare}\mathbb{A})/\Omega'$. We thus finish the proof of the main theorem with

Lemma 6.2. *If Ω' is a pointed subobject of Ω that is choice, then the forgetful functor $({}^{\blacksquare}\mathbb{A})/\Omega' \to \mathbb{A}$ preserves well-orderability.*

$({}^{\blacksquare}\mathbb{A})/\Omega'$ may be constructed by first slicing by Ω' and then forming the topos of closed sheaves. Hence 6.2 will be established by the following three lemmas:

Lemma 6.3. *For any subterminator, U, in any topos, $\mathbb{B}$, the inclusion functor $\mathbb{B}: U \to B$ preserves well-orderability.*

Lemma 6.4. *In any topos $\mathbb{B}$, if B is a well-ordered object, then the forgetful functor $\mathbb{B}/B \to \mathbb{B}$ preserves well-orderability.*

Lemma 6.5. *In any topos any pointed subobject of Ω is well-ordered if it is choice.*

Proof. The proof of 6.3 is as immediate as that of 5.3 and for the same reason: the inclusion functor preserves P^+.

The proof of 6.5 appears in the next section.

For 6.4 let $f:A\to B$ be an object in $\mathbb{B}/B$, $\leqslant_f$ a well-ordering thereon. We may apply the forgetful functor and regard $\leqslant_f$ as a partial ordering in $\mathbb{B}$. Let $\leqslant_B$ be a well-ordering on B. We obtain a well-ordering, $\leqslant_A$ on A by

$$x\leqslant_A y \quad \textit{iff} \quad f(x)\leqslant_B f(y) \text{ and } [(f(x)=f(y)) \text{ implies } (x\leqslant_f y)].$$

It is easy to check that this is a partial ordering on A. As noted in Section 1 it suffices to show that every well-supported subobject of A has a unique minimum. Let A' be a well-supported subobject and let $B'\subset B$ be the image of f restricted to A'. By moving to the topos $\mathbb{B}/B'$ we obtain a well-supported subobject $A'\to B'$ and there exists a minimal point $g:B'\to A'$. Returning to the ambient topos $\mathbb{B}$ we note that g is a monomorphism with a 'co-final' image: for any $x\in A'$ it is the case that $g(f(x))\leqslant_f x$. Now use the fact that $\leqslant_B$ is a well-ordering to obtain a minimal point, y_1 of B'. Then $g(y)$ is a minimum point in A' for $\leqslant_A$. □

Now that $\leqslant_A$ is known to be a total ordering it may be easily shown that:

Lemma 6.6. *$\leqslant_A$ is the unique total ordering on A that extends $\leqslant_f$ and such that $A\to B$ is order-preserving.* □

7. Choice subobjects of Ω in general: the Higgs object in particular

There is a largest choice subobject among the pointed subobjects of Ω and it is it that we will prove is well-ordered. We will define it, however, by indirection. In any topos, $\mathbb{B}$, there is a largest subterminator $B\subset\mathbf{1}$ such that $\mathbb{B}/B$ is boolean. The quickest definition of B is as the universal quantification of the natural inclusion of $\mathbf{1}+\mathbf{1}$ into Ω. B is easily characterized by:

Lemma 7.1. *B is the largest subterminator such that all objects whose supports are contained in B have boolean lattices of subobjects.* □

Define $H\subset\Omega$ as the largest pointed subobject such that $(^{\blacksquare}\mathbb{A})/H$ is boolean. By Lemma 6.1 if Ω' is choice, then $(^{\blacksquare}\mathbb{A})/\Omega'$ is boolean, hence $\Omega'\subset H$. It suffices for the proof of 6.5, and hence for the proof of the main theorem, to prove:

Lemma 7.2. *H is well-ordered.*

(Thus the proof devolves to this one special case. We could go further: it suffices to consider H in the free topos on no generators).

The well-ordering on H is natural. Ω is partially ordered by containment. The universal subobject $t:\mathbf{1}\to\Omega$ is its top element. We will see that the induced ordering on H is linear and that when we reverse it to make t the bottom it becomes a well-ordering. It is even more natural: as will become evident this well-ordering is the unique total ordering on H that puts t on the bottom. We need first, however, a better handle on H.

In any distributive lattice an element is CO-DISCRETE if "the lattice from there up is boolean": x is co-discrete iff for all $y \geqslant x$ there exists z such that $y \wedge z = x$ and $y \vee z = 1$. (In the lattice of open subsets of a topological T_0 space an open subset is co-discrete iff its complement is discrete.) A direct translation of 7.1 says:

Lemma 7.3. *H is the largest pointed subobject of Ω such that the only pointed objects it classifies are those in which the point, when viewed as a subobject, is co-discrete in the entire lattice of subobjects.* □

Not just points but arbitrary subobjects:

Lemma 7.4. *H is the largest pointed subobject of Ω that classifies only co-discrete subobjects.*

Proof. Given $A' \subset A$ let

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d] \\ \mathbf{1} \arrow[r] & A/A' \end{tikzcd}$$

be a pushout. The inverse-image map from Sub(A/A') to Sub(A) establishes an isomorphism from the lattice of pointed subobjects of A/A' to the lattice of subobjects in A which contain A'. A' is co-discrete in A iff $\mathbf{1}$ is co-discrete in A/A'. Moreover, $A/A' \to \Omega$ is the characteristic map of $\mathbf{1}$ iff $A \to A/A' \to \Omega$ is the characteristic map of A'. Hence if the characteristic map of A' lies in H, then A' is a co-discrete subobject and H is the largest subobject with that property. □

We have not yet characterized the subobjects classified by H. We have said that H classifies only co-discrete subobjects. We have said that H is the largest subobject with that property. Say that a subobject is STABLY CO-DISCRETE if its inverse image under any map is co-discrete.

Lemma 7.5. *A subobject is classified by H iff it is stably co-discrete.*

Proof. Clearly every subobject classified by H is stably co-discrete. For the converse suppose that $A' \subset A$ is stably co-discrete, let $A \to \Omega$ be its characteristic map, and let $\Omega' \subset \Omega$ be the image. We wish to show that $\Omega' \subset H$. That is, we wish to show that every subobject classified by Ω' is co-discrete. Suppose, then, that $B' \subset B$ is classified by Ω'.

Let

$$\begin{tikzcd}
C' \arrow[rr] \arrow[dr] \arrow[dd] & & B' \arrow[dr] \arrow[dd] & \\
& C \arrow[rr] \arrow[dd] & & B \arrow[dd] \\
A' \arrow[rr] \arrow[dr] & & \mathbf{1} \arrow[dr] & \\
& A \arrow[rr] & & \Omega'
\end{tikzcd}$$

be a diagram of pullbacks. $C' \subset C$ is co-discrete because $A' \subset A$ is stably co-discrete. $C \to B$ is epic because $A \to \Omega'$ is. We obtain an embedding of the Heyting algebra $\mathrm{Sub}(B)$ into $\mathrm{Sub}(C)$ and it carries B' to C', hence it embeds the Heyting algebra of subobjects in B above B' into the Boolean algebra of subobjects in C above C'. □

The product of a pair of pointed objects is naturally a pointed object. This ordinary product is not, however, the product as defined in ${}^{\blacksquare}\mathbb{A}$. The critical fact about H:

Lemma 7.6. *The ordinary product $H \times H$ is an object in $({}^{\blacksquare}\mathbb{A})/H$.*

Proof. It suffices to show that $\langle t, t \rangle : \mathbf{1} \to H \times H$ is stably co-discrete. In any distributive lattice the meet of two co-discrete elements is co-discrete. Since inverse images preserve intersections it follows that stably co-discrete subobjects are closed under finite intersection. Each 'axis' of $H \times H$ is stably co-discrete since it is the inverse image (under a projection map) of $t : \mathbf{1} \to H$. The 'base point' of a product is the intersection of the axes. □

The forgetful functor ${}^{\blacksquare}\mathbb{A} \to \mathbb{A}$ preserves pullbacks but not products. (The empty product is sent to Ω.) Given two pointed objects we may construct their 'pointed product' via the pullback diagram

$$\begin{tikzcd}
A \underset{\bullet}{\times} B \arrow[r] \arrow[d] & B \arrow[d] \\
A \arrow[r] & \Omega
\end{tikzcd}$$

revealing the pointed product as a subobject of the ordinary product. It may be characterized as the largest subobject of the ordinary product such that the projection maps restricted thereto are strict maps. More to our purpose, it may be re-characterized as the largest subobject which meets the union of the two axes in the base point.

If the ordinary product $A \times B$ lives in the boolean topos $^{\blacksquare}\mathbb{A}/H$, then it decomposes there as the 'pointed co-product' of the axes and their joint complement. Their complement must be the product as defined in $^{\blacksquare}\mathbb{A}/H$, thus $H \times H$ is the pointed co-product of two copies of H together with the pointed product of H with itself. But H is a subterminator and the latter must be another copy of H. We thus obtain:

Lemma 7.7. *Let $H \vee H \vee H$ be the threefold wedge of H, defined by the pushout*

$$\begin{array}{ccc} \mathbf{1}+\mathbf{1}+\mathbf{1} & \longrightarrow & H+H+H \\ \downarrow & & \downarrow \\ \mathbf{1} & \longrightarrow & H \vee H \vee H. \end{array}$$

The natural map

$$H \vee H \vee H \to H \times H,$$

the components of which are the injections of the two axes and the diagonal, is an isomorphism. □

This isomorphism in $\mathbb{A}$, $H \vee H \vee H \approx H \times H$, is equivalent to the first-order condition

$$(t=x) \quad or \quad (x=y) \quad or \quad (y=t),$$

(This condition on H may be translated as saying that it has at most two elements one of which is t. It certainly says that in the category of sets and the condition is preserved by any exact functor. Caution: H need not be K-finite nor is it always imbeddable in a K-finite object.)

This condition easily says that the following is a total ordering in H:

$$(x \leqslant y) \quad iff \quad (t=x) \text{ or } (x=y).$$

It remains only to show that this is a well-ordering. Let H' be well-supported subobject of H. Let $U = H' \cap \{t\}$. $\{t\}$ is stably co-discrete in H hence U is co-discrete in H'. There exists $U \subset H'' \subset H'$ such that $H'' \cup (H' \times U) = H'$ and $H'' \cap (H' \times U) = U$. ($H' \times U$ is a subobject of H' since U is a subterminator). We will show that H'' is the minimum point of H'. First, it is well-supported because support preserves unions and $\operatorname{spt}(H'') \cup \operatorname{spt}(H' \times U) = \operatorname{spt}(H'') \cup U = \operatorname{spt}(H') = \mathbf{1}$ and $U \subset \operatorname{spt}(H'')$. From $H'' \times U = U$ we may infer for any $x \in H''$ and $y \in H'$ that

$$(y=t) \quad implies \quad (x=t).$$

We still have

$$(t=x) \quad or \quad (x=y) \quad or \quad (y=t).$$

Hence

$$(t = x) \quad or \quad (x = y).$$

That is,

$$x \in H'' \; and \; y \in H' \quad imply \quad x \leqslant y.$$

Which together with the fact that H'' is well-supported says, of course, that H'' is the minimum point of H'. □

8. A diversion with two lemmas

We began the ordering of a choice object by choosing the bottom point, (just as is usually done in the classical case). We could have chosen any point. The first of our two diversionary lemmas says, in a particularly strong way, that the points of a choice object are indistinguishable.

Lemma 8.1. *For any pair of maps from* $\mathbf{1}$ *to a choice object,* $x, y : \mathbf{1} \to C$, *there is a 'transposition' on* C *that interchanges* x *and* y; *that is, there is an automorphsim* $\theta : C \to C$ *such that*

$$\theta^2 = 1_C, \qquad x\theta = y, \qquad y\theta = x,$$
$$C = \mathrm{Im}(x) \cup \mathrm{Im}(y) \cup \mathrm{Equalizer}(\theta, 1_C).$$

Proof. We have seen that any point of a choice object is co-discrete in its lattice of subobjects and that the intersection of two co-discrete subobjects is still co-discrete. If U is the intersection of the points, then C decomposes 'over U' as the union of the two points and a 'complement'. □

In order to state the second diversionary lemma, let C be a choice object with a chosen bottom point $0 : \mathbf{1} \to C$. View C as an object in $^{\blacksquare}\mathbb{A}$ well-order it there and, apply the forgetful functor $^{\blacksquare}\mathbb{A} \to \mathbb{A}$ to obtain a relation, $^{\blacksquare}\leqslant$, in $\mathbb{A}$. The forgetful functor preserves composition and intersection of relations, hence $^{\blacksquare}\leqslant$ is a partial ordering on C. It is not a total ordering. But

Lemma 8.2. *The relation defined by*

$$(x \leqslant y) \quad iff \quad (x = 0) \; or \; (x \, ^{\blacksquare}\!\leqslant y)$$

is a well-ordering on C.

The union of $^{\blacksquare}\leqslant$ and its reciprocal is not the maximal relation but it is the relation tabulated by the pointed product. If we use the decomposition of $C \times C$ (as described in the argument leading to 7.7) as the union of the two axes and the pointed product we obtain that $\leqslant$ is a total ordering. It is clearly the only total ordering that extends ${}^{\blacksquare}\leqslant$ and has 0 as bottom. By 6.6 it must therefore be the result of the construction used for 6.3. □

9. The classification of well-orderings in Grothendieck topoi

Throughout this section all topoi will be complete and locally small (as is any Grothendieck topos). Completeness, of course, is with respect to the underlying topos of sets. So is local smallness (all hom sets are sets). Note that in a topos, local smallness implies well-powered (all lattices of subobjects are sets). We consider first the boolean case.

The natural inclusion of the topos of sets into the given complete topos preserves power-objects in the boolean case, hence carries well-ordered sets to well-ordered objects. Put another way:

Lemma 9.1. *In a boolean complete topos any well-ordered co-power of* $\mathbf{1}$ *is a well-ordered object.* □

Any subobject of a well-ordered object is well-ordered which leads us to the definition of a *boolean canonical well-ordered object* as an object of the form $\Sigma_{\alpha<\beta} U_\alpha$ where $\{U_\alpha\}_{\alpha<\beta}$ is a well-ordered descending sequence of non-empty subterminators.

Lemma 9.2. *In a boolean complete locally small topos any well-ordered object is isomorphic to a unique boolean canonical well-ordered object.*

Proof. Given a well-ordered object C, recursively define $U_\alpha \to C$ as the minimum of the complement of $(\Sigma_{\gamma<\alpha} U_\gamma) \to C$. □

If we interpret this result in ${}^{\blacksquare}\mathbb{A}/H$ we are led to the following construction. Let $\{H_\alpha\}_{\alpha<\beta}$ be a descending sequence of non-trivial pointed subobjects of H. Define their 'wedge' W via the pushout diagram

$$\begin{array}{ccc} \Sigma_\beta \mathbf{1} & \longrightarrow & \Sigma_{\alpha<\beta} H_\alpha \\ \downarrow & & \downarrow \\ \mathbf{1} & \longrightarrow & W. \end{array}$$

An object of this form will be called *a well-supported canonical well-ordered object.*

Lemma 9.3. *In a complete locally small topos any well-supported well-ordered object is isomorphic to a unique canonical well-supported well-ordered object.* □

10. The classification of well-ordered objects in spatial topoi

Let X be a space. For each point $x \in X$ let α_x be a non-empty well-ordered set. If x is not a locally closed point, set $\alpha_x = 1$. Topologize the disjoint union, Y, of the α_x's so that the bottom section is open and its complement is discrete. The topology is unique. For a neighborhood basis of $\beta \in \alpha_x$ take those sections $U \to Y$ that send each $y \in U\backslash\{x\}$ to $0 \in \alpha_y$. If β is not 0, then x must be a closed point of U.

There is a sense in which all the 'glue' of Y lies along the base section. This is misleading: by Lemma 8.1 the same could be said for any global section. If X is T_1, then any point of Y lies in a global section.

Every well-supported well-ordered sheaf over X is uniquely of this form as will be readily seen by interpreting 9.3 in light of the forthcoming description of H.

The isomorphism of Lemma 7.7 is preserved by any exact functor, in particular by the stalk functors: each stalk, therefore, of H has at most two points, one of which is the bottom point. (In the category of sets it is clear that a pointed object whose cartesian square decomposes as the union of the two axes and the diagonal has at most two elements.) The bottom section is co-discrete. As above, the topology is unique. If $x \in X$ is not locally closed, then the stalk H_x is a single point. We need only one more fact: if $x \in X$ is locally closed, then H_x has more than one point. But H must classify all stably co-discrete subobjects and in a spatial topos all co-discrete subobjects are stably so (the inverse image under a local homeomorphism of a discrete subset is always discrete). Let Y be the sheaf as constructed above where α_x has two or one points depending on whether x is locally closed or not. The map to H that classifies the zero section is an isomorphism.

11. The classification of well-ordered objects for M-sets

Let M be a monoid. If M is a group, then all well-ordered M-sets have the trivial M-action. If M is a monoid with half-invertible elements, that is, if M has pairs of elements a, b such that $ab = 1$ but not $ba = 1$, then the terminator is the largest well-ordered M-set. If M is a monoid which is not a group but in which $ab = 1$ implies $ba = 1$ then all non-empty well-ordered M-sets arise as follows: let α be a well-ordered set; let each unit of M act trivially on α; let each non-unit act as the constant function with $0 \in \alpha$ as its constant value. This description follows immediately from 9.3 and the forthcoming description of H.

The forgetful functor from M-sets to sets is exact and just as for the stalk functors in the last section it must carry H to a set with at most two elements one of which is the base point. The base point is a sub-M-set, that is, it is a fixed point under the action of M. Clearly any unit in M must therefore act trivially on H. In the last section co-discrete and stably co-discrete coincided. Quite the opposite is the case for M-sets ($\mathbf{0}$ is co-discrete in $\mathbf{1}$ but stably co-discrete iff M is a group).

To explicate the structure of H in the case that M is not a group let $M \twoheadrightarrow H$ be an epimorphism of left M-sets (such must exist since H can not have more than one fixed point). Let $M' \subset M$ be the inverse image of the base point $t \in H$. M' must be a co-discrete subobject of M, that is, it must be a co-discrete element in the lattice of left-ideals. That lattice has a maximal proper element, namely the left-ideal $\mathfrak{A}$ of all elements that do not have left inverses. It has, therefore, precisely two co-discrete elements: M and $\mathfrak{A}$. H has two elements iff $\mathfrak{A}$ is stably co-discrete and in that case H is $M/\mathfrak{A}$, the result of collapsing $\mathfrak{A}$ to a point.

Lemma 11.1. *The maximal proper ideal of a monoid is stably co-discrete iff the monoid satisfies the condition*:

$$ab = 1 \quad \textit{implies} \quad ba = 1.$$

If the condition is violated, that is, if $a, b \in M$ are such that $ab = 1$ and not $ba = 1$, then consider the map $M \to M$ that sends x to xb. The inverse image of $\mathfrak{A}$ is a left-ideal strictly smaller (hence not co-discrete) than $\mathfrak{A}$: it does not contain the element a. But $a \in \mathfrak{A}$: if not, then it would have both a left and right inverse; hence it would be a unit; hence b would be a unit.

Conversely, if the condition holds, that is if $\mathfrak{A}$ is the set of non-units, then for any map of left-M-sets $A \to M$ the inverse image of $\mathfrak{A}$ contains $\mathfrak{A}A$ and it suffices to show that $\mathfrak{A}A$ is co-discrete in A. Define the relation $\equiv$ on the complement of $\mathfrak{A}A$ by $x \equiv y$ if there is a unit $u \in M$ such that $ux = y$. The lattice of subobjects above $\mathfrak{A}A$ is the lattice of $\equiv$-invariant subsets.

12. The classification of well-ordered objects in functor categories

Let $\mathbb{A}$ be a small category. Say that an object $A \in \mathbb{A}$ is *firm* if it does not appear as a proper retract, that is, if

$$A \to B \to A = 1_A \quad \textit{implies} \quad B \to A \to B = 1_B.$$

For each object A of $\mathbb{A}$ let α_A be a non-zero well-ordered set. If A and B are isomorphic, set $\alpha_A = \alpha_B$. If A is not firm, set $\alpha_A = 1$. Turn this assignment into a functor as follows: any isomorphism in $\mathbb{A}$ is sent to an order-isomorphism; any non-isomorphism is sent to a constant map with 0 as its constant value.

All well-ordered well-supported objects in the functor category are uniquely of this form. The argument is the usual conversion of the case for M-sets, that is, the usual conversion from domain categories with one object to those with many.

13. H lore

The object $H \subset \Omega$ was first described by Denis Higgs, albeit in the following very different way. Consider, first, the object, Σ, of permutations of Ω. Not the structure-preserving permutations, but all the permutations. Σ is constructed as a subobject of Ω^{Ω}. (There is only one structure-preserving permutation, indeed, there is only one permutation that preserves the universal subobject $t:\mathbf{1}\to\Omega$.) The uniqueness property on Ω easily implies that the map $\Sigma\to\Omega$ obtained by evaluating at t is monic. Σ is thus isomorphic with the 'orbit' of t. That orbit is, in fact, H. The isomorphism is, in fact, an isomorphism of groups: H is closed with respect to the double Heyting arrow operation defined on Ω; the very definition of 'co-discrete' says, in fact, that this binary operation on H is a group operation, indeed, one for which each element is its own inverse. As Higgs knew, H is an elementary 2-group.

The isomorphism of 7.7 said, in a particular way, that H has at most two elements. That particular way is a way that is preserved by exact functors (indeed, by near-exact functors). From any topos the collection of exact functors to well-pointed topoi is collectively faithful. In a well-pointed topos (one in which the terminator generates) the isomorphism of 7.7 says that the group is a quotient of $\mathbf{1}+\mathbf{1}$ and certainly that implies that the group is an elementary 2-group.

This reduction to the well-pointed case provides easy proofs for a number of other things. As an example, any map $H\to H$ that preserves the constant t is a group endomorphism. Since this is true in all topoi, the internally defined monoid of t-preserving endomorphisms $\mathscr{B}\subset H^H$ is not just a monoid but a ring (the addition coming from the group operation on H). It is a boolean ring. $\mathscr{B}$ is a fundamental boolean algebra in the topos. It is complete. There are two natural maps $H\to\mathscr{B}$ each of which generates $\mathscr{B}$ as a ring. (One of these maps sends H to the subobject of atoms, the other to co-atoms.)

There is a natural map from Ω to $\mathscr{B}$, the lambda-convert of the characteristic map of the co-discrete subobject $\Omega\vee H\subset\Omega\times H$. Does this reveal $\mathscr{B}$ as the reflection of the locale Ω into atomically generated locales? The condition that $\Omega{:}\to\mathscr{B}$ be faithful is an interesting one. For functor categories it is equivalent to the condition that every object in the domain category appear as a retract of a firm object, for spatial topoi it is equivalent to the condition (enjoyed by Spec(R) for any commutative ring R) that the locally closed points in the base space form a super-dense subset (one that is dense in each closed subset). In the first case it says that the functor category is equivalent to one for which each object in the domain category is firm. In the second case it says that the topos of sheaves is equivalent to one for which each point in the base space is locally closed (a $T_{1/2}$ space?).

The faithfulness of $\Omega\to\mathscr{B}$ is equivalent to the faithfulness of the co-geometric functor $\mathbb{A}\to({}^{\blacksquare}\mathbb{A}/H)$. We always have the faithful co-geometric representation $\mathbb{A}$ to the topos of double-negation-dense sheaves in ${}^{\blacksquare}\mathbb{A}$. There is a representation of topoi from there into $(^{\blacksquare}\mathbb{A})/H$. The faithfulness of $\Omega \to \mathcal{B}$ seems to be saying that the topos is more initimately bound to a boolean topos than usual.

$\mathcal{B}$ can be huge. Consider the case of M-sets where M is the monoid of natural numbers. We may regard the objects to be sets with successor functions, maps to be functions that preserve successors. $\mathcal{B}$ has a continuum of elements. Each element has precisely two predecessors (one of which may be itself). Its connected components are therefore characterized by the presence of finite orbits. There are a continuum of connected components with no finite orbits. For each natural number n, there are precisely $n^{-1} \sum_{d/n} \mu(n/d)2^d$ components each with its unique finite orbit of size n. (μ is the Euler μ-function.)

Let M be the free monoid, A^*, generated by a set A. The objects may be regarded as sets with an A-indexed family of successor functions. Cut down to the full subcategory, $\mathbb{A}$ of locally finite A^*-sets, that is, those sets in which each orbit is finite. This full subcategory is a topos, and if one is willing to remove initial states and outputs from the notion of finite automata, it is the topos of locally finite automata with A as input. $\mathcal{B}$ may be constructed as the family of Kleene-regular subsets of A^*. $\mathbb{A}/\mathcal{B}$ may be regarded as the topos of locally finite automata, this time each with a distinguished subset of 'successful' states.